# *On zeta functions of ternary zero forms*

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*To the memory of Takuro Shintani*

**Introduction.**

In this paper, we shall prove functional equations and the existence of analytic continuations of zeta functions attached to ternary zero forms (non-degenerate rational ternary quadratic forms with non-trivial rational zeros) and calculate the principal parts of the Laurent expansions at their poles.

Zeta functions of indefinite quadratic forms were introduced and closely investigated by C. L. Siegel [7]. In order to explain our problem more precisely, let us recall the definition of his zeta functions in a form slightly different from the original one (cf. [5, p. 155]). Let $Q(x)$ be a rational non-degenerate indefinite quadratic form on a $\boldsymbol{Q}$-vector space $V$ and $\mathfrak{S}$ be the matrix of $Q(x)$ with respect to a fixed basis of $V$. Denote by $O(Q)$ the orthogonal group of $Q$. The unit group of $Q$ is denoted by $\Gamma$: $\Gamma=O(Q)_{\boldsymbol{Z}}$. For any $x$ in $V_{\boldsymbol{Q}}$, put $G_x=\{g\in O(Q);\ gx=x\}$ and $\Gamma_x=\Gamma\cap G_x$. If $Q$ is not a ternary zero form, for an $x\in V_{\boldsymbol{Q}}$ such that $Q(x)\neq 0$, a fundamental domain of $\Gamma_x$ in $G_{x\boldsymbol{R}}$ has a finite volume $\mu(x)$ with respect to a Haar measure on $G_{x\boldsymbol{R}}$. Under a suitable normalization of Haar measures on $G_{x\boldsymbol{R}}$ ($x\in V_{\boldsymbol{Z}}$, $Q(x)\neq 0$), the Siegel zeta functions are defined by

$$\zeta_i(\mathfrak{S};\ s)=\sum_x{}'\mu(x)/|Q(x)|^s \qquad (i=1,\ 2) \tag{0-1}$$

where $x$ runs through all $\Gamma$-orbits in $\{x\in V_{\boldsymbol{Z}};\ \operatorname{sgn} Q(x)=(-1)^{i-1}\}$. However, if $Q$ is a ternary zero form, $\mu(x)$ fails to be finite for any $x$ in $V_{\boldsymbol{Q}}$ which satisfies the following condition:

(0-2) $\qquad -Q(x)\det\mathfrak{S}$ *is a square of some rational number.*

This causes a serious difficulty in the study of zeta functions of ternary zero forms. Because of this fact, Siegel restricted his consideration to $\zeta_i(\mathfrak{S};s)$ such that $\operatorname{sgn}(\det\mathfrak{S})=(-1)^{i-1}$. So it is a natural question to ask if it is possible to define $\mu(x)$ also for $x$ in $V_{\boldsymbol{Z}}$ satisfying (0-2) and $Q(x)\neq 0$ so that both of $\zeta_1(\mathfrak{S};s)$ and $\zeta_2(\mathfrak{S};s)$ defined by (0-1) have good analytic properties. The aim of the present paper is to give an affirmative answer to this question. The first attack in this direction was made by T. Shintani in [6] where he treated the special case $Q(x)=x_1x_3-x_2^2$. His success is based on his discovery that, for $Q(x)=x_1x_3-x_2^2$, $\zeta_i(\mathfrak{S}; s)$ can be regarded as residues of zeta functions in two variables associated with certain prehomogeneous vector space. This is the case also in the general setting and the method of partial Fourier transforms used in [4][1] enables us to generalize Shintani's result to arbitrary ternary zero forms.

This paper consists of two sections. In § 1, we shall present a definition of zeta functions of ternary zero forms and formulate our main result (Theorem 1). The second section is devoted to an investigation of certain Dirichlet series in two variables whose properties are summerized in Theorem 2. Theorem 1 will be easily derived from Theorem 2.

*Notation.* As usual we denote by $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$ and $\boldsymbol{Z}$ the field of complex numbers, the field of real numbers, the field of rational numbers and the ring of rational integers, respectively. For any non-zero real number $x$, $\operatorname{sgn} x$ is $x/|x|$. For any complex number $z$, we put $e[z]=\exp 2\pi\sqrt{-1}\, z$. The Riemann zeta function and the gamma function are denoted by $\zeta(s)$ and $\Gamma(s)$ respectively. For any finite dimensional real vector space $E$, $\mathcal{S}(E)$ is the space of all rapidly decreasing functions on $E$.

## § 1. Zeta functions of ternary zero forms.

**1.1.** Let $V$ be the vector space of $2\times 2$ (complex) symmetric matrices. Put $G=GL_2(\boldsymbol{C})$. Denote by $\rho$ the rational representation of $G$ on $V$ defined by $\rho(g)x=gx{}^tg$ $(g\in G,\ x\in V)$. We put

$$G_{\boldsymbol{R}}=GL_2(\boldsymbol{R}),\quad V_{\boldsymbol{R}}=V\cap M(2\,;\,\boldsymbol{R}),$$

$$G_{\boldsymbol{Q}}=GL_2(\boldsymbol{Q}),\quad V_{\boldsymbol{Q}}=V\cap M(2\,;\,\boldsymbol{Q}).$$

In the following we identify $V$ with its dual vector space via the symmetric bilinear form

$$\langle x,\ x^*\rangle=\operatorname{tr}(xJx^{*t}J),\quad J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\quad (x,\ x^*\in V). \tag{1-1}$$

Note that $\langle x,\ x\rangle=2\det x$.

Fix a lattice $M$ in $V_{\boldsymbol{Q}}$ and denote by $M^*$ the lattice dual to $M$ with respect to the inner product (1-1):

$$M^*=\{x^*\in V_{\boldsymbol{Q}}\,;\ \langle x,\ x^*\rangle\in\boldsymbol{Z}\quad \text{for all}\quad x\in M\}.$$

Let $\Gamma$ be the subgroup of $SL_2(\boldsymbol{Z})$ given by

$$\Gamma=\{\gamma\in SL_2(\boldsymbol{Z})\,;\ \rho(\gamma)M=M\}=\{\gamma\in SL_2(\boldsymbol{Z})\,;\ \rho(\gamma)M^*=M^*\}.$$

1) A summary of [4] is found in Proc. Japan Acad., 57A, 74-79, (1981).

Then the index of $\Gamma$ in $SL_2(\boldsymbol{Z})$ is finite.

Let $G_+$ be the connected component of the identity element of $G_{\boldsymbol{R}}$: $G_+=\{g\in GL_2(\boldsymbol{R});\ \det g>0\}$. We normallize a Haar measure $dg$ on $G_+$ by setting

$$dg=(\det g)^{-2}\prod_{i,j=1,2} dg_{ij} \qquad (g=(g_{ij}))\,.$$

Set

$$V_i=\{x\in V_{\boldsymbol{R}};\ \operatorname{sgn}(\det x)=(-1)^i\},\quad V_{i\boldsymbol{Q}}=V_i\cap V_{\boldsymbol{Q}} \qquad (i=1, 2)\,,$$

$$V'_{1\boldsymbol{Q}}=\{x\in V_{1\boldsymbol{Q}};\ \sqrt{-\det x}\notin\boldsymbol{Q}\}$$

and $$V''_{1\boldsymbol{Q}}=\{x\in V_{1\boldsymbol{Q}};\ \sqrt{-\det x}\in\boldsymbol{Q}\}\,.$$

Notice that $V_1$ is an open $G_+$-orbit in $V_{\boldsymbol{R}}$ and $V_2$ is a union of two open $G_+$-orbits in $V_{\boldsymbol{R}}$. Let $dx$ be the Euclidean measure normalized by $dx=dx_1dx_2dx_3$ $\left(x=\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3\end{pmatrix}\right)$. Then $\omega(x)=|\det x|^{-3/2}dx$ defines a $G_+$-invariant measure on $V_1$ and $V_2$.

For an $x\in V_{\boldsymbol{R}}$, put $G_{+x}=\{g\in G_+;\ \rho(g)x=x\}$ and $\Gamma_x=\Gamma\cap G_{+x}$. For an $x$ in $V_1\cup V_2$, we normalize a Haar measure $d\mu_x$ on $G_{+x}$ by the formula

$$\text{(1-2)}\qquad \int_{G_+}F(g)dg=\int_{G_+/G_{+x}}\omega(\rho(g)x)\int_{G_{+x}}F(gh)d\mu_x(h) \qquad (F\in L^1(G))\,.$$

If $x\in V'_{1\boldsymbol{Q}}\cup V_{2\boldsymbol{Q}}$, the volume

$$\text{(1-3)}\qquad \mu(x)=\int_{G_{+x}/\Gamma_x} d\mu_x$$

is finite. On the other hand, when $x$ is in $V''_{1\boldsymbol{Q}}$, since the integral (1-3) is divergent, we have to modify the definition of $\mu(x)$. An element $x=\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3\end{pmatrix}$ in $V_{\boldsymbol{Q}}$ is said to be *primitive* if $x_1$, $2x_2$, $x_3$ are integers and $(x_1, 2x_2, x_3)=1$. For any $x\in V''_{1\boldsymbol{Q}}$, express $x$ as $x=q\bar{x}$ where $q$ is a non-zero rational number and $\bar{x}$ is a primitive element in $V_{\boldsymbol{Q}}$. Set

$$\text{(1-4)}\qquad \mu(x)=2^{-3}\log(4|\det\bar{x}|) \qquad (x\in V''_{1\boldsymbol{Q}})\,.$$

It is clear that $\mu(\rho(\gamma)x)=\mu(x)$ for any $x\in V_{1\boldsymbol{Q}}\cup V_{2\boldsymbol{Q}}$ and any $\gamma\in\Gamma$.

Put $M_i=M\cap V_i$ and $M_i^*=M^*\cap V_i$ $(i=1, 2)$. The set $M_i$ and $M_i^*$ are $\rho(\Gamma)$-stable. Denote by $\Gamma\backslash M_i$ and $\Gamma\backslash M_i^*$ the set of all $\rho(\Gamma)$-orbits in $M_i$ and $M_i^*$ respectively. Set

$$\text{(1-5)}\qquad \zeta_i(M;\ s)=2^{1-i}\sum_{x\in\Gamma\backslash M_i}\mu(x)/|\det x|^s$$

$$\text{(1-5)*}\qquad \zeta_i(M^*;\ s)=2^{1-i}\sum_{x\in\Gamma\backslash M_i^*}\mu(x)/|\det x|^s \qquad (i=1, 2)\,.$$

These Dirichlet series are absolutely convergent for $\operatorname{Re} s>3/2$.

In order to describe the functional equations of $\zeta_i(M;s)$ and $\zeta_i(M^*;s)$, we introduce some notations. An element $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ in $G_+$ acts on the upper half plane $\mathfrak{H}$ and the Riemann sphere $\boldsymbol{C}\cup\{\infty\}$ by means of the fractional linear transformation $z\mapsto\sigma z=(dz-c)/(-bz+a)$. Since $\Gamma$ has a finite index in $SL_2(\boldsymbol{Z})$, the number of inequivalent cusps is finite. Let $\kappa_1,\cdots,\kappa_\nu$ ($\kappa_i\in\boldsymbol{Q}\cup\{\infty\}$) be a complete set of $\Gamma$-inequivalent cusps. For each $i$ ($1\leqq i\leqq\nu$), take a $\sigma_i\in SL_2(\boldsymbol{Z})$ such that $\kappa_i=\sigma_i\infty$ and set $y^{(i)}={}^t\sigma_i^{-1}\begin{pmatrix}1\\0\end{pmatrix}$. Put

$$\Gamma_\infty^{(i)}=\{\gamma\in\Gamma;\ {}^t\gamma^{-1}y^{(i)}=y^{(i)}\}.$$

For an integer $m$, we set

$$\Gamma_\infty(m)=\left\{\begin{pmatrix}1 & 0\\ mn & 1\end{pmatrix};\ n\in\boldsymbol{Z}\right\}. \tag{1-6}$$

Then there exists a positive integer $\delta_i$ such that $\Gamma_\infty^{(i)}=\sigma_i\Gamma_\infty(\delta_i)\sigma_i^{-1}$. Let $\lambda_i$ (resp. $\lambda_i^*$) the smallest positive rational number which is expressed as ${}^ty^{(i)}xy^{(i)}$ for some $x$ in $M$ (resp. $M^*$). Moreover we put $v(\Gamma\backslash\mathfrak{H})=\int_{\Gamma\backslash\mathfrak{H}}v^{-2}dudv$ ($z=u+iv$).

Now we can state our main theorem.

THEOREM 1. (1) *The functions $\zeta_i(M;s)$ and $\zeta_i(M^*;s)$ ($i=1,2$) have analytic continuations to meromorphic functions of $s$ in $\boldsymbol{C}$. They are holomorphic except at $s=1$ and $3/2$.*

(2) *They satisfy the following functional equations:*

$$\begin{aligned}\begin{pmatrix}\zeta_1(M^*;3/2-s)\\ \zeta_2(M^*;3/2-s)\end{pmatrix}&=v(M)2^{2-2s}\pi^{-2s+1/2}\Gamma(s-1/2)\Gamma(s)\\ &\quad\times\begin{pmatrix}\sin\pi s & 1\\ 0 & \cos\pi s\end{pmatrix}\begin{pmatrix}\zeta_1(M;s)\\ \zeta_2(M;s)\end{pmatrix}\\ &\quad+v(M)^{2-2s}2^{-2s}\pi^{-2s+1/2}\Gamma(s-1/2)\Gamma(s)\zeta(2s-1)\\ &\quad\times\left\{\sum_{i=1}^{\nu}\delta_i\lambda_i(\lambda_i^*/\lambda_i)^{2(1-s)}\right\}\begin{pmatrix}\left(\dfrac{\Gamma'}{\Gamma}(s)-\dfrac{\Gamma'}{\Gamma}(s-1/2)\right)\sin\pi s\\ -\pi\end{pmatrix}\end{aligned} \tag{1-7}$$

*where* $v(M)=\int_{V_R/M}dx$.

(3) *The principal parts of the Laurent expansions of $\zeta_1(M;s)$ and $\zeta_2(M;s)$ at $s=1$ and $s=3/2$ are given in the following table:*

| | | |
|---|---|---|
| $s=1$ | $\zeta_1(M;s)$ | $-2^{-3}v(M)^{-1}\left(\sum_{i=1}^{\nu}\delta_i\lambda_i\right)(s-1)^{-2}$ $+2^{-2}v(M)^{-1}\left\{\sum_{i=1}^{\nu}\delta_i\lambda_i\log(\lambda_i/4\pi)\right\}(s-1)^{-1}$ |
| | $\zeta_2(M;s)$ | $-2^{-3}\pi v(M)^{-1}\left(\sum_{i=1}^{\nu}\delta_i\lambda_i\right)(s-1)^{-1}$ |
| $s=\frac{3}{2}$ | $\zeta_1(M;s)$ | $2^{-2}\pi v(M)^{-1}v(\Gamma\backslash\mathfrak{H})(s-3/2)^{-1}$ |
| | $\zeta_2(M;s)$ | $2^{-2}\pi v(M)^{-1}v(\Gamma\backslash\mathfrak{H})(s-3/2)^{-1}$ |

REMARK. If $M=V_Q\cap M(2;\boldsymbol{Z})$, then $\Gamma=SL_2(\boldsymbol{Z})$, $\nu=1$, $\lambda_1=\lambda_1^*=\delta_1=1$ and $v(M)=1$. Moreover the zeta functions $\zeta_1(M^*;s)$, $\zeta_2(M^*;s)$, $\zeta_1(M;s)$ and $\zeta_2(M;s)$ coincide with $2^{2s-2}\xi_+(s)$, $2^{2s-2}\pi\xi_-(s)$, $2^{2s-2}\xi_+^*(s)$ and $2^{2s-2}\pi\xi_-^*(s)$, respectively, where $\xi_\pm(s)$ and $\xi_\pm^*(s)$ are the Dirichlet series studied by T. Shintani in [6] and our result is consistent with his.

**1.2.** We shall briefly indicate the relation between our zeta functions and the Siegel zeta functions of ternary zero forms. Take a $\boldsymbol{Z}$-basis $\{x^{(1)}, x^{(2)}, x^{(3)}\}$ of the lattice $M$ and let $a$ be a positive rational number. Put

$$\mathfrak{S}=\frac{a}{2}\begin{pmatrix}\langle x^{(1)}, x^{(1)}\rangle & \langle x^{(1)}, x^{(2)}\rangle & \langle x^{(1)}, x^{(3)}\rangle\\ \langle x^{(2)}, x^{(1)}\rangle & \langle x^{(2)}, x^{(2)}\rangle & \langle x^{(2)}, x^{(3)}\rangle\\ \langle x^{(3)}, x^{(1)}\rangle & \langle x^{(3)}, x^{(2)}\rangle & \langle x^{(3)}, x^{(3)}\rangle\end{pmatrix}.$$

The ternary quadratic form defined by the matrix $\mathfrak{S}$ is a zero form with signature (1, 2). Conversely every ternary zero form with signature (1, 2) can be obtained in this manner. Identify $V$ with $\boldsymbol{C}^3$ via the basis $\{x^{(1)}, x^{(2)}, x^{(3)}\}$. We may consider the group $\rho(\Gamma)\cong\Gamma/\{\pm1\}$ as a subgroup of $GL_3(\boldsymbol{C})$. Then $\rho(\Gamma)$ is contained in the unit group $O(\mathfrak{S})_{\boldsymbol{Z}}$ of $\mathfrak{S}$ and the index $j$ of $\rho(\Gamma)$ in $O(\mathfrak{S})_{\boldsymbol{Z}}$ is finite. It is easy to check that, for any $x\in V_{2Q}$, $\Gamma_x$ is a finite group and $\mu(x)=\frac{\pi}{2\sharp(\Gamma_x)}$. Hence we get

$$\zeta_2(M, s)=\frac{j}{2^4}a^{2s}\pi\zeta_1(\mathfrak{S};s)$$

and

$$\zeta_2(M^*;s)=\frac{j}{2^4}(2/a)^{2s}\pi\zeta_1(\mathfrak{S}^{-1};s)$$

where $\zeta_1(\mathfrak{S};s)$ (resp. $\zeta_1(\mathfrak{S}^{-1};s)$) is the Siegel zeta function of the ternary zero form with the matrix $\mathfrak{S}$ (resp. $\mathfrak{S}^{-1}$) (cf. [7], I).

## § 2. Certain Dirichlet series in two variables attached to the vector space of 2 by 2 symmetric matrices.

**2.1.** We keep the notation in § 1. Put $W=\boldsymbol{C}^2$. We consider elements in $W$ as column vectors. Set $V^\sim=V\oplus W$ and $G^\sim=GL_2(\boldsymbol{C})\times GL_1(\boldsymbol{C})$. Let $\rho^\sim$ be the representation of $G^\sim$ on $V^\sim$ defined by

$$\rho^\sim(g, t)(x, y)=(gx^tg, {}^tg^{-1}yt) \quad (g\in GL_2(\boldsymbol{C}),\ t\in GL_1(\boldsymbol{C}),\ x\in V,\ y\in W).$$

Put

$$P_1(x, y)={}^tyxy,\ P_2(x, y)=P_2(x)=\det x$$

and

$$S^\sim=\{(x, y)\in V^\sim;\ P_1(x, y)P_2(x, y)=0\}.$$

Then the triple $(G^\sim, \rho^\sim, V^\sim)$ is a prehomogeneous vector space with the singular set $S^\sim$ and the polynomials $P_1$ and $P_2$ are irreducible relative invariants of $(G^\sim, \rho^\sim, V^\sim)$ corresponding to the characters $\chi_1(g, t)=\chi_1(t)=t^2$ and $\chi_2(g, t)=\chi_2(g)=\det g^2$ respectively.

We consider the standard $\boldsymbol{Q}$-structure on $(G^\sim, \rho^\sim, V^\sim)$:

$$G_{\boldsymbol{R}}^\sim=GL_2(\boldsymbol{R})\times GL_1(\boldsymbol{R}),\ V_{\boldsymbol{R}}^\sim=V_{\boldsymbol{R}}\oplus W_{\boldsymbol{R}}=V_{\boldsymbol{R}}\oplus \boldsymbol{R}^2,$$

$$G_{\boldsymbol{Q}}^\sim=GL_2(\boldsymbol{Q})\times GL_1(\boldsymbol{Q}),\ V_{\boldsymbol{Q}}^\sim=V_{\boldsymbol{Q}}\oplus W_{\boldsymbol{Q}}=V_{\boldsymbol{Q}}\oplus \boldsymbol{Q}^2.$$

The identity component of $G_{\boldsymbol{R}}^\sim$ is denoted by $G_+^\sim$:

$$G_+^\sim=\{(g, t)\in G_{\boldsymbol{R}}^\sim;\ \det g>0,\ t>0\}.$$

Set

$$V_i^\sim=\{(x, y)\in V_{\boldsymbol{R}}^\sim-S_{\boldsymbol{R}}^\sim;\ \operatorname{sgn} P_2(x)=(-1)^i\}.$$

Let $M$ and $M^*$ be as in § 1. We define two lattices $L$ and $L^*$ in $V_{\boldsymbol{Q}}^\sim$ by $L=M\oplus \boldsymbol{Z}^2$ and $L^*=M^*\oplus \boldsymbol{Z}^2$. Put $L_i=L\cap V_i^\sim$ and $L_i^*=L^*\cap V_i^\sim$ $(i=1, 2)$. These sets are $\rho^\sim(\Gamma^\sim)$-stable subsets in $V_{\boldsymbol{Q}}^\sim$ where $\Gamma^\sim=\{(\gamma, 1)\in G_+^\sim;\ \gamma\in\Gamma\}$. The zeta functions associated with $(G^\sim, \rho^\sim, V^\sim)$ and $L$ are defined by the formula

$$\xi_i(L;\ s_1, s_2)=2^{-1}\sum_{(x,y)\in\Gamma^\sim\backslash L_i}|P_1(x, y)|^{-s_1}|P_2(x)|^{-s_2} \qquad (i=1, 2)$$

where we denote by $\Gamma^\sim\backslash L_i$ the set of all $\rho^\sim(\Gamma^\sim)$-orbits in $L_i$ (for the general theory of zeta functions associated with prehomogeneous vector spaces, see [4]). The Dirichlet series $\xi_i(L^*;\ s_1, s_2)$ $(i=1, 2)$ are defined in the same manner.

For any $\rho^\sim(\Gamma^\sim)$-stable subset $A$ of $L$ or $L^*$ and for an $F$ in $\mathcal{S}(V_{\boldsymbol{R}}^\sim)$, set

$$Z(F, A;\ s_1, s_2)=\int_{G_+^\sim/\Gamma^\sim}\chi_1(t)^{s_1}\chi_2(g)^{s_2}\sum_{(x,y)\in A-S^\sim}F(\rho^\sim(g, t)(x, y))dgd^\times t \tag{2-1}$$

where $dg$ is the Haar measure on $G_+$ normalized as in § 1 and $d^\times t=t^{-1}dt$. We normalize a Euclidean measure $dx$ on $V_{\boldsymbol{R}}$ as in § 1 and denote by $dy$ the standard Euclidean measure on $W_{\boldsymbol{R}}=\boldsymbol{R}^2$. For an $F\in\mathcal{S}(V_{\tilde{\boldsymbol{R}}})$ and for $i=1, 2$, put

$$\Psi_i(F;\, s_1,\, s_2)=\int_{V_{\tilde{i}}}|P_1(x,\, y)|^{s_1}|P_2(x)|^{s_2}F(x,\, y)dxdy\,.$$

The integrals $\Psi_i(F;\, s_1,\, s_2)$ $(i=1, 2)$ are absolutely convergent for $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>0$ and have analytic continuations to meromorphic functions of $(s_1, s_2)$ in $\boldsymbol{C}^2$ (cf.[1]).

The next lemma can be easily proved and we omit the proof.

LEMMA 2.1. *The integrals* $Z(F,\, L\,;\, s_1,\, s_2)$, $Z(F,\, L^*;\, s_1,\, s_2)$ *and the series* $\xi_i(L\,;\, s_1,\, s_2)$, $\xi_i(L^*;\, s_1,\, s_2)$ $(i=1, 2)$ *are convergent absolutely for* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>1$ *and the following identities hold*:

$$Z(F,\, L\,;\, s_1,\, s_2)=4^{-1}\sum_{i=1}^{2}\xi_i(L\,;\, s_1,\, s_2)\Psi_i(F;\, s_1-1,\, s_2-1)\,,$$

$$Z(F,\, L^*;\, s_1,\, s_2)=4^{-1}\sum_{i=1}^{2}\xi_i(L^*;\, s_1,\, s_2)\Psi_i(F;\, s_1-1,\, s_2-1)\,.$$

The analytic properties of $\xi_i(L\,;\, s_1,\, s_2)$ and $\xi_i(L^*;\, s_1,\, s_2)$ are given in the following theorem, on which the proof of Theorem 1 is based.

THEOREM 2. (1) *The functions* $\xi_i(L\,;\, s_1, s_2)$ *and* $\xi_i(L^*;\, s_1, s_2)$ $(i=1, 2)$ *have analytic continuations to meromorphic functions of* $(s_1,\, s_2)$ *in* $\boldsymbol{C}^2$ *which satisfy the following functional equations*:

$$\text{(2-2)}\quad \begin{pmatrix}\xi_1\!\left(L^*;\, s_1,\, \dfrac{3}{2}-s_1-s_2\right)\\ \xi_2\!\left(L^*;\, s_1,\, \dfrac{3}{2}-s_1-s_2\right)\end{pmatrix}=v(M)2^{2-s_1-2s_2}\pi^{-s_1-2s_2+1/2}\Gamma(s_2)\Gamma(s_1+s_2-1/2)$$
$$\times\begin{pmatrix}\sin(s_1+2s_2)\pi/2 & \sin \pi s_1/2\\ \cos \pi s_1/2 & \cos(s_1+2s_2)\pi/2\end{pmatrix}\begin{pmatrix}\xi_1(L\,;\, s_1,\, s_2)\\ \xi_2(L\,;\, s_1,\, s_2)\end{pmatrix},$$

$$\text{(2-3)}\quad \begin{pmatrix}\xi_1(L\,;\, 1-s_1,\, s_1+s_2-1/2)\\ \xi_2(L\,;\, 1-s_1,\, s_1+s_2-1/2)\end{pmatrix}$$
$$=2\pi^{-2s_1}\Gamma(s_1)^2\cos \pi s_1/2\begin{pmatrix}\cos \pi s_1/2 & 0\\ 0 & \sin \pi s_1/2\end{pmatrix}\begin{pmatrix}\xi_1(L\,;\, s_1,\, s_2)\\ \xi_2(L\,;\, s_1,\, s_2)\end{pmatrix}.$$

(2) *The functions*

$$(s_1-1)^2(s_2-1)(s_1+s_2-3/2)\xi_i(L\,;\, s_1,\, s_2)\qquad (i=1,\, 2)$$

*are entire functions of* $(s_1,\, s_2)$ *and*

$$\lim_{s_2\to 1}(s_2-1)\xi_1(L;\ s_1,\ s_2)=\lim_{s_2\to 1}(s_2-1)\xi_2(L;\ s_1,\ s_2)$$

$$=v(M)^{-1}\zeta(s_1)\zeta(2s_1)\sum_{i=1}^{\nu}\delta_i\lambda_i^{1-s_1}, \tag{2-4}$$

$$\lim_{s_2\to -s_1+3/2}(s_1+s_2-3/2)\begin{pmatrix}\xi_1(L;\ s_1,\ s_2)\\ \xi_2(L;\ s_1,\ s_2)\end{pmatrix}=v(M)^{-1}2^{1-s_1}\pi^{-s_1+1/2}$$

$$\times\Gamma(s_1-1/2)\zeta(s_1)\zeta(2s_1-1)\Big(\sum_{i=1}^{\nu}\delta_i\lambda_i^{s_1}\Big)\begin{pmatrix}\sin\pi s_1/2\\ \cos\pi s_1/2\end{pmatrix}, \tag{2-5}$$

$$\lim_{s_1\to 1}(s_1-1)^2\begin{pmatrix}\xi_1(L;\ s_1,\ s-s_1/2)\\ \xi_2(L;\ s_1,\ s-s_1/2)\end{pmatrix}$$

$$=v(M)^{1-2s}\zeta(2s-1)\Big(\sum_{i=1}^{\nu}\delta_i\lambda_i(\lambda_i^*/\lambda_i)^{2-2s}\Big)\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{2-6}$$

*Moreover the Dirichlet series* $\zeta_1(M;\ s)$ *and* $\zeta_2(M;\ s)$ *have analytic continuations to meromorphic functions of* $s$ *in* $\boldsymbol{C}$ *and the following formula holds*:

$$\lim_{s_1\to 1}\frac{\partial}{\partial s_1}\left\{(s_1-1)^2\begin{pmatrix}\xi_1(L;\ s_1,\ s-s_1/2)\\ \xi_2(L;\ s_1,\ s-s_1/2)\end{pmatrix}\right\}$$

$$=2\begin{pmatrix}\zeta_1(M;\ s)+\dfrac{2C-\log 2}{2}v(M)^{1-2s}\zeta(2s-1)\displaystyle\sum_{i=1}^{\nu}\delta_i\lambda_i(\lambda_i^*/\lambda_i)^{2-2s}\\ \zeta_2(M;\ s)\end{pmatrix} \tag{2-7}$$

*where* $C$ *is the Euler constant.*

REMARKS. (1) By replacing $L$, $M$, $\lambda_i$ and $\lambda_i^*$ by $L^*$, $M^*$, $\lambda_i^*$ and $\lambda_i$ respectively, we obtain the residue formulas for $\xi_i(L^*;\ s_1,\ s_2)$ $(i=1,\ 2)$.

(2) The first assertion of the theorem was proved in [4] §7 if $\Gamma=SL_2(\boldsymbol{Z})$.

(3) In [6], Shintani treated the special case $M=V\cap M(2;\ \boldsymbol{Z})$ by a method somewhat different from ours and showed Theorem 2 except the assertion that $(s_1-1)^2(s_2-1)(s_1+s_2-3/2)\xi_i(L;\ s_1,\ s_2)$ are entire founctions.

PROOF OF THEOREM 1. Since $(s_1-1)^2(s-s_1/2-1)(s+s_1/2-3/2)\times\xi_i(L;\ s_1,\ s-s_1/2)$ are entire functions of $(s_1,\ s)$, the formula (2-7) implies that $(s-1)^2\cdot(s-3/2)\zeta_i(M;\ s)\ (i=1,\ 2)$ are entire functions of $s$ in $\boldsymbol{C}$. It follows from (2-4), (2-6) and (2-7) that the residues of $\zeta_1(M;\ s)$ and $\zeta_2(M;\ s)$ at $s=3/2$ are equal to $(\pi^2/12)v(M)^{-1}\sum_{i=1}^{\nu}\delta_i$. Since $\sum_{i=1}^{\nu}\delta_i=[SL_2(\boldsymbol{Z}):\Gamma]$ and $v(\Gamma\backslash\mathfrak{H})=(\pi/3)[SL_2(\boldsymbol{Z}):\Gamma]$, we get $\lim_{s\to 3/2}(s-3/2)\zeta_i(M;\ s)=(\pi/4)v(M)^{-1}v(\Gamma\backslash\mathfrak{H})$. The formulas of the principal parts of the Laurent expansions of $\zeta_1(M;\ s)$ and $\zeta_2(M;\ s)$ at $s=1$ are immediate consequences of (2-5), (2-6), (2-7) and $\Gamma'(1/2)/\Gamma(1/2)=-\log 4-C$. By taking the residues at $s_1=1$ of both sides of the equality obtained from (2-2) by substituting $(s_1, s_2)$ by $(s_1, s-s_1/2)$, we get the functional equation (1-7). q.e.d.

**2.2.** The rest of this paper is devoted to the proof of Theorem 2.

Let $Z_+^{(1)}(F, A; s_1, s_2)$, $\hat{Z}_+^{(1)}(F, A; s_1, s_2)$ and $Z_+^{(2)}(F, A; s_1, s_2)$ be the integrals obtained from $Z(F, A; s_1, s_2)$ by restricting the domain of integration in (2-1) to $\{(g, t)\in G_+^{\sim}/\Gamma^{\sim};\ \chi_1(t)\geqq 1\}$, $\{(g, t)\in G_+^{\sim}/\Gamma^{\sim};\ \chi_1(t)\geqq\chi_2(g)\}$ and $\{(g, t)\in G_+^{\sim}/\Gamma^{\sim};\ \chi_2(g)\geqq 1\}$, respectively. Then the following lemma is an immediate consequence of Lemma 2.1.

LEMMA 2.2. *Let $A$ be a $\rho^{\sim}(\Gamma^{\sim})$-stable subset of $L$ or $L^*$ and $F$ be a function in $\mathcal{S}(V_{\boldsymbol{R}}^{\sim})$.*

(1) *When* $\mathrm{Re}\, s_2>1$, $Z_+^{(1)}(F, A; s_1, s_2)$ *is absolutely convergent.*

(2) *When* $\mathrm{Re}\, s_1+\mathrm{Re}\, s_2>1$ *and* $\mathrm{Re}\, s_2>1$, $\hat{Z}_+^{(1)}(F, A; s_1, s_2)$ *is absolutely convergent.*

(3) *When* $\mathrm{Re}\, s_1>1$, $Z_+^{(2)}(F, A; s_1, s_2)$ *is absolutely convergent.*

Let $B$ be the subgroup of $G$ consisting of all non-degenerate lower triangular matrices. The restriction of the representation $\rho$ of $G$ to $B$ is also denoted by the same symbol $\rho$. Then the triple $(B, \rho, V)$ is a prehomogeneous vector space with the singular set $S=\{x\in V;\ x_1P_2(x)=0\}$. This prehomogeneous vector space was closely investigated by T. Shintani [6]. The next two integrals were introduced by him:

$$\Phi_i(f;\ s_1, s_2)=\int_{V_i}|x_1|^{s_1}|P_2(x)|^{s_2}f(x)dx\,,$$

$$\Sigma(f;\ s)=\int_{R^2}|a|^{s-1}f\left(\begin{pmatrix} a & b \\ b & b^2/a\end{pmatrix}\right)da\,db \qquad (f\in\mathcal{S}(V_R),\ i=1, 2)\,.$$

The integral $\Sigma(f;\ s)$ (resp. $\Phi_i(f;\ s_1, s_2)$) converges absolutely for $\mathrm{Re}\, s>1$ (resp. $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>0$) and is continued to a meromorphic function of $s$ (resp. $(s_1, s_2)$) in $\boldsymbol{C}$ (resp. $\boldsymbol{C}^2$). Let $B_+$ be the connected component of the identity element of $B_R=B\cap GL_2(\boldsymbol{R})$. We normalize a right invariant measure $db$ on $B_+$ by setting

$$db=t_1^{-2}t_2^{-1}dt_1dt_2du \qquad \left(b=\begin{pmatrix} t_1 & 0 \\ u & t_2\end{pmatrix}\in B_+\right).$$

Take a positive integer $\delta$ and let $\mathfrak{M}$ be a $\rho(\Gamma_\infty(\delta))$-invariant lattice in $V_Q$ (for the definition of $\Gamma_\infty(\delta)$, see (1-6)). Denote by $\mathfrak{M}^*$ the lattice dual to $\mathfrak{M}$ with respect to the inner product (1-1). Set

$$I(f, \mathfrak{M};\ s_1, s_2)=\int_{B_+/\Gamma_\infty(\delta)}t_1^{2(s_1+s_2)}t_2^{2s_2}\sum_{x\in\mathfrak{M}-S}f(\rho(b)x)db$$

($f\in\mathcal{S}(V_{\boldsymbol{R}})$, $(s_1, s_2)\in\boldsymbol{C}^2$). This integral is absolutely convergent for $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>1$ (cf. Shintani [6], Lemma 3). Also set

$$I_+(f, \mathfrak{M}\,;\, s_1, s_2)=\int_{B_+/\Gamma_\infty(\delta),\, t_1t_2\geqq 1} t_1^{2(s_1+s_2)}t_2^{2s_2}\sum_{x\in\mathfrak{M}-S} f(\rho(b)x)db\,.$$

Then $I_+(f, \mathfrak{M}\,;\, s_1, s_2)$ defines a holomorphic function of $(s_1, s_2)$ in the domain $\{(s_1, s_2)\in\boldsymbol{C}^2\,;\ \mathrm{Re}\, s_1>1\}$.

Put $\mathfrak{M}_0=\{x\in\mathfrak{M}\,;\ x_1\neq 0,\ P_2(x)=0\}$. The set $\mathfrak{M}_0$ is $\rho(\Gamma_\infty(\delta))$-stable. Denote by $\Gamma_\infty(\delta)\backslash\mathfrak{M}_0$ the set of all $\rho(\Gamma_\infty(\delta))$-orbits in $\mathfrak{M}_0$. Set

$$\zeta_0(\mathfrak{M}\,;\, s)=\sum_{x\in\Gamma_\infty(\delta)\backslash\mathfrak{M}_0}|x_1|^{-s}\,.$$

It is easy to check that $\zeta_0(\mathfrak{M}\,;\, s)$ converges absolutely for $\mathrm{Re}\, s>1$ and coincides with $\zeta(s)\zeta(2s-1)/\zeta(2s)$ up to an elementary factor expressed in terms of exponetial functions. Hence $\zeta_0(\mathfrak{M}\,;\, s)$ has an analytic continuation to a meromorphic function of $s$ in $\boldsymbol{C}$.

We define the Fourier transform $f^*$ of $f\in\mathcal{S}(V_{\boldsymbol{R}})$ by the formula

$$f^*(x^*)=\int_{V_{\boldsymbol{R}}} f(x)\mathrm{e}[\langle x, x^*\rangle]dx$$

where the inner product $\langle\ ,\ \rangle$ is given by (1-1).

Put

$$\lambda=\min\{|x_1|\,;\ x\in\mathfrak{M},\ x_1\neq 0\}$$

and

$$\lambda^*=\min\{|x_1|\,;\ x\in\mathfrak{M}^*,\ x_1\neq 0\}\,.$$

The following lemma is a slight generalization of Lemma 4 of [6] and one can prove it in the same manner as in [6].

LEMMA 2.3. *If* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>1$, *then*

$$\begin{aligned}I(f, \mathfrak{M}\,;\, s_1, s_2)&=I_+(f, \mathfrak{M}\,;\, s_1, s_2)+v(\mathfrak{M})^{-1}I_+(f^*, \mathfrak{M}^*\,;\, s_1, 3/2-s_1-s_2)\\&+\frac{v(\mathfrak{M})^{-1}}{8(s_1+s_2-3/2)}\zeta_0(\mathfrak{M}^*\,;\, s_1)\Sigma(f^*\,;\, s_1-1)-\frac{1}{8s_2}\zeta_0(\mathfrak{M}\,;\, s_1)\Sigma(f\,;\, s_1-1)\\&+\frac{v(\mathfrak{M})^{-1}\delta}{4(s_2-1)}\lambda^{1-s_1}\zeta(s_1)\{\Phi_1(f\,;\, s_1-1, 0)+\Phi_2(f\,;\, s_1-1, 0)\}\\&-\frac{\delta}{2(s_1+s_2-1/2)}\lambda^{*1-s_1}\zeta(s_1)\{\Phi_1(f^*\,;\, s_1-1, 0)+\Phi_2(f^*\,;\, s_1-1, 0)\}\,.\end{aligned}$$

Notice that any $x\in V_{\boldsymbol{R}}$ of rank 1 is written uniquely as

$$x=k_\theta\begin{pmatrix} v & 0\\ 0 & 0\end{pmatrix}{}^t k_\theta,\ k_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\qquad (v\in\boldsymbol{R}^\times,\ 0\leqq\theta<\pi)\,.$$

Set

$$\Omega=\{(x,\ y)\in V_{\boldsymbol{R}}^{\sim};\ \mathrm{rank}\ x=1,\ P_1(x,\ y)\neq 0\}$$

and

$$\Sigma^{\sim}(F;\ s)=\int_{\Omega}|P_1(x,\ y)|^s F(x,\ y)dvd\theta dy \qquad (F\in\mathcal{S}(V_{\boldsymbol{R}}^{\sim})).$$

When $\mathrm{Re}\, s>0$, the integral $\Sigma^{\sim}(F;\ s)$ converges absolutely and has an analytic continuation to a meromorphic function of $s$ in $\boldsymbol{C}$.

Define a partial Fourier transform $F^*$ of $F\in\mathcal{S}(V_{\boldsymbol{R}}^{\sim})$ with respect to $V_{\boldsymbol{R}}$ by setting

$$F^*(x^*,\ y)=\int_{V_{\boldsymbol{R}}}F(x,\ y)\,\mathrm{e}[\langle x,\ x^*\rangle]dx.$$

LEMMA 2.4. *Let $\lambda_i$, $\lambda_i^*$ and $\sigma_i$ be as in* § 1. *If* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>1$ *and* $F\in\mathcal{S}(V_{\boldsymbol{R}}^{\sim})$, *then*

$$\begin{aligned}Z(F,\ L\,;\ s_1,\ s_2)&=Z_+^{(2)}(F,\ L\,;\ s_1,\ s_2)+v(M)^{-1}Z_+^{(2)}(F^*,\ L^*;\ s_1,\ 3/2-s_1-s_2)\\&+\frac{v(M)^{-1}\zeta(2s_1)}{8(s_1+s_2-3/2)}\left\{\sum_{i=1}^{\nu}\zeta_0(\rho(\sigma_i)^{-1}M^*;\ s_1)\right\}\Sigma^{\sim}(F^*;\ s_1-1)\\&-\frac{\zeta(2s_1)}{8s_2}\left\{\sum_{i=1}^{\nu}\zeta_0(\rho(\sigma_i)^{-1}M;\ s_1)\right\}\Sigma^{\sim}(F;\ s_1-1)\\&+\frac{v(M)^{-1}}{4(s_2-1)}\zeta(2s_1)\zeta(s_1)\left(\sum_{i=1}^{\nu}\delta_i\lambda_i^{1-s_1}\right)\{\Psi_1(F;\ s_1-1,\ 0)\\&+\Psi_2(F;\ s_1-1,\ 0)\}-\frac{\zeta(2s_1)\zeta(s_1)}{2(s_1+s_2-1/2)}\left(\sum_{i=1}^{\nu}\delta_i\lambda_i^{*1-s_1}\right)\\&\times\{\Psi_1(F^*;\ s_1-1,\ 0)+\Psi_2(F^*;\ s_1-1,\ 0)\}.\end{aligned}$$

PROOF. Since $\boldsymbol{Z}^2-\{0\}=\bigcup_{i=1}^{\nu}\{n^t\gamma^{-1}y^{(i)};\ n\in\boldsymbol{Z},\ n>0,\ \gamma\in\Gamma/\Gamma_{\infty}^{(i)}\}$, we have

$$Z(F,\ L\,;\ s_1,\ s_2)=\zeta(2s_1)\sum_{i=1}^{\nu}\int_{G_+^{\sim}/\Gamma_{\infty}^{(i)}}\chi_1(t)^{s_1}\chi_2(g)^{s_2}\sum_x{}'F(\rho^{\sim}(g,\ t)(x,\ y^{(i)}))dgd^{\times}t$$

where the summation is taken over all $x$ in $M$ such that $P_1(x,\ y^{(i)})P_2(x)\neq 0$. Put

$$F_0(x,\ y)=\int_0^{2\pi}F(\rho^{\sim}(k_\theta,\ 1)(x,\ y))d\theta,\quad k_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

Then

$$Z(F,\ L\,;\ s_1,\ s_2)=\zeta(2s_1)\sum_{i=1}^{\nu}\int_0^{\infty}t^{2s_1}I(F_0(*,\ \begin{pmatrix}t\\0\end{pmatrix}),\ \rho(\sigma_i)^{-1}M;\ s_1,\ s_2)d^{\times}t.$$

It is easy to check that

$$\int_0^\infty t^{2s_1}\Sigma(F_0(*, \begin{pmatrix} t \\ 0 \end{pmatrix}); s_1-1)d^\times t = \Sigma^\sim(F; s_1-1)$$

and

$$\int_0^\infty t^{2s_1}\Phi_j(F_0(*, \begin{pmatrix} t \\ 0 \end{pmatrix}); s_1-1, 0)d^\times t = \Psi_j(F; s_1-1, 0).$$

Moreover the lattice dual to $\rho(\sigma_i)^{-1}M$ coincides with $\rho(\sigma_i)^{-1}M^*$ and $(F^*)_0(x, y) = (F_0)^*(x, y)$. Now the lemma follows immediately from Lemma 2.3.

q. e. d.

REMARK. It can be proved that

$$\left.\begin{array}{l} \sum_{i=1}^{\nu}\zeta_0(\rho(\sigma_i)^{-1}M; s) \\ \\ \sum_{i=1}^{\nu}\zeta_0(\rho(\sigma_i)^{-1}M^*; s) \end{array}\right\} = 2\zeta(s)\zeta(2s-1)/\zeta(2s)\times\left\{\begin{array}{l} \sum_{i=1}^{\nu}\delta_i\lambda_i^{*s} \\ \\ \sum_{i=1}^{\nu}\delta_i\lambda_i^s. \end{array}\right.$$

Put $M_1'=M\cap V_{1Q}'$, $M_1''=M\cap V_{1Q}''$, $M_2'=M_2$, $L_1'=(M_1'\oplus \boldsymbol{Z}^2)\cap V_1^\sim$, $L_1''=(M_1''\oplus \boldsymbol{Z}^2)\cap V_1^\sim$ and $L_2'=L_2$. It is obvious that $L_1'$ and $L_1''$ are $\rho^\sim(\Gamma^\sim)$-stable subsets of $L$. Set

$$\zeta_i(M_i'; s) = 2^{1-i}\sum_{x\in\Gamma\backslash M_i'}\mu(x)/|\det x|^s \qquad (i=1, 2)$$

and

$$\zeta_1(M_1''; s) = \sum_{x\in\Gamma\backslash M_1''}\mu(x)/|\det x|^s.$$

Then $\zeta_1(M; s)=\zeta_1(M_1'; s)+\zeta_1(M_1''; s)$ and $\zeta_2(M; s)=\zeta_2(M_2'; s)$. For an $F\in\mathcal{S}(V_{\boldsymbol{R}}^\sim)$, let $\hat{F}$ be the partial Fourier transform of $F$ with respect to $W_R=\boldsymbol{R}^2$ which is defined by

$$\hat{F}(x, y^*) = \int_{\boldsymbol{R}^2}F(x, y)\,\mathrm{e}[\langle y, y^*\rangle]dy$$

where $\langle y, y^*\rangle = {}^t yJy^*$ (for the definition of $J$, see (1-1)).

LEMMA 2.5. (1) *When* $\mathrm{Re}\, s_1>1$ *and* $\mathrm{Re}\, s_2>3/2$,

$$\begin{aligned} Z(F, L_i'; s_1, s_2) &= Z_+^{(1)}(F, L_i'; s_1, s_2) + \hat{Z}_+^{(1)}(\hat{F}, L_i'; 1-s_1, s_1+s_2-1/2) \\ &\quad + \frac{1}{2(s_1-1)}\zeta_i(M_i'; s_2+1/2)\Psi_i(F; 0, s_2-1) \\ &\quad - \frac{1}{2s_1}\zeta_i(M_i'; s_2)\Psi_i(\hat{F}; 0, s_2-3/2) \qquad (i=1, 2). \end{aligned}$$

(2) *The formula obtained by replacing* $F$, $\hat{F}$, $s_1$ *and* $s_2$ *in the right hand side of the equality above by* $\hat{F}$, $F$, $1-s_1$ *and* $s_1+s_2-1/2$, *respectively, holds for* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>1$.

PROOF. We prove only the first assertion. Since $L_i'-S^{\sim}=L_i'=M_i'\times(\boldsymbol{Z}^2-\{0\})$, we have, by the Poisson summation formula,

$$\sum_{(x,y)\in L_i'} F(\rho^{\sim}(g,t)(x,y))=t^{-2}\det g\sum_{(x,y)\in L_i'}\hat{F}(\rho^{\sim}(g,(\det g)/t)(x,y))$$
$$+t^{-2}\det g\sum_{x\in M_i'}\hat{F}(gx{}^tg,0)-\sum_{x\in M_i'}F(gx{}^tg,0).$$

Hence

$$Z(F,L_i';s_1,s_2)=Z_+^{(1)}(F,L_i';s_1,s_2)+\hat{Z}_+^{(1)}(\hat{F},L_i';1-s_1,s_1+s_2-1/2)$$
$$+\int_{G_+/\Gamma}(\det g)^{2s_2+1}\sum_{x\in M_i'}\hat{F}(gx{}^tg,0)dg\int_0^1 t^{2(s_1-1)}d^{\times}t$$
$$-\int_{G_+/\Gamma}(\det g)^{2s_2}\sum_{x\in M_i'}F(gx{}^tg,0)\,dg\int_0^1 t^{2s_1}d^{\times}t.$$

By (1-2) and (1-3), the last two terms of the right hand side of this equality are rewritten as

$$\frac{1}{2(s_1-1)}\zeta_i(M_i';s_2+1/2)\Phi_i(\hat{F}(*,0);0,s_2-1)$$

and

$$-\frac{1}{2s_1}\zeta_i(M_i';s_2)\Phi_i(F(*,0);0,s_2-3/2).$$

Therefore the first part of the lemma is an immediate consequence of the formulas

$$\Psi_i(F;0,s_2-1)=\Phi_i(\hat{F}(*,0);0,s_2-1)$$

and

$$\Psi_i(\hat{F};0,s_2-3/2)=\Phi_i(F(*,0);0,s_2-3/2).$$

q.e.d.

Next consider the integral $Z(F,L_1'';s_1,s_2)$ $(F\in\mathcal{S}(V_{\tilde{R}}))$. For a lattice $N$ in $W_{\boldsymbol{Q}}=\boldsymbol{Q}^2$, we introduce an integral $J(f,N;s)$ $(f\in\mathcal{S}(W_{\boldsymbol{R}}),\ s\in\boldsymbol{C})$ which plays an important role in the proof of a formula for $Z(F,L_1'';s_1,s_2)$ analogous to that given in Lemma 2.5. The integral $J(f,N;s)$ is defined by

$$J(f,N;s)=\int_0^{\infty}\int_0^{\infty}|uv|^s\sum_{y\in N'}f(uy_1,vy_2)d^{\times}ud^{\times}v$$

where $N'=\{y\in N;\ y_1y_2\neq 0\}$. This integral converges absolutely for $\mathrm{Re}\,s>1$. Set

$$J_+(f,N;s)=\iint_{\substack{u>0,v>0\\ uv\geq 1}}|uv|^s\sum_{y\in N'}f(uy_1,vy_2)d^{\times}ud^{\times}v.$$

Then $J_+(f, N; s)$ is absolutely convergent for any $s$ and represents an entire function of $s$.

For an $f\in\mathcal{S}(W_{\boldsymbol{R}})$, the Fourier transform $\hat{f}$ of $f$ is defined by

$$\hat{f}(y^*)=\int_{W_{\boldsymbol{R}}} f(y)\,\mathrm{e}[\langle y, y^*\rangle]dy, \quad \langle y, y^*\rangle={}^t yJy^*.$$

Put

$$\hat{N}=\{y^*\in W_{\boldsymbol{Q}}\,;\ {}^t yJy^*\in\boldsymbol{Z} \quad \text{for all} \quad y\in N\}.$$

Set

$$\rho_1(N)=\min\left\{u\in\boldsymbol{Q}\,;\ u>0,\ \begin{pmatrix}u\\0\end{pmatrix}\in N\right\}$$

and

$$\rho_2(N)=\min\left\{u\in\boldsymbol{Q}\,;\ u>0,\ \begin{pmatrix}0\\u\end{pmatrix}\in N\right\}.$$

We also define $\rho_1(\hat{N})$ and $\rho_2(\hat{N})$ similarly.

LEMMA 2.6. *If* $\operatorname{Re} s>1$ *and* $f\in\mathcal{S}(W_{\boldsymbol{R}})$,

$$\begin{aligned}J(f, N; s)=&J_+(f, N; s)+v(N)^{-1}J_+(\hat{f}, \hat{N}; 1-s)+\frac{v(N)^{-1}}{(s-1)^2}\hat{f}(0)+\frac{f(0)}{s^2}\\&+\frac{v(N)^{-1}}{s-1}[\hat{f}(0)\{\log(\rho_1(\hat{N})\rho_2(\hat{N}))+2C\}+\langle\log|uv|, f\rangle]\\&-\frac{1}{s}[f(0)\{\log(\rho_1(N)\rho_2(N))+2C\}+\langle\log|uv|, \hat{f}\rangle]\end{aligned}$$

*where* $v(N)=\int_{W_{\boldsymbol{R}}/N} dy$, *$C$ is the Euler constant and*

$$\langle\log|uv|, f\rangle=\iint_{\boldsymbol{R}^\times\boldsymbol{R}^\times}\log|uv|f(u, v)\,du\,dv.$$

PROOF. We have, by the Poisson summation formula,

$$\begin{aligned}J(f, N; s)=&J_+(f, N; s)+v(N)^{-1}J_+(\hat{f}, \hat{N}; 1-s)\\&+\iint_{uv\le 1}|uv|^s S(f; u, v)\,d^\times u\,d^\times v\end{aligned}$$

where

$$S(f; u, v)=v(N)^{-1}(uv)^{-1}\sum_{y\in\hat{N}-\hat{N}'}\hat{f}(u^{-1}y_1, v^{-1}y_2)-\sum_{y\in N-N'}f(uy_1, vy_2).$$

We shall calculate the last integral of the equality above by dividing it to the following three integrals:

$$J_1=\iint_{0<u,v\leq 1},\ J_2=\iint_{1\leq u\leq v^{-1}},\ J_3=\iint_{1\leq v\leq u^{-1}}.$$

For $f\in\mathcal{S}(W_R)$, put

$$f^{(1)}(y_1^*,\ y_2)=\int_R f(y_1,\ y_2)\,\mathrm{e}[y_1y_1^*]dy_1$$

and

$$f^{(2)}(y_1,\ y_2^*)=\int_R f(y_1,\ y_2)\,\mathrm{e}[y_2y_2^*]dy_2.$$

Then we get

$$\begin{aligned}
&S(f;\ u,\ v)\\
&\quad=v(N)^{-1}(uv)^{-1}\Big\{\sum_{m\in Z-\{0\}}\hat{f}(u^{-1}\rho_1(\hat{N})m,\ 0)+\sum_{m\in Z-\{0\}}\hat{f}(0,\ v^{-1}\rho_2(\hat{N})m)+\hat{f}(0)\Big\}\\
&\qquad-\{u^{-1}\rho_1(N)^{-1}\sum_{m\in Z}f^{(1)}(u^{-1}\rho_1(N)^{-1}m,\ 0)\\
&\qquad+v^{-1}\rho_2(N)^{-1}\sum_{m\in Z}f^{(2)}(0,\ v^{-1}\rho_2(N)^{-1}m)-f(0)\},
\end{aligned}$$

hence,

$$\begin{aligned}
J_1=&\frac{v(N)^{-1}}{s-1}\Big\{\int_1^\infty u^{1-s}\sum_{m\in Z-\{0\}}\hat{f}(u\rho_1(\hat{N})m,\ 0)d^\times u\\
&+\int_1^\infty v^{1-s}\sum_{m\in Z-\{0\}}\hat{f}(0,\ v\rho_2(\hat{N})m)d^\times v\Big\}\\
&-\frac{1}{s}\Big\{\rho_1(N)^{-1}\int_1^\infty u^{1-s}\sum_{m\in Z-\{0\}}f^{(1)}(u\rho_1(N)^{-1}m,\ 0)d^\times u\\
&+\rho_2(N)^{-1}\int_1^\infty v^{1-s}\sum_{m\in Z-\{0\}}f^{(2)}(0,\ v\rho_2(N)^{-1}m)d^\times v\Big\}\\
&+\frac{v(N)^{-1}}{(s-1)^2}\hat{f}(0)+\frac{1}{s^2}f(0)-\frac{1}{(s-1)s}\{\rho_1(N)^{-1}f^{(1)}(0)+\rho_2(N)^{-1}f^{(2)}(0)\}.
\end{aligned}$$

We can calculate $J_2$ and $J_3$ in the same manner, and obtain

$$\begin{aligned}
J(f,\ N;\ s)=&J_+(f,\ N;\ s)+v(N)^{-1}J_+(\hat{f},\ \hat{N};\ 1-s)+\frac{v(N)^{-1}}{(s-1)^2}\hat{f}(0)+\frac{1}{s^2}f(0)\\
&+\frac{v(N)^{-1}}{s-1}\Big\{\int_1^\infty\sum_{m\neq 0}\hat{f}(0,\ v\rho_2(\hat{N})m)d^\times v\\
&+\rho_2(\hat{N})^{-1}\int_1^\infty v\sum_{m\neq 0}f^{(2)}(v\rho_2(\hat{N})^{-1}m,\ 0)d^\times v-\rho_2(\hat{N})^{-1}f^{(2)}(0)\}\\
&+\frac{v(N)^{-1}}{s-1}\Big\{\int_1^\infty\sum_{m\neq 0}\hat{f}(u\rho_1(\hat{N})m,\ 0)d^\times u
\end{aligned}$$

$$+\rho_1(\hat{N})^{-1}\int_1^\infty u\sum_{m\neq 0} f^{(1)}(0,\ u\rho_1(\hat{N})^{-1}m)d^\times u-\rho_1(\hat{N})^{-1}f^{(1)}(0)\Big\}$$

$$-\frac{1}{s}\Big\{\int_1^\infty \sum_{m\neq 0} f(0,\ v\rho_2(N)m)d^\times v$$

$$+\rho_2(N)^{-1}\int_1^\infty v\sum_{m\neq 0} f^{(2)}(0,\ v\rho_2(N)^{-1}m)d^\times v-\rho_2(N)^{-1}f^{(2)}(0)\Big\}$$

$$-\frac{1}{s}\Big\{\int_1^\infty \sum_{m\neq 0} f(u\rho_1(N)m,\ 0)d^\times u$$

$$+\rho_1(N)^{-1}\int_1^\infty u\sum_{m\neq 0} f^{(1)}(u\rho_1(N)^{-1}m,\ 0)d^\times u-\rho_1(N)^{-1}f^{(1)}(0)\Big\}.$$

Here we use the relation $v(N)=\rho_1(N)/\rho_1(\hat{N})=\rho_2(N)/\rho_2(\hat{N})$. Recall the following formula which gives an analytic continuation of the Riemann zeta function:

(2-8)
$$\zeta(s)\int_{R^\times}|u|^{s-1}\phi(u)du$$
$$=\int_1^\infty u^s\sum_{m\neq 0}\phi(um)d^\times u+\int_1^\infty u^{1-s}\sum_{m\neq 0}\hat{\phi}(um)d^\times u+\frac{\hat{\phi}(0)}{s-1}-\frac{\phi(0)}{s}$$

where $\hat{\phi}(u^*)=\int_R \phi(u)\,\mathrm{e}[uu^*]du$ $(\phi\in\mathcal{S}(\boldsymbol{R}))$. We consider the integral $\int_{R^\times}|u|^{s-1}\cdot\phi(u)du$ as a meromorphic function of $s$ in $\boldsymbol{C}$ by analytic continuation. Then it is known that the following functional equation holds for any $\phi\in\mathcal{S}(\boldsymbol{R})$:

(2-9)
$$\int_{R^\times}|u|^{s-1}\hat{\phi}(u)du=2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}\int_{R^\times}|u|^{-s}\phi(u)du$$

(cf. [2], p. 359). It follows from (2-8) and (2-9) that

$$\int_1^\infty \sum_{m\neq 0}\hat{f}(0,\ v\rho_2(\hat{N})m)d^\times v$$
$$+\rho_2(\hat{N})^{-1}\int_1^\infty v\sum_{m\neq 0} f^{(2)}(v\rho_2(\hat{N})^{-1}m,\ 0)d^\times v-\rho_2(\hat{N})^{-1}f^{(2)}(0)$$
$$=2\frac{d}{ds}\Big[(2\pi\rho_2(\hat{N}))^{-s}\Gamma(s+1)\cos\frac{\pi s}{2}\zeta(s)\Big(\int_{R^\times\times R}|u|^{-s}f(u,\ v)du\,dv\Big)\Big]_{s=0}.$$

Since $\zeta(0)=-1/2$, $\zeta'(0)=-\log 2\pi/2$ and $\Gamma'(1)=-C$, the right hand side is equal to

$$(\log\rho_2(\hat{N})+C)\hat{f}(0)+\int_{R^\times\times R}\log|u|f(u,\ v)du\,dv.$$

The last three terms of the above expression of $J(f,\ N;\ s)$ can be calculated in the same manner and we get the lemma. q. e. d.

Set

$$\eta(M;\ s)=\sum_{x\in\Gamma\backslash M_1''}|\det x|^{-s}$$

where $\Gamma\backslash M_1''$ stands for the set of all $\rho(\Gamma)$-orbits in $M_1''$. It is easy to check the following lemma.

LEMMA 2.7. *When* $\mathrm{Re}\, s>1$, *the series* $\eta(M;\ s)$ *converges absolutely and*

$$\eta(M;\ s)=2v(M)^{1-2s}\zeta(2s-1)\sum_{i=1}^{\nu}\delta_i\lambda_i(\lambda_i^*/\lambda_i)^{2-2s}.$$

LEMMA 2.8. (1) *If* $\mathrm{Re}\, s_1>1$, $\mathrm{Re}\, s_2>3/2$ *and* $F\in\mathcal{S}(V_{\tilde{R}})$,

$$\begin{aligned}
Z(F,\ L_1'';\ s_1,\ s_2) &= Z_+^{(1)}(F,\ L_1'';\ s_1,\ s_2)+\hat{Z}_+^{(1)}(\hat{F},\ L_1'';\ 1-s_1,\ s_1+s_2-1/2)\\
&+\frac{2^{-3}}{(s_1-1)^2}\eta(M;\ s_2+1/2)\Psi_1(F;\ 0,\ s_2-1)\\
&+\frac{2^{-3}}{s_1^2}\eta(M;\ s_2)\Psi_1(\hat{F};\ 0,\ s_2-3/2)\\
&+\frac{1}{2(s_1-1)}\Big[\Big\{\zeta_1(M_1'';\ s_2+1/2)+\frac{2C-\log 2}{4}\eta(M;\ s_2+1/2)\\
&+\frac{1}{8}\eta'(M;\ s_2+1/2)\Big\}\Psi_1(F;\ 0,\ s_2-1)\\
&+\frac{1}{4}\eta(M;\ s_2+1/2)\frac{\partial\Psi_1}{\partial s_1}(F;\ 0,\ s_2-1)\Big]\\
&-\frac{1}{2s_1}\Big[\Big\{\zeta_1(M_1'';\ s_2)+\frac{2C-\log 2}{4}\eta(M;\ s_2)\\
&-\frac{1}{8}\eta'(M;\ s_2)\Big\}\Psi_1(\hat{F};\ 0,\ s_2-3/2)\\
&+\frac{1}{4}\eta(M;\ s_2)\Big\{\frac{\partial\Psi_1}{\partial s_1}(\hat{F};\ 0,\ s_2-3/2)-\frac{\partial\Psi_1}{\partial s_2}(\hat{F};\ 0,\ s_2-3/2)\Big\}\Big].
\end{aligned}$$

(2) *The formula obtained by replacing* $F$, $\hat{F}$, $s_1$ *and* $s_2$ *in the right hand side of the equality above by* $\hat{F}$, $F$, $1-s_1$ *and* $s_1+s_2-1/2$, *respectively, holds for* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2>1$.

PROOF. For each $x\in M_1''$, take a positive rational number $q$ such that $x=q\bar{x}$ where $\bar{x}$ is a primitive element in $V_Q$. Let $U=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ be an integral matrix such that

$$\bar{x}=Ux_0{}^tU,\ x_0=\begin{pmatrix} 0 & 1/2\\ 1/2 & 0\end{pmatrix},\quad \det U>0$$

and $(a, c)=(b, d)=1$. Denote by $N_x$ the lattice in $W_Q=Q^2$ given by $N_x=\{{}^tUy\,;\ y\in Z^2\}$. Since $G_{+x_0}=\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix};\ a\in R^\times\right\}$ and the Haar measure $d\mu_{x_0}$ on $G_{+x_0}$ normalized by (1-2) is equal to $4^{-1}d^\times a$, $Z(F, L_1''\,;\ s_1, s_2)$ is written as

$$2^{-3}\sum_{x\in\Gamma\backslash M_1''}\int_{G_+/G_{+x_0}}\chi_2(g)^{s_2}\omega(\rho(g)x_0)\int_{R^\times}d^\times a\int_0^\infty t^{2s_1}\sum_y F_{Ug}\left(qx_0, \begin{pmatrix} t/a & 0 \\ 0 & at\end{pmatrix}y\right)d^\times t$$

where $F_{Ug}(x, y)=F(\rho^\sim(Ug, 1)(x, y))$ and the summation with respect to $y$ is taken over all $y={}^t(y_1, y_2)\in N_x$ such that $y_1y_2\neq 0$. Hence, applying Lemma 2.6 to the integral

$$\int_{R^\times}d^\times a\int_0^\infty t^{2s_1}\sum_y F_{Ug}\left(qx_0, \begin{pmatrix} t/a & 0 \\ 0 & at\end{pmatrix}y\right)d^\times t=J(F_{Ug}(qx_0, *), N_x\,;\ s_1),$$

we have

$$\begin{aligned}
Z(F, L_1''\,;\ s_1, s_2)&=Z_+^{(1)}(F, L_1''\,;\ s_1, s_2)+\hat{Z}_+^{(1)}(\hat{F}, L_1''\,;\ 1-s_1, s_1+s_2-1/2)\\
&+\frac{2^{-3}}{(s_1-1)^2}\eta(M\,;\ s_2+1/2)\Psi_1(F\,;\ 0, s_2-1)+\frac{2^{-3}}{s_1^2}\eta(M\,;\ s_2)\Psi_1(\hat{F}\,;\ 0, s_2-3/2)\\
&+\frac{1}{2(s_1-1)}\left\{\phi_1(s_2+1/2)\Psi_1(F\,;\ 0, s_2-1)+\frac{1}{4}\eta(M\,;\ s_2+1/2)\frac{\partial\Psi_1}{\partial s_1}(F\,;\ 0, s_2-1)\right\}\\
&-\frac{1}{2s_1}\left[\phi_2(s_2)\Psi_1(\hat{F}\,;\ 0, s_2-3/2)\right.\\
&\left.+\frac{1}{4}\eta(M\,;\ s_2)\left\{\frac{\partial\Psi_1}{\partial s_1}(\hat{F}\,;\ 0, s_2-3/2)-\frac{\partial\Psi_1}{\partial s_2}(\hat{F}\,;\ 0, s_2-3/2)\right\}\right]
\end{aligned}$$

where

$$\phi_1(s)=\frac{C}{2}\eta(M\,;\ s)+\frac{1}{4}\sum_{x\in\Gamma\backslash M_1''}\log(\rho_1(\hat{N}_x)\rho_2(\hat{N}_x)/q)|\det x|^{-s}$$

and

$$\phi_2(s)=\frac{C}{2}\eta(M\,;\ s)+\frac{1}{4}\sum_{x\in\Gamma\backslash M_1''}\log(q\rho_1(N_x)\rho_2(N_x)/4)|\det x|^{-s}.$$

Now the first part of the lemma follows immediately from the formulas $\rho_1(N_x)=\rho_2(N_x)=\det U$ and $\rho_1(\hat{N}_x)=\rho_2(\hat{N}_x)=1$. We omit the similar proof of the second part. q.e.d.

LEMMA 2.9. *The functions $\Psi_i(F\,;\ s_1, s_2)\,(i=1, 2)$ satisfy the following functional equations for any $F\in\mathcal{S}(V_{\tilde{R}})$:*

$$\text{(2-10)}\quad \begin{pmatrix}\Psi_1(F^*\,;\ s_1, s_2)\\ \Psi_2(F^*\,;\ s_1, s_2)\end{pmatrix}=\Gamma(s_2+1)\Gamma(s_1+s_2+3/2)2^{-s_1-2s_2-2}\pi^{-5/2-s_1-2s_2}$$

$$\times\begin{pmatrix} -\cos(s_1+2s_2)\pi/2 & -\sin\pi s_1/2 \\ \cos\pi s_1/2 & \sin(s_1+2s_2)\pi/2 \end{pmatrix}\begin{pmatrix} \Psi_1(F;\, s_1,\, -3/2-s_1-s_2) \\ \Psi_2(F;\, s_1,\, -3/2-s_1-s_2) \end{pmatrix},$$

$$\text{(2-11)}\quad \begin{pmatrix} \Psi_1(\hat{F};\, s_1,\, s_2) \\ \Psi_2(\hat{F};\, s_1,\, s_2) \end{pmatrix} = \pi^{-2(s_1+1)}\Gamma(s_1+1)^2$$

$$\times\begin{pmatrix} 2\sin^2(\pi s_1/2) & 0 \\ 0 & -\sin\pi s_1 \end{pmatrix}\begin{pmatrix} \Psi_1(F;\, -s_1-1,\, s_1+s_2+1/2) \\ \Psi_2(F;\, -s_1-1,\, s_1+s_2+1/2) \end{pmatrix}.$$

PROOF (cf. [4], § 7). The functional equation (2-10) follows immediately from Lemma 1 (i) of [6]. We can easily reduce the functional equation (2-11) to the formulas for the Fourier transforms of $|x^2 \pm y^2|^s$ ([2], Chap. III 2.6.).

q. e. d.

Now we are ready to prove Theorem 2.

PROOF OF THEOREM 2. Lemma 2.1, Lemma 2.2 (3), Lemma 2.4 and the formula (2-10) of the above lemma imply that $(s_2-1)(s_1+s_2-3/2)\xi_i(L;\, s_1,\, s_2)$ $(i=1, 2)$ are extended to holomorphic functions of $(s_1, s_2)$ in $D_1=\{(s_1, s_2)\in \boldsymbol{C}^2;\ \mathrm{Re}\, s_1>1\}$. Lemma 2.1, Lemma 2.2 (1), (2), Lemma 2.5 (1), Lemma 2.8 (1) and the formula (2-11) show that $(s_1-1)^2\xi_i(L;\, s_1,\, s_2)$ $(i=1, 2)$ have analytic continuations to holomorphic functions of $(s_1, s_2)$ in $D_2=\{(s_1, s_2)\in\boldsymbol{C}^2;\ \mathrm{Re}\, s_1+\mathrm{Re}\, s_2>3/2,\ \mathrm{Re}\, s_2>3/2\}$. Hence $(s_1-1)^2(s_2-1)(s_1+s_2-3/2)\xi_i(L;\, s_1,\, s_2)$ $(i=1, 2)$ can be continued analytically to holomorphic functions in the tube domain $D_1\cup D_2$. It is obvious that the convex hull of $D_1\cup D_2$ coincides with $\boldsymbol{C}^2$. Therefore the functions $(s_1-1)^2(s_2-1)(s_1+s_2-3/2)\xi_i(L;\, s_1,\, s_2)$ $(i=1, 2)$ have analytic continuations to entire functions of $(s_1, s_2)$ (cf. [3], Theorem 2.5.10). The functional equation

$$\text{(2-12)}\qquad Z(F,\, L;\, s_1,\, s_2)=v(M)^{-1}Z(F^*,\, L^*;\, s_1,\, 3/2-s_1-s_2)$$

follows from Lemma 2.4. Moreover we have, by Lemma 2.5 and Lemma 2.8,

$$\text{(2-13)}\qquad Z(F,\, L;\, s_1,\, s_2)=Z(\hat{F},\, L;\, 1-s_1,\, s_1+s_2-1/2).$$

The functional equation (2-2) (resp. (2-3)) is an immediate consequence of Lemma 2.1, (2-10) (resp. (2-11)), and (2-12) (resp. (2-13)). The residue formula (2-4) is obtained from Lemma 2.4. The formula (2-5) follows from (2-3) and (2-4). Finally we easily derive (2-6) and (2-7) from Lemma 2.5, Lemma 2.7 and Lemma 2.8.

q. e. d.

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