Hecke characters and models of abelian varieties with complex multiplication

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To the memory of Takuro Shintani

Introduction. Let A be an abelian variety of dimension n defined over $\boldsymbol{C}$, $\mathcal{C}$ be a polarization of A and K be a CM-field of degree $2n$. We assume that there exists an isomorphism θ of K into the endomorphism algebra $\mathrm{End}(A)\otimes\boldsymbol{Q}$ of A. Let (K, Φ) be the CM-type determined by (A, θ) and (K', Φ') be the reflex of (K, Φ). For every subfield D of K, we put $\theta_D=\theta|D$ and consider the structure $(A, \mathcal{C}, \theta_D)$ of the polarized abelian variety with endomorphisms $\theta(D)$ taken into account. Let M_D be the field of moduli of $(A, \mathcal{C}, \theta_D)$ and let k_0 be a finite algebraic number field which contains M_D. We set $k=k_0K'$.

When K is cyclic over D, G. Shimura [7], [10] proved some criterions by which the existence of a model of $(A, \mathcal{C}, \theta_D)$ over k_0 becomes equivalent to the existence of a Hecke character of $k_A^\times$ which satisfies a few simple conditions. Here $k_A^\times$ denotes the idèle group of k. For applications obtained from these criterions, we should refer the reader to [7], §5 and [10], §3.

In the first half of this paper (§1~§3), we shall try to generalize Shimura's criterions for an arbitrary subfield D of K, under the assumption that A is simple and that $\mathrm{End}(A)$ contains the maximal order of K. By virtue of the results on the zeta function of A given in [7], [10], we obtain basic necessary conditions which the Hecke character ψ of $k_A^\times$ should satisfy for the existence of a model of $(A, \mathcal{C}, \theta_D)$ over k_0. Also, as is explained in [7] (see §1 of this paper), we shall lose no generality by assuming that K is normal over D. Under these conditions and the existence of ψ, the obstruction for the existence of a model over k_0 can be described by a cohomology class ξ in $H^2(\mathrm{Gal}(K/D), Z_K)$. Here Z_K denotes the group of all roots of unity in K. We can give a condition, in terms of ψ, which guarantees that ξ splits locally at every place of K (Theorem 1). Then this condition leads to a certain descent data in which isomorphisms are replaced by isogenies. In our case, we can prove that this actually leads to the descent so that there exists a structure (A', θ'_D) which is rational over k_0 and is *isogenous* to (A, θ_D) (Theorem 2). This result can be regarded as an affirmative answer, though only up to isogeny, to the problem of generalization of Shimura's criterions, in the case A is simple.

In §4, we shall prove that $(A, \mathcal{C})$ has a model over its field of moduli in certain special cases by applying Shimura's criterion for the case $D=F$, where F denotes the maximal real subfield of K. On the other hand, in §5, we shall present examples of $(A, \mathcal{C})$ which have no models over their fields of moduli[1]. Our results seem to suggest that the existence of a model of $(A, \mathcal{C}, \theta_F)$ over M_F depends delicately on the ramification of 2 in K and M_K.

Notation and terminology.

By an *algebraic number field*, we understand an algebraic extension of $\boldsymbol{Q}$ in $\boldsymbol{C}$. We denote by ρ the complex conjugation in $\boldsymbol{C}$. Let F be an algebraic number field of finite degree. The maximal order, the unit group, the group of principal ideals, the ideal group and the ideal class group of F are denoted by $\mathfrak{O}_F$, E_F, P_F, I_F and C_F respectively. For $\mathfrak{x}\in I_F$, $N(\mathfrak{x})$ denotes the absolute norm of $\mathfrak{x}$. The maximal abelian extension of F in $\boldsymbol{C}$ is denoted by F_{ab}. By $F_A^\times$ and $F_\infty^\times$, we denote the idèle group of F and the archimedean part of $F_A^\times$ respectively. Let $x\in F_A^\times$. The element of $\mathrm{Gal}(F_{ab}/F)$, which is the image of x under the Artin map, is denoted by $[x, F]$. By x_∞ (resp. $x_{\infty 1}$), we denote the image of x under the projection to $F_\infty^\times$ (resp. to the first archimedean component of $F_\infty^\times$ which corresponds to the identical injection of F into $\boldsymbol{C}$). We denote by $|x|_F$ (resp. $|x|_0$) the idèle norm of x (resp. of the finite part of x). By $\mathrm{div}(x)\in I_F$, we denote the divisor of x which is canonically obtained. Let v be a place of F. Then F_v and $(\mathfrak{O}_F)_v$ stand for the completions of F and $\mathfrak{O}_F$ at v respectively. If ω is a quasicharacter of $F_A^\times$, ω_v denotes the quasicharacter of $F_v^\times$ obtained from ω. If K is a finite extension of F, $\mathfrak{d}_{K/F}$, $D(K/F)$ and $N_{K/F}$ denote the relative different, the relative discriminant and the norm map from K to F respectively. As to the terminology concerning abelian varieties of CM-type, we shall follow that in [7] and [11].

§1. Review of known results.

We use the same notation as in the introduction. Throughout the paper, for the sake of simplicity, we assume

$$\mathrm{End}(A)=\theta(\mathfrak{O}_K). \tag{1.1}$$

This implies $\mathrm{End}(A)\otimes\boldsymbol{Q}\cong K$; hence A is absolutely simple. Let Φ be the representation of K by $n\times n$ complex matrices which is realized on the tangent space of A at the origin through θ. Let (K', Φ') be the reflex of (K, Φ). We shall recall briefly several basic results given in [7]. Let $\mathcal{C}$ be any polarization

1) For the generic case, see [6].

of A and M be the field of moduli of $(A, \mathcal{C})$. Then $M_K = MK'$ and K' is normal over $K' \cap M$. Therefore we have $\mathrm{Gal}(M_K/M) \cong \mathrm{Gal}(K'/K' \cap M)$. If $\sigma \in \mathrm{Aut}(\boldsymbol{C}/M)$, there is an isomorphism μ_σ of $(A, \mathcal{C})$ to $(A^\sigma, \mathcal{C}^\sigma)$. Then, for every $a \in \mathfrak{O}_K$, we have $\mu_\sigma^{-1}\theta(a)^\sigma \mu_\sigma = \theta(b)$ with $b \in \mathfrak{O}_K$. We see easily that b does not depend on the choice of μ_σ. The map $a \to b$ defines a ring automorphism of $\mathfrak{O}_K$ and it extends to the field automorphism $\pi(\sigma)$ of K. Thus we get

$$\mu_\sigma^{-1}\theta(a)^\sigma \mu_\sigma = \theta(a^{\pi(\sigma)}), \qquad a \in \mathfrak{O}_K. \tag{1.2}$$

Clearly we have $\pi(\sigma)=1$ if and only if σ induces the identity on M_K. Let D_0 be the fixed field of $\pi(\mathrm{Gal}(M_K/M))$. Then we have

$$\mathrm{Gal}(K/D_0) \cong \mathrm{Gal}(MK'/M) \cong \mathrm{Gal}(K'/K' \cap M). \tag{1.3}$$

The following lemma is now easy to prove (cf. [9], Prop. 2):

LEMMA 1. *The field of moduli M of $(A, \mathcal{C})$ coincides with the field of moduli M_{D_0} of $(A, \mathcal{C}, \theta_{D_0})$.*

LEMMA 2. *We have* $\det(\Phi'(x^\sigma)) = (\det \Phi'(x))^{\pi(\sigma)}$ *and* $\mathrm{trace}(\Phi'(x^\sigma)) = (\mathrm{trace}(\Phi'(x)))^{\pi(\sigma)}$ *for $x \in K'$, $\sigma \in \mathrm{Gal}(MK'/M)$.*

PROOF. Take $\sigma \in \mathrm{Aut}(\boldsymbol{C}/M)$ and let μ_σ be an isomorphism of $(A, \mathcal{C})$ to $(A^\sigma, \mathcal{C}^\sigma)$. Let $\mathcal{D}(A)$ and $\mathcal{D}(A^\sigma)$ denote the space of invariant differential 1-forms on A and on A^σ respectively. By (1.2), we have

$$\mathrm{trace}(\theta(a^{\pi(\sigma)}) \mid \mathcal{D}(A)) = \mathrm{trace}(\theta(a)^\sigma \mid \mathcal{D}(A^\sigma)), \qquad a \in \mathfrak{O}_K. \tag{1.4}$$

Suppose that Φ is equivalent to $\sum_{i=1}^{n} \sigma_i$ with the isomorphisms σ_i's of K into $\boldsymbol{C}$. Then, by (1.4), we get

$$\sum_{i=1}^{n} a^{\pi(\sigma)\sigma_i} = \sum_{i=1}^{n} a^{\sigma_i \sigma} \qquad \text{for} \quad a \in K. \tag{1.5}$$

A theorem of Artin on the linear independence of isomorphisms implies that $\pi(\sigma)\sigma_i = \sigma_{j(i)}\sigma$ as isomorphisms of K into $\boldsymbol{C}$ for every i, where the map $i \to j(i)$ is a permutation on n-letters.

Let L be a finite Galois extension of $\boldsymbol{Q}$ which contains K and M. Let S be the set of all extensions of σ_i's to elements of $\mathrm{Gal}(L/\boldsymbol{Q})$. Put $S' = S^{-1}$ and $H' = \{\gamma \in \mathrm{Gal}(L/\boldsymbol{Q}) \mid \gamma S' = S'\}$. Then K' is the subfield of L which corresponds to H' and Φ' consists of all the different restrictions of the elements of S' to K' (cf. [5], p. 126). Clearly we may assume that $\sigma \in \mathrm{Gal}(L/M)$. Let $\widetilde{\pi(\sigma)}$ denote an extension of $\pi(\sigma) \in \mathrm{Gal}(K/D_0)$ to an element of $\mathrm{Gal}(L/\boldsymbol{Q})$. From (1.5), we get $\widetilde{\pi(\sigma)}S = S\sigma$. Therefore we have

(1.6) $$S'\widetilde{\pi(\sigma)}=\sigma S',$$

for every $\sigma\in\mathrm{Gal}(L/M)$ and every extension $\widetilde{\pi(\sigma)}$ of $\pi(\sigma)$. Suppose that Φ' is equivalent to $\sum_{j=1}^{t}\tau_j$ with isomorphisms τ_j's of K' into $\boldsymbol{C}$. We may consider τ_j as an element of $\mathrm{Gal}(L/\boldsymbol{Q})$. Then (1.6) implies that $\sigma\tau_j=\tau_{i(j)}\widetilde{\pi(\sigma)}$ on K' with some $\tau_{i(j)}$ for every j. Therefore we obtain

$$\prod_{j=1}^{t}x^{\sigma\tau_j}=\prod_{j=1}^{t}x^{\tau_j\widetilde{\pi(\sigma)}}=\Bigl(\prod_{j=1}^{t}x^{\tau_j}\Bigr)^{\pi(\sigma)}\qquad\text{for}\quad x\in K',$$

and the latter equality in the same way. This completes the proof.

Now assume that $(A, \mathcal{C})$ is defined over a finite algebraic number field k_0. We put $k=k_0K'$. Since $k_0\supseteq M$, we have $\mathrm{Gal}(k/k_0)\cong\mathrm{Gal}(M_K/k_0\cap M_K)\subseteq\mathrm{Gal}(M_K/M)$. For the homomorphism of $\mathrm{Gal}(k/k_0)$ into $\mathrm{Aut}(K)$ which is canonically obtained from π, we use the same letter π. Let ψ be the Hecke character of $k_A^\times$ determined by (A, θ) over k[2]. Define a homomorphism g of $k_A^\times$ into $K_A^\times$ by $g=\det\Phi'\circ N_{k/K'}$. Then ψ satisfies the following properties (cf. [7], (1.12), (1.13)).

(A) $\psi(x)=1/g(x)_{\infty 1}$ for $x\in k_\infty^\times$.

(B) If $x\in k_A^\times$ and $x_\infty=1$, then $\psi(x)\in K^\times$, $\psi(x)\psi(x)^\rho=|x|_k^{-1}$ and $\psi(x)\mathfrak{O}_K=g(x)\mathfrak{O}_K$.

(C) $\psi(x^\sigma)=\psi(x)^{\pi(\sigma)}$ for every $\sigma\in\mathrm{Gal}(k/k_0)$ if $x\in k_A^\times$, $x_\infty=1$.

(For (C), see (6.9) and (6.10) of [7] or [10], § 2).

Furthermore, assume that $(A, \mathcal{C}, \theta_F)$ is rational over h and that $h\not\supseteq K'$. Then ψ satisfies (cf. [10], Theorem 1)

(D) $\psi(x)=\chi(x)|x|_h^{-1}$ for $x\in h_A^\times$,

where χ is the character of $h_A^\times$ which corresponds to the quadratic extension $k=hK'$ of h.

The following converse theorems are obtained by Shimura and Casselman[3]. Let (A, θ) be a structure of type (K, Φ) and $\mathcal{C}$ be a polarization of A.

THEOREM I ([7], Theorem 6). *Let k be an algebraic number field which contains K' and assume that there exists a Hecke character ψ of $k_A^\times$ which satisfies* (A) *and* (B). *Then there exists a structure (A', θ') which is rational over k, is isomorphic to (A, θ) and determines ψ over k.*

THEOREM II ([10], Theorem 2). *Let h be an algebraic number field such that $h\supseteq M_F$, $h\not\supseteq K'$. Put $k=hK'$ and assume that there exists a Hecke character ψ of $k_A^\times$ which satisfies* (A), (B) *and* (D). *Then the structure (A', θ') in Theorem* I *can be taken so as to satisfy one more condition that (A', θ'_F) is rational over h.*

2) For this terminology, we refer to [7], p. 510.

3) Here we state their results under the restrictive condition (1.1).

In Theorem I (resp. II), we note that any polarization C' of A' is rational over k (resp. h) (cf. [7], Prop. 4).

§ 2. Formulation of problems.

One of our purposes in this paper is to generalize Theorems I and II. First we shall explain precise formulation of the problem. Let (A, θ) be a structure of type (K, Φ) and k_0 be an algebraic number field of finite degree. We take any polarization C of A and assume that

(E) k_0 contains M.

Here M denotes the field of moduli of (A, C) as before. Put $k=k_0K'$. As is explained in § 1, we have an injective homomorphism π of $\mathrm{Gal}(k/k_0)$ into $\mathrm{Aut}(K)$ such that

$$\mu_\sigma^{-1}\theta(a)^\sigma\mu_\sigma=\theta(a^{\pi(\sigma)}), \qquad a\in\mathfrak{O}_K \tag{2.1}$$

holds for any $\sigma\in\mathrm{Gal}(k/k_0)$ and for any isomorphism μ_σ of (A, C) to (A^σ, C^σ). Let D denote the fixed field of $\pi(\mathrm{Gal}(k/k_0))$. Then π gives the isomorphism $\mathrm{Gal}(k/k_0)\cong\mathrm{Gal}(K/D)$. We assume that there is given a Hecke character ψ of $k_A^\times$ which satisfies (A), (B) and (C). Now let us consider the following problem.

(P) Give a criterion, in terms of ψ, which guarantees the existence of a model (A', θ') satisfying the following conditions.

(P1) (A', θ') is isomorphic to (A, θ).
(P2) (A', θ'_D) is rational over k_0.
(P3) (A', θ') determines ψ over k.

We note that ψ depends only on the k-isogeny class of (A, θ)[4]. Therefore, it may be too optimistic to expect an affirmative solution, which controls the descent only by ψ, of this problem in the general case. Thus we formulate also a weaker version (P′) of this problem replacing (P1) by the following condition.

(P′1) (A', θ') is isogenous to (A, θ).

In this and the following sections, we shall be concerned with these problems. First, by Theorem I stated in § 1, we may assume that (A, C, θ) is rational over k.

REMARK 1. Suppose that there exists a structure (A', θ') which satisfies (P1) (resp. (P′1)), (P2), (P3). By [11], p. 74, Prop. 30 and (P2), (A', θ') is rational over k. Then (P1) (resp. (P′1)) and (P3) imply that (A', θ') is isomor-

4) To see this, it is sufficient, for example, to trace back the proof of [7], Theorem 5.

phic (resp. isogenous) to (A, θ) over k (cf. [7], Theorem 5).

LEMMA 3. *For $\sigma\in\mathrm{Gal}(k/k_0)$, let μ_σ be an isomorphism of $(A, \mathcal{C})$ to $(A^\sigma, \mathcal{C}^\sigma)$. Then μ_σ is defined over k.*

PROOF. Put $\Phi^*(a)=\Phi(a^{\pi(\sigma^{-1})})^\sigma$ and $\theta^*(a)=\theta(a^{\pi(\sigma^{-1})})^\sigma$ for $a\in K$. Applying Prop. 1 of [7], we see that (A^σ, θ^*) is of type (K, Φ^*). Let ψ' be the Hecke character of $k_A^\times$ determined by (A^σ, θ^*) over k. By (1.5), we see that Φ^* is equivalent to Φ. Using Prop. 1 of [7] and assumption (C), we have $\psi'(x)=\psi(x)$ if $x\in k_A^\times$ and $x_\infty=1$. Hence we have $\psi'=\psi$. On the other hand, we have $\mu_\sigma\theta^*(a)=\theta(a)\mu_\sigma$ for $a\in K$. Therefore, by Theorem 5 of [7], μ_σ is rational over k.

Let Z_K denote the group of all roots of unity in K. Since we have $\mathrm{Aut}((A, \mathcal{C}))=\theta(Z_K)$ (cf. [11], p. 117), we get $\mu_{\sigma\tau}^{-1}\mu_\sigma^\tau\mu_\tau=\theta(\zeta_{\sigma,\tau})$ with $\zeta_{\sigma,\tau}\in Z_K$ for any $\sigma, \tau\in\mathrm{Gal}(k/k_0)$. For $\alpha, \beta\in\mathrm{Gal}(K/D)$, we put $\xi_{\alpha,\beta}=\zeta_{\pi^{-1}(\alpha),\pi^{-1}(\beta)}$.

PROPOSITION 1. *The map $(\alpha, \beta)\to\xi_{\alpha,\beta}$ of $\mathrm{Gal}(K/D)\times\mathrm{Gal}(K/D)$ to Z_K defines a 2-cocycle. The cohomology class ξ of $\{\xi_{\alpha,\beta}\}$ in $H^2(\mathrm{Gal}(K/D), Z_K)$ does not depend on the choice of isomorphisms μ_σ. Let φ be the canonical homomorphism of $H^2(\mathrm{Gal}(K/D), Z_K)$ to $H^2(\mathrm{Gal}(K/D), E_K)$. There exists a model (A', θ') such that A' is rational over k_0 and that (A', θ') is isomorphic to (A, θ) over k if and only if $\varphi(\xi)=1$. Furthermore, if this is the case, (A', θ') can be taken so that (A', θ'_D) is rational over k_0.*

PROOF. Note that $\mathrm{Gal}(K/D)$ acts on Z_K on the right. The cocycle condition is written as

$$\xi_{\alpha\beta,\gamma}\xi_{\alpha,\beta}^{\gamma}=\xi_{\alpha,\beta\gamma}\xi_{\beta,\gamma} \quad \text{for} \quad \alpha, \beta, \gamma\in\mathrm{Gal}(K/D). \tag{2.2}$$

We can verify (2.2) by a direct computation using (2.1). It is also immediate to verify that the cohomology class of $\xi_{\alpha,\beta}$ does not depend on the choice of $\{\mu_\sigma\}$. Suppose that $\varphi(\xi)=1$. Then we can find a 1-cochain $\{a_\alpha\}$, $a_\alpha\in E_K$ so that $\xi_{\alpha,\beta}=a_{\alpha\beta}^{-1}a_\alpha^\beta a_\beta$ for any $\alpha, \beta\in\mathrm{Gal}(K/D)$. Define an isomorphism η_σ of A to A^σ by $\eta_\sigma=\mu_\sigma\theta(a_{\pi(\sigma)}^{-1})$ for any $\sigma\in\mathrm{Gal}(k/k_0)$. Then we get

$$\begin{aligned}\eta_{\sigma\tau}^{-1}\eta_\sigma^\tau\eta_\tau&=\theta(a_{\pi(\sigma\tau)})\mu_{\sigma\tau}^{-1}\mu_\sigma^\tau\theta(a_{\pi(\sigma)}^{-1})^\tau\mu_\tau\theta(a_{\pi(\tau)}^{-1})\\&=\theta(a_{\pi(\sigma\tau)})\mu_{\sigma\tau}^{-1}\mu_\sigma^\tau\mu_\tau\theta((a_{\pi(\sigma)}^{\pi(\tau)})^{-1}(a_{\pi(\sigma)}^{-1})=1\end{aligned}$$

for any $\sigma, \tau\in\mathrm{Gal}(k/k_0)$. Therefore, by Weil's criterion of descent, we can find an abelian variety A' rational over k_0 which is isomorphic to A over k. Let η be this isomorphism from A' to A and set $\theta'(a)=\eta^{-1}\theta(a)\eta$, $a\in K$. Put $\mu_\sigma=\eta^\sigma\circ\eta^{-1}$ for $\sigma\in\mathrm{Gal}(k/k_0)$. Then μ_σ is an isomorphism of A to A^σ and (2.1) also holds for this μ_σ. Then we get $\eta\theta'(a)^\sigma\eta^{-1}=\theta(a^{\pi(\sigma)})$; hence $\theta'(a)^\sigma=\theta'(a)$ if

$a \in D$. Therefore (A', θ'_D) is rational over k_0. Conversely, if there exists a model A' rational over k_0 which is isomorphic to A over k, we can get $\varphi(\xi)=1$ by a direct computation. This completes the proof.

REMARK 2. By Prop. 1, we see that the problem (P) has an affirmative solution if and only if $\varphi(\xi)=1$.

§3. A solution of the problem in the sense of isogeny.

In this section, let us abbreviate $\mathfrak{O}_K$ to $\mathfrak{O}$ and set $U_0=\prod_w \mathfrak{O}_w^\times$, where w runs over all finite places of K. For $\sigma \in \mathrm{Gal}(k/k_0)$, let $\tilde{\sigma}$ be an extension of σ to an element of $\mathrm{Gal}(k_{ab}/k_0)$. For any $\sigma, \tau \in \mathrm{Gal}(k/k_0)$, we have $(\widetilde{\sigma\tau})^{-1}\tilde{\sigma}\tilde{\tau} \in \mathrm{Gal}(k_{ab}/k)$. Take $y \in k_A^\times$ so that $[y, k]=(\widetilde{\sigma\tau})^{-1}\tilde{\sigma}\tilde{\tau}$, $y_\infty=1$. Since $\psi(y)\mathfrak{O}_K=g(y)\mathfrak{O}_K$ by (B), we get $\psi(y)g(y)^{-1} \in U_0$. We put $\delta_{\sigma,\tau}=\psi(y)g(y)^{-1}$. By Theorem 2, (ii) of [7], we see easily that $\delta_{\sigma,\tau}$ does not depend on the choice of y, though it depends on the choice of $\tilde{\sigma}$ and $\tilde{\tau}$. For $\alpha, \beta \in \mathrm{Gal}(K/D)$, set $\eta_{\alpha,\beta}=\delta_{\pi^{-1}(\alpha),\pi^{-1}(\beta)}$.

LEMMA 4. *The map $(\alpha, \beta) \to \eta_{\alpha,\beta}$ from $\mathrm{Gal}(K/D)\times\mathrm{Gal}(K/D)$ to U_0 defines a 2-cocycle. The cohomology class of $\eta_{\alpha,\beta}$ does not depend on the choice of $\{\tilde{\sigma}\}$.*

PROOF. It is sufficient to show

$$\delta_{\sigma\tau,\nu}\delta_{\sigma,\tau}^{\pi(\nu)}=\delta_{\sigma,\tau\nu}\delta_{\tau,\nu} \qquad \text{for} \quad \sigma, \tau, \nu \in \mathrm{Gal}(k/k_0). \tag{3.1}$$

By Lemma 2, we can easily see that $g(x^\sigma)=g(x)^{\pi(\sigma)}$ for $\sigma \in \mathrm{Gal}(k/k_0)$. Then we can verify (3.1) by a direct computation using (C). The second assertion can be verified in a straightforward way.

Let η denote the cohomology class of $\{\eta_{\alpha,\beta}\}$ in $H^2(\mathrm{Gal}(K/D), U_0)$. Roughly speaking, η is the image of the cohomology class which defines the extension $1 \to \mathrm{Gal}(k_{ab}/k) \to \mathrm{Gal}(k_{ab}/k_0) \to \mathrm{Gal}(k/k_0) \to 1$ by ψg^{-1}.

PROPOSITION 2. *Let ι_0 denote the canonical homomorphism $H^2(\mathrm{Gal}(K/D), Z_K) \to H^2(\mathrm{Gal}(K/D), U_0)$ which is obtained from the injection $Z_K \to U_0$. Then we have $\iota_0(\xi)=\eta$.*

PROOF. Define an onto map ω of $K\otimes_Q \boldsymbol{R}$ to A as in [7], p. 508~9. Since $\mathrm{End}(A)=\theta(\mathfrak{O})$, ω induces an isomorphism $K\otimes_Q \boldsymbol{R}/\mathfrak{A} \cong A$ with a fractional ideal $\mathfrak{A}$ of K. Hence we can write $\mathfrak{A}=s\mathfrak{O}$ with $s \in K_A^\times$. Take any $\sigma \in \mathrm{Gal}(k/k_0)$ and put $r_\sigma=s/s^{\pi(\sigma^{-1})}$. We set

$$\varphi(v)=\omega^{-1}(\mu_\sigma^{-1}(\omega(r_\sigma v^{\pi(\sigma^{-1})})^{\tilde{\sigma}})) \qquad \text{for} \quad v \in K/\mathfrak{A}. \tag{3.2}$$

Then φ induces an $\mathfrak{O}$-linear automorphism of $K/\mathfrak{A}$. Therefore there exists an

$e_{\tilde{\sigma}} \in U_0$ such that

$$\varphi(v)=e_{\tilde{\sigma}}v \qquad \text{for} \quad v\in K/\mathfrak{A}. \tag{3.3}$$

By a direct computation using (3.3), we get

$$\mu_{\sigma\tau}^{-1}\mu_{\sigma}^{\tau}\mu_{\tau}(\omega(v))=\mu_{\sigma\tau}^{-1}(\omega(r_{\sigma}(e_{\tilde{\sigma}}^{\pi(\sigma^{-1})})^{-1}(r_{\tau}(e_{\tilde{\tau}}^{\pi(\tau^{-1})})^{-1}v^{\pi(\tau^{-1})})^{\pi(\sigma^{-1})})^{\tilde{\sigma}\tilde{\tau}(\widetilde{\sigma\tau})^{-1}\widetilde{\sigma\tau}}) \tag{3.4}$$

for $v\in K/\mathfrak{A}$. Take $x\in k_A^{\times}$ so that $x_{\infty}=1$, $[x, k]=\tilde{\sigma}\tilde{\tau}(\widetilde{\sigma\tau})^{-1}$. By Theorem 2, (i) and (1.11) of [7], we have

$$\omega(v)^{[x,k]}=\omega(\psi(x)g(x)^{-1}v) \qquad \text{for} \quad v\in K/\mathfrak{A}. \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\mu_{\sigma\tau}^{-1}\mu_{\sigma}^{\tau}\mu_{\tau}(\omega(v))=\omega((\psi(x)g(x)^{-1})^{\pi(\sigma\tau)}e_{\widetilde{\tau\sigma}}(e_{\tilde{\sigma}}^{-1})^{\pi(\tau)}e_{\tilde{\tau}}^{-1}v), \qquad v\in K/\mathfrak{A}. \tag{3.6}$$

We have $(\psi(x)g(x)^{-1})^{\pi(\sigma\tau)}=\psi(x^{\sigma\tau})g(x^{\sigma\tau})^{-1}$ and $[x^{\sigma\tau}, k]=(\widetilde{\sigma\tau})^{-1}\tilde{\sigma}\tilde{\tau}$. Therefore we obtain

$$\iota_0(\xi_{\alpha,\beta})=\eta_{\alpha,\beta}d_{\alpha,\beta}(d_{\alpha}^{\beta})^{-1}d_{\beta}^{-1} \qquad \text{for} \quad \alpha, \beta\in \mathrm{Gal}(K/D), \tag{3.7}$$

where $d_{\alpha}=e_{\widetilde{\pi^{-1}(\alpha)}}$. This completes the proof.

By Shapiro's lemma, we have

$$H^2(\mathrm{Gal}(K/D), U_0)\cong \bigoplus_{v} H^2(\mathrm{Gal}(K_w/D_v), \mathfrak{O}_w^{\times}), \tag{3.8}$$

where v extends over all finite places of D and w denotes a place lying over v. Thus the condition

$$\eta=1 \tag{F}$$

implies that the image of ξ in $H^2(\mathrm{Gal}(K_w/D_v), \mathfrak{O}_w^{\times})$ is trivial for every finite place w of K. On the other hand, Shimura's Theorem II quoted in §1 can be interpreted as giving the condition for the splitting of ξ at the archimedean places of K.

PROPOSITION 3. *With the notation as above, we assume $D\subsetneqq F$. Let ι_{ϕ} denote the homomorphism $H^2(\mathrm{Gal}(K/D), Z_K)\to H^2(\mathrm{Gal}(\boldsymbol{C}/\boldsymbol{R}), \boldsymbol{C}^{\times})$ associated with an isomorphism ϕ of K into $\boldsymbol{C}$. Then $\iota_{\phi}(\xi)=1$ if and only if ϕ satisfies* (D) *for the subfield h of k which corresponds to $\pi^{-1}(\mathrm{Gal}(K/F))$.*

PROOF. Let Res denote the restriction map of $H^2(\mathrm{Gal}(K/D), Z_K)$ to $H^2(\mathrm{Gal}(K/F), Z_K)$. By virtue of the above quoted result of Shimura and Prop. 1, we see that $\mathrm{Res}(\xi)=1$ if and only if ϕ satisfies (D). The isomorphism ϕ of K into $\boldsymbol{C}$ induces a homomorphism $\phi_*: H^2(\mathrm{Gal}(K/F), Z_K)\to H^2(\mathrm{Gal}(\boldsymbol{C}/\boldsymbol{R}), \boldsymbol{C}^{\times})$. Our assertion would follow immediately if we could show that ϕ_* is an isomorphism. By taking a cyclic factor set as a representative for every cohomology class, we get $H^2(\mathrm{Gal}(K/F), Z_K) \cong \{\pm 1\}$, $H^2(\mathrm{Gal}(\boldsymbol{C}/\boldsymbol{R}), \boldsymbol{C}^\times) \cong \boldsymbol{R}^\times/\boldsymbol{R}_+^\times$ and $\phi_*(-1) = -1 \bmod \boldsymbol{R}_+^\times$. This completes the proof.

Summing up these considerations, we obtain the following Theorem.

THEOREM 1. *Let the notation be as above. We assume that k_0 satisfies* (E) *and $k_A^\times$ has a Hecke character ψ which satisfies* (A), (B), (C), *where $k = k_0 K'$. Define a subgroup U of $K_A^\times$ by $U = U_0 \times K_\infty^\times$ and let ι be the canonical homomorphism of $H^2(\mathrm{Gal}(K/D), Z_K)$ to $H^2(\mathrm{Gal}(K/D), U)$. Then we have $\iota(\xi) = 1$ if and only if ψ satisfies* (F) *and also* (D) *in the case $D \subseteq F$.*

Thus we have obtained necessary and sufficient conditions (A)$\sim$(F) for ξ (or for $\varphi(\xi)$) to be "everywhere locally trivial", in terms of ψ. The naturality of our arguments seems to suggest that this would be the best information about ξ which could be obtained from ψ.

Suppose that $\iota(\xi) = 1$. Then, by the Hasse principle, there exists a 1-cochain $\{a_\alpha\}$, $a_\alpha \in K^\times$ such that $\xi_{\alpha,\beta} = a_{\alpha\beta}^{-1} a_\alpha^\beta a_\beta$ for every $\alpha, \beta \in \mathrm{Gal}(K/D)$. Put $\mu''_\sigma = \mu_\sigma \circ \theta(a_{\pi(\sigma)}^{-1})$, which can be considered as an element of $\mathrm{Hom}(A, A^\sigma) \otimes \boldsymbol{Q}$. Then we have

$$\mu''_{\sigma\tau} = \mu''^{\tau}_{\sigma} \mu''_{\tau} \qquad \text{for every} \quad \sigma, \tau \in \mathrm{Gal}(k/k_0), \tag{3.9}$$

where the equality is understood in the category of abelian varieties whose morphisms are extended from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(A, B) \otimes \boldsymbol{Q}$. The formula (3.9) can be interpreted as "descent data in the sense of isogeny".

THEOREM 2. *Let the notation and the assumptions be the same as in Theorem* 1. *We assume $\iota(\xi) = 1$. Then there exists a structure (A', θ') of type (K, Φ) which satisfies* (P′1), (P2) *and* (P3).

PROOF. We take two exact sequences and shall make similar considerations as in Iwasawa [3].

$$1 \longrightarrow E_K \longrightarrow U \longrightarrow U/E_K \longrightarrow 1. \tag{3.10}$$

$$1 \longrightarrow K^\times U/K^\times \longrightarrow K_A^\times/K^\times \longrightarrow K_A^\times/K^\times U \longrightarrow 1. \tag{3.11}$$

We abbreviate $\mathrm{Gal}(K/D)$ to G, and $H^i(G, B)$ to $H^i(B)$ for any G-module B. By ${}^G B$, we denote the module of G-invariants of B. Note that $K^\times U/K^\times \cong U/E_K$ as G-modules. Then we get long exact sequences of cohomology groups:

$$\cdots \longrightarrow H^1(U) \longrightarrow H^1(U/E_K) \xrightarrow{\delta_1} H^2(E_K) \xrightarrow{\iota} H^2(U) \longrightarrow \cdots \tag{3.12}$$

$$\cdots \longrightarrow H^0(K_A^\times/K^\times U) \xrightarrow{\delta_2} H^1(U/E_K) \longrightarrow 1 \longrightarrow \cdots \tag{3.13}$$

Since $\iota(\varphi(\xi))=1$, we have $\varphi(\xi)=\delta_1(\delta_2(c))$ for some $c\in H^0(K_A^\times/K^\times U)$. It is well known that $K_A^\times/K^\times U\cong C_K$ and $H^0(C_K)\cong {}^G C_K/N_{K/D}(C_K)$. Let $\bar{\mathfrak{b}}\in {}^G C_K$ be an element which represents c^{-1}. Take $\mathfrak{b}\in I_K$ and $b\in K_A^\times$ so that $\mathfrak{b} \bmod P_K=\bar{\mathfrak{b}}$, $\mathrm{div}(b)=\mathfrak{b}$. We get

$$b^\alpha=ba_\alpha f_\alpha, \qquad \alpha\in G, \tag{3.14}$$

with $a_\alpha\in K^\times$ and $f_\alpha\in U$. Then, by definition, $f_\alpha \bmod E_K$ defines a 1-cocycle taking values in U/E_K. We may assume

$$\varepsilon_{\alpha,\beta}=(f_{\alpha\beta}^{-1}f_\alpha^\beta f_\beta)^{-1}, \qquad \alpha,\ \beta\in G, \tag{3.15}$$

where $\varepsilon_{\alpha,\beta}$ is a suitable 2-cocycle which represents $\varphi(\xi)$. Then we have

$$\varepsilon_{\alpha,\beta}=a_{\alpha\beta}^{-1}a_\alpha^\beta a_\beta, \qquad \alpha,\ \beta\in G. \tag{3.16}$$

Clearly we may assume that $\mathfrak{b}$ is an integral ideal of K. We may also assume that an isomorphism μ_σ of A to A^σ is chosen for every $\sigma\in\mathrm{Gal}(k/k_0)$ so as to satisfy $\mu_{\sigma\tau}^{-1}\mu_\sigma^\tau\mu_\tau=\theta(\varepsilon_{\pi(\sigma),\pi(\tau)})$. Let $\mathfrak{G}$ be the group of $\mathfrak{b}$-section points of (A,θ). Define an abelian variety A^* by $A^*=A/\mathfrak{G}$ and let η be the canonical isogeny from A to A^* with the kernel $\mathfrak{G}$. Since $\mathfrak{G}$ is rational over k, A^* and η can be defined over k. Take $t\in \boldsymbol{Z}$ so that $ta_\alpha^{-1}\in\mathfrak{O}_K$ for every $\alpha\in G$. Put $b_\alpha=ta_\alpha^{-1}$ and define an isogeny of A to A^σ by $\mu'_\sigma=\mu_\sigma\circ\theta(b_{\pi(\sigma)})$. The kernel of the isogeny $\eta\theta(t)$ coincides with the group of $(t)\mathfrak{b}$-section points of (A,θ). We have

$$\mathrm{Ker}(\eta^\sigma\circ\mu'_\sigma)=\{x\in A\,|\,\theta(b_{\pi(\sigma)})x\in\mu_\sigma^{-1}(\mathfrak{G}^\sigma)\}. \tag{3.17}$$

Clearly $\mathfrak{G}^\sigma$ is the group of $\mathfrak{b}$-section points of (A^σ,θ^σ). Hence, using (2.1), we see that $\mu_\sigma^{-1}(\mathfrak{G}^\sigma)$ is contained in the group of $\mathfrak{b}^{\pi(\sigma)}$-section points of (A,θ). Comparing their orders (cf. [11], p. 61), we see that they coincide. Since $(b_{\pi(\sigma)})\mathfrak{b}^{\pi(\sigma)}=(t)\mathfrak{b}$ by (3.14), we obtain $\mathrm{Ker}(\eta^\sigma\circ\mu'_\sigma)=\mathrm{Ker}(\eta\theta(t))$. Therefore there exists an isomorphism ϕ_σ rational over k of A^* to $(A^*)^\sigma$ which makes the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "\eta\theta(t)"] \arrow[d, "\mu'_\sigma"'] & A^* \arrow[d, "\phi_\sigma"] \\ A^\sigma \arrow[r, "\eta^\sigma"'] & (A^*)^\sigma \end{tikzcd} \tag{3.18}$$

Then we can verify in a straightforward manner that $\phi_{\sigma\tau}=\phi_\sigma^\tau\phi_\tau$ for every $\sigma,\tau\in\mathrm{Gal}(k/k_0)$. Hence there exists an abelian variety A' defined over k_0 which is isomorphic to A^* over k. We can define an isomorphism θ^* of K into $\mathrm{End}(A^*)\otimes\boldsymbol{Q}$ so that $\eta\circ\theta(a)=\theta^*(a)\circ\eta$ for $a\in K$ and that $\theta^*(\mathfrak{O}_K)=\mathrm{End}(A^*)$ (cf. [11], §7, Prop. 7 and 8). Then (A^*,θ^*) is of type (K,Φ) and we can verify $\phi_\sigma^{-1}\theta^*(a)\phi_\sigma=\theta^*(a^{\pi(\sigma)})$, $a\in K$. Therefore we can define an isomorphism θ' of

K into $\mathrm{End}(A')\otimes \boldsymbol{Q}$ so that (A', θ') is isomorphic to (A^*, θ^*) and that (A', θ'_D) is rational over k_0. Thus (P′1) and (P2) are satisfied for (A', θ'). Since (A', θ') is isogenous to (A, θ) over k, (P3) is also satisfied. This completes the proof.

COROLLARY. *We fix a structure* $(A, \mathcal{C}, \theta)$ *as above. Let* k_0 *be an algebraic number field which contains* M. *Then, in the isogeny class of* (A, θ), *there exists a structure* (A', θ') *such that* (A', θ'_D) *is rational over* k_0 *and that* $\mathrm{End}(A')=\theta'(\mathfrak{O}_K)$ *if and only if there exists a Hecke character* ϕ *of* $k_A^\times$ *which satisfies* (A), (B), (C), (F) *and also* (D) *in the case* $D\subseteqq F$.

PROOF. It is enough to prove the "only if" part. Let (A', θ') be the structure as in the statement and η be an isogeny from (A', θ') to (A, θ). For $\sigma\in \mathrm{Aut}(\boldsymbol{C}/k_0)$, we have $\eta^\sigma=\mu_\sigma\theta(a_\sigma)\eta$ with some $a_\sigma\in K$. Also we have $\eta\theta'(a)=\theta(a)\eta$. From these relations, we obtain $\theta'(a)^\sigma=\theta'(a^{\pi(\sigma)})$ immediately. This shows that the Hecke character ϕ of $k_A^\times$ determined by (A', θ') over k satisfies (A), (B), (C), (D), (F).

REMARK 3. In a similar way as in the above proof, we can prove $\varphi(\xi)=1$, if the order of the group of ambiguous ideal classes of K modulo the ideal classes which can be represented by an ambiguous ideal of K is relatively prime to $|Z_K|$.

REMARK 4. Suppose that $D\subseteqq F$. Since ρ belongs to the center of $\mathrm{Gal}(K/D)$, we have $H^i(\mathrm{Gal}(K/D), Z_K)\cong(\boldsymbol{Z}/2\boldsymbol{Z})^{t_i}$ for $i>0$ with some non-negative integer t_i, by [2], p. 113, Cor. In this case, we see easily that assumptions (A), (B), (C), (D), (E) imply $\xi=1$ if a 2-Sylow subgroup of $\mathrm{Gal}(K/D)$ is cyclic.

§4. Construction of certain Hecke characters of M_K.

In this section, we shall examine the possibility of the construction of a Hecke character ϕ of M_K which satisfies (A), (B) and (D). To do so, we must of course assume that

$$M_F \not\supseteq K'. \tag{4.1}$$

We note that assumption (1.1) is still in force throughout the following. Put $k=M_K$ and $h=M_F$. Then $[k:h]=2$ by (4.1). Let F' be the maximal real subfield of K' and let $\mathfrak{f}_0$ be the conductor of K' as a class field over F'. Let ω be the character of $F_A'^\times$ which corresponds to K'. We have

$$\omega(x)=\prod_{\mathfrak{p}|\mathfrak{f}_0}\omega_{\mathfrak{p}}(x_{\mathfrak{p}})\,\mathrm{sgn}(x_{\infty 1})\cdots\mathrm{sgn}(x_{\infty n}), \tag{4.2}$$

if $x\in F_A'^\times$ and $x_v\in\mathfrak{O}_{F'_v}^\times$ for every finite place v of F' which does not divide $\mathfrak{f}_0$.

Here sgn denotes the signature function on $\boldsymbol{R}^{\times}$.

To construct ψ, we first impose the following assumptions.

(I) Every prime ideal of F' which divides (2) is unramified in K'.

(II) $E_K=E_F$.

(II') $E_{K'}=E_{F'}$.

(III) $P_{K'}\cap I_{F'}=P_{F'}$.

Hereafter we shall abbreviate $\mathfrak{O}_{K'}$ to $\mathfrak{O}'$. Let $\mathfrak{p}$ be a prime ideal of F' which divides $\mathfrak{f}_0$ and let $\mathfrak{P}$ be the prime divisor of $\mathfrak{p}$ in K'. By (I), $\omega_{\mathfrak{p}}|\mathfrak{O}_{F'_{\mathfrak{p}}}^{\times}$ coincides with the quadratic residue symbol mod $\mathfrak{p}$ in $\mathfrak{O}_{F'_{\mathfrak{p}}}^{\times}$ and it extends to the quadratic residue symbol mod $\mathfrak{P}$ in $\mathfrak{O}'^{\times}_{\mathfrak{P}}$, which we shall denote by $\tilde{\omega}_{\mathfrak{P}}$. Put $\mathfrak{f}=\prod_{\mathfrak{P}|\mathfrak{f}_0}\mathfrak{P}$. We first define a quasicharacter ψ_0 of $K'^{\times}\prod_{w}\mathfrak{O}'^{\times}_{w}K'^{\times}_{\infty}$ by

$$\psi_0(xyz)=\prod_{w|\mathfrak{f}}\tilde{\omega}_w(y_w)(f(z))_{\infty 1}^{-1}\quad\text{for } x\in K'^{\times}_{\infty},\ y\in\prod_{w}\mathfrak{O}'^{\times}_{w},\quad z\in K'^{\times}. \tag{4.3}$$

Here f denotes the map $\det\Phi'$ from $K'^{\times}_{A}$ to $K^{\times}_{A}$. By (4.2) and (II'), we get $\prod_{w|\mathfrak{f}}\tilde{\omega}_w(\varepsilon)(f(\varepsilon))_{\infty 1}^{-1}=\prod_{v|\mathfrak{f}_0}\omega_v(\varepsilon)N_{F'/Q}(\varepsilon)=\omega(\varepsilon)=1$ if $\varepsilon\in E_{K'}$. Hence we see that (4.3) is well-defined. We set

$$\psi_0(x)=\omega(x)|x|_{F'}^{-1}\qquad\text{for}\quad x\in F'^{\times}_{A}. \tag{4.4}$$

Put $T=F'^{\times}_{A}K'^{\times}\prod_{w}\mathfrak{O}'^{\times}_{w}K'^{\times}_{\infty}$ and $V=F'^{\times}_{A}\cap K'^{\times}\prod_{w}\mathfrak{O}'^{\times}_{w}K'^{\times}_{\infty}$. To see that ψ_0 extends to a quasicharacter of T, it is sufficient to verify that (4.3) and (4.4) are consistent on V. Take $x\in V$. Since $\operatorname{div}(x)\in I_{F'}\cap P_{K'}=P_{F'}$, we have $\operatorname{div}(x)=(a)$ with $a\in F'^{\times}$. Then we have $a^{-1}x\in F'^{\times}_{A}\cap\prod_{w}\mathfrak{O}'^{\times}_{w}K'^{\times}_{\infty}=\prod_{v}(\mathfrak{O}_{F'})^{\times}_{v}\times F'^{\times}_{\infty}$. Put $y=a^{-1}x$. It is enough to show

$$\prod_{w|f}\tilde{\omega}_w(y_w)(f(y))_{\infty 1}^{-1}=\omega(y)|y|_{F'}^{-1}. \tag{4.5}$$

As $(f(y))_{\infty 1}=|y|_{F'}\omega_{\infty}(y_{\infty})$, (4.5) follows immediately. Set

$$P=\{z\in K'^{\times}_{A}\mid f(z)\mathfrak{O}_K=(\beta),\ \beta\beta^{\rho}=|z|_0^{-1}\text{ for some }\beta\in K^{\times}\}, \tag{4.6}$$

$$I_0(\Phi')=\{\mathfrak{A}\in I_{K'}\mid \mathfrak{A}^{\Phi'}=(\beta),\ \beta\beta^{\rho}=N(\mathfrak{A})\text{ for some }\beta\in K^{\times}\}. \tag{4.7}$$

Then P (resp. $I_0(\Phi')$) is the subgroup of $K'^{\times}_{A}$ (resp. $I_{K'}$) which corresponds to the unramified class field k of K'. The map $x\to\operatorname{div}(x)$ gives an isomorphism $P/\prod_{w}\mathfrak{O}'^{\times}_{w}\times K'^{\times}_{\infty}\cong I_0(\Phi')$. We see easily that $P\supseteq T$. If

$$P=T\ (=F'^{\times}_{A}K'^{\times}\prod_{w}\mathfrak{O}'^{\times}_{w}K'^{\times}_{\infty}), \tag{4.8}$$

we see that $\psi=\psi_0\circ N_{k/K'}$ satisfies (A), (B), (D) (see below).

To weaken these assumptions and to look at the situation more closely, let us fix our attention to a specific CM-type studied by Shimura in [9]. Let G be the dihedral group of order $4n$ $(n\geqq 2)$ generated by α and β subject to the relations $\alpha^{2n}=\beta^2=1$, $\beta\alpha\beta=\alpha^{-1}$. Let L be a Galois extension of $\boldsymbol{Q}$ such that $\mathrm{Gal}(L/\boldsymbol{Q})\cong G$. We assume that K and F are subfields of L which correspond to the subgroups $\langle 1, \beta\rangle$ and $\langle 1, \beta, \alpha^n, \beta\alpha^n\rangle$ of G respectively. Moreover we assume that $\rho=\alpha^n$ on L and that Φ is equivalent to $\sum_{k=0}^{n-1}\alpha^k$. Then: (K, Φ) is primitive; K' is the subfield of L which corresponds to $\langle 1, \beta\alpha^{n-1}\rangle$; Φ' is equivalent to $\sum_{k=0}^{n-1}\alpha^{-k}$.

We assume

(IV) n is even.

(The odd dimensional case would be simpler; see [8], Theorem 9.5.) By (IV), K does not contain any imaginary quadratic field.

REMARK 5. If we assume

(I$'$) $\mathfrak{d}_{K'/F'}$ does not divide (2),

then we get (II$'$) and (III) by virtue of Prop. A.7. (iii) of [9].

LEMMA 5. *The finite abelian group P/T is an elementary abelian 2-group. Moreover we can take a representative for every coset of P/T in the form $\pi_1\cdots\pi_t$, where π_i is a prime element of $K'_{\mathfrak{P}_i}$ for a prime ideal $\mathfrak{P}_i|\mathfrak{f}$ considered as an element of $K_A'^{\times}$.*

PROOF. Take $x\in P$ and put $\mathfrak{x}=\mathrm{div}(x)$. Then $\mathfrak{x}\in I_0(\Phi')$. Set

$$I_0(K'/F')=P_{K'}\{\mathfrak{A}\in I_{K'}\,|\,\mathfrak{A}^{\rho}=\mathfrak{A}\}. \tag{4.9}$$

By Prop. A.7 (ii) of [9], we have $I_0(\Phi')\subseteqq I_0(K'/F')$. Hence we have $\mathfrak{x}^2\in P_{K'}I_{F'}$ and $\mathfrak{x}=(a)\mathfrak{b}\mathfrak{P}_1\cdots\mathfrak{P}_t$. Here $a\in K'^{\times}$, $\mathfrak{b}\in I_{F'}$ and $\mathfrak{P}_i$'s are prime ideals of K' which ramify in K'/F'. Then we see immediately that $x^2\in T$ and that $a^{-1}b^{-1}x\in \pi_1\cdots\pi_t\prod_w\mathfrak{O}_w'^{\times}K_\infty'^{\times}$, with an idèle b of $F_A'^{\times}$ such that $\mathrm{div}(b)=\mathfrak{b}$.

We use the notation in Lemma 5. Since $\mathfrak{P}_1\cdots\mathfrak{P}_t\in I_0(\Phi')$, we have $(\mathfrak{P}_1\cdots\mathfrak{P}_t)^{\Phi'}=(c)$, $cc^{\rho}=N(\mathfrak{P}_1\cdots\mathfrak{P}_t)$, $c\in K^{\times}$. By $\mathfrak{P}_i^{\rho}=\mathfrak{P}_i$, we get $(c^{\rho})=(c)$. Hence $c^{\rho}=\varepsilon c$, $\varepsilon\in E_F$ by (II). Then we get $c=\varepsilon^{\rho}c^{\rho}=\varepsilon^2 c$; i.e. $\varepsilon=\pm 1$. If $c^{\rho}=-c$, we get $cc^{\rho}=-c^2=N(\mathfrak{P}_1\cdots\mathfrak{P}_t)$. This implies that K contains an imaginary quadratic field, which is a contradiction. Therefore we have $c\in F$ and $c^2=N(\mathfrak{P}_1\cdots\mathfrak{P}_t)$.

Let $\mathfrak{p}_i$ be the prime ideal of F' such that $\mathfrak{p}_i=\mathfrak{P}_i^2$. We can write $K'_{\mathfrak{P}_i}=F'_{\mathfrak{p}_i}(\sqrt{\varpi_i})$ with $\varpi_i\in F'_{\mathfrak{p}_i}$. Since $\mathfrak{p}_i$ does not divide (2) and $K'_{\mathfrak{P}_i}$ is ramified over $F'_{\mathfrak{p}_i}$, we can take ϖ_i as a prime element of $F'_{\mathfrak{p}_i}$. Then $\sqrt{\varpi_i}$ is a prime element of $K'_{\mathfrak{P}_i}$ so that we may put $\pi_i=\sqrt{\varpi_i}$. We have $\psi_0((\pi_1\cdots\pi_t)^2)=\psi_0(\varpi_1\cdots\varpi_t)=$

$\omega(\varpi_1 \cdots \varpi_t)|\varpi_1 \cdots \varpi_t|_{F'}^{-1}=\omega(\varpi_1 \cdots \varpi_t)N(\mathfrak{P}_1 \cdots \mathfrak{P}_t)=\omega(\varpi_1 \cdots \varpi_t)c^2$. Note that $\omega_{\mathfrak{p}_i}(\varpi_i)=\omega_{\mathfrak{p}_i}(-1)$ since $-\varpi_i$ is a norm from $K'_{\mathfrak{P}_i}$. Therefore we obtain

$$(4.10) \qquad \psi_0((\varpi_1 \cdots \varpi_t)^2)=\Big(\prod_{i=1}^{t}\Big(\frac{-1}{\mathfrak{p}_i}\Big)\Big)c^2,$$

where $\Big(\frac{-1}{\mathfrak{p}_i}\Big)=\omega_{\mathfrak{p}_i}(-1)$. Here we make one more assumption.

(V) Every prime ideal $\mathfrak{p}$ of F' which ramifies in K' satisfies $\Big(\frac{-1}{\mathfrak{p}}\Big)=1$. Then we set

$$(4.11) \qquad \psi_0(\varpi_1 \cdots \varpi_t)=c\,.$$

Clearly, this assignment of the value of $\psi_0(\varpi_1 \cdots \varpi_t)$ for a complete set of representatives of a set of generators of P/T defines an extension of ψ_0 to a quasicharacter of P. Since $P=K'^{\times}N_{k/K'}(k_A^{\times})$, we can define a quasicharacter ψ of $k_A^{\times}$ by

$$(4.12) \qquad \psi=\psi_0 \circ N_{k/K'}\,.$$

Now let (A, θ) be a structure of type (K, Φ) and C be a polarization of A. The field of moduli M of (A, C) coincides with the field of moduli M_F of (A, C, θ_F) (cf. Lemma 1). We assume that A is isomorphic to $\boldsymbol{C}^n/\omega(\mathfrak{A})$ as a complex torus with an ideal $\mathfrak{A}$ of K. (Here ω is the map of $\boldsymbol{C}^n \cong K\otimes_{\boldsymbol{Q}}\boldsymbol{R}$ onto A used in §3). Then we have $M_F=\{x \in MK' \mid x^{\sigma}=x\}$, where $\sigma=\rho[\mathfrak{A}\mathfrak{A}^{\alpha^{n-1}}, MK'/K']^{-1}\in \mathrm{Gal}(MK'/F')$ (cf. [9], Prop. 6). Here $[\mathfrak{x}, MK'/K']\in \mathrm{Gal}(MK'/K')$ denotes the Artin symbol for $\mathfrak{x}\in I_{K'}$. Note that $[M_K : M_F]=2$.

PROPOSITION 4. *With the notation as above, assume that* (I), (II), (IV), (V) *are satisfied. Moreover assume either* (II′) *and* (III) *or* (I′). *Then* (A, C) *has a model over its field of moduli.*

PROOF. It is sufficient to verify that the Hecke character ψ of $k_A^{\times}$ constructed above satisfies (A), (B) and (D). By the definition (4.3), (A) is clearly satisfied. If $x\in h_A^{\times}$, we have $\psi(x)=\psi_0(N_{k/K'}(x))=\omega(N_{k/K'}(x))|N_{k/K'}(x)|_{F'}^{-1}=\chi(x)|x|_h^{-1}$, since $\chi=\omega\circ N_{k/K'}$ and $N_{k/K'}(x)=N_{h/F'}(x)$. Therefore we have proved (D). Take $x\in K_A'^{\times}$ such that $x_\infty=1$. Obviously $\psi_0(x)\in K^{\times}$. We have

$$(4.13) \qquad \psi_0(x)\psi_0(x)^{\rho}=|x|_{K'}^{-1}$$

for $x\in T$, and also for $x\in P$ since every representative of P/T as above satisfies (4.13). Let us show

$$(4.14) \qquad \psi_0(x)\mathfrak{O}_K=f(x)\mathfrak{O}_K\,.$$

We have (4.14) if $x \in T$. Let $x=\pi_1 \cdots \pi_t$ be a representative of a coset of P/T as in Lemma 5. Then we have $\psi_0(x)\mathfrak{O}_K=c\mathfrak{O}_K$ and $f(x)\mathfrak{O}_K=(\mathfrak{P}_1 \cdots \mathfrak{P}_t)^{\Phi'}=c\mathfrak{O}_K$ in the notation as before. This finishes the verification of (A), (B), (D) and completes the proof.

If $n=2$, which is the classical case treated by Hecke, we can considerably weaken the assumptions. To perform explicit computations below, let us remark the following. This example is studied in [11], p. 74, *c*); we can identify it with our present notation by putting $\alpha=\sigma$, $\beta=\sigma\tau$.

THEOREM 3. *With the notation and assumptions as above, we assume* $n=2$ *and* (I). *Then* $(A, \mathcal{C})$ *has a model over its field of moduli.*

PROOF. By Prop. A.7 of [9], we see that (II), (II′) and (III) are satisfied. Therefore, in view of the proof of Prop. 4, it is sufficient to obtain a contradiction from $(\mathfrak{P}_1 \cdots \mathfrak{P}_t)^{\Phi'}=(c)$, $c^2=N(\mathfrak{P}_1 \cdots \mathfrak{P}_t)$, $c \in F$ and

$$\prod_{i=1}^{t}\left(\frac{-1}{\mathfrak{p}_i}\right)=-1\,. \tag{4.15}$$

Here we have used the same notation as before. By Prop. A.7 of [9], we get

$$N_{F/\boldsymbol{Q}}(D(K/F))D(F/\boldsymbol{Q})=N_{F'/\boldsymbol{Q}}(D(K'/F'))D(F'/\boldsymbol{Q})\,. \tag{4.16}$$

By (4.15), we have $F=\boldsymbol{Q}(\sqrt{d})$, $d \in \boldsymbol{Z}$, $d \equiv 3 \bmod 4$. Hence (2) is ramified in F. By the assumption (I), (2) is ramified in F'. We can write $K=F(\sqrt{-(x+y\sqrt{d})})$ with $x, y \in \boldsymbol{Z}$ such that $x+y\sqrt{d}$ is totally positive. Then we have $F'=\boldsymbol{Q}(\sqrt{d'})$, $d'=x^2-y^2d$. Using $d \equiv 3 \bmod 4$, we get either $d' \equiv 1$ or $2 \bmod 4$ or $d'/4 \equiv 1$ or $2 \bmod 4$, since we can exclude the case where $4|x$ and $4|y$. Therefore we obtain $8 \| D(F'/\boldsymbol{Q})$. By (4.16), we have $2 \| N_{F/\boldsymbol{Q}}(D(K/F))$. Let $\mathfrak{q}$ be the prime factor of (2) in F. Since $(2)=N_{F/\boldsymbol{Q}}(\mathfrak{q})$, we must have $\mathfrak{q} \| D(K/F)$. On the other hand, we have $\mathfrak{q}^2 | D(K/F)$ since $[K:F]=2$, $\mathfrak{q}$ divides (2). This is a contradiction and we complete the proof of Theorem 3.

§ 5. Examples of abelian varieties of CM-type which have no models over their fields of moduli.

We shall show that there are "counter examples" even in Hecke's case if we drop the assumption (I).

PROPOSITION 5. *Let p be a rational prime such that* $p \equiv 1 \bmod 4$ *and let* $F'=\boldsymbol{Q}(\sqrt{p})$. *Suppose that the class number of* F' *is* 1 *and that* (2) *remains prime in* F'. *Take a rational prime q such that* $q \equiv 3 \bmod 8$, $\left(\frac{p}{q}\right)=1$. *Let* $q=\mathfrak{a}\mathfrak{a}'$

with a totally positive element $a\in\mathfrak{O}_{F'}$, where a' denotes the conjugate of a. Let $K'=F'(\sqrt{-a})$ and let (K, Φ) be the CM-*type of dihedral type for $n=2$ studied in* §4 *which has K' as the field of reflex. Let (A, θ) be a structure of type (K, Φ) and $\mathcal{C}$ be a polarization of A. Then $(A, \mathcal{C})$ has no model over its field of moduli.*

PROOF. Note that we have $F=\boldsymbol{Q}(\sqrt{q})$, $K=F(\sqrt{-2(x+\sqrt{q})})$ if $a=x+y\sqrt{p}$, $x, y\in\boldsymbol{Q}$. It is clear that the prime ideal (a) of F' ramifies in K' and that all the other prime ideals of F' except (2) do not ramify in K'. Let ω be the character of $F_A'^{\times}$ which corresponds to K'. By (4.2) and $q\equiv 3 \bmod 4$, we see that

$$\omega_2(-1)=-1\,. \tag{5.1}$$

Hence (2) is ramified in K'. Let $\mathfrak{q}$ be the prime factor of (2) in K'. By (4.16), we see that the prime factor of (2) in F ramifies in K. This implies that there is the unique prime factor of $2\boldsymbol{Z}$ in the normal closure of K over $\boldsymbol{Q}$. Therefore we have $\mathfrak{q}^{\Phi'}=(2)$; i.e. $\mathfrak{q}\in I_0(\Phi')$. As $\left(\frac{2}{q}\right)=-1$, we have $2\notin N_{K'/F'}(K'^{\times})$. From this, we see easily that $\mathfrak{q}$ is not a principal ideal of K'. By Prop. A.1 of [9], we have $[I_0(K'/F'):P_{K'}]=2$. Since $I_0(\Phi')\subseteqq I_0(K'/F')$, we obtain

$$[I_0(\Phi'):P_{K'}]=2\,. \tag{5.2}$$

By a similar method as in the genus theory of quadratic number fields (cf. [1], p. 271, Aufgabe 25), we can prove that the class number of K' divided by 2 is an odd integer using $\left(\frac{2}{q}\right)=-1$. Therefore, by (5.2), $[M_K:K']$ is odd.

Now put $k=M_K$, $h=M_F$ and assume that there exists a Hecke character ψ of $k_A^{\times}$ which satisfies (A), (B) and (D). As $[h:F']$ is odd and $\mathfrak{q}$ is completely decomposed in k, there must exist a prime factor $\mathfrak{p}$ of (2) in h such that $\mathfrak{p}$ is unramified in h/F' and that the relative degree of $\mathfrak{p}$ over F' is 1. Then $\mathfrak{p}$ ramifies in k; so let $\mathfrak{P}$ be the prime factor of $\mathfrak{p}$ in k. By (D), we have

$$\psi_{\mathfrak{P}}(x)=\chi_{\mathfrak{p}}(x) \qquad \text{if} \quad x\in\mathfrak{O}_{h_{\mathfrak{p}}}^{\times}\,. \tag{5.3}$$

Take $x\in\mathfrak{O}_{k_{\mathfrak{P}}}^{\times}$ and put $y=\psi_{\mathfrak{P}}(x)$. Then, by (B), we have $y\in K^{\times}$, $yy^{\rho}=1$ and $y\mathfrak{O}_K=\mathfrak{O}_K$. Hence we get $y\in Z_K=\{\pm 1\}$. Thus we obtain

$$\psi_{\mathfrak{P}}(x)=\pm 1 \qquad \text{for} \quad x\in\mathfrak{O}_{k_{\mathfrak{P}}}^{\times}\,. \tag{5.4}$$

We see that $k_{\mathfrak{P}}/h_{\mathfrak{p}}$ and $K_{\mathfrak{q}}'/F_2'$ are isomorphic as quadratic extensions of local fields. Therefore we may assume that there exists a character $\tilde{\omega}$ of $\mathfrak{O}_{K_{\mathfrak{q}}'}^{\times}$ such that

$$\tilde{\omega}(x)=\omega_2(x) \quad \text{if} \quad x\in\mathfrak{O}_{F'_2}^{\times}, \tag{5.5}$$

$$\tilde{\omega}(x)=\pm 1 \quad \text{if} \quad x\in\mathfrak{O}_{K'_q}^{\times}. \tag{5.6}$$

We have $\omega_2(a)=1$ since a is a norm. By (5.1), we get $\omega_2(-a)=-1$. Then $\omega_2(-a)=\tilde{\omega}(-a)=(\tilde{\omega}(\sqrt{-a}))^2=1$ by (5.5) and (5.6). This is a contradiction and completes the proof.

Numerical Example. Take $F'=\boldsymbol{Q}(\sqrt{5})$, $K'=F'(\sqrt{-(8+3\sqrt{5})})$. Then we have $F=\boldsymbol{Q}(\sqrt{19})$ and $K=F(\sqrt{-2(8+\sqrt{19})})$. We see that the class number of K' is 2, using either Minkowski's bound or the Shintani class number formula [12]. Since $[I_0(\Phi'): P_{K'}]=2$ as above, we have $M_K=K'$ and $M_F=F'$ (for every $(A, \mathcal{C})$). Then the examination of the extensibility of ω_2 with value group $\{\pm 1\}$ shows that there is no ψ satisfying (A), (B), (D).

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