# *Congruences between Hilbert cusp forms and units in quartic fields*

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*To the memory of Takuro Shintani*

## § 0. Introduction.

In previous papers (Doi, Saito, Yamauchi [1], [4], [5]), we studied cusp forms of level $q^{\nu}$ for a prime $q$ in the case of elliptic modular forms and especially showed that their coefficients have certain congruence properties. In this paper, we begin the study of Hilbert cusp forms of level $\mathfrak{q}^{\nu}$ for a prime ideal $\mathfrak{q}$. The purpose of this paper is to report an interesting example of Fourier coefficients of a Hilbert cusp form over a real quadratic field. It will be shown that these Fourier coefficients satisfy a congruence relation modulo a certain prime ideal which can be related to a unit in a quartic field. We note this quartic field has one complex prime and two real primes, and quartic fields of this type were discussed in Shintani [8]. More examples of this type and other types of examples will be discussed in a subsequent paper.

In § 1, we collect some results on the operator $U_{\chi}$ which was introduced in [5] in the case of one variable, and in § 2 we discuss our example.

**§ 1.** Let $F$ be a totally real field with $[F:\boldsymbol{Q}]=g$, and $\mathfrak{o}$ its ring of integers. For the sake of simplicity, we assume the class number of $F$ is one and $[E, E_{+}]=2^{g}$, where $E$ is the group of units of $F$ and $E_{+}$ is its subgroup consisting of totally positive elements. Let $GL_2(F)^{+}$ be the subgroup of $GL_2(F)$ consisting of elements with totally positive determinants, and $GL_2(\mathfrak{o})^{+}=GL_2(F)^{+}\cap GL_2(\mathfrak{o})$. For a prime ideal $\mathfrak{q}$ of $F$ and a positive integer $\nu$, we denote by $\Gamma_0(\mathfrak{q}^{\nu})$ the congruence subgroup given by

$$\Gamma_0(\mathfrak{q}^{\nu})=\left\{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(\mathfrak{o})^{+}\,\middle|\, c\equiv 0 \mod \mathfrak{q}^{\nu}\right\}.$$

For an even positive integer $k$, let $S_k(\mathfrak{q}^{\nu})$ denote the space of cusp forms with respect to $\Gamma_0(\mathfrak{q}^{\nu})$ of weight $k$, namely the space of holomorphic functions on the $g$-fold product $H^{g}$ of the complex upper half planes satisfying (i) $f|[\gamma]_k=f$ for $\gamma\in\Gamma_0(\mathfrak{q}^{\nu})$ and (ii) $f$ vanishes at each cusp of $\Gamma_0(\mathfrak{q}^{\nu})$. Here for a functions $f$ on

$H^g$ and $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(F)^+$, we put

$$(f|[\gamma]_k)(z)=\prod_{i=1}^{g}(c^{(i)}z_i+d^{(i)})^k(\det\gamma^{(i)})^{-k/2}f(\gamma(z)),$$

where $z=(z_1,\cdots,z_g)\in H^g$ and for $x\in F$, $x^{(i)}$ denotes the $i$-th embedding of $x$ into $\boldsymbol{R}$.

For a prime ideal $\mathfrak{p}$, let $F_{\mathfrak{p}}$ denote the completion of $F$ at $\mathfrak{p}$ and $\mathfrak{o}_{\mathfrak{p}}$ its ring of integers. We choose and fix a totally positive generator $\alpha$ of $\mathfrak{q}$. Let $\chi$ be a character of $\mathfrak{o}$ with the conductor $\mathfrak{q}^{\mu}$. For $\chi$, we define

$$\Xi_{\mathfrak{q}}(U_\chi)=\left\{g\in\begin{pmatrix} \alpha^{\nu+2\mu}\mathfrak{o}_{\mathfrak{q}} & \alpha^{\nu+\mu}\mathfrak{o}_{\mathfrak{q}}^{\times} \\ \alpha^{2\nu+\mu}\mathfrak{o}_{\mathfrak{q}}^{\times} & \alpha^{\nu+2\mu}\mathfrak{o}_{\mathfrak{q}}^{\times} \end{pmatrix}\middle| v_{\mathfrak{q}}(\det g)=2\nu+4\mu\right\},$$

where $v_{\mathfrak{p}}$ denotes the additive valuation of $\mathfrak{o}_{\mathfrak{p}}$ normalized by $v_{\mathfrak{p}}(\alpha)=1$. For $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put

$$\tilde{\chi}_{\mathfrak{q}}(g)=\chi(-bc/\alpha^{3\nu+2\mu})\bar{\chi}(\det g/\alpha^{2\nu+4\mu}).$$

Let $\Xi(U_\chi)=GL_2(F)^+\cap(\prod_{\mathfrak{p}\neq\mathfrak{q}}GL_2(\mathfrak{o}_{\mathfrak{p}})\times\Xi_{\mathfrak{q}}(U_\chi))$ and for $g\in\Xi(U_\chi)$, we put $\tilde{\chi}(g)=\tilde{\chi}_{\mathfrak{q}}(g_{\mathfrak{q}})$. Here $g_{\mathfrak{q}}$ denotes the $\mathfrak{q}$-component of $g$. For $\chi$ with $1\leqq\mu\leqq\nu/2$, we define an operator $U_\chi$ on $S_k(\mathfrak{q}^{\nu})$ by

$$f|U_\chi=\frac{1}{\mathfrak{g}(\bar{\chi})^2}\sum_{g\in\Gamma_0(\mathfrak{q}^{\nu})\backslash\Xi(U_\chi)}\tilde{\chi}(g)f|[g]_k \qquad \text{for} \quad f\in S_k(\mathfrak{q}^{\nu}),$$

where $\mathfrak{g}(\bar{\chi})=\sum_{a \bmod \mathfrak{q}^{\mu}}\bar{\chi}(a)e^{2\pi\sqrt{-1}\,\mathrm{Tr}\,a/(\mathfrak{d}\alpha^{\mu})}$. Here $\mathfrak{d}$ is a totally positive generator of the different of $F$ over $\boldsymbol{Q}$, which is fixed in the following.

For the trivial character $\chi_1$, we put $f|U_{\chi_1}=f$. For $f\in S_k(\mathfrak{q}^{\nu})$, we put

$$f|W=f\left|\left[\begin{pmatrix} 0 & -1 \\ \alpha^{\nu} & 0 \end{pmatrix}\right]_k\right..$$

For an integral ideal $\mathfrak{n}$ prime to $\mathfrak{q}$, the Hecke operator $T(\mathfrak{n})$ is defined in the usual way. Let $S_k^0(\mathfrak{q}^{\nu})$ denote the space of new forms in $S_k(\mathfrak{q}^{\nu})$. Then we see $U_\chi$, $W$ and $T(\mathfrak{n})$ satisfy the followings; (i) If $f$ is an old form in $S_k(\mathfrak{q}^{\nu})$, then $f|U_\chi=0$, (ii) $f|U_\chi U_{\chi'}=U_{\chi\chi'}$ for $f\in S_k^0(\mathfrak{q}^{\nu})$, if the conductors $\mathfrak{f}_\chi$ and $\mathfrak{f}_{\chi'}$ of $\chi$ and $\chi'$ satisfy $v_{\mathfrak{q}}(\mathfrak{f}_\chi)\leqq\frac{\nu}{3}$ and $v_{\mathfrak{q}}(\mathfrak{f}_{\chi'})\leqq\frac{\nu}{3}$, (iii) $f|U_\chi W=f|WU_\chi$ for $f\in S_k^0(\mathfrak{q}^{\nu})$, (iv) $f|U_\chi T(\mathfrak{n})=f|T(\mathfrak{n})U_\chi$ for $f\in S_k^0(\mathfrak{q}^{\nu})$. Using the above properties, we can define a decomposition of $S_k^0(\mathfrak{q}^{\nu})$ as follows;

$$S_k^0(\mathfrak{q}^\nu)=\bigoplus_{a\in(\mathfrak{o}/\mathfrak{q}^{[\nu/3]})^\times}(S^+(a)\oplus S^-(a))$$

$$S^\pm(a)=\{f\in S_k(\mathfrak{q}^\nu)\mid f|U_\chi=\bar{\chi}(a)f \text{ for } \chi\in X(q^{[\nu/3]}),\ f|W=\pm f\}\ ,$$

where $X(q^{[\nu/3]})$ denotes the group of characters of $(\mathfrak{o}/q^{[\nu/3]})^\times$. As in [5], we can give a formula of $\mathrm{tr}\, T(\mathfrak{n})$ on $S^\pm(a)$. In the calculation of the example in § 2, we need the formula in the case where $\nu=3$, $N\mathfrak{q}=q$ with a prime $q$ and $\left(\frac{a}{\mathfrak{q}}\right)=1$, which takes a rather simple form, and we give it in the following as Theorem 1.

THEOREM 1. *The notation being as above, assume* $\left(\frac{a}{\mathfrak{q}}\right)=1$ *and* $\left(\frac{n}{\mathfrak{q}}\right)=1$ *for a totally positive generator* $n$ *of* $\mathfrak{n}$. *Then we have*

$$\mathrm{tr}\, T(\mathfrak{n})\mid S^\pm(a)=\frac{1}{2q-2}(A\pm 2B)\,,$$

$$\begin{aligned}A=&\delta(n)\frac{(q+1)(q-1)^2}{2}(k-1)^g\zeta_F(-1)\\&+\frac{1}{2}(-1)^g\sum_{s,f}(q-1)\Big(\sum_{lm\equiv(4an)^{-1}D_1/\alpha \bmod \mathfrak{q}}\zeta^{l+m}\Big)\varphi_1(s,\ n)h(D_1/f^2)/w(D_1/f^2)\\&+\frac{1}{2}(-1)^g\sum_{s,f}2(1-q)\varphi_1(s,\ n)h(D_1/f^2)/w(D_1/f^2)\\&+\frac{1}{2}(-1)^g\sum_{s,f}\varphi_1(s,\ n)(h(D_1/\alpha^2f^2)/w(D_1/\alpha^2f^2)-qh(D_1/f^2)/w(D_1/f^2))\end{aligned}$$

$$B=\frac{1}{2}(-1)^g\sum_{s,f}\frac{q-1}{2}(\zeta^m+\zeta^{-m})\varphi_2(s,\ n)h(D_2/f^2)/w(D_2/f^2)\,,$$

*where* $\delta(n)$ *is* 0 *or* 1 *according as* $n$ *is a square or not, and* $\zeta_F(s)$ *is the Dedekind zeta function of* $F$. $\zeta=e^{2\pi\sqrt{-1}\,\mathrm{Tr}\,(1/\mathfrak{d}\alpha)}$, *where* $\mathfrak{d}$ *is the totally positive generator of the different of* $F$ *over* $\boldsymbol{Q}$. *In the term* $A$, $s$ *runs through all integers which satisfy the condition that* $D_1=s^2-4n$ *is totally negative and* $v_\mathfrak{q}(D_1)=1$ *in the first sum,* $v_\mathfrak{q}(D_1)=2$ *and* $\left(\frac{D_1/\alpha^2}{\mathfrak{q}}\right)=-1$ *in the second sum, and* $v_\mathfrak{q}(D_1)=3$ *in the third sum respectively. For* $x$ *in* $F$, *let* $x^{(i)}$ $(1\leqq i\leqq g)$ *denote all the conjugates of* $x$ *over* $\boldsymbol{Q}$. $\varphi_1(s,\ n)=\prod_{i=1}^{g}\frac{\xi(i)^{k-1}-\eta(i)^{k-1}}{\xi(i)-\eta(i)}$, *where* $\xi(i)$ *and* $\eta(i)$ *are the roots of the equation* $X^2-s^{(i)}X+n^{(i)}=0$. *In the term* $B$, $s$ *runs through all integers such that* $D_2=(\alpha s^2-4n)\alpha$ *is totally negative, and* $\varphi_2(s,\ n)=q^{1-k/2}\prod_{i=1}^{g}\frac{\xi(i)^{k-1}-\eta(i)^{k-1}}{\xi(i)-\eta(i)}$, *where* $\xi(i)$ *and* $\eta(i)$ *are the roots of* $X^2-\alpha^{(i)}s^{(i)}X+\alpha^{(i)}n^{(i)}=0$. $m$ *is a solution of* $(an)^{-1}s^2\equiv m^2 \bmod \mathfrak{q}$. *In* $A$ (*resp.* $B$), $f$ *runs through all totally positive integers* *modulo* $E_+$ *which satisfy* $(f, \mathfrak{q})=1$, $f^2|D_1$ (*resp.* $f^2|D_2$) *and the condition that the congruence equation* $X^2\equiv D_1/f^2 \bmod 4$ (*resp.* $X^2\equiv D_2/f^2 \bmod 4$) *has a solution in* $\mathfrak{o}$. *For such* $f$, *there exists a unique* $\mathfrak{o}$-*order* $\Lambda$ *of* $F(\sqrt{D_i})$ *with* $\mathfrak{o}+f\Lambda =\mathfrak{o}\left[\frac{s+\sqrt{D_1}}{2}\right]$ *for* $i=1$, $\mathfrak{o}\left[\frac{\alpha s+\sqrt{D_2}}{2}\right]$ *for* $i=2$, *and* $h(D_i/f^2)$ *is its class number and* $w(D_i/f^2)=[\Lambda^\times : E]$.

The details of the result in this section will appear in [6].

## § 2. The Example.

We take $F=\boldsymbol{Q}(\sqrt{5})$, $\mathfrak{q}=(4+\sqrt{5})$, $\alpha=4+\sqrt{5}$, $\nu=3$, $k=2$ and $\mathfrak{d}=\frac{5-\sqrt{5}}{2}$. Then we find $N\mathfrak{q}=11$, and $\dim S^+(9)=2$, $\dim S^-(9)=0$. For a prime ideal $\mathfrak{p}$ prime to $\mathfrak{q}$, we denote by $f_{T_\mathfrak{p}}(X)$ the characteristic polynomial of $T_\mathfrak{p}$ on $S^+(9)$. Let $C(\mathfrak{p})$ be the eigenvalue for $T_\mathfrak{p}$ of $f\in S^+(9)$, then $f_{T_\mathfrak{p}}(X)=(X-C(\mathfrak{p}))^2$ or $X^2-C(\mathfrak{p})^2$ according as $\left(\frac{\beta}{\mathfrak{q}}\right)=1$ or $-1$, where $\beta$ is a totally positive generator of $\mathfrak{p}$. We give $f_{T_\mathfrak{p}}(X)$ for some prime ideals $\mathfrak{q}$ in the following table.

| $\mathfrak{p}=(\beta)$ | $\left(\frac{\beta}{\mathfrak{q}}\right)$ | $f_{T_\mathfrak{p}}(X)$ |
|---|---|---|
| (2) | $-1$ | $X^2-(7-2\alpha_1+2\alpha_2-2\alpha_4)$ |
| (3) | 1 | $(X-2-2\alpha_1-2\alpha_3)^2$ |
| $((5-\sqrt{5})/2)$ | $-1$ | $X^2-(8-\alpha_2+4\alpha_4)$ |
| (23) | 1 | $(X+18+2\alpha_1+4\alpha_2-12\alpha_3-4\alpha_4)^2$ |
| $((11-\sqrt{5})/2)$ | 1 | $(X+2\alpha_2+\alpha_4)^2$ |
| (67) | 1 | $(X+40+26\alpha_1+8\alpha_2-42\alpha_3-2\alpha_4)^2$ |

Here $\alpha_i=e^{2\pi i\sqrt{-1}/11}+e^{-2\pi i\sqrt{-1}/11}$ and $F_{11}=\boldsymbol{Q}(\alpha_i)$ is the maximal real subfield of $K_{11}=\boldsymbol{Q}(e^{2\pi\sqrt{-1}/11})$. We note $N_{F_{11}/\boldsymbol{Q}}(C((2)))=N_{F_{11}/\boldsymbol{Q}}(C(((5-\sqrt{5})/2)))=1541=23\cdot 67$. The characteristic polynomials in the above table were obtained using the formula in § 1 and the class number formula by Shintani [7].

Now let us consider the quadratic extension $K=F(\sqrt{3+2\sqrt{5}})$ of $F$, where $N(3+2\sqrt{5})=-11$. This field $K$ has two real primes and one complex prime, hence the rank of the group of units in $K$ is two. Let $E_K$ denote the group of units in $K$, and $E_K^1$ its subgroup consisting of elements which satisfy $N_{K/F}\varepsilon=1$. We denote the fundamental unit $\frac{1+\sqrt{5}}{2}$ of $F$ by $\varepsilon_F$ and put

$$\varepsilon_0=\frac{1+\sqrt{3+2\sqrt{5}}}{2}$$

$$\varepsilon_1=\frac{-3-\sqrt{5}+(1-\sqrt{5})\sqrt{3+2\sqrt{5}}}{4}.$$

Then we have

LEMMA 2. *The notation being as above, it holds*

$$E_K^1=\langle\pm 1,\ \varepsilon_1\rangle$$

$$E_K=\langle\pm 1,\ \varepsilon_F,\ \varepsilon_0\rangle=\langle\pm 1,\ \varepsilon_1,\ \varepsilon_0\rangle.$$

PROOF. If we prove $E_K^1=\langle\pm 1, \varepsilon_1\rangle$, then the assertion on $E_K$ follows from it by noting that $N_{K/F}\varepsilon_0=-\varepsilon_F$ and $\varepsilon_0^2=-\varepsilon_1\varepsilon_F$. Let $\varepsilon$, $\varepsilon'$ be elements in $E_K^1$ such that $\mathrm{Tr}_{K/F}\varepsilon=s>0$, $\mathrm{Tr}_{K/F}\varepsilon'=s'>0$. Since $s$, $s'>2$, we see $\mathrm{Tr}_{K/F}\varepsilon\varepsilon'>\mathrm{Tr}_{K/F}\varepsilon$ and $\mathrm{Tr}_{K/F}\varepsilon\varepsilon'>\mathrm{Tr}_{K/F}\varepsilon'$. By some calculation, it can be shown that $-\varepsilon_1$ is the unit in $E_K^1$ which has the smallest positive $s$. Our assertion follows from this.

We note that $\varepsilon_1$ can be characterized as a generator of $E_K^1$ which satisfies $\varepsilon_1\equiv 1 \bmod \mathfrak{p}_{11}$, where $\mathfrak{p}_{11}=(\sqrt{3+2\sqrt{5}})$. Taking the 11-th power of $\varepsilon_1$, we find

$$\varepsilon_1^{11}=\frac{-4623-2069\sqrt{5}+(1691+757\sqrt{5})\sqrt{3+2\sqrt{5}}}{4}.$$

Here we remark $4623=3\cdot 23\cdot 67$. The primes 23 and 67 coincide with the primes which appear in $N_{F_{11}/Q}(C((2)))$. In the following, we shall explain this relation.

First, we see the class number of $K$ is one by the Minkowski constant, and the primes $\mathfrak{p}_{23}=(23)$ and $\mathfrak{p}_{67}=(67)$ in $F$ decompose in $K$. Let us denote the primes in $K$ over $\mathfrak{p}_{23}$ and $\mathfrak{p}_{67}$ by $\tilde{\mathfrak{p}}_{23}$, $\tilde{\mathfrak{p}}_{23}'$ and $\tilde{\mathfrak{p}}_{67}$, $\tilde{\mathfrak{p}}_{67}'$ respectively. Then for example, we can choose

$$\tilde{\mathfrak{p}}_{23}=\left(\frac{1+9\sqrt{5}+(1+\sqrt{5})\sqrt{3+2\sqrt{5}}}{4}\right)$$

$$\tilde{\mathfrak{p}}_{67}=\left(\frac{-15-9\sqrt{5}+(11-7\sqrt{5})\sqrt{3+2\sqrt{5}}}{4}\right).$$

Let $\mathfrak{p}_\infty$ be the infinite prime of $F$ which satisfies $\mathrm{sgn}_{\mathfrak{p}_\infty}(\varepsilon_F)=1$, and $\tilde{\mathfrak{p}}_\infty$ be the infinite prime of $K$ over $\mathfrak{p}_\infty$ such that $\mathrm{sgn}_{\tilde{\mathfrak{p}}_\infty}(\varepsilon_0)=1$. We define ideal class characters $\lambda_{23}$ and $\lambda_{67}$ of $K$ with the conductors $\mathfrak{f}(\lambda_{23})=\tilde{\mathfrak{p}}_{11}^2\tilde{\mathfrak{p}}_{23}\tilde{\mathfrak{p}}_\infty$ and $\mathfrak{f}(\lambda_{67})=\tilde{\mathfrak{p}}_{11}^2\tilde{\mathfrak{p}}_{67}\tilde{\mathfrak{p}}_\infty$. Since the class number of $K$ is one, it is enough to define their values for principal ideals. Let $\mathfrak{P}_{23}$ be the prime ideal of $F_{11}$ which divides $C((2))^2$ or the constant term of $f_{T(2)}(X)$, and $\omega_{23}$ the character of $(\boldsymbol{Z}/23)^\times$ such that $\omega_{23}(a)\equiv a \bmod \tilde{\mathfrak{P}}_{23}$, where $\tilde{\mathfrak{P}}_{23}=(e^{2\pi\sqrt{-1}/11}-4,\ 23)\subset K_{11}$, which lies over $\mathfrak{P}_{23}$. Under the natural isomorphism $(\mathfrak{o}_K/\tilde{\mathfrak{p}}_{11}^2)^\times \cong (\mathfrak{o}_F/\mathfrak{p}_{11})^\times \times ((1+\mathfrak{p}_{11})/(1+\mathfrak{p}_{11}^2))$, let $(a, 1+b\sqrt{3+2\sqrt{5}})$ $(a, b\in \boldsymbol{Z})$ be the element which corresponds to $x\in(\mathfrak{o}_K/\tilde{\mathfrak{p}}_{11}^2)^\times$. We put $\psi_{11}(x)=\left(\frac{a}{11}\right)(-1)^b\omega(15^b)$, and for $(\alpha)$ prime to $\tilde{\mathfrak{p}}_{11}\tilde{\mathfrak{p}}_{23}$, we define

$$\lambda_{23}((\alpha))=\psi_{11}(\alpha)\omega(N_{F/Q}(\tilde{\alpha}))\,\mathrm{sgn}_{\,\mathfrak{p}_\infty}(\alpha)\,,$$

where $\tilde{\alpha}$ is an element of $\mathfrak{o}_F$ such that $\tilde{\alpha}\equiv\alpha \bmod.\tilde{\mathfrak{p}}_{23}$. Then we have

PROPOSITION 3. *The notation being as above, $\lambda_{23}$ defines an ideal class character of $K$ with the conductor $\tilde{\mathfrak{p}}_{11}^2\tilde{\mathfrak{p}}_{23}\tilde{\mathfrak{p}}_\infty$.*

PROOF. It is enough to show that $\lambda_{23}(E_K)=1$. For $\varepsilon_0$, we have $\varepsilon_0 \equiv -5+\sqrt{5} \bmod.\tilde{\mathfrak{p}}_{23}$ and $N_{F/Q}(-5+\sqrt{5})=20$. Then we see

$$\lambda(\varepsilon_0)=\left(\frac{2}{11}\right)(-\omega(15))\omega(20)\,\mathrm{sgn}_{\,\tilde{\mathfrak{p}}_\infty}(\varepsilon_0)$$
$$=1\,.$$

For $\varepsilon_F$ and $-1$, we can check the condition easily.

Let $\tilde{\varepsilon}_1$ be an element of $\mathfrak{o}_F$ such that $\tilde{\varepsilon}_1\equiv\varepsilon_1 \bmod.\tilde{\mathfrak{p}}_{23}$. We remark that $\tilde{\varepsilon}_1$ gives a generator of $(1+\mathfrak{p}_{11})/(1+\mathfrak{p}_{11}^2)$ and the fact that 23 divides $\mathrm{Tr}_{K/Q}\varepsilon_1$ implies that the order of $N_{F/Q}\varepsilon_1$ in $(\boldsymbol{Z}/23)^\times$ is 22.

Next, we take the prime 67. Let $\mathfrak{P}_{67}$ be the prime ideal in $F_{11}$ which divides $C((2))$ and $\omega_{67}$ the character of $(\boldsymbol{Z}/67)^\times$ which satisfies $\omega(a)\equiv a \bmod.\mathfrak{P}_{67}$, where $\tilde{\mathfrak{P}}_{67}=(e^{2\pi\sqrt{-1}/33}-23, 67)$ in $K_{33}=\boldsymbol{Q}(e^{2\pi\sqrt{-1}/33})$, and $\tilde{\mathfrak{P}}_{67}$ lies over $\mathfrak{P}_{67}$. In this case, we put $\psi_{11}(x)=\left(\frac{a}{11}\right)(-1)^b\omega(45^b)$ in the above notation. For $(\alpha)$ prime to $\tilde{\mathfrak{p}}_{11}\tilde{\mathfrak{p}}_{67}$, we define

$$\lambda_{67}((\alpha))=\psi_{11}(\alpha)\omega_{67}(N_{F/Q}(\tilde{\alpha}))\,\mathrm{sgn}_{\,\tilde{\mathfrak{p}}_\infty}(\alpha)\,,$$

where $\tilde{\alpha}$ is an element in $\mathfrak{o}_F$ such that $\tilde{\alpha}\equiv\alpha \bmod.\tilde{\mathfrak{p}}_{67}$. Then as above, we have

PROPOSITION 4. *$\lambda_{67}$ gives an ideal class character of $K$ with the conductor $\tilde{\mathfrak{p}}_{11}^2\tilde{\mathfrak{p}}_{67}\tilde{\mathfrak{p}}_\infty$.*

By a result of Jacquet-Langlands [2], the $L$-functions of $\lambda_{23}$ and $\lambda_{67}$ correspond to Hilbert cusp forms $f_{23}$ and $f_{67}$, which are of weight 1 and have levels $\mathfrak{p}_{11}^3\mathfrak{p}_{23}$ and $\mathfrak{p}_{11}^3\mathfrak{p}_{67}$ respectively.

Now let us compare these cusp forms with the cusp form given in the example. Let $a(\mathfrak{p})$ be the eigenvalue of $f_{23}$ for $T(\mathfrak{p})$. For several $\mathfrak{p}$, $a(\mathfrak{p})$ are given as follows;

$$a((2))=0\,,\quad a((3))=\zeta_{11}^{3}+\zeta_{11}^{5}\,,\quad a((5-\sqrt{5})/2)=0\,,$$
$$a((23))=-\zeta_{11}^{2}\,,\quad a((11-\sqrt{5})/2)=-\zeta_{11}-\zeta_{11}^{9}\,,$$
$$a((67))=-\zeta_{11}^{5}-\zeta_{11}^{7}\ (\zeta_{11}=e^{2\pi\sqrt{-1}/11})\,.$$

Then for these values, we find

$$f_{T_{\mathfrak{p}}}(a(\mathfrak{p}))\equiv 0 \qquad \mathrm{mod.}\,\tilde{\mathfrak{P}}_{23}\,.$$

For $f_{67}$, let $b(\mathfrak{p})$ be the eigenvalue for $T(\mathfrak{p})$, then

$$b((2))=0\,,\quad b((3))=-\zeta_{33}^{17}-\zeta_{33}^{7}\,,\quad b((5-\sqrt{5})/2)=0$$
$$b((23))=-\zeta_{33}^{17}-\zeta_{33}^{18}\,,\quad b((11-\sqrt{5})/2)=\zeta_{33}^{9}+\zeta_{33}^{2}$$
$$b((67))=-\zeta_{33}^{3}\ (\zeta_{33}=e^{2\pi\sqrt{-1}/33})\,.$$

and we also find

$$f_{T_{\mathfrak{p}}}(b(\mathfrak{p}))\equiv 0 \qquad \mathrm{mod.}\,\tilde{\mathfrak{P}}_{67}\,.$$

In fact, we can prove

THEOREM 5. *Let* $\bar{\mathfrak{P}}_{23}$ *(resp.* $\bar{\mathfrak{P}}_{67}$*) be the prime ideal in* $K_{11}(\sqrt{C(2)})$ *(resp.* $K_{33}(\sqrt{C(2)})$*) lying over* $\tilde{\mathfrak{P}}_{23}$ *(resp.* $\tilde{\mathfrak{P}}_{23}$*). There exists a primitive form f in* $S^{+}(9)$ *which satisfies*

$$a(\mathfrak{p})\equiv C(\mathfrak{p}) \qquad \mathrm{mod.}\,\bar{\mathfrak{P}}_{23}$$
$$b(\mathfrak{p})\equiv C(\mathfrak{p}) \qquad \mathrm{mod.}\,\bar{\mathfrak{P}}_{67}\,,$$

*where* $C(\mathfrak{p})$ *is the eigenvalue of f for* $T(\mathfrak{p})$.

This theorem can be shown by means of the same idea as in M. Koike [3]. Here we omit a proof and remark that in the proof, we need the fact

$$L\Big(0,\ \bar{\omega}_{23}\Big(\frac{}{5}\Big)\Big)\not\equiv 0 \qquad \mathrm{mod.}\,\tilde{\mathfrak{P}}_{23}$$
$$L\Big(0,\ \bar{\omega}_{67}\Big(\frac{}{5}\Big)\Big)\not\equiv 0 \qquad \mathrm{mod.}\,\tilde{\mathfrak{P}}_{67}\,.$$

In a subsequent paper, we shall discuss in a more general setting the property of Fourier coefficients of Hilbert cusp forms of level $\mathfrak{q}^{\nu}$ with more examples. The authors have found that the phenomena heavily depend on the quadratic extension $K$ of $F$ with the conductor $\mathfrak{q}$. In the case where $\mathrm{rk}\,E_K\leqq \mathrm{rk}\,E_F+1$, we find analogous results to the case of elliptic modular forms. However, when $\mathrm{rk}\,E_K\geqq \mathrm{rk}\,E_F+2$, the situation seems to be different.

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