On the Picard number of a Fermat surface

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To the memory of Takuro Shintani

0. The Picard number of an algebraic surface is an interesting invariant of arithmetic nature, but it is in general not so easy to determine it. In this paper, we consider the case of the (complex) Fermat surface of degree m in $\boldsymbol{P}^3$ defined by

$$X_m^2:\ x_0^m+x_1^m+x_2^m+x_3^m=0\,.$$

Given m, there is an algorithm for computing the Picard number $\rho(X_m^2)$ of X_m^2 (see below), but an explicit formula for $\rho(X_m^2)$ has been known only for m prime. Inspired by some observation of Weidner [6], we shall generalize this to a certain extent by using the related results on a Fermat curve due to Koblitz and Rohrlich [1]. The main results are Theorems 6, 7 and 8.

1. First we review the algorithm for computing $\rho(X_m^2)$, fixing the relevant notation for Fermat varieties (cf. [5]).

Fix $m>1$. For an integer a, let $\langle a\rangle$ be the least non-negative integer congruent to a modulo m. For any integer $n\geqq 0$, let

$$\mathfrak{A}_m^n=\left\{\alpha=(a_0,\ a_1,\ \cdots,\ a_{n+1})\mid 1\leqq a_i\leqq m-1,\ \sum_{i=0}^{n+1}a_i\equiv 0\ (m)\right\}.$$

The group $(\boldsymbol{Z}/m)^\times$ acts on this set by the rule: if $t\in(\boldsymbol{Z}/m)^\times$ and $\alpha=(a_i)\in\mathfrak{A}_m^n$, let $t\cdot\alpha=(\langle ta_i\rangle)$. For $\alpha=(a_i)\in\mathfrak{A}_m^n$, set $|\alpha|=\sum_{i=0}^{n+1}\langle a_i\rangle/m$; $|\alpha|$ is a natural number $\leqq n+1$ such that $|\alpha|+|-\alpha|=n+2$. When n is even, define

$$\mathfrak{B}_m^n=\{\alpha\in\mathfrak{A}_m^n\mid |t\cdot\alpha|=n/2+1,\ \forall t\in(\boldsymbol{Z}/m)^\times\}.$$

Then, by the well known characterization of Hodge classes on a Fermat variety (cf. [5, Th. I]), the cardinality $|\mathfrak{B}_m^n|$, added by 1, gives the dimension of Hodge classes of middle codimension on the n-dimensional Fermat variety of degree m

$$X_m^n:\ x_0^m+x_1^m+\cdots+x_{n+1}^m=0\,.$$

In particular, we have

$$\rho(X_m^2)=|\mathfrak{B}_m^2|+1 \tag{1}$$

by Lefschetz' theorem. There is a similar description of Hodge classes on a product of several Fermat varieties; in particular, we have

$$\rho(X_m^1 \times X_m^1) = |\mathfrak{B}_m^4 \cap (\mathfrak{A}_m^1 * \mathfrak{A}_m^1)| + 2, \tag{2}$$

where $*$ stands for juxtaposition.

2. An element $\alpha=(a_0, a_1, a_2, a_3)$ of $\mathfrak{B}_m^2$ is called *decomposable* if $a_i+a_j\equiv 0 \pmod m$ for some $i\neq j$; otherwise it is called *indecomposable*. Let $\mathfrak{D}_m^2$ (or $\mathfrak{J}_m^2$) denote the set of decomposable (or indecomposable) elements of $\mathfrak{B}_m^2$. The number of decomposable elements is easily computed and is given by the formula:

$$|\mathfrak{D}_m^2| = 3(m-1)(m-2)+\delta_m, \qquad \delta_m = \begin{cases} 0 & (m: \text{odd}) \\ 1 & (m: \text{even}). \end{cases} \tag{3}$$

It is known that the subspace of the Néron-Severi group $NS(X_m^2)$ spanned by the cohomology classes of lines lying on X_m^2 has the dimension $|\mathfrak{D}_m^2|+1$ (cf. the proof of Theorem 7 below).

For each $d|m$, let $\mathfrak{J}_m^2(d)$ denote the set of $\alpha=(a_0, \cdots, a_3)\in\mathfrak{J}_m^2$ such that $\mathrm{GCD}(\alpha)$ $(=\mathrm{GCD}(a_0, \cdots, a_3))=d$. Note that the map $\alpha\mapsto d^{-1}\alpha$ is a bijection from $\mathfrak{J}_m^2(d)$ to $\mathfrak{J}_{m/d}^2(1)$. Now, for any $\alpha=(a_i)\in\mathfrak{J}_m^2$, the coefficients $a_0, \cdots, a_3$ are either all distinct or at most two of them coincide. (For, if $a_0=a_1$ and $a_2=a_3$, α becomes decomposable. If $a_0=a_1=a_2$, we may assume first $\alpha\in\mathfrak{J}_m^2(1)$ and further $a_0=1$. Then we should have $a_3=2m-3$, which is impossible.) We set $w_\alpha=1$ or $1/2$ according as all the coefficients of α are distinct or otherwise. Given $\alpha\in\mathfrak{J}_m^2$, there are $(4!)w_\alpha$ elements of $\mathfrak{J}_m^2$ which are permutations of α. Thus, if we set

$$g(m) = \sum_{\substack{\alpha\in\mathfrak{J}_m^2(1) \\ \text{up to permutation}}} w_\alpha, \tag{4}$$

then $|\mathfrak{J}_m^2(1)|=24g(m)$. Note that $g(m)$ is a non-negative integer, since $(-1)\cdot\alpha$ is not a permutation of α and $w_\alpha+w_{-\alpha}$ is either 2 or 1. Therefore we can write

$$\rho(X_m^2) = 3(m-1)(m-2)+\delta_m+1+24\sum_{\substack{d|m \\ d<m}} g(m/d). \tag{5}$$

3. By the formula (5) (or even (1)), one can compute $\rho(X_m^2)$, in principle, for any given value of m. Weidner [6] computed $\rho(X_m^2)$ for $m\leqq 272$, and made some interesting observation. Namely, let

$$\Delta(m) = g(m) - \{\varphi(m/3)+2\varphi(m/2)\}, \tag{6}$$

where $\varphi(x)$ is the Euler function and $\varphi(x)$ is defined to be $=0$ if x is not a natural number. Then the main observation in [6] is:

(i) $\Delta(m)=0$ for all values of $m \leqq 272$ except for 28 values of m ($m=2, 3, \cdots, 180$) listed below;

(ii) in particular, $g(m)=0$ if GCD$(m, 6)=1$ (and $m \leqq 272$). It was known that (ii) holds for any prime m (cf. [5, p. 181]), and we shall see later that (ii) is true for all m with GCD$(m, 6)=1$ (Theorem 6(a)). On the other hand, we wondered if (i) could be true for larger values of m, and found by computer, helped by N. Maruyama, that

(i′) $\Delta(m)=0$ for all values of $m \leqq 672$ with the *same* exception as in (i).

Table of m with $\Delta(m) \neq 0$ $(m \leqq 672)$

m	prime decomposition	$g(m)$	$\varphi(m/3)+2\varphi(m/2)$	$\Delta(m)$
2	2	0	2	−2
3	3	0	1	−1
4	2^2	0	2	−2
6	$2\cdot 3$	1	5	−4
8	2^3	2	4	−2
10	$2\cdot 5$	6	8	−2
12	$2^2\cdot 3$	12	6	6
14	$2\cdot 7$	14	12	2
15	$3\cdot 5$	12	4	8
18	$2\cdot 3^2$	32	14	18
20	$2^2\cdot 5$	34	8	26
21	$3\cdot 7$	18	6	12
24	$2^3\cdot 3$	50	12	38
28	$2^2\cdot 7$	22	12	10
30	$2\cdot 3\cdot 5$	114	20	94
36	$2^2\cdot 3^2$	34	16	18
40	$2^3\cdot 5$	32	16	16
42	$2\cdot 3\cdot 7$	196	30	166
48	$2^4\cdot 3$	40	24	16
60	$2^2\cdot 3\cdot 5$	228	24	204
66	$2\cdot 3\cdot 11$	80	50	30
72	$2^3\cdot 3^2$	44	32	12
78	$2\cdot 3\cdot 13$	92	60	32
84	$2^2\cdot 3\cdot 7$	102	36	66
90	$2\cdot 3^2\cdot 5$	80	56	24
120	$2^3\cdot 3\cdot 5$	120	48	72
156	$2^2\cdot 3\cdot 13$	96	72	24
180	$2^2\cdot 3^2\cdot 5$	88	64	24

Suggested by this, we consider the following question:

(Q) Is $\Delta(m)=0$ for any $m>180$?

If true, it will give a closed formula of the Picard number of a Fermat surface as follows. By (6), we have

$$\sum_{\substack{d\mid m\\ d>1}} g(d)=\sum_{\substack{d\mid m\\ d>1}} \varphi(d/3)+2\sum_{\substack{d\mid m\\ d>1}} \varphi(d/2)+\sum_{\substack{d\mid m\\ d>1}} \Delta(d) \tag{7}$$

$$=(m/3)^{*}+2(m/2)^{*}+\sum_{\substack{d\mid m\\ d>1}} \Delta(d),$$

where we define, for a positive rational number x,

$$x^{*}=\begin{cases} x & \text{if } x \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\varepsilon(m)=\sum_{1<d\mid m} \Delta(d). \tag{8}$$

By (5) and (7), we have

$$\rho(X_m^2)=3(m-1)(m-2)+\delta_m+1+24(m/3)^{*}+48(m/2)^{*}+24\varepsilon(m). \tag{9}$$

Now, if (Q) is true, then only those divisors d of m with $d\leqq 180$ (more precisely, $d\in\{2, 3, \cdots, 180\}$ in the table) will contribute to $\varepsilon(m)$, and the above (9) will give a desired closed formula for $\rho(X_m^2)$ for all m.

4. In the rest of this paper, we note certain results which seem to support the affirmative answer of (Q) and hence the validity of the formula (9) in its closed form.

First we exhibit some indecomposable elements of $\mathfrak{B}_m^2$ in case GCD$(m, 6)>1$, which implies the inequality $\Delta(m)\geqq 0$ for all $m>12$.

LEMMA 1. (a) *Assume that* $m=2m'$ *is even, and let*

$$\alpha_i=(i, m'+i, m-2i, m'), \qquad 1\leqq i\leqq m'-1,\ i\neq m'/2$$

$$\beta_i=\begin{cases} (i, m'+i, m'+2i, m-4i), & 1\leqq i\leqq (m'-1)/2,\ i\neq m'/3 \\ (i, m'+i, 2i-m', 2m-4i), & (m'+1)/2\leqq i\leqq m'-1,\ i\neq m/3. \end{cases}$$

Then α_i *and* β_i *are indecomposable elements of* $\mathfrak{B}_m^2$, *which belong to* $\mathfrak{Z}_m^2(1)$ *if and only if* GCD$(i, m')=1$.

(b) *Assume that* m *is divisible by* 3, $m=3m''$, *and let*

$$\gamma_j=(j, m''+j, 2m''+j, m-3j), \qquad 1\leqq j\leqq m''-1,\ j\neq m''/2.$$

Then γ_j is an indecomposable element of $\mathfrak{B}_m^2$, which belongs to $\mathfrak{J}_m^2(1)$ if and only if GCD$(j, m'')=1$.

LEMMA 2. *With the same notation as above,*

(a) *the four coefficients of each of indecomposable elements α_i, β_i or γ_j in $\mathfrak{J}_m^2(1)$ are all distinct except for the following cases*: 1) $m=6$, α_1, α_2; 2) $m=10$, β_1, β_2, β_3, β_4; 3) $m=12$, β_1, β_5, γ_1, γ_3.

(b) *If $m>8$, then α_i, β_i $(1\leqq i\leqq m'-1, (i, m')=1)$ and γ_j $(1\leqq j\leqq m''-1, (j, m'')=1)$ form $2\varphi(m')+\varphi(m'')$ distinct elements of $\mathfrak{J}_m^2(1)$; moreover there are no pairs $\{\alpha, \alpha'\}$ among these elements such that α' is a permutation of α.*

The verification of these lemmas is straightforward, and so it will be omitted.

PROPOSITION 3. *For any $m>12$, we have*

$$g(m)\geqq\varphi(m/3)+2\varphi(m/2), \qquad i.e., \ \Delta(m)\geqq 0. \tag{10}$$

PROOF. This is an immediate consequence of Lemmas 1 and 2 in view of (4) and (6). q. e. d.

Considering the case $\Delta(m)=0$, we have

PROPOSITION 4. *The question* (Q) *is affirmative if and only if the following statement* (Q′) *is true*:

(Q′) *For any $m>180$, the set $\mathfrak{J}_m^2(1)$ of indecomposable elements α of $\mathfrak{B}_m^2$ with* GCD$(\alpha)=1$ *consists exactly of $\varphi(m/3)+2\varphi(m/2)$ elements α_i, β_i $(1\leqq i\leqq m'-1, (i, m')=1)$ and γ_j $(1\leqq j\leqq m''-1, (j, m'')=1)$ defined in Lemma* 1.

As a consequence of Proposition 3, we obtain (at least) the *lower estimate* of the Picard number $\rho(X_m^2)$:

$$\rho(X_m^2)\geqq 3(m-1)(m-2)+\delta_m+1+24(m/3)^*+48(m/2)^*+24\varepsilon'(m) \tag{9′}$$

where $\varepsilon'(m)=\sum_{\substack{d\mid m\\ 1<d\leqq 180}}\Delta(d)$.

5. Now we translate our problem into a problem about a Fermat curve. We fix the following notation: let $\mathfrak{A}_m^{1,0}\subset\mathfrak{A}_m^1$ be the set of $\beta\in\mathfrak{A}_m^1$ such that $|\beta|=1$, i. e.

$$\mathfrak{A}_m^{1,0}=\left\{(a_0, a_1, a_2)\mid 1\leqq a_i\leqq m-1, \ \sum_{i=0}^{2}a_i=m\right\}.$$

For $\beta\in\mathfrak{A}_m^{1,0}$, we set

$$H_\beta=\{t\in(\boldsymbol{Z}/m)^\times \mid |t\cdot\beta|=1\}.$$

Obviously H_β is a full set of representatives of $(\boldsymbol{Z}/m)^\times \bmod\{\pm1\}$.

PROPOSITION 5. *The set of indecomposable elements $\alpha=(a_0, a_1, a_2, a_3)$ of $\mathfrak{B}_m^2$ with $a_0+a_1<m$ is in one-to-one correspondence with the set of pairs (β, γ) of elements β, γ of $\mathfrak{A}_m^{1,0}$ such that* (i) *$H_\beta=H_\gamma$ and* (ii) *β and γ have only the last coefficient in common (i.e. if $\beta=(b_0, b_1, b_2)$, $\gamma=(c_0, c_1, c_2)$, then $b_2=c_2$ and $\{b_0, b_1\}\cap\{c_0, c_1\}=\varnothing$).*

PROOF. Given $\alpha=(a_0, \cdots, a_3)\in\mathfrak{B}_m^2$ as above, we let

$$\beta=(a_0, a_1, m-a_0-a_1), \qquad \gamma=(m-a_2, m-a_3, m-a_0-a_1).$$

Obviously $\beta, \gamma\in\mathfrak{A}_m^{1,0}$. Now, since $\alpha\in\mathfrak{B}_m^2$, we have

$$|t\cdot\beta|+|-t\cdot\gamma|=|t\cdot\alpha|+1=3 \qquad (\forall t\in(\boldsymbol{Z}/m)^\times),$$

which implies $|t\cdot\beta|=|t\cdot\gamma|$ for all $t\in(\boldsymbol{Z}/m)^\times$. Thus we have $H_\beta=H_\gamma$, and the pair (β, γ) satisfies the condition (i). The condition (ii) is clear because α is indecomposable. Conversely, given $\beta=(b_0, b_1, b_2)$ and $\gamma=(c_0, c_1, c_2)$ satisfying the conditions (i), (ii), we let $\alpha=(b_0, b_1, m-c_0, m-c_1)$, which obviously defines the inverse correspondence. q.e.d.

Although it is implicit in the above proof, Proposition 5 is based on a special case of the "inductive structure" of Fermat varieties, i.e. on the connection of X_m^2 and $X_m^1\times X_m^1$ (cf. [3], [4], [5]).

6. The study of pairs (β, γ) of elements of $\mathfrak{A}_m^{1,0}$ satisfying $H_\beta=H_\gamma$ is related to the decomposition of the Jacobian variety $J(X_m^1)$ of the Fermat curve X_m^1 into certain isogeny factors. For this and for what follows, we refer the reader to Koblitz-Rohrlich [1]. We quote a part of their main results, relevant to our problem, in the following form:

THEOREM K-R. *Suppose $\beta, \gamma\in\mathfrak{A}_m^{1,0}$ and $H_\beta=H_\gamma$. Then*

(a) *if* GCD$(m, 6)=1$, *then β is a permutation of γ.*

(b) *Assume further that β and γ have only the last coefficient in common and* GCD$(\beta, \gamma)=1$. (b_1) *If $m=3^n$ $(n\geqq2)$, then for some $t\in(\boldsymbol{Z}/m)^\times$, $t\cdot\beta$ and $t\cdot\gamma$ are permutations of* $(1, 2\cdot3^{n-1}+1, 3^{n-1}-2)$ *and* $(3, 2\cdot3^{n-1}-1, 3^{n-1}-2)$. ($\mathrm{c}_1$) *If $m=2^n$ $(n\geqq4)$, then for some $t\in(\boldsymbol{Z}/m)^\times$, $t\cdot\beta$ and $t\cdot\gamma$ are permutations of one of the following pairs:*

1) $(1, m-4, 3)$, $\quad(m/2-1, m/2-2, 3)$

2) $(1, m-2, 1)$, $(m/2, m/2-1, 1)$

3) $(1, m/2, m/2-1)$, $(2, m/2-1, m/2-1)$

4) $(1, m/2+1, m/2-2)$, $(2, m/2, m/2-2)$

5) $(2, m-4, 2)$, $(m/4-1, 3m/4-1, 2)$

6) $(m/2, m/2-2, 2)$, $(m/4-1, 3m/4-1, 2)$.

7. From Proposition 5 and Theorem K-R, we can immediately deduce the following:

THEOREM 6. (a) *If* $\mathrm{GCD}(m, 6)=1$, *then there are no indecomposable elements of* $\mathfrak{B}_m^2$. (b) *If* $m=3^n$ $(n\geqq 2)$, *then any indecomposable element of* $\mathfrak{B}_m^2$ *with* $\mathrm{GCD}(\alpha)=1$ (*i.e. any element of* $\mathfrak{J}_m^2(1)$) *is a permutation of one of* γ_j $(1\leqq j\leqq m/3, (j, 3)=1)$ *of Lemma* 1. (c) *If* $m=2^n$ $(n\geqq 4)$, *then any element of* $\mathfrak{J}_m^2(1)$ *is a permutation of one of* α_i *or* β_i $(1\leqq i\leqq m/2, i: odd)$ *of Lemma* 1.

This shows that the statement (Q′) of Proposition 4 is true in the case where m is either relatively prime to 6 or m is a power of 2 or 3. Therefore we can state the following results on the Picard number $\rho(X_m^2)$.

THEOREM 7. *Assume that* m *is relatively prime to* 6. *Then the Picard number of the* (*complex*) *Fermat surface* X_m^2 *of degree* m *is given by the formula*:

$$\rho(X_m^2)=3(m-1)(m-2)+1. \tag{11}$$

Furthermore the Néron-Severi group $\mathrm{NS}(X_m^2)\otimes \boldsymbol{Q}$ *is spanned by the cohomology classes of lines* (=1-*dimensional subspaces of* $\boldsymbol{P}^3$) *lying on* X_m^2.

PROOF. The formula (11) follows from (9) by what we have seen above. The second statement is a consequence of a more general result, valid for higher dimensional Fermat varieties (cf. [2], [5, Th. III]). More directly, we can show it in the following way. Consider the $3m^2$ lines in $\boldsymbol{P}^3$ defined by

$$\begin{aligned} &L_{\zeta,\eta}^{(1)}: \zeta x_0+x_1=0, \quad \eta x_2+x_3=0 \\ &L_{\zeta,\eta}^{(2)}: \zeta x_0+x_2=0, \quad \eta x_1+x_3=0 \quad (\zeta^m=\eta^m=1). \\ &L_{\zeta,\eta}^{(3)}: \zeta x_0+x_3=0, \quad \eta x_1+x_2=0 \end{aligned} \tag{12}$$

It is easy to see that these lines lie on X_m^2 (note that m is odd). Let $c(L_{\zeta,\eta}^{(i)})$ denote the class of $L_{\zeta,\eta}^{(i)}$ in $\mathrm{NS}(X_m^2)\subset H^2(X_m^2, \boldsymbol{Z})$. It suffices to show that $\{c(L_{\zeta,\eta}^{(i)})\}$ spans $\mathrm{NS}(X_m^2)\otimes \boldsymbol{C}$. For each $\alpha=(a_0, a_1, a_2, a_3)\in\mathfrak{D}_m^2$ with $a_0+a_1\equiv 0 \pmod m$, define

(13) $$\omega_\alpha=\sum_{\zeta,\eta}\zeta^{a_1}\eta^{a_3}c(L^{(1)}_{\zeta,\eta}).$$

Then the following facts can be seen without difficulty: first

(14) $$g^*(\omega_\alpha)=\alpha(g)\omega_\alpha$$

where $g=[\zeta_0, \cdots, \zeta_3]\in \mathrm{Aut}(X^2_m)$ $(\zeta_j^m=1)$ and $\alpha(g)=\prod_{j=0}^{3}\zeta_j^{a_j}$. Looking at the intersection properties of lines $L^{(i)}_{\zeta,\eta}$, we have

(15) $$\omega_\alpha\cdot\bar{\omega}_\alpha=-m^3,$$

which implies that $\omega_\alpha\neq 0$ in $H^2(X^2_m, \boldsymbol{C})$. Thus ω_α spans the eigenspace $V(\alpha)\subset H^2(X^2_m, \boldsymbol{C})$ with "character" α (cf. [5, Th. I]). If we denote by H a hyperplane section of X^2_m, then

(16) $$c(H)=\sum_\zeta c(L^{(i)}_{\zeta,\eta})=\sum_\eta c(L^{(i)}_{\zeta,\eta}) \qquad (i=1, 2, 3),$$

and

(17) $$m^2c(L^{(1)}_{\zeta,\eta})=m\cdot c(H)+\sum_{\substack{\alpha=(a_0,a_1,a_2,a_3)\\ a_0+a_1\equiv 0\,(m)}}\zeta^{a_0}\eta^{a_2}\omega_\alpha.$$

The similar relations hold also for $L^{(2)}_{\zeta,\eta}$ and $L^{(3)}_{\zeta,\eta}$, and it follows that the space $\bigoplus_{\alpha\in\mathfrak{D}^2_m} V(\alpha)\oplus\boldsymbol{C}c(H)$ is nothing but the subspace of $H^2(X^2_m, \boldsymbol{C})$ spanned by $c(L^{(i)}_{\zeta,\eta})$'s. Since $\mathfrak{B}^2_m=\mathfrak{D}^2_m$ in our case $(\mathrm{GCD}(m, 6)=1)$, this proves that $\mathrm{NS}(X^2_m)\otimes\boldsymbol{C}$ is spanned by $c(L^{(i)}_{\zeta,\eta})$'s. q. e. d.

Question. Suppose m is relatively prime to 6. Is $\mathrm{NS}(X^2_m)$ spanned *over* $\boldsymbol{Z}$ by $c(L^{(i)}_{\zeta,\eta})$'s?

From Theorem 6 (b), (c), we also obtain

THEOREM 8. (i) *Assume that* $m=2^n$. *Then*

(18) $$\rho(X^2_m)=\begin{cases} 3(m-1)(m-2)+2+24(m-6) & (n\geqq 3)\\ 20 & (n=2)\\ 2 & (n=1).\end{cases}$$

(ii) *Assume that* $m=3^n$. *Then*

(19) $$\rho(X^2_m)=3(m-1)(m-2)+1+8(m-3).$$

It should be remarked that the Picard number of the Fermat surface X^2_m in characteristic $p>0$ can become larger than the value of $\rho(X^2_m)$ in (1) or (9); for instance, if $m\geqq 4$, it is equal to the Betti number

$$b_2(X_m^2)=(m-1)(m^2-3m+3)+1$$

if and only if $p^\nu \equiv -1 \pmod{m}$ for some ν (cf. [4, Th. 4.3]).

8. Finally we note that, reversing the above arguments, we can reformulate the question (Q) or (Q′) in terms of data for a Fermat curve, which suggests an analogue of Theorem K-R for the values of m not treated in [1].

As an illustration, we state the case where $m=3m''$ with m'' odd. By Lemma 1 (b) and Proposition 5, we see that each of the following pairs of elements of $\mathfrak{A}_m^{1;0}$ satisfies the conditions (i), (ii) of Proposition 5:

$$(20) \quad \begin{cases} (j,\, m-3j,\, 2j),\ (m''-j,\, 2m''-j,\, 2j) & (1 \leqq j < m'',\ (j,\, m'')=1)\,; \\ (j,\, 2m''+j,\, m''-2j),\ (3j,\, 2m''-j,\, m''-2j) & (1 \leqq j < m''/2,\ (j,\, m'')=1)\,. \end{cases}$$

Now the question (Q) (for the above m), is equivalent to the following:

$(\mathrm{Q}'')_m$ Let (β, γ) be a pair of elements of $\mathfrak{A}_m^{1;0}$ such that (i) $H_\beta=H_\gamma$, (ii) β, γ have only the last coefficient in common, and (iii) GCD$(\beta, \gamma)=1$. Assume $m=3m''$ (m'' : odd) and $m>21$. Is the pair (β, γ) among the pairs listed in (20) up to permutations?

This question $(\mathrm{Q}'')_m$ and similar one for m even might be handled by modifying the methods of Koblitz and Rohrlich [1]. In order to determine all pairs (β, γ) satisfying $H_\beta=H_\gamma$, it is also necessary to consider the case where β and γ have no coefficients in common. This is a question about the "semi-decomposable" indecomposable elements of $\mathfrak{B}_m^4$ (cf. (2)), which is related to the Hodge Conjecture for the 4-dimensional Fermat variety X_m^4 (cf. [5, § 4]). There remains much to be clarified here too.

Note added in proof.

1) The same problem as in this paper has been considered in a recent article of W. Meyer and W. Neutsch: "Fermatquadrupel", Math. Ann. 256 (1981), 51-62. Our Theorem 6 (a) answers their Vermutung 2.

2) Recently, N. Aoki has completely solved the question (Q) by verifying the statement (Q′) of Proposition 4 for all $m>180$. Thus we have the closed formula for the Picard number $\rho(X_m^2)$ for arbitrary m. (cf. Aoki's Master thesis at Univ. of Tokyo, in preparation.)

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