Quasi-unipotent constructible sheaves

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Dedicated to the memory of Professor Takuro Shintani

It is known that the local monodromy of the constructible sheaves obtained by Gauss-Manin connection is always quasi-unipotent. The purpose of this paper is to study the properties of such constructible sheaves that their local monodromies are quasi-unipotent. More precisely, we say that a constructible sheaf of $\boldsymbol{C}$-vector spaces on a complex analytic space X is quasi-unipotent if, for any analytic map φ from the unit disc $D=\{z\in\boldsymbol{C}\,;\,|z|<1\}$ into X, the monodromy $\varphi^{-1}F$ along the path $\{z\,;\,|z|=\varepsilon\}$ is quasi-unipotent for $0<\varepsilon\ll 1$. Recall that an endomorphism of a finite-dimensional vector space is called quasi-unipotent if any of its eigenvalues is a root of the unity.

We shall show the quasi-unipotency is stable under the pull-back and the proper direct image. Above all, the most remarkable property that the quasi-unipotency enjoys is that this is a generic property in the following sense. Let X be an analytic space, Y a closed analytic subset and Z a closed analytic subset of codimension $\geqq 2$. Let F be a sheaf on X such that $F|_{X-Y}$ and $F|_Y$ are locally constant sheaves of finite rank. Under these conditions, if $F|_{X-Z}$ is quasi-unipotent, then F is also quasi-unipotent. In order to prove this result, we first reduce this to the case when X is a neighbourhood of the origin of $\boldsymbol{C}^2$, Y a union of non-singular curves and Z the origin. In this case one can describe the monodromy group of $X-Y$ by using the graph associated Y.

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§1. Quasi-unipotent sheaf.

1. We shall recall the definition of constructible sheaves. We refer [1] for further properties of constructible sheaves.

Let X be an analytic space and let F be a sheaf of $\boldsymbol{C}$-vector spaces. We say that F is *constructible* if there exists a decreasing sequence $\{X_j\}_{j=0,1,\dots}$ of closed analytic subsets of X satisfying the following conditions:

$$X=X_0\,, \qquad \cap X_j=\varnothing\,. \tag{1.1.1}$$

(1.1.2) $F|_{X_j - X_{j+1}}$ is a locally constant sheaf of finite rank for $j=0, 1, \cdots$.

This is a local property; i.e., if there exists an open covering $\{\Omega_k\}$ of X such that $F|\Omega_k$ is constructible, then F is constructible. If F is a constructible sheaf, then the set of points which do not have a neighbourhood where F is locally constant is a nowhere dense closed analytic subset of X.

The following propositions are known.

PROPOSITION 1.1. (1) *If F and G are constructible sheaves and if $f: F \to G$ is a homomorphism, then the kernel, the image and the cokernel of f are constructible.* (2) *If $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves on X and if F' and F'' are constructible, then F is also constructible.* (3) *If F is a constructible sheaf on X and if Y is a closed analytic subset of X, then $\mathcal{H}_Y^j(F)$ is constructible for any j.*

PROPOSITION 1.2. *Let $f: X \to Y$ be an analytic map.* (1) *If G is a constructible sheaf on Y, then $f^{-1}G$ is a constructible sheaf on X.* (2) *If f is a proper map and if F is a constructible sheaf on X, then $R^j f_*(F)$ is a constructible sheaf on Y for any j.*

2. Let X be an arcwise connected topological space and x_0 a point of X. Let F be a locally constant sheaf on X. Let I denote the unit interval. Then, for a continuous map $\varphi: I \to X$, $\varphi^{-1}(F)$ is a constant sheaf on I and we obtain an isomorphism from $(\varphi^{-1}F)_0 = F_{\varphi(0)}$ onto $(\varphi^{-1}F)_1 = F_{\varphi(1)}$. Hence if $\varphi(0) = \varphi(1) = x_0$, we obtain the automorphism of F_{x_0}, which we shall call the *monodromy* of F along the path φ. This depends only on the homotopy type of φ and we obtain the group homomorphism

$$\pi = \pi_1(X, x_0) \longrightarrow \mathrm{Aut}(F_{x_0}).$$

Conversely, if V is a representation of π, then there exists a unique locally constant sheaf F such that F_{x_0} is isomorphic to V as a representation of π.

Recall that for any sheaf F on X, we have

$$\boldsymbol{R}\Gamma(X; F) = \boldsymbol{R}\,\mathrm{Hom}_{\boldsymbol{C}[\pi]}(\boldsymbol{C}, \boldsymbol{R}\Gamma(\tilde{X}; p^{-1}F)),$$

where $p: \tilde{X} \to X$ is the universal covering of X, $\boldsymbol{C}[\pi]$ the group ring and $\boldsymbol{C}$ is the trivial representation of π. Hence, in particular, if $\tilde{X}$ is contractible and if F is a locally constant sheaf on X, then we have

$$\boldsymbol{R}\Gamma(X; F) = \boldsymbol{R}\,\mathrm{Hom}_{\boldsymbol{C}[\pi]}(\boldsymbol{C}, F_{x_0}).$$

3. Let D denote the unit disc $\{z \in \boldsymbol{C}; |z| < 1\}$ in $\boldsymbol{C}$ and let D^* denote $D - \{0\}$. For a constructible sheaf F on D, there exists $\delta > 0$ such that $F|_{D_\delta^*}$ is a locally constant sheaf. Here, D_δ^* denotes the set $\{z \in \boldsymbol{C}; 0<|z|<\delta\}$. Hence the monodromy of F along $\{z; |z|=\varepsilon\}$ does not depend on ε if $0<\varepsilon \ll 1$. We shall call it the *monodromy* of F around the origin.

For a linear endomorphism T of a finite-dimensional vector space, we say that T is *quasi-unipotent* if any eigenvalue of T is a root of unity. This condition is equivalent to the existence of two integers $m, l \geqq 1$ such that $(T^m-1)^l=0$.

Now, let X be an analytic space.

DEFINITION 1.3. A constructible sheaf F on X is called *quasi-unipotent* at a point x of X if, for any analytic map $\varphi: D \to X$ with $\varphi(0)=x$, the monodromy of $\varphi^{-1}F$ around the origin is quasi-unipotent. If F is quasi-unipotent at any point of X, we say that F is quasi-unipotent.

The following two propositions are obvious by the definition.

PROPOSITION 1.4. (1) *A locally constant sheaf of finite rank is quasi-unipotent.* (2) *Let $F' \to F \to F''$ be an exact sequence of constructible sheaves. If F' and F'' are quasi-unipotent, then so is F.*

PROPOSITION 1.5. *Let $f: X \to Y$ be an analytic map. Then, for any quasi-unipotent sheaf F on Y, $f^{-1}F$ is quasi-unipotent.*

PROPOSITION 1.6. *Let F be a constructible sheaf on D. If the monodromy of F around the origin is quasi-unipotent, then F is quasi-unipotent at the origin.*

PROOF. Let φ be an analytic map from D into D such that $\varphi(0)=0$. If $\varphi(D)=\{0\}$, then $\varphi^{-1}F$ is a constant sheaf and hence its monodromy around the origin is the identity. If $\varphi(D) \neq \{0\}$, then $\varphi: (D, 0) \to (D, 0)$ is equivalent to z^m for some integer $m \geqq 1$. Hence, if we denote by M the monodromy of F around the origin, then the monodromy of $\varphi^{-1}F$ around the origin is M^m, which is quasi-unipotent. Q.E.D.

PROPOSITION 1.7. *Let $f: X \to Y$ be an analytic map. Assume that the topology of Y equals the quotient topology of X. Let F be a constructible sheaf on Y. Then F is quasi-unipotent if and only if $f^{-1}F$ is quasi-unipotent.*

PROOF. Assume that $f^{-1}F$ is quasi-unipotent. We shall prove that, for an analytic map, $\varphi: D \to Y$, $\varphi^{-1}F$ is quasi-unipotent at the origin. Set $y=\varphi(0)$. If $\varphi(D)=\{y\}$, then this is obvious. If not, the origin is a discrete point of $\varphi^{-1}(0)$. Hence, we may assume, by shrinking D and Y, that φ is finite and $\varphi^{-1}(y)=\{0\}$. Set $X'=X \times_Y D$ and let φ' and f' denote the projections from X' to X and D, respectively. We shall prove that the closure of $f'^{-1}(D^*)$ contains some point of $f'^{-1}(0)$. If $f'^{-1}(D^*)$ were a closed subset of X', then $\varphi' f'^{-1}(D^*)=f^{-1}\varphi(D^*)$ would be a closed subset of Y. Hence $\varphi(D^*)$ would be closed in Y. Therefore

$\varphi(D^*)$ would contain $y=\varphi(0)$, which contradicts $\varphi^{-1}(y)=\{0\}$. Thus, $f'^{-1}(D^*)$ is not a closed subset of X'. Let x be a point of $f'^{-1}(0)$ which belongs to the closure of $f'^{-1}(D^*)$. Then, there exists a holomorphic map $\psi: D\to X'$ such that $\psi(0)=x$ and $\psi(D^*)\subset f'^{-1}(D^*)$. Since $f^{-1}F$ is quasi-unipotent, $\psi^{-1}\varphi'^{-1}f^{-1}F=(f'\circ\psi)^{-1}\cdot(\varphi^{-1}F)$ is quasi-unipotent. On the other hand, we have $(f'\circ\psi)^{-1}(D^*)\subset D^*$, and hence $f'\circ\psi:(D, 0)\to(D, 0)$ is isomorphic to $g(z)=z^m$ for an integer $m\geqq 1$. Hence $g^{-1}\varphi^{-1}F$ is quasi-unipotent at the origin. If M denotes the monodromy of $\varphi^{-1}F$ around the origin, then the monodromy of $g^{-1}\varphi^{-1}F$ around the origin equals M^m. Therefore M^m is quasi-unipotent, which implies the quasi-unipotency of M.

Q.E.D.

PROPOSITION 1.8. *Let $f: X\to Y$ be a finite map and F a constructible sheaf on X. Then F is quasi-unipotent if and only if f_*F is quasi-unipotent.*

PROOF. Note that F is a quotient of $f^{-1}f_*F$. Hence, if f_*F is quasi-unipotent, then F is quasi-unipotent by Proposition 1.5 and Proposition 1.4 (2). Conversely assume that F is quasi-unipotent. In order to show the quasi-unipotency of f_*F, let φ be an analytic map from D into Y. Set $X'=X\times_Y D$ and let X'' be the normalization of X'. Let $\psi: X''\to X$ and $g: X''\to D$ denote the projections. Then $\varphi^{-1}f_*F$ is isomorphic to $g_*\psi^{-1}F$ on $D_\delta^*=\{z\in D: 0<|z|<\delta\}$ for $0<\delta\ll 1$. Hence it is sufficient to show the quasi-unipotency of $g_*\psi^{-1}F$. Thus, Proposition 1.8 is reduced to the following: if X is non-singular and if $f: X\to D$ is a finite map, then, for any quasi-unipotent sheaf F on X, f_*F is quasi-unipotent at the origin. We may assume that $f^{-1}(0)$ consists of a single point, say 0. Then $f:(X, 0)\to(D, 0)$ is isomorphic to z^m for an integer $m\geqq 1$. Let M and M' denote the monodromy of F and f_*F around 0, respectively. Then one can easily show that

$$M'^m=M\oplus\cdots\oplus M \qquad (m\text{-times}).$$

Hence, if M is quasi-unipotent, M' is also quasi-unipotent. Q.E.D.

4. Let X be a non-singular analytic space and Y a connected non-singular hypersurface of X. Then, for any point y of Y, there exists a sufficiently small neighbourhood U of y such that $\pi_1(U-Y)$ is generated by the one element γ, which we can obtain as follows. Take a holomorphic map $\varphi: D\to U$ such that $\varphi^{-1}(Y)=\{0\}$ as an analytic space. Then we take as γ the path $\varphi(\varepsilon e^{2\pi it})$ $(0\leqq t\leqq 1)$ for $0<\varepsilon<1$. Remark that this element determines a unique conjugacy class of $\pi_1(X-Y)$.

For a locally constant sheaf F on $X-Y$, the monodromy of F along γ is called the monodromy of F around Y.

PROPOSITION 1.9. *Let Y be a normally crossing hypersurface of a non-singular analytic space X and F a sheaf on X such that $F|_{X-Y}$ and $F|_Y$ are locally constant sheaves. Let X be a point of X. Then the following conditions are equivalent*

(1) *F is quasi-unipotent on some neighbourhood of x.*
(2) *The monodromy of F around any irreducible components containing x is quasi-unipotent.*
(3) *F is quasi-unipotent at x.*

PROOF. The implications (1)$\Rightarrow$(2) and (3) are evident. We shall show (3)$\Rightarrow$(2)$\Rightarrow$(1). Let us take a local coordinate system $(t_1, \cdots, t_n)$ and a small neighbourhood U of x such that $U=\{t\in \boldsymbol{C}^n\,;\, |t_j|<\varepsilon\}$, $x=\{0\}$ and $Y\cap U=\bigcup_{j=1}^{p} H_j$ with $H_j=\{t\in u\,;\, t_j=0\}$. Then $\pi_1(U-Y)$ is the free abelian group generated by p elements $\gamma_1, \cdots, \gamma_p$ where γ_j is the path around H_j. Since F_Y is quasi-unipotent, by replacing F with F_{X-Y}, we may assume from the beginning $F_Y=0$. Now, we shall assume (3). Let ρ be the monodromy representation. For any pair $m=(m_1, \cdots, m_p)$ of positive integers, let $\varphi_m: D\to U$ be the map defined by $\varphi_m(t)=(\varepsilon t^{m_1}, \cdots, \varepsilon t^{m_p}, 0, \cdots, 0)$. Then if we denote by γ the path in D around the origin, we have $\varphi_m(\gamma)=\gamma_1^{m_1}\cdots\gamma_p^{m_p}$. If F is quasi-unipotent at x, then $\rho(\varphi_m(\gamma))=\rho(\gamma_1)^{m_1}\cdots \rho(\gamma_p)^{m_p}$ is quasi-unipotent for any m. This implies the quasi-unipotency of $\rho(\gamma_j)$. Finally, we shall show (2)$\Rightarrow$(1). Let $\varphi: D\to U$ be an analytic map. If $\varphi(D)\subset Y$, then $\varphi^{-1}(F)$ is quasi-unipotent. If $\varphi(D)\not\subset Y$, we may assume $\varphi(D^*)\subset U-Y$. Hence, $\varphi(\gamma)=\gamma_1^{m_1}\cdots\gamma_p^{m_p}$ for some integers $m_1, \cdots, m_p$. Since $\rho(\gamma_j)$ are quasi-unipotent by (2), $\rho(\gamma)$ is also quasi-unipotent. Q. E. D.

PROPOSITION 1.10. *Let F be a constructible sheaf on an analytic space X. Then the set of the points at which F is not quasi-unipotent is a nowhere dense closed analytic subset of X.*

PROOF. We shall prove this by the induction on the dimension of X. We shall denote by $R(F)$ the set of the points at which F is not quasi-unipotent. Let us take a closed nowhere dense analytic subset Y of X such that $F|_{X-Y}$ is locally constant. Then we have

$$R(F)=R(F_{X-Y})\cup R(F|_Y)\,.$$

By the hypothesis of the induction, $R(F|_Y)$ is a closed analytic subset of Y and we have $R(F_{X-Y})\subset Y$. Hence, by replacing F with F_{X-Y}, it is enough to show that $R(F)$ is a closed analytic subset under the assumption: $F|_Y=0$ and $F|_{X-Y}$ is locally constant. By the desingularization theorem of Hironaka, there exists a proper surjective map $f: X'\to X$ such that X' is non-singular and $f^{-1}(Y)$ is a normally crossing hypersurface. Let $\{H_j; j \in J\}$ denote the set of irreducible components of $f^{-1}(Y)$ and let I denote the set of $j \in J$ such that the monodromy of $f^{-1}F$ around H_j is not quasi-unipotent. Then the preceding proposition implies

$$R(f^{-1}F) = \bigcup_{j \in I} H_j$$

and Propositions 1.7 and 1.9 imply

$$R(F) = f(R(f^{-1}F)).$$

Hence, $R(F)$ is a closed analytic subset of X. Q. E. D.

§ 2. Proper direct image of quasi-unipotent sheaf.

1. We shall prove in this section that the proper direct image of a quasi-unipotent sheaf is also quasi-unipotent. In the course of the proof, we reduce this global problem to a local problem.

THEOREM 2.1. *Let $f: X \to Y$ be a proper analytic map. If F is a quasi-unipotent sheaf on X, then $R^p f_*(F)$ is quasi-unipotent for any p.*

In order to prove this theorem, let φ be an analytic map from D into Y. It is enough to show that $\varphi^{-1} R^p f_*(F)$ is quasi-unipotent at the origin. Set $X' = X \times_Y D$ and let f' and φ' denote the projections from X' to D and X, respectively. Then $\varphi^{-1} R^p f_*(F)$ is isomorphic to $R^p f'_*(\varphi'^{-1} F)$. Hence, it is sufficient to show the following

LEMMA 2.2. *If $f: X \to D$ is a proper map and if F is a quasi-unipotent sheaf on X, then $R^p f_*(F)$ is quasi-unipotent at the origin.*

We shall prove this by the induction on the dimension of X. Since $R^p f_*(F_{X - f^{-1}(0)}) = R^p f_*(F)$ on D^*, we may assume from the beginning $F_{f^{-1}(0)} = 0$, by replacing F with $F_{f^{-1}(D^*)}$. We may also assume that $\overline{\mathrm{Supp}\, F} = X$. Hence, $f^{-1}(0)$ is nowhere dense in X. Let Y_1 denote the set of the points of X where f is not smooth. Then, $\dim Y_1 < \dim X$ holds on a neighbourhood of $f^{-1}(0)$. Let us take a nowhere dense closed analytic subset Y_2 of X such that $F|_{X - Y_2}$ is locally constant. Set $Y = f^{-1}(0) \cup Y_1 \cup Y_2$. Then, $R^p f_*(F|_Y)$ is quasi-unipotent at the origin by the hypothesis of the induction. Thus, by using the exact sequence

$$R^p f_*(F_{X-Y}) \longrightarrow R^p f_*(F) \longrightarrow R^p f_*(F_Y),$$

it is enough to show that $R^p f_*(F_{X-Y})$ is quasi-unipotent. By replacing F with F_{X-Y}, we may assume further $F_Y = 0$. By the desingularization theorem of Hironaka, there exists a proper map $g: X' \to X$ satisfying the following conditions

(2.1.1) $X'-g^{-1}(Y) \longrightarrow X-Y$ is an isomorphism.

(2.1.2) X' is non-singular.

(2.1.3) $g^{-1}(Y)$ is a normally crossing hypersurface.

Now, we have $R^p f_*(F)=R^p(f\circ g)_*(g^{-1}F)$ because $F_Y=0$ implies $\boldsymbol{R}g_*(g^{-1}F)=F$. Hence, by replacing X, Y, f, F with $X', g^{-1}(Y), f\circ g, g^{-1}F$, we may assume

(2.1.4) X is non-singular, and

(2.1.5) There exists a normally crossing hypersurface Y of X such that $Y\supset f^{-1}(0)$, $F|_Y=0$ and $F|_{X-Y}$ is locally constant.

2. In order to prove Lemma 2.2, we shall prepare several lemmas. For $\lambda\in \boldsymbol{C}-\{0\}$, let us denote by $\boldsymbol{C}_\lambda$ the locally constant sheaf on D^* of rank 1 whose monodromy around the origin is λ.

LEMMA 2.3. *Let $F^{\cdot}$ be a bounded complex of sheaves on D^* such that $\mathcal{H}^j(F^{\cdot})$ is locally constant for any j, and let λ be a non-zero complex number. Then, the following conditions are equivalent.*

(1) *For any j, no eigenvalue of the monodromy of $\mathcal{H}^j(F^{\cdot})$ around the origin is equal to λ^{-1}.*

(2) *$H^j(D^*; F^{\cdot}\otimes \boldsymbol{C}_\lambda)=0$ for any j.*

PROOF. In order to prove (1)$\Rightarrow$(2), we can assume that $F^{\cdot}$ is a sheaf, i.e. $F^j=0$ for $j\neq 0$. Let π denote the fundamental group of D^*, which is generated by the single element γ. Hence $\boldsymbol{C}[\pi]=\boldsymbol{C}[\gamma, \gamma^{-1}]$ and $\boldsymbol{C}=\boldsymbol{C}[\pi]/\boldsymbol{C}[\pi](\gamma-1)$. Hence, as indicated in § 1.2, we have

$$H^j(D^*; F)=\mathrm{Ext}^j_{\boldsymbol{C}[\pi]}(\boldsymbol{C}, F_{x_0})=\begin{cases}\mathrm{Ker}(\gamma-1; F_{x_0}) & \text{for } j=0\\ \mathrm{Coker}(\gamma-1; F_{x_0}) & \text{for } j=1\\ 0 & \text{for } j\neq 0, 1.\end{cases} \tag{2.2.1}$$

Here, $x_0\in D^*$ and π acts on F_{x_0} by the monodromy representation. The implication of (2) from (1) is then obvious.

Now, we shall prove (2)$\Rightarrow$(1) by the induction on j. By replacing $F^{\cdot}$ with $F^{\cdot}\otimes \boldsymbol{C}_\lambda$, we may assume $\lambda=1$ from the beginning. Assuming that no eigenvalue of the monodromy of $\mathcal{H}^j(F^{\cdot})$ is 1 for $j<p$, we shall prove that this holds for $j=p$ also. We have the triangle

$$\begin{tikzcd} & \tau_{<p}(F^{\cdot}) \arrow[dl] & \\ F^{\cdot} \arrow[rr] & & \tau_{\geqq p}(F^{\cdot}) \arrow[ul, "+1"'] \end{tikzcd} .$$

Here $\tau_{<p}$ and $\tau_{\geqq p}$ denote the truncation operators which conserve the cohomology groups of degree $<p$ and $\geqq p$, respectively. Since $\boldsymbol{R}\Gamma(D^*; F^{\cdot}) = \boldsymbol{R}\Gamma(D^*; \tau_{<p}(F^{\cdot})) = 0$, we have $\boldsymbol{R}\Gamma(D^*; \tau_{\geqq p}(F^{\cdot})) = 0$. By taking its p-th cohomology group, we obtain $H^0(D^*; \mathcal{H}^p(F^{\cdot})) = H^p(D^*; \tau_{\geqq p}(F^{\cdot})) = 0$. Hence (2.2.1) implies that the monodromy of $\mathcal{H}^p(F^{\cdot})$ does not have 1 as its eigenvalue. Q.E.D.

LEMMA 2.4. *Let X be a non-singular manifold and Y a normally crossing hypersurface and $j: X-Y \hookrightarrow X$ the inclusion. Let x_0 be a point of Y and Y_1 an irreducible component of Y which contains x_0. Let F be a locally constant sheaf of finite rank on $X-Y$. If no eigenvalue of the monodromy of F around Y_1 equals* 1, *then we have $R^p j_*(F)_{x_0} = 0$ for any p.*

PROOF. Let us take a local coordinate system $(t_1, \cdots, t_n)$ around x_0 such that $x_0 = 0$ and $Y = \bigcup_{j=1}^{l} Y_j$ with $Y_j = \{t_j = 0\}$. If we take a small ball U centered at the origin, we have $H^p(U-Y; F) = R^p j_*(F)_{x_0}$. Let π be the fundamental group of $U-Y$, which is the free abelian group generated by $\gamma_1, \cdots, \gamma_p$. Here γ_j is the cycle around Y_j. As shown in § 1.2, we have

$$H^p(U-Y; F) = \mathrm{Ext}^p_{\boldsymbol{C}[\pi]}(\boldsymbol{C}, F_{x_1})$$

for $x_1 \in U-Y$. Since $\gamma_1 - 1$ is invertible on F_{x_1} and zero on $\boldsymbol{C}$, this cohomology group must vanish. Q.E.D.

3. Now, we resume the proof of Lemma 2.2 under the assumptions (2.1.4) and (2.1.5). We shall set $D_\varepsilon^* = \{z \in \boldsymbol{C}; 0 < |z| < \varepsilon\}$. By Lemma 2.3, it is enough to show

$$\lim_{\varepsilon \to 0} \boldsymbol{R}\Gamma(D_\varepsilon^*; \boldsymbol{R}f_*(F) \otimes \boldsymbol{C}_\lambda) = 0$$

for any λ which is not a root of unity. On the other hand, letting $j: X-Y \hookrightarrow X$ be the inclusion, we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \boldsymbol{R}\Gamma(D_\varepsilon^*; \boldsymbol{R}f_*(F) \otimes \boldsymbol{C}_\lambda) &= \lim_{\varepsilon \to 0} \boldsymbol{R}\Gamma(f^{-1}(D_\varepsilon^*); F \otimes f^{-1}\boldsymbol{C}_\lambda) \\ &= \boldsymbol{R}\Gamma(f^{-1}(0); \boldsymbol{R}j_*(F \otimes f^{-1}\boldsymbol{C}_\lambda|_{X-Y})). \end{aligned}$$

If λ is not a root of unity, the monodromy of $F \otimes f^{-1}\boldsymbol{C}_\lambda$ around any irreducible component of $f^{-1}(0)$ does not have 1 as its eigenvalue. Hence, Lemma 2.4 implies $\boldsymbol{R}j_*(F\otimes f^{-1}\boldsymbol{C}_\lambda|_{X-Y})|_{f^{-1}(0)}=0$. This completes the proof of Lemma 2.2 and at the same time that of Theorem 2.1.

§ 3. Generic property of quasi-unipotency.

1. The purpose of this section is to prove the following theorem.

THEOREM 3.1. *Let X be an analytic space, Y a closed analytic subset of X and Z a closed analytic subset of X of codimension $\geqq 2$. Let F be a sheaf on X such that $F|_{X-Y}$ and $F|_Y$ are locally constant of finite rank. If $F|_{X-Z}$ is quasi-unipotent, then F is quasi-unipotent.*

It is easy to see that we may assume $F_Y=0$ from the beginning. We shall first reduce the problem to the case when X is a non-singular two-dimensional manifold. Let $\varphi: D\to X$ be an analytic map and we shall show that $\varphi^{-1}F$ is quasi-unipotent at the origin. If $\varphi(0)$ does not belong to $Z\cap Y$, this is clear. Hence, we may assume $\varphi(0)\in Z\cap Y$. If $\varphi(D)\subset Y$, then this is also clear. Hence we may assume that $\varphi(0)\in Z\cap Y$ and $\varphi^{-1}(Y)=\{0\}$.

In this case there exist a holomorphic map ψ from a two-dimensional normal analytic space X' into X and an analytic map $\varphi': D\to X'$ such that $\psi\circ\varphi'=\varphi$ and $\psi^{-1}(Z)$ has codimension 2. By replacing X and F with X' and $\psi^{-1}F$, we may assume from the beginning that X is a two-dimensional normal analytic space. Now, there exists a finite map p from $(X, \varphi(0))$ to $(\boldsymbol{C}^2, 0)$. We may assume that p is a finite map from X onto an open neighbourhood U of the origin and $p^{-1}(0)=Z$. Let Y' be a closed curve in U such that $Y'\supset p(Y)$ and $X-p^{-1}(U)\to U-Y'$ is a local isomorphism. By Proposition 1.8, p_*F is quasi-unipotent on $U-\{0\}$. Let j denote the inclusion map $U-Y'\hookrightarrow U$ and we put $F'=j_!j^{-1}p_*F$. Then $F'|_{U-Y'}$ is locally constant and $F'|_{U-\{0\}}$ is quasi-unipotent.

If we can prove the quasi-unipotency of F', then $j_*j^{-1}F'$ is quasi-unipotent. Since p_*F is a subsheaf of $j_*j^{-1}F'$, p_*F is quasi-unipotent, and hence F is quasi-unipotent by Proposition 1.8. Thus we may assume that

(3.1.1) $\qquad 0\in X\subset \boldsymbol{C}^2$

(3.1.2) $\qquad Y$ is a curve

(3.1.3) $\qquad Z=\{0\}$

(3.1.4) $\qquad F_Y=0$.

Now let $\varphi: D\to X$ be an analytic map such that $\varphi(0)=0$ and $\varphi^{-1}(Y)=\{0\}$. Then there exists an analytic map $\psi: D\times D\to X$ such that $\psi(0, y)=\varphi(y)$. Hence, by replacing X and F with $D\times D$ and $\psi^{-1}F$, we may assume further

$$Y\cap\{x=0\}=\{0\} \tag{3.1.5}$$

and it is enough to show that $F|_{x=0}$ is quasi-unipotent at the origin under these conditions. If we take a suitable integer m and define $f: \boldsymbol{C}^2\to\boldsymbol{C}^2$ by $f(x, y)=(x^m, y)$, then $f^{-1}(Y)$ is a union of non-singular curves transversal to the y-axis. Thus we reduce the theorem to the following lemma.

LEMMA 3.2. *Let J be a finite set, X a neighbourhood of the origin in $\boldsymbol{C}^2$ and let Y be a union of non-singular curves C_j $(j\in J)$ in X which are transversal to the y-axis at the origin. Let F be a sheaf on X such that $F_Y=0$ and $F|_{X-Y}$ is a locally constant sheaf of finite rank. Under these conditions, if $F|_{X-\{0\}}$ is quasi-unipotent, then $F|_{\{x=0\}}$ is quasi-unipotent at the origin.*

2. We can translate Lemma 3.2 in terms of representations of the fundamental group. Let us take a sufficiently small ball U centered at the origin. The $G=\pi_1(U-Y)$ is generated by the path γ_j around C_j $(j\in J)$. Let γ_0 be the path around the origin in the y-axis. Then Lemma 3.2 is equivalent to the following proposition by using the correspondence between the representations of G and the locally constant sheaves on $U-Y$.

PROPOSITION 3.3. *Let ρ be a finite-dimensional representation of G. If $\rho(\gamma_j)$ is quasi-unipotent for any $j\in J$, then $\rho(\gamma_0)$ is also quasi-unipotent.*

If G is abelian, then this proposition is obvious, because the product of quasi-unipotent matrices commuting to each other is also quasi-unipotent. As we shall see in the next section, G is not very far from abelian group, which permits us to prove Proposition 3.3.

§ 4. Description of the fundamental group.

1. Let J be a finite set and let $\{C_j\}_{j\in J}$ be a set of germs of non-singular curve in $(\boldsymbol{C}^2, 0)$. Assume that $C_j\cap C_k=\{0\}$ for $j\neq k$. Set $Y=\bigcup_{j\in J} C_j$. We shall describe the fundamental group of $U-Y$ for a sufficiently small ball U centered at the origin. This problem has been studied for a long time, but here we shall describe the fundamental group in the term of a tree.

Let $m(j, k)$ denote the intersection number of C_j and C_k at the origin for $j\neq k\in J$. Then it is known that $\pi_1(U-Y)$ depends only on the data $(J, (m(j, k))_{j, k\in J})$.

We set, as convention,

$$m(j, j)=\infty\,. \tag{4.1.1}$$

Then $\{m(j, k)\}$ enjoys the following properties.

(4.1.2) $\quad m(j, k)$ is a positive integer for $j \neq k$.

(4.1.3) $$m(j, k)=m(k, j)$$

(4.1.4) $$m(i, k) \geqq \min(m(i, j), m(j, k)) \quad \text{for} \quad i, j, k \in J.$$

2. More generally let J be a finite set and $m(j, k)$ the map from $J \times J$ into $\boldsymbol{Z} \cup \{\infty\}$ which satisfies (4.1.1), (4.1.2), (4.1.3) and (4.1.4).

We define $m(i)=\max_{j \neq i} m(i, j)$. (If $\sharp J=1$, $m(i)$ means 0.) Let $\mathfrak{S}$ be the set of the pairs $\sigma=(J, p)$ of a non-empty subset J of I and an integer $p \geqq 0$ which satisfies the following property:

(4.2.1) $\quad$ For $i \in I$, $p \leqq m(i)+1$ and $I=\{j \in J ; m(i, j) \geqq p\}$.

We denote $|\sigma|$ for I and $p(\sigma)$ for p. Set $\sigma_0=(J, 0)$, which is a unique element of $\mathfrak{S}$ satisfying $p(\sigma_0)=0$. For $\sigma \in \mathfrak{S}$ such that $\sigma \neq \sigma_0$, we denote by $A(\sigma)$ the unique element of $\mathfrak{S}$ which satisfies $|A(\sigma)| \supset |\sigma|$ and $p(A(\sigma))=p(\sigma)-1$. By connecting σ and $A(\sigma)$, we provide $\mathfrak{S}$ with the structure of a tree (i.e. a connected graph without circuit), with a specific element σ_0. Then $p(\sigma)$ is nothing but the distance from σ_0 to σ. For any j, we denote by σ_j the element $(\{j\}, m(j)+1)$. Then it is easy to see that $\{\sigma_j ; j \in J\} \cup \{\sigma_0\}$ is the set of the end points of $\mathfrak{S}$. Thus the data $(J ; (m(j, k))_{j, k \in J})$ is completely described by the tree $\mathfrak{S}$ with the specific end point σ_0.

3. We shall return to the original problem to describe $G=\pi_1(U-Y)$. Let $\mathfrak{S}$ be the tree given by J and $m(j, k)$ in the preceding section. By a coordinate transformation we may assume that C_j is transversal to the y-axis. Hence C_j can be written by $y=a_j(x)$ for a holomorphic function $a_j(x)$ defined on a neighbourhood of the origin satisfying $a_j(0)=0$. Let us develop $a_j(x)$ by the Taylor series

(4.3.1) $$a_j(x)=\sum_{\nu=0}^{\infty} a_{j,\nu} x^{\nu}, \quad \text{with} \quad a_{j,0}=0.$$

We have

(4.3.2) $$a_{j,\nu}=a_{k,\nu} \quad \text{for} \quad \nu<m(j, k)$$

and

$$a_{j,\nu} \neq a_{k,\nu} \quad \text{for} \quad \nu=m(j, k).$$

We shall take $\delta>0$ small enough so that $a_j(x)$ is defined on $D_\delta=\{x \in \boldsymbol{C} ; |x|<\delta\}$ and $a_j(x) \neq a_k(x)$ for $x \in D_\delta-\{0\}$ and $j \neq k$. Then we have $\pi_1(U-Y)=\pi_1(D_\delta \times \boldsymbol{C}-Y)$. Since $D_\delta^* \times \boldsymbol{C}-Y$ is locally trivial over D_δ^*, $\pi_1(\{x\} \times \boldsymbol{C}-Y)$ is locally constant in $x \in D_\delta^*$. Therefore, by moving x to $e^{2\pi\sqrt{-1}}x$, we obtain the automorphism T of

$\pi_1(\{x\}\times C-Y)$, i.e. the monodromy. It is well-known that $\pi_1(D_\delta\times C-Y)$ is equal to the quotient of $\pi_1(\{x\}\times C-Y)$ by the normal subgroup generated by $\gamma^{-1}T(\gamma)$ $(\gamma\in\pi_1(\{x\}\times C-Y))$. On the other hand, $\{x\}\times C-Y$ is the space C deleted by $\sharp J$ elements. Hence $\pi_1(\{x\}\times C-Y)$ is the free group generated by γ_j $(j\in J)$, where γ_j is the path in $\{x\}\times C$ around C_j. For $x\in D_\delta^*$, we set $Y(x)=\{y\in C;(x, y)\in Y\}=\{a_j(x);j\in J\}$. Let us take a complex number c $(\mathrm{Re}\, c\gg 1)$ and we shall describe $\pi_1(C-Y(\varepsilon e^{2\pi i\theta}), c)$ for $0<\varepsilon\ll 1$.

Now, we take $a(\sigma)\in C$, which satisfies the following conditions:

(4.3.3) $$\mathrm{Re}\, a(\sigma)>|a_{j,p(\sigma)}| \quad \text{for} \quad j\in|\sigma|.$$

(4.3.4) For $j\in|\sigma|$, we denote by l_j the segment joining $a(\sigma)$ and $a_{j,p(\sigma)}$. Then either l_j and l_k intersect only at $a(\sigma)$ or $a_{j,p(\sigma)}=a_{k,p(\sigma)}$.

We define $a_{\sigma,\nu}$ by

(4.3.5) $$\begin{aligned} a_{\sigma,\nu}&=a_{j,\nu} &&\text{for}\quad \nu<p(\sigma),\ j\in|\sigma|,\\ &=a(\sigma) &&\text{for}\quad \nu=p(\sigma),\\ &=0 &&\text{for}\quad \nu>p(\sigma).\end{aligned}$$

For σ, we set $L'(\sigma)=A^{-1}(\sigma)$. We give the linear order on $L'(\sigma)$ by $\tau<\tau'$ if $\arg(a_{\tau,p(\sigma)}-a(\sigma))>\arg(a_{\tau',p(\sigma)}-a(\sigma))$. Here, $\theta=\arg(a_{\tau,p(\sigma)}-a(\sigma))$ is a real number such that $0<\theta<1$ and $a_{\tau,p(\sigma)}-a(\sigma)=\mathrm{r}e^{2\pi i\theta}$ for $r>0$. By (4.3.3) and (4.3.4) such an ordering exists uniquely.

We set, for $0<\varepsilon\ll 1$,

(4.3.6) $$a_\sigma(\theta)=\sum_\nu a_{\sigma,\nu}(\varepsilon e^{2\pi i\theta})^\nu$$

and

(4.3.7) $$b_\sigma(\theta)=\sum_{\nu<p(\sigma)} a_{\sigma,\nu}(\varepsilon e^{2\pi i\theta})^\nu+a(\sigma)\varepsilon^{p(\sigma)}(e^{2\pi i\theta})^{p(\sigma)-1}.$$

Setting $c=a(\sigma_0)$, we shall describe $\pi_1(\{\varepsilon e^{2\pi i\theta}\}\times C-Y, c)$. We shall denote by $\alpha_\sigma(\theta)$ the straight path from $a_{A(\sigma)}(\theta)$ to $b_\sigma(\theta)$, by $\beta_\sigma(\theta)$ the path from $b_\sigma(\theta)$ to $a_\sigma(\theta)$ defined by

$$\beta_\sigma(\theta)(t)=\sum_{\nu<p(\sigma)} a_{\sigma,\nu}(\varepsilon e^{2\pi i\theta})^\nu+a(\sigma)\varepsilon^{p(\sigma)}(e^{2\pi i\theta})^{p(\sigma)-1}e^{2\pi i\theta t} \qquad (0\leqq t\leqq 1),$$

and by $\varepsilon_\sigma(\theta)$ the path from $b_\sigma(\theta)$ to itself defined by

$$\beta_\sigma(\theta)(t)=\sum_{\nu<p(\sigma)} a_{\sigma,\nu}(\varepsilon e^{2\pi i\theta})^\nu+a(\sigma)\varepsilon^{p(\sigma)}(e^{2\pi i\theta})^{p(\sigma)-1}e^{2\pi i t} \qquad (0\leqq t\leqq 1).$$

LEMMA 4.1. *For $0<\varepsilon\ll 1$, $\alpha_\sigma(\theta)$, $\beta_\sigma(\theta)$ and $\varepsilon_\sigma(\theta)$ do not pass $Y(\varepsilon e^{2\pi i\theta})$.*

PROOF. Since we can prove this for $\beta_\sigma(\theta)$ and $\varepsilon_\sigma(\theta)$ in the same way, we shall prove this only for $\alpha_\sigma(\theta)$. Set $p=p(\sigma)$ and $\tau=A(\sigma)$. If $\alpha_\sigma(\theta)$ passes $Y(\varepsilon e^{2\pi i\theta})$, there are $j\in J$ and t $(0\leqq t\leqq 1)$ such that

$$\sum_\nu a_{j,\nu}(\varepsilon e^{2\pi i\theta})^\nu=\alpha_\sigma(\theta)(t)=\sum_{\nu<p-1}\alpha_{\sigma,\nu}(\varepsilon e^{2\pi i\theta})^\nu$$
$$+((1-t)a(\tau)+t(a_{\sigma,p-1}+\varepsilon a(\sigma)))(\varepsilon e^{2\pi i\theta})^{p-1}.$$

We shall show first $a_{j,\nu}=a_{\sigma,\nu}$ for $\nu<p-1$ by the induction on ν. Suppose $a_{j,\mu}=a_{\sigma,\mu}$ for $\mu<\nu$. Then $|a_{j,\nu}-a_{\sigma,\nu}|$ is majorated by $M\varepsilon$ for some $M>0$. Hence if ε is small enongh, we obtain $a_{j,\nu}=a_{\sigma,\nu}$. Thus we obtain $a_{j,\nu}=a_{\sigma,\nu}=a_{\tau,\nu}$ for $\nu<p-1$, which means $|\tau|\ni j$. In the same way, $|(1-t)a(\tau)+ta_{\sigma,p-1}-a_{j,p-1}|$ is majorated by $M\varepsilon$ for some $M>0$. Hence if ε is small enough, this implies that $a_{j,p-1}$ is on the segment joining $a(\tau)$ and $a_{\sigma,p-1}$. If we take $k\in|\sigma|\subset|\tau|$, we have $a_{\sigma,p-1}=a_{k,p-1}$. Hence, (4.3.4) for τ shows $a_{j,p-1}=a_{k,p-1}$ and hence we have $a_{j,\nu}=a_{\sigma,\nu}$ for $\nu<p$ and $j\in|\sigma|$. Therefore, we obtain

$$|(1-t)(a(\tau)-a_{j,p-1})+t\varepsilon a(\sigma)-\varepsilon a_{j,p}e^{2\pi i\theta})|\leqq M\varepsilon^2$$

for some $M>0$, which does not depend on ε. Hence, $1-t$ is majorated by $M'\varepsilon$ for some $M'>0$ and hence $(1-t)\varepsilon a(\sigma)$ is majorated by $M''\varepsilon^2$ for $M''>0$. Thus we obtain

$$|(1-t)(a(\tau)-a_{j,p-1})+\varepsilon(a(\sigma)-a_{j,p}e^{2\pi i\theta})|\leqq M'''\varepsilon^2 \qquad \text{for some} \quad M'''>0.$$

Hence we obtain

$$M'''\varepsilon^2\geqq(1-t)(\mathrm{Re}\, a(\tau)-|a_{j,p-1}|)+\varepsilon(\mathrm{Re}\, a(\sigma)-|a_{j,p}|)$$
$$\geqq\varepsilon(\mathrm{Re}\, a(\sigma)-|a_{j,p}|),$$

which contradicts (4.3.3) for $0<\varepsilon\ll 1$. Q.E.D.

The following lemma is also obvious.

LEMMA 4.2. 1) *For* $\sigma\neq\sigma_0$, $a_j(\varepsilon e^{2\pi i\theta})$ *is inside of* $\varepsilon_\sigma(\theta)$ *for* $j\in|\sigma|$ *and outside of* $\varepsilon_\sigma(\theta)$ *for* $j\notin|\sigma|$. 2) *For an inner point* σ, *set* $L'(\sigma)=\{\tau_1,\cdots,\tau_N\}$ *with* $\tau_1<\cdots<\tau_N$. *Then* $|\sigma|$ *is a disjoint union of* $|\tau_1|,\cdots,|\tau_N|$ *and we have*

$$\varepsilon_\sigma(\theta)=\beta_\sigma(\theta)^{-1}(\alpha_{\tau_1}(\theta)^{-1}\varepsilon_{\tau_1}(\theta)\alpha_{\tau_1}(\theta))(\alpha_{\tau_2}(\theta)^{-1}\varepsilon_{\tau_2}(\theta)\alpha_{\tau_2}(\theta))$$
$$\cdots(\alpha_{\tau_N}(\theta)^{-1}\varepsilon_{\tau_N}(\theta)\alpha_{\tau_N}(\theta))\beta_\sigma(\theta).$$

See Fig. 1.

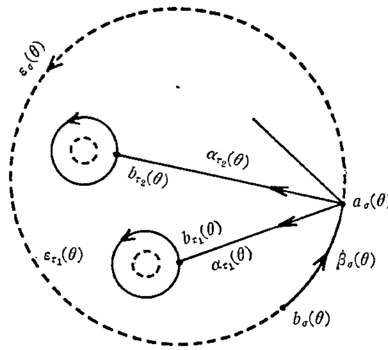

Fig. 1

Let $\gamma_\sigma(\theta)$ denote the path

(4.3.8) $$\gamma_\sigma(\theta)=(\alpha_{\sigma_p}(\theta)\beta_{\sigma_{p-1}}(\theta)\cdots\alpha_{\sigma_1}(\theta))^{-1}\varepsilon_\sigma(\theta)(\alpha_{\sigma_p}(\theta)\beta_{\sigma_{p-1}}(\theta)\cdots\alpha_{\sigma_1}(\theta)),$$

where $p=p(\sigma)$ and $\sigma=\sigma_p$, $\sigma_{p-1}=A(\sigma)$, $\sigma_{p-2}=A(\sigma_{p-1})$, $\cdots$. This $\gamma_\sigma(\theta)$ is the path in $\boldsymbol{C}-Y(\varepsilon e^{2\pi i\theta})$ starting from $c=a(\sigma_0)$ ending at the same point and surrounding $\{a_j(\varepsilon e^{2\pi i\theta});j\in|\sigma|\}$. Set $\gamma_\sigma=\gamma_\sigma(0)$ and $\gamma_j=\gamma_{\sigma_j}$. Then it is easy to see by Lemma 4.2

(4.3.9) $$\gamma_\sigma=\gamma_{\tau_1}\cdots\gamma_{\tau_N},$$

where $L'(\sigma)=\{\tau_1,\cdots,\tau_N\}$ with $\tau_1<\cdots<\tau_N$.

Since $\gamma_\sigma(\theta)$ moves continuously in θ, we have

$$T(\gamma_\sigma)=\gamma_\sigma(1).$$

We have $\alpha_\sigma(0)=\alpha_\sigma(1)$, $\beta_\sigma(0)=1$, $\beta_\sigma(1)=\varepsilon(\sigma)$. Hence if we set $\alpha_\sigma=\alpha_\sigma(0)$, we have

$$\gamma_\sigma=(\alpha_{\sigma_p}\alpha_{\sigma_{p-1}}\cdots\alpha_{\sigma_1})^{-1}\varepsilon_\sigma(\alpha_{\sigma_p}\cdots\alpha_{\sigma_1}),$$

and

$$\gamma_\sigma(1)=(\alpha_{\sigma_p}\varepsilon_{\sigma_{p-1}}\alpha_{\sigma_{p-1}}\cdots\varepsilon_{\sigma_1}\alpha_{\sigma_1})^{-1}\varepsilon_\sigma(\alpha_{\sigma_p}\varepsilon_{\sigma_{p-1}}\cdots\alpha_{\sigma_1})$$
$$=(\varepsilon_{\sigma_p}\alpha_{\sigma_p}\varepsilon_{\sigma_{p-1}}\cdots\varepsilon_{\sigma_1}\alpha_{\sigma_1})^{-1}\varepsilon_\sigma(\varepsilon_{\sigma_p}\alpha_{\sigma_p}\cdots\alpha_{\sigma_1}).$$

Hence, we obtain

$$\gamma_\sigma(1)=(\gamma_{\sigma_p}\cdots\gamma_{\sigma_1})^{-1}\gamma_\sigma(\gamma_{\sigma_p}\cdots\gamma_{\sigma_1}).$$

In fact we have $\gamma_{\sigma_p}\cdots\gamma_{\sigma_1}=(\alpha_{\sigma_p}\cdots\alpha_{\sigma_1})^{-1}(\varepsilon_{\sigma_p}\alpha_{\sigma_p}\varepsilon_{\sigma_{p-1}}\alpha_{\sigma_{p-1}}\cdots\varepsilon_{\sigma_1}\alpha_{\sigma_1})$.

We shall define g_σ by $\gamma_{\sigma_p}\gamma_{\sigma_{p-1}}\cdots\gamma_{\sigma_1}$. Then, we have

(4.3.10) $$g_{\sigma_0}=1 \quad\text{and}\quad g_\sigma=\gamma_\sigma g_{A(\sigma)} \quad\text{for}\quad \sigma\neq\sigma_0.$$

By using g_σ, we obtain

$$T(\gamma_\sigma)=\mathrm{Ad}(g_\sigma^{-1})\gamma_\sigma.$$

Therefore $G=\pi_1(U-Y)$ is the group generated by $\{\gamma_j;j\in J\}$ with the fundamental relation

(4.3.11) $\quad \gamma_\sigma g_\sigma = g_\sigma \gamma_\sigma$ where γ_σ and g_σ are given by (4.3.9) and (4.3.10).

Since $\gamma_\sigma = g_\sigma g_{A(\sigma)}^{-1}$, (4.3.11) is equivalent to

(4.3.12) $$g_\sigma g_\tau = g_\tau g_\sigma \quad \text{if } \sigma \text{ and } \tau \text{ is connected.}$$

Let σ be an inner point of $\mathfrak{S}$ and let $L(\sigma)$ be the set of the points of $\mathfrak{S}$ connected with σ. Then $L(\sigma) = L'(\sigma) \cup \{A(\sigma)\} = \{\tau_1, \cdots, \tau_N, A(\sigma)\}$. The set $|\sigma|$ is a disjoint union of $|\tau_1|, \cdots, |\tau_N|$. Suppose that $\tau_1, \cdots, \tau_N$ are so aranged that $\tau_1 < \cdots < \tau_N$. Then (4.3.9) implies $\gamma_\sigma = \gamma_{\tau_1} \cdots \gamma_{\tau_N}$. We have $g_\sigma = \gamma_\sigma g_{A(\sigma)} = \gamma_{\tau_1} \cdots \gamma_{\tau_N} g_{A(\sigma)}$. Since g_σ commutes with $\gamma_{\tau_1}, \cdots, \gamma_{\tau_N}$ and $g_{A(\sigma)}$, we obtain by multiplying g_σ^N from the right

$$g_\sigma^{1+N} = (\gamma_{\sigma_1} g_\sigma) \cdots (\gamma_{\sigma_N} g_\sigma) g_{A(\sigma)} = g_{\tau_1} \cdots g_{\tau_N} g_{A(\sigma)} .$$

Hence if we order $L(\sigma)$ by $\tau_1, \cdots, \tau_N, A(\sigma)$, we obtain

(4.3.13) $\quad g_\sigma^{l(\sigma)} = \prod_{\tau \in L(\sigma)} g_\tau$ where $l(\sigma) = \# L(\sigma)$ and the product is taken according to the order of $L(\sigma)$.

Conversely, it is easy to see that $G = \pi_1(U-Y)$ is the group generated by $\{g_\sigma ; \sigma \in \mathfrak{S}\}$ with the fundamental relations (4.3.12), (4.3.13) and $g_{\sigma_0} = 1$.

4. More generally, we say a tree $\mathfrak{S}$ is *oriented* if, for any inner point σ of $\mathfrak{S}$, the set $L(\sigma)$ of the points connected with σ is cyclically ordered. Remark that any oriented tree can be embedded into the oriented $\boldsymbol{R}^2$. This means that $\mathfrak{S}$ is regarded as such a subset of $\boldsymbol{R}^2$ that the open segments joining two connected points of $\mathfrak{S}$ do not intersect to each other. Moreover, for any inner point σ, the cyclic order of $L(\sigma)$ is given by the orientation of $\boldsymbol{R}^2$. Let us remark also such an embedding is unique up to deformation. We call a *rooted* tree a pair of a tree $\mathfrak{S}$ and an end point of $\mathfrak{S}$. This point is called a root of $\mathfrak{S}$.

For an oriented rooted tree $\mathfrak{S}$, we denote by $G_\circ(\mathfrak{S})$ the group generated by $\{g_\sigma ; g \in \mathfrak{S}\}$ with the fundamental relations (4.3.12), (4.3.13) and $g_{\sigma_0} = 1$ for the root σ_0.

Note that the relation (4.3.13) does depend only on the cyclic order of $L(\sigma)$. In fact, if $L(\sigma) = \{\tau, \cdots, \tau_l\}$ and $g_\sigma^l = g_{\tau_1} \cdots g_{\tau_l}$, then we have $g_\sigma^l = \mathrm{Ad}(g_{\tau_l}) g_\sigma^l = g_{\tau_l} g_{\tau_1} \cdots g_{\tau_{l-1}}$ because g_{τ_l} and g_σ commute by (4.3.12).

Note that the isomorphic class of the group $G_\circ(\mathfrak{S})$ does not depend on the orientation of $\mathfrak{S}$.

By using these terminologies, we can summarize the result of § 4.3 by the following theorem.

THEOREM 4.3. $\pi_1(U-Y) = G_\circ(\mathfrak{S})$.

Here $\mathfrak{S}$ is the rooted tree obtained by J and $(m(j, k))$.

5. Now, we shall prove Proposition 3.3. We have $\gamma_j = g_{\sigma_j} g^{-1}_{A(\sigma_j)}$ and $\gamma_0 = g_{A^{-1}(\sigma_0)} g^{-1}_{\sigma_0}$. Hence Proposition 3.3 is a corollary of the following Theorem 4.4 and Corollary 4.5.

For an oriented tree $\mathfrak{S}$, we shall define $G(\mathfrak{S})$ the group generated by $\{g_\sigma ; \sigma \in \mathfrak{S}\}$ with the fundamental relations (4.3.12) and (4.3.13). For an end point σ of $\mathfrak{S}$, let γ_σ denote $g_\sigma g_\tau^{-1}$, where τ is the unique element of $\mathfrak{S}$ connected with σ.

THEOREM 4.4. *Let $\mathfrak{S}$ be an oriented tree. Let σ be a finite-dimensional representation of $G(\mathfrak{S})$. If $\rho(\gamma_\sigma)$ is quasi-unipotent for any end point σ, then, for any two connected points σ and τ, $\rho(g_\sigma g_\tau^{-1})$ is quasi-unipotent.*

PROOF. We may assume without loss of generality that ρ is irreducible. Let U denote the group of the roots of unity. Then $\boldsymbol{C}^*/U$ is an abelian group without torsion. Hence $\boldsymbol{C}^*/U$ is regarded as a vector space over $\boldsymbol{Q}$. Let I denote the set of the eigenvalues of g_σ's. Since I is a finite set, there exists a linear map from $\boldsymbol{C}^*/U$ into $\boldsymbol{Q}$ which separates IU/U; i.e. there exists a map φ from $\boldsymbol{C}^*$ into $\boldsymbol{Q}$ such that

$$\varphi(a, b) = \varphi(a) + \varphi(b) \quad \text{for} \quad a, b \in \boldsymbol{C}^*. \tag{4.5.1}$$

$$\text{For } a, b \in I \text{ such that } ab^{-1} \notin U, \text{ we have } \varphi(a) \neq \varphi(b). \tag{4.5.2}$$

If we denote by I' the set of the eigenvalues of g_σ's for the inner points σ, then by the assumption, we have $\varphi(I) = \varphi(I')$. In order to prove the theorem, it is enough to show that IU/U consists of a single point or $\varphi(I)$ consists of a single point. Set $c = \sup \varphi(I) = \varphi(I')$. For $\sigma \in \mathfrak{S}$, let $V(\sigma)$ denote the direct sum of the generalized eigenspaces of g_σ with eigenvalues in $\varphi^{-1}(c)$. In another word, we have

$$V(\sigma) = \{v \in V \,;\, \text{there exists } a \in \boldsymbol{C}[x] \text{ such that } a(\rho(g_\sigma))v = 0 \text{ and } a^{-1}(0) \subset \varphi^{-1}(c)\}.$$

Here V denotes the representation space of ρ.

If σ and τ are connected, then g_τ commutes with g_σ, and hence

$$\rho(g_\tau)V(\sigma) = V(\sigma) \quad \text{if } \tau \text{ and } \sigma \text{ are connected.} \tag{4.5.3}$$

Let σ be an inner point and $\{\tau_1, \cdots, \tau_N\}$ the set of the points connected with σ. Then we have

$$g_\sigma^N = \prod_{j=1}^N g_{\tau_j}.$$

Therefore, we obtain $\det(\rho(g_\sigma)|_{V(\sigma)})^N = \prod_j \det(\rho(g_{\tau_j})|_{V(\sigma)})$. Set $d = \dim V(\sigma)$ and let $\{\lambda_1, \cdots, \lambda_d\}$ and $\{\mu_{j,1}, \cdots, \mu_{j,d}\}$ denote the set of the eigenvalues of $\rho(g_\sigma)|_{V(\sigma)}$ and $\rho(g_{\tau_j})|_{V(\sigma)}$, respectively. Then we have $\det(\rho(g_\sigma)|_{V(\sigma)})=\prod_{\lambda=1}^{d}\lambda_\nu$ and $\det(\rho(g_{\tau_j})|_{V(\sigma)})=\prod_\nu \mu_{j,\nu}$. Hence we obtain

$$N\sum_{\nu=1}^{d}\varphi(\lambda_\nu)=\sum_{j=1}^{N}\sum_{\nu=1}^{d}\varphi(\mu_{j,\nu}).$$

On the other hand, $\varphi(\lambda_\nu)=c$ and $\varphi(\mu_{j,\nu})\leqq c$ for any j and ν. This implies $\varphi(\mu_{j,\nu})=c$ and hence we obtain $V(\sigma)\subset V(\tau_j)$. Thus we have obtained

(4.5.4) If σ and τ are connected and σ is an inner point, then we have $V(\sigma)\subset V(\tau)$.

Hence, $V(\sigma)$ does not depend on an inner point σ. Set $V'=V(\sigma)$ for an inner point σ. Then (4.5.4) implies $V'\subset V(\sigma)$ for any point σ of $\mathfrak{S}$. On the other hand, (4.5.3) implies that V' is invariant by all g_σ. Hence V' is invariant by $G(\mathfrak{S})$. Since $\varphi(I')\ni c$, we have $V'\neq 0$. Hence we obtain $V'=V=V(\sigma)$ for any σ. This shows that $\varphi(I)=\{c\}$. Q.E.D.

COROLLARY 4.5. *Let $\mathfrak{S}$ be an oriented and rooted tree with a root σ_0 and let ρ be a finite-dimensional representation of $G_0(\mathfrak{S})$. If $\rho(\gamma_\sigma)$ is quasi-unipotent for any end point $\sigma\neq\sigma_0$, then, for any two connected points σ and τ, $\rho(g_\sigma g_\tau^{-1})$ is quasi-unipotent.*

PROOF. Let $\mathfrak{S}'$ be the oriented and rooted tree obtained by $\mathfrak{S}$ with the opposite orientation and let $\mathfrak{S}''$ be the tree which is the union of $\mathfrak{S}$ and $\mathfrak{S}'$ identified at the end points. Hence the underlying set of $\mathfrak{S}''$ is $\mathfrak{S}'\cup(\mathfrak{S}-\sigma_0)$. Then one has a homomorphism

$$\varphi:\ G(\mathfrak{S}'')\longrightarrow G_0(\mathfrak{S})$$

$$\text{by}\quad g_\sigma\longmapsto g_\sigma\quad\text{for}\quad \sigma\in\mathfrak{S}\quad\text{and}\quad g_\sigma\longmapsto g_\sigma^{-1}\quad\text{for}\quad\sigma\in\mathfrak{S}'.$$

The preceding theorem for $\mathfrak{S}''$ implies immediately the desired result.

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