Primitive forms for a universal unfolding of a function with an isolated critical point

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To the memory of Takuro Shintani

§ 0. Introduction.

This is a brief summary of the note [6] on primitive integrals. The proofs of statements in the present note will be found there. Unfortunately, our theory is still incomplete in its key point (see § 3 (3.4)).

First let us describe our aim in this note.

Just as the integration of the differential form $dx/\sqrt{4x^3-g_2x-g_3}$ plays a basic role in the study of elliptic integrals associated to the family of elliptic curves $F(x, y, g)=y^2-(4x^3-g_2x-g_3)=0$, there should be a specific differential n-form $\zeta^{(0)}$ which should play a basic role in the study of periods of integrals over vanishing cycles of a family $X\to S$ associated to a universal unfolding $F(x, t)$ of a hypersurface isolated singular point of dimension n.

This will be done in this note by a primitive form $\zeta^{(0)}$ (Definition (3.2)). To state exactly the integrability condition which $\zeta^{(0)}$ enjoys (3.2), one needs to introduce the higher residue pairings $K^{(k)}$, $k=0, 1, 2, \cdots$ on the de-Rham cohomology group $\mathcal{H}_F^{(0)}$ (see § 2 (2.1)).

As consequences of the introduction of such a $\zeta^{(0)}$ one gets several structures, namely, i) a definition of exponents $\alpha_1, \cdots, \alpha_\mu$ with the duality property $\{\alpha_1, \cdots, \alpha_\mu\}=\{n+1-\alpha_1, \cdots, n+1-\alpha_\mu\}$ in § 6 ii) a flat structure of the base space S of the family (i.e. an embeddings of S into an Euclidean space Ω_f) in § 5, iii) an algebraic representation of the intersection form I of vanishing homology classes in § 7, and iv) the τ-function.

Furthermore the integrals of the form $\zeta^{(0)}$ over vanishing cycles induce a period map $\tilde{S}\to H^n$, where $\tilde{S}$ is a branched monodromy covering of S and H^n is the middle cohomology group of the general fiber of $X\to S$. Due to the duality of the exponents, the period map is everywhere biregular. Thus one is naturally led to the consideration of the inverse mapping from the image domain in H^n to a domain in Ω_f.

The study of the inverse map (in other words, the modular functions) is a fascinating theme. In case of the family of elliptic curves, the inverse map is described by the elliptic modular functions. In the case of the family of rational double points, it is described by the invariants of a Weyl group. Except for these classical examples, almost nothing is yet known about the inverse map.

I am indebted to many mathematicians with whom I discussed these subjects. Among all of them I single out Professor Egbert Brieskorn, to whom I would like to express my sincerest gratitude.

§ 1. Hamiltonian system.

We introduce the concept of a Hamiltonian system $F(x, t)$ over a frame (Z, X, S, T) (Def. (1.3)), which gives a convenient terminology by which we describe the universal unfolding of a function $f(x)$. (For more details see § 1, ..., § 6 [6].)

(1.1) *Primitive vector field* δ_1 *and an energy function.* Let S be a complex manifold of dimension m with a base point $0 \in S$, and let δ_1 be a holomorphic vector field on S which is non-singular at 0. Then there exists an $m-1$ dimensional manifold with a base point $0 \in T$ and a local holomorphic submersion $\pi:(S, 0) \to (T, 0)$ such that $\pi^{-1}\mathcal{O}_T = \{g \in \mathcal{O}_S ; \delta_1 g = 0\}$, where $\mathcal{O}_S$, $\mathcal{O}_T$ denotes the sheaf of germs of holomorphic functions on S or T respectively.

Put

(1.1.1) $$\mathcal{G} := \{\delta \in \pi_* \mathrm{Der}_S : [\delta_1, \delta] = 0\},$$

where Der_S is the sheaf of germs of holomorphic vector fields on S, and $[\cdot, \cdot]$ is the bracket product of vector fields. Then $\mathcal{G}$ is an $\mathcal{O}_T$-free module of rank m and is closed under the bracket product. Hence one gets an exact sequence of $\mathcal{O}_T$-Lie algebras,

(1.1.2) $$0 \longrightarrow \mathcal{O}_T \delta_1 \longrightarrow \mathcal{G} \longrightarrow \mathrm{Der}_T \longrightarrow 0 .$$

In this situation, the following are equivalent.

i) to give a holomorphic function t_1 on S such that

$$\delta_1 t_1 = 1 \quad \text{and} \quad t_1(0) = 0$$

ii) to give a section σ of the projection $\pi:(S, 0) \to (T, 0)$

iii) to give an $\mathcal{O}_T$-Lie algebra splitting of (1.1.2.).

We refer to t_1 of i) above as an energy function w.r.t. δ_1, which has ambiguity of adding a function of $\mathcal{O}_T$.

(1.2) *Frame.* Let X be a smooth $n+m$ dimensional manifold with a base point $0 \in X$ and let $q:(X, 0) \to (T, 0)$ be a smooth surjective map. Let the following be the diagram of the fiber product of X and S over T.

(1.2.1)
$$\begin{array}{ccc} (Z, 0) & \xrightarrow{\hat{\pi}} & (X, 0) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ (S, 0) & \xrightarrow{\pi} & (T, 0) \end{array}$$

There exists a unique lifting $\hat{\partial}_1$ on Z of the vector field ∂_1 on S such that $\hat{\pi}: Z \to X$ coincides with the projection to the parameter space (orbit space) of $\hat{\partial}_1$.

For convenience, we introduce a coordinate system $(x, t)=(x, t_1, t')=(x_0, \cdots, x_n, t_1, t_2, \cdots, t_m)$ of the frame (1.2.1) as follows, $(t')=(t_2, \cdots, t_m)$ is a local coordinate system for T, $(t)=(t_1, t')$ is one for S, $(x, t')=(x_0, \cdots, x_n, t')$ is a local coordinate system for X and $(x, t)=(x, t_1, t')$ is one for Z.

(1.3) *Hamiltonian system.*

Definition. A Hamiltonian system over a frame (1.2.1) is one of the following equivalent set of data.

i) An energy function $F(x, t)$ on Z w.r.t. $\hat{\partial}_1$,

$$\text{i.e.} \quad \hat{\partial}_1 F=0 \quad \text{and} \quad F(0)=0\,.$$

ii) A holomorphic map $\varphi: (X, 0) \to (S, 0)$ which commutes with the morphisms of the diagram (1.2.1).

iii) A holomorphic section $\sigma: (X, 0) \hookrightarrow (Z, 0)$ w.r.t. the projection $\hat{\pi}$.

By fixing an energy t_1 of (1.1), we can write $F(x, t)=t_1-F_1(x, t')$ for a suitable function F_1 on X. Then the map φ and the section σ is given by $t_1=F_1(x, t')$.

Let us put $f(x):=F_1(x, 0)=\varphi(x, 0)$ and assume that $f(x)$ has an isolated critical point at 0.

Definition. The system $F(x, t)$ is called a universal unfolding of $f(x)$, if the correspondence $\delta \in T_{S,0}$ (= the tangent space of S at $0) \mapsto (\delta F)|_{t=0} \in \mathcal{O}_{q^{-1}(0),0}/(\partial f/\partial x_0, \cdots, \partial f/\partial x_n)$ is bijective.

In this note we shall always assume that F is a universal unfolding. Thus $m=\dim_C S=\dim_C \mathcal{O}_{q^{-1}(0),0}/(\partial f/\partial x_0, \cdots, \partial f/\partial x_n)=\mu$.

(1.4) *Critical set C and the discriminant D.* Let C be the critical set variety of φ in X defined by the ideal $(\partial F_1/\partial x_0, \cdots, \partial F_1/\partial x_n)\mathcal{O}_X$, which is a complete intersection. The restriction $q|_C: C \to T$ is a μ-sheeted branched covering and $q_*\mathcal{O}_C$ is an $\mathcal{O}_T$-free module of rank μ. Let A be an $\mathcal{O}_T$-endomorphism of $q_*\mathcal{O}_C$ defined by the multiplication of t_1. Then the characteristic polynomial $\Delta(t_1, t')=$ $(\det(t_1 I - A(t'))$ is called the discriminant; it defines a divisor D in S such that $D=\varphi(C)$.

(1.5) *Residual product.* Let us consider the $\mathcal{O}_S$-homomorphism,

(1.5.1) $$r: \mathrm{Der}_S \longrightarrow \mathcal{O}_C, \qquad \delta \in \mathrm{Der}_S \longmapsto (\hat{\delta}F)|_{\sigma(C)} \in \mathcal{O}_C,$$

where $\hat{\delta}$ is a lifting of δ to a vector field on Z.

The fact that F is a universal unfolding means that the morphism r of (1.5.1) induces an $\mathcal{O}_T$-bijection

(1.5.2) $$r: \mathcal{G} \cong q_* \mathcal{O}_C.$$

Hence $\mathcal{G}$ obtains naturally an $\mathcal{O}_C$-algebra structure. We shall denote by * the product and call it the residual product. (For instance $\delta * \delta'$, $t_1 * \delta$, etc. mean by definition the following relations $(\delta \hat{*} \delta')F|_{\sigma(C)} = (\hat{\delta}F \cdot \hat{\delta}'F)|_{\sigma(C)}$, $t_1 \hat{*} \delta F|_{\sigma(C)} = t_1 \hat{\delta} F|_C$, etc.)

(1.6) *Logarithmic vector fields, Euler vector field E on S.* Let us put

$$\mathrm{Der}_S(\log \Delta) := \{\delta \in \mathrm{Der}_S : \delta\Delta \in \Delta \mathcal{O}_S\},$$

which is an $\mathcal{O}_S$-module of rank μ, closed under bracket product. One sees that $\ker(r) = \mathrm{Der}_S(\log \Delta)$ (1.5.1), so that one obtains an $\mathcal{O}_T$-direct sum decomposition, and an $\mathcal{O}_T$-homomorphism w,

(1.6.1) $$\pi_* \mathrm{Der}_S = \mathcal{G} \oplus \pi_* \mathrm{Der}_S(\log \Delta),$$

(1.6.2) $$w: \mathcal{G} \longrightarrow \pi_* \mathrm{Der}_S(\log \Delta), \qquad \delta \in \mathcal{G} \mapsto \text{the second factor of } t_1\delta.$$

(Using the residual product in (1.5), $w(\delta) = t_1\delta - t_1 * \delta$.) One sees $\mathrm{Der}_S(\log \Delta) \cong \mathcal{O}_S \underset{\mathcal{O}_T}{\otimes} w(\mathcal{G})$ since the determinant of the coefficients of $w(\partial/\partial t_1), \cdots, w(\partial/\partial t_\mu)$ is Δ.

Among all logarithmic vector fields of $\mathrm{Der}_S(\log \Delta)$, there exists a specific one $E := w(\delta_1)$ which we call the Euler vector field. The discriminant Δ is homogeneous of degree μ, w.r.t. E, i.e.,

(1.6.1) $$E\Delta = \mu\Delta.$$

In general, using a standard energy $\hat{t}_1 := (1/\mu)\,\mathrm{tr.}(t_1 I - A)$,

(1.6.2) $$w(\delta)\Delta = \mu(\delta \hat{t}_1)\Delta$$

and

$$E\hat{t}_1 = \hat{t}_1.$$

(1.7) *Relative de-Rham cohomology $\mathcal{H}_F^{(0)}$.* Let us define

(1.7.1) i) $\mathcal{H}_F^{(0)} := \varphi_* \Omega_{X/T}^{n+1} / dF_1 \wedge d\varphi_* \Omega_{X/T}^{n-1}$,

ii) $\mathcal{H}_F^{(-1)} := \varphi_* \Omega_{X/T}^{n} / dF_1 \wedge \varphi_* \Omega_{X/T}^{n-1} + d(\varphi_* \Omega_{X/T}^{n-1})$,

where $\Omega_{X/T}^p := \Omega_X^p / q^{-1}\Omega_T^1 \wedge \Omega_X^{p-1}$ is the sheaf of germs of holomorphic relative p-forms on X relative to q.

Then $\mathcal{H}_F^{(0)}$, $\mathcal{H}_F^{(-1)}$ are $\mathcal{O}_S$-free modules of rank μ and the wedge product by dF_1 induces an inclusive $\mathcal{O}_S$-homomorphism

$$0 \longrightarrow \mathcal{H}_F^{(-1)} \underset{dF_1}{\hookrightarrow} \mathcal{H}_F^{(0)} \overset{r^{(0)}}{\longrightarrow} \varphi_* \Omega_F \longrightarrow 0, \tag{1.7.2}$$

where we denote by Ω_F the sheaf $\Omega_{X/S}^{n+1} := \Omega_X^{n+1} / \sum_{i=1}^{\mu} dt_i \wedge \Omega_X^n$, and $r^{(0)}$ is the natural homomorphism induced by $\Omega_{X/T}^{n+1} \to \Omega_{X/S}^{n+1}$.

Since $\Omega_F = \mathcal{O}_C dx_0 \wedge \cdots \wedge dx_n$ is an $\mathcal{O}_C$-free module of rank 1, $\varphi_* \Omega_F$ is concentrated in $D \subsetneqq S$ and hence $\mathcal{H}_F^{(-1)}|_{S-D} \cong \mathcal{H}_F^{(0)}|_{S-D}$ defines a vector bundle on $S-D$, which corresponds to $R^n \varphi_* \mathbf{C}_{X-\varphi^{-1}(D)}$.

(1.8) *The Gauß-Manin connection.* The Gauß-Manin connection ∇ is an integrable covariant differentiation,

$$\nabla : \mathrm{Der}_S \times \mathcal{H}_F^{(-1)} \longrightarrow \mathcal{H}_F^{(0)}, \qquad (\delta, \omega) \mapsto \nabla_\delta \omega, \tag{1.8.1}$$

where $\nabla_{\partial/\partial t_i}[\zeta] = (-1)^{i+1}[(dt_2 \wedge \cdots \wedge dt_\mu)^{-1} dt_1 \wedge \cdots \hat{dt_i} \cdots \wedge dt_\mu \wedge d\zeta]$. (Here $[\cdot]$ means a de-Rham cohomology class defined by a differential form $\cdot$.)

In particular one has an $\mathcal{O}_T$-bijection,

$$\nabla_{\delta_1} : \mathcal{H}_F^{(-1)} \cong \mathcal{H}_F^{(0)}, \qquad [\zeta] \mapsto [d\zeta]. \tag{1.8.2}$$

Thus combining the inclusion of (1.7.2) and (1.8.2), one gets a decreasing $\mathcal{O}_S$-free module filtration of $\mathcal{H}_F^{(0)}$,

$$\mathcal{H}_F^{(-k)} := \{\omega \in \mathcal{H}_F^{(0)} : (\nabla_{\delta_1})^i \omega \in \mathcal{H}_F^{(0)}, \quad i=1, \cdots, k\},\ k \in \mathbf{N}. \tag{1.8.3}$$

Then one gets easily,

(1.8.4) i) $0 \longrightarrow \mathcal{H}_F^{(-k-1)} \longrightarrow \mathcal{H}_F^{(-k)} \overset{r^{(k)}}{\longrightarrow} \Omega_* \Omega_F \longrightarrow 0$ (where $r^{(k)} = r^{(0)} \nabla_{\delta_1}^k$),

ii) $\nabla : \mathrm{Der}_S \times \mathcal{H}_F^{(-k-1)} \longrightarrow \mathcal{H}_F^{(-k)}$.

Note. Let $F(x, t) + \sum_{i=1}^{2m} y_i^2$ a Hamiltonian obtained by adding $2m$ new variables. Then one has a natural Der_S-isomorphism ([6] §6 (6.4))

$$\mathcal{H}_F^{(-k-m)} \cong \mathcal{H}_{F+\sum_{i=1}^{2m} y^2}^{(-k)} \qquad \text{for } k \in \mathbf{Z}.$$

(1.9) *Regularity of the Gauß-Manin connection.* In (1.8.4), one has a relation $r^{(k)}(\nabla_\delta\zeta)=r(\delta)r^{(k)}(\nabla_{\delta_1}\zeta)$ for $(\delta, \zeta)\in \mathrm{Der}_S\times\mathcal{H}_F^{(-k-1)}$. (One needs to check it only for $k=0$.) Thus we get,

(1.9.1) $$\nabla : \mathrm{Der}_S(\log \Delta)\times\mathcal{H}_F^{(-k)} \longrightarrow \mathcal{H}_F^{(-k)}, \qquad k=0, 1, 2, \cdots .$$

This implies that the connection ∇ has logarithmic poles. Hence the connection ∇ is regular singular.

(1.10) *A connection ∇_w on Ω_F.* Since $\nabla_{w(\delta)}$ for $\delta\in\mathcal{G}$ preserves the filtration of (1.8.3), it induces a differential operator on Ω_F

(1.10.1) $$\tilde{\nabla}_w : \mathcal{G}\times\Omega_F \longrightarrow \Omega_F,$$
$$\tilde{\nabla}_{w(\delta)}(r^{(0)}(\zeta)) := r^{(0)}(\nabla_{w(\delta)}\zeta) \qquad \text{for } \zeta\in\mathcal{H}_F^{(0)} .$$

On the other hand, the vector field $w(\delta)|_D$ can be lifted to a vector field on C, which is tangent to $q^{-1}(\mathrm{Sing}\, D)\cap C$. Hence one may regard (1.10.1) as an integrable connection on Ω_F with logarithmic poles.

We shall call the covariant differentiation by E,

(1.10.2) $$\tilde{\nabla}_E : \Omega_F \longrightarrow \Omega_F,$$

the degree operator.

(1.11) *The second order differentiation $u(\delta, \delta')$.* Let us define a differential operator u of order 2, and a covariant differentiation by u, by

(1.11.1) $$u(\delta, \delta')=\delta\delta'-(\delta*\delta')\delta_1 \qquad \text{for } \delta, \delta'\in\mathcal{G},$$

(1.11.2) $$\nabla_{u(\delta,\delta')} := \nabla_\delta\nabla_{\delta'}-\nabla_{\delta*\delta'}\nabla_{\delta_1} : \mathcal{H}_F^{(-k-1)} \longrightarrow \mathcal{H}_F^{(-k)}, \qquad k\in \boldsymbol{N}.$$

Obviously from (1.11.2) an operator

(1.11.3) $$\tilde{\nabla}_u : \mathcal{G}\times\mathcal{G}\times\Omega_F \longrightarrow \Omega_F$$

is defined by $\tilde{\nabla}_{u(\delta,\delta')}(r^{(-1)}(\zeta)) := r^{(0)}(\nabla_{u(\delta,\delta')}\zeta)$ for $\zeta\in\mathcal{H}_F^{(-1)}$.

§ 2. The higher residue pairings $K^{(k)}$.

(2.1) The aim of this paragraph is to prove the following theorem. (cf. §§ 9, 10 [6])

THEOREM. *There exists an infinite sequence of $\mathcal{O}_T$-bilinear forms*

(2.1.1) $$K^{(k)} : \pi_*\mathcal{H}_F^{(0)}\times\pi_*\mathcal{H}_F^{(0)} \longrightarrow \mathcal{O}_T, \qquad k=0, 1, 2, \cdots$$

with the following properties.

i) $K^{(k)}$ *is symmetric or skew-symmetric, when* k *is even or odd respectively.*

ii) $K^{(0)}([\psi_1 dx], [\psi_2 dx]) = \mathrm{Res}_{X/T}\begin{bmatrix} \psi_1\psi_2 dx \\ \dfrac{\partial F}{\partial x_0} \cdots \dfrac{\partial F}{\partial x_n}\end{bmatrix}$ *for* $[\psi_i dx] \in \pi_* \mathcal{H}_F^{(0)}$.

iii) $K^{(k)}(\tilde{\omega}_1, \omega_2) = K^{(k-1)}(\nabla_{\delta_1}\omega_1, \omega_2)$ *for* $\omega_1 \in \pi_*\mathcal{H}_F^{(-1)}$, $\omega_2 \in \pi_*\mathcal{H}_F^{(0)}$.

iv) $K^{(k)}(t_1\omega_1, \omega_2) - K^{(k)}(\omega_1, t_1\omega_2) = (n+k)K^{(k-1)}(\omega_1, \omega_2)$ *for* $\omega_i \in \pi_*\mathcal{H}_F^{(0)}$.

v) $\delta K^{(k)}(\omega_1, \omega_2) = K^{(k)}(\nabla_\delta\omega_1, \omega_2) + K^{(k)}(\omega_1, \nabla_\delta\omega_2)$ *for* $\omega_i \in \pi_*\mathcal{H}_F^{(-1)}$ *and* $\delta \in \mathcal{G}$.

Such a $K^{(k)}$ with properties i)$\sim$v) is unique up to a constant factor.

(2.2) *Quantization.* Let us consider the following complex

(2.2.1) $$(\Omega^{\cdot}_{X/T}[[\delta_1^{-1}]], \hat{d}), \qquad \text{where } \hat{d} = \delta_1^{-1}d - dF_1\wedge .$$

PROPOSITION. *The complex* (2.2.1) *is pure* $n+1$*-dimensional. Furthermore, the natural inclusion* $\Omega^{n+1}_{X/T} \to \Omega^{n+1}_{X/T}[[\delta_1^{-1}]]$ *induces a map*

(2.2.2) $$\pi_*\mathcal{H}_F^{(0)} \longrightarrow H^{n+1}((q_*\Omega^{\cdot}_{X/T}[[\delta_1^{-1}]], \hat{d}))$$

so that the right hand side of (2.2.2) $(= q_*\Omega^{n+1}_{X/T}[[\delta_1^{-1}]]/dq_*\Omega^{n}_{X/T}[[\delta^{-1}]])$ *with its natural filtration by the order of the power of* δ_1^{-1} *is a formal completion of* $\pi_*\mathcal{H}_F^{(0)}$ w.r.t. *the filtration of* (1.8.3). *Furthermore,* (2.2.2) *is a* $\mathcal{G}[\delta_1^{-1}]$*-homomorphism.*

(2.3) *Double complex.* For a Stein open set $\mathcal{V} \subset T$ let us consider a Stein covering $\mathfrak{U} = \{\mathcal{V}_0, \cdots, \mathcal{V}_n\}$ of $(X - C)\cap q^{-1}(\mathcal{V})$ given by $\mathcal{V}_i = \{x \in X \cap q^{-1}(\mathcal{V}) : \partial F/\partial x_i \neq 0\}$.

Consider the double Čech complex

(2.3.1) $$\Omega^{p,q} := C^p(\mathfrak{U}, \Omega^q_{X/T}) \underset{\mathcal{O}_T}{\otimes} [[\delta_1^{-1}]] \qquad 0 \leqq p \leqq n,\ 0 \leqq q \leqq n+1$$

with $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ the Čech coboundary and $\hat{d} : \Omega^{p,q} \to \Omega^{p,q+1}$ the morphism of (2.2.1).

PROPOSITION. *The single complex associated to* (2.3.1) *with the coboundary operator* $\hat{\partial} = \partial + (-1)^{n+p+q}\hat{d}$ *is acyclic.*

(2.4) *Symbolic derivations* $\nabla^{(k)}$ *and* $K^{(k)}$. As a consequence of the proposition of (2.3), we obtain the construction of $K^{(k)}$ as follows.

The images of the natural inclusion $\Gamma(\mathcal{V}, q_*\Omega^{n+1}_{X/T}) \to C^0(\mathfrak{U}, \Omega^{n+1}_{X/T})$ are $\hat{\partial}$-closed. Hence by the proposition, for any $\omega \in q_*\Omega^{n+1}_X$, there exists $L = (L^i) \in \bigoplus_{i=0}^{n} C^i(\mathfrak{U}, \Omega^{n-i}_{X/T}) \underset{\mathcal{O}_T}{\otimes} [[\delta_1^{-1}]]$ such that $\omega = \hat{\partial} L$. Thus, let us associate $L^n \in C^n(\mathfrak{U}, \mathcal{O}_X) \otimes [[\delta_1^{-1}]]$ to ω.

If $\omega = dF_1 \wedge d\eta \in dF_1 \wedge dq_* \Omega^{n-1}_{X/T}$, then $\omega = \hat{\partial}(d\eta)$ so that $\hat{\partial}(L - d\eta) = 0$. Again by the proposition, $L - d\eta = \hat{\partial}\xi$ for some ξ and $L^n = \partial\xi^{n-1} \in \partial C^{n-1}(\mathfrak{U}, \mathcal{O}_X) \otimes [[\partial_1^{-1}]]$. Thus we obtain a well defined $\mathcal{O}_T$-homomorphism

$$\pi_* \mathcal{H}_F^{(0)} \ni [\omega] \longrightarrow [L^n] \in \mathcal{R}_C^{n-1} q_* \mathcal{O}_X \otimes [[\partial_1^{-1}]] . \tag{2.4.1}$$

We denote the image of (2.4.1) by $\sum_{k=0}^{\infty} \nabla^{(k)}(\omega)\partial_1^{-k}$. One checks easily that $\nabla^{(k)}(\omega)$ lies in $\hat{\mathcal{H}}_F^{(0)} := \{\xi \in \mathcal{R}_C^{n+1} q_* \mathcal{O}_X : dF_1 \wedge d\xi = 0 \text{ in } \mathcal{R}_C^{n+1} q_* \Omega^2_{X/T}\}$.

DEFINITION. For $\omega, \omega' \in \pi_* \mathcal{H}_F^{(0)}$, we define

$$K^{(k)}(\omega, \omega') := \mathrm{Res}_{X/T}[(\nabla^{(k)}\omega)\omega'] \in \mathcal{O}_T \qquad k = 0, 1, 2, \cdots . \tag{2.4.2}$$

Example. For $\omega = [\varphi dx]$, $\omega' = [\psi dx] \in \pi_* \mathcal{H}_F^{(0)}$,

$$K^{(0)}(\omega, \omega') := \mathrm{Res}_{X/T}\begin{bmatrix} \varphi\psi dx \\ \dfrac{\partial F}{\partial x_0} \cdots \dfrac{\partial F}{\partial x_n} \end{bmatrix} \tag{2.4.3}$$

$$K^{(1)}(\omega, \omega') := \frac{1}{2}\sum_{i=0}^{n} \mathrm{Res}_{X/T}\begin{bmatrix} \left(\dfrac{\partial\varphi}{\partial x_i}\psi - \varphi\dfrac{\partial\psi}{\partial x_i}\right)dx \\ \dfrac{\partial F}{\partial x_0} \cdots \left(\dfrac{\partial F}{\partial x_i}\right)^2 \cdots \dfrac{\partial F}{\partial x_n} \end{bmatrix} . \tag{2.4.4}$$

As one sees directly from the formula (2.4.3), $K^{(0)}$ is factrized by $r^{(0)}$ of (1.7.2) so that it induces an $\mathcal{O}_T$-bilinear form

$$J : q_* \Omega_F \times q_* \Omega_F \longrightarrow \mathcal{O}_T . \tag{2.4.5}$$

Due to the usual duality theorem for residues, J is $\mathcal{O}_T$-non-degenerate.

(2.5) *Note.* One may construct $K^{(k)}$ slightly differently as follows.

Consider the Hamiltonian $\hat{F}$ over the following frame

$$\begin{array}{ccc} \hat{Z} = Z \times_T Z/\sim & \longrightarrow & \hat{X} = X \times_T X \\ \downarrow & & \downarrow \\ \hat{S} = S \times_T S/\sim & \xrightarrow{\hat{\pi}} & \hat{T} = T , \end{array}$$

where the relation $\sim$ means to consider the orbit space of the vector field $p_1^{-1}\partial_1 + p_2^{-1}\partial_2$ where the p_i's are the projections of $Z \times_T Z$ and $S \times_T S$ to their factors, and we put $\hat{\partial}_1 = (p_1^{-1}\partial_1 - p_2^{-1}\partial_1)/2$ and $\hat{F} = p_1^{-1}F - p_2^{-1}F$.

Let $\Delta : X \to \hat{X}$ be the diagonal embedding. For any $\omega \in \hat{\pi}_* \mathcal{H}_{\hat{F}}^{(0)}$, as in (2.4) there exists $L = (L^i) \in \bigoplus_{i=0}^{2n+1} C^i(\hat{\mathfrak{U}}, \Omega^{2n+1-i}_{\hat{X}/T}) \otimes [[\hat{\partial}_1^{-1}]]$ such that $\omega = \hat{\partial} L$. Note that

$\Delta^* L^n(\omega)\in C^n(\mathfrak{U}, \Omega^{n+1}_{X/T})\otimes[[\delta_1^{-1}]]$. Then we have a formula

$$\sum_{k=1}^{\infty} K^{(k)}(\omega, \omega')\delta_1^{-k}=\frac{1}{2^{n+1}}\operatorname{Res}_{X/T}[\Delta^* L^n(p_1^*\omega\wedge p_2^*\omega')]. \tag{2.5.1}$$

§ 3. The primitive form $\zeta^{(0)}$.

(3.1) *Notation.* For an element ζ of $\mathcal{H}_F^{(0)}$, we shall denote by $\zeta^{(-k)}$ the element of $\mathcal{H}_F^{(-k)}$ defined by $(\nabla_{\delta_1})^{-k}\zeta$.

(3.2) DEFINITION. An element $\zeta=\zeta^{(0)}\in\Gamma(S, \mathcal{H}_F^{(0)})$ is called a primitive form if it satisfies the following 5 conditions.

i) Invertibility. $r^{(0)}(\zeta^{(0)})\in\Gamma(C, \Omega_F)$ is an $\mathcal{O}_C$-free base of Ω_F.
ii) The first integrability. $K^{(1)}(\nabla_\delta\zeta^{(-1)}, \nabla_{\delta'}\zeta^{(-1)})=0$ for $\delta, \delta'\in\mathcal{G}$.
iii) Homogeneity. $\nabla_E\zeta^{(0)}=(r-1)\zeta^{(0)}$ for a constant r.
(The number r will be called the minimal exponent.)
iv) $K^{(k)}(\nabla_\delta\nabla_{\delta'}\zeta^{(-2)}, \nabla_{\delta''}\zeta^{(-1)})=0$ for $k\geqq 2$ and $\delta, \delta', \delta''\in\mathcal{G}$.
v) $K^{(k)}(t_1\nabla_\delta\zeta^{(-1)}, \nabla_{\delta'}\zeta^{(-1)})=0$, for $k\geqq 2$ and $\delta, \delta'\in\mathcal{G}$.

(3.3) *Example* 1. Let $F(x, t)$ be a universal unfolding of a simple singularity. Then there exists a unique primitive form up to a constant factor. In weighted homogeneous coordinates, it is given by

$$\zeta^{(0)}=[dx_0\wedge dx_1\wedge\cdots\wedge dx_n]. \tag{3.3.1}$$

In this case, the minimal exponent r is equal to $n/2+1/h$, where h is the Coxeter number.

Example 2. Let $F(x, t)$ be a universal unfolding of one of the simple elliptic singularities given as follows:

$$\tilde{E}_6: yz^2-x(x-y)(x-\lambda y)+\sum_{i=1}^{8-1}\varphi_i t_i,$$

$$\tilde{E}_7: xy(x-y)(x-\lambda y)+z^2+\sum_{i=1}^{9-1}\varphi_i t_i,$$

$$\tilde{E}_8: x(x-y^2)(x-\lambda y^2)+z^2+\sum_{i=1}^{10-1}\varphi_i t_i.$$

Then the primitive forms up to constant factors form a one-dimensional projective space $\boldsymbol{P}^1$. Explicitly they are given by

$$\zeta^{(0)}=[dx\,dy\,dz/u(\lambda)], \tag{3.3.2}$$

where $u(\lambda)$ is a solution of the Gauß-Legendre equation;

$$(3.3.3)\qquad \lambda(1-\lambda)\frac{d^2}{d\lambda^2}u-(2\lambda-1)\frac{d}{d\lambda}u-\frac{1}{4}u=0\,.$$

In this case, the minimal exponent r is equal to $1=n/2$.

(3.4) *Conjecture.* For any universal unfolding $F(x, t)$ of a function with an isolated critical point, the set of all primitive forms (modulo constant factors) forms a finite dimensional, non-void projective manifold.

The construction of $\zeta^{(0)}$ is divided into two steps, the linear part of the problem and the non-linear part of the problem. The main difficulty lies in the non-linear part.

i) Non-linear part. If $\zeta^{(0)}$ is a primitive form. Then the leading term $r^{(0)}(\zeta^{(0)})\in\Gamma(C, \Omega_F)$ should satisfy a system of non-linear (integrability) conditions (3.2.1) (4.1.2) (4.3.1) (4.6.1) (4.9.1).

Thus, first, one should find the solution space of these equations.

ii) Linear part. Theorem. Suppose and element $\zeta\in\Gamma(C, \Omega_F)$ satisfies the conditions of i) above. Then there exists a unique primitive form $\zeta^{(0)}$, such that $\zeta=r^{(0)}(\zeta^{(0)})$.

We shall discuss this problem in more detail in [6].

§ 4. Flat function z, and N and ∇.

(4.1) *Flat function z and the first integrability condition.* Since the multiplication by an element of $q_*\mathcal{O}_C$ is self-adjoint w.r.t. the bilinear form J of (2.4.5), one gets a symmetric bilinear map

$$(4.1.1)\qquad q_*\Omega_F\times q_*\Omega_F\longrightarrow(q_*\mathcal{O}_C)^\vee\,,\qquad(\omega_1, \omega_2)\mapsto J(\cdot\omega_1, \omega_2)\,.$$

(Here $^\vee$ means the $\mathcal{O}_T$-dual module.) Combining the dual homomorphism $r^\vee$ of (1.5.2), the target space of (4.1.1) may be considered as the space of 1-forms $\sum_{i=1}^{\mu}\mathcal{O}_T dt_i$. Let a form $\omega\in\mathcal{G}^\vee$ be the image by (4.1.1) of $r^{(0)}(\zeta^{(0)})\times r^{(0)}(\zeta^{(0)})$ for a $\zeta^{(0)}\in\Gamma(S, \mathcal{H}_F^{(0)})$. Then one has an equality

$$2K^{(1)}(\nabla_\delta\zeta^{(-1)}, \nabla_{\delta'}\zeta^{(-1)})=\delta\langle\delta', \omega\rangle-\delta'\langle\delta, \omega\rangle-\langle[\delta, \delta'], \omega\rangle\,,$$

where $\langle\,,\rangle$ is the usual pairing of 1-forms and vector fields on S.

Hence there exists a function z on S such that

$$(4.1.2)\qquad dz=\omega$$

if and only if

$$(4.1.3)\qquad K^{(1)}(\nabla_\delta\zeta^{(-1)}, \nabla_{\delta'}\zeta^{(-1)})=0\qquad\text{for }\forall\delta, \delta'\in\mathcal{G}\,.$$

DEFINITION. i) The condition (4.1.3) will be called the first integrability condition on $r^{(0)}(\zeta^{(0)})$.

ii) The function z of (4.1.2) will be called the flat function of $r^{(0)}(\zeta^{(0)})$.

(4.2) *Example* 1. The flat function associated to the primitive form (3.3.1) of the universal unfolding of a simple singularity is the Cartan-Killing form restricted to a Cartan algebra under a suitable identification of the unfolding and the simple Lie algebra by Brieskern [1].

Example 2. The flat function associated to the primitive form (3.3.2) of the universal unfolding of a simple elliptic singularity is a ratio $\mathcal{U}_1(\lambda)/\mathcal{U}_2(\lambda)$ of two linearly independent solutions of (3.3.3).

(4.3) *The homogeneity*. Let an element $\zeta^{(0)}$ satisfy the condition (3.2) ii), then by Definition of $\tilde{\nabla}$ (1.10.1), $r^{(0)}(\zeta^{(0)})$ should satisfy a homogenity condition

(4.3.1) $$\tilde{\nabla}_E r^{(0)}(\zeta^{(0)})=(r-1)r^{(0)}(\zeta^{(0)}).$$

If $\zeta^{(0)}$ satisfies the first integrability condition (4.1.3) then (4.3.1) is equivalent to the following homogeneity condition on the associated flat function z

(4.3.2) $$d(Ez-(1-s)z)=0,$$

where $s:=n+1-2r$ is the spector radius (cf. [10] § 3) and d is the exterior differentiation operator.

(4.4) *The connection* ∇ *on* $\mathcal{G}$. Let $\zeta^{(0)}$ satisfy the invertibility (3.2) i) and the first integrability condition (4.1.3).

Since $r^{(0)}(\zeta^{(0)})$ is invertible, there is an $\mathcal{O}_T$-isomorphism

(4.4.1) $$\mathcal{G}\cong q_*\Omega_F \quad (\delta\mapsto \delta F|_C r^{(0)}(\zeta^{(0)})).$$

For short we shall denote $\delta*\zeta^{(0)}$ instead of $\delta F|_C r^{(0)}(\zeta^{(0)})$.

Then depending only on $r^{(0)}(\zeta^{(0)})$, we define a connection

(4.4.2) $$\nabla: \mathrm{Der}_T\times\mathcal{G}\longrightarrow\mathcal{G}$$

by the relation

(4.4.3) $$K^{(1)}(\nabla_\delta\nabla_{\delta'}\zeta^{(-2)}, \nabla_{\delta''}\zeta^{(-1)})=(J((\nabla_\delta\delta')*\zeta^{(0)}, \delta''*\zeta^{(0)})$$

(equivalently, by $(\nabla_\delta\delta')*\zeta^{(0)}=\tilde{\nabla}_{u(\delta,\delta')}\zeta^{(0)}$ cf. (1.11.3)).

(4.5) *Metric property and torsion freeness of* ∇. From (2.1) v) follows the metric property of ∇.

(4.5.1) $$\delta J(\delta'*\zeta^{(0)}, \delta''*\zeta^{(0)})=J(\nabla_{\delta}\delta'*\zeta^{(0)}, \delta''*\zeta^{(0)})+J(\delta'*\zeta^{(0)}, \nabla_{\delta}\delta'*\zeta^{(0)}).$$

From the integrability of ∇ follows the "torsion freeness" of ∇

(4.5.2) $$\nabla_{\delta}(\delta'*\delta'')-\delta'*\nabla_{\delta}\delta''-\nabla_{\delta'}(\delta*\delta'')+\delta*\nabla_{\delta'}\delta''=[\delta, \delta']*\delta''.$$

By putting $\delta''=\delta_1$, one obtains

(4.5.3) $$\nabla_{\delta}\delta'-\nabla_{\delta'}\delta=[\delta, \delta'].$$

(4.6) *Integrability of* ∇. If $\zeta^{(0)}$ is a primitive form (enough only (3.2) iii)) then the connection ∇ is integrable:

(4.6.1) $$[\nabla_{\delta}, \nabla_{\delta'}]=\nabla_{[\delta,\delta']} \quad \text{for } \forall\delta, \delta'\in\mathcal{G}.$$

(4.7) *The $\mathcal{O}_T$-endomorphism N on $\mathcal{G}$.* Let $\zeta^{(0)}$ as in (4.4) be invertible and satisfy the first integrability condition.

One defines an $\mathcal{O}_T$-endomorphism

(4.7.1) $$N: \mathcal{G} \longrightarrow \mathcal{G}$$

by the relation

(4.7.2) $$K^{(1)}(t_1\nabla_{\delta}\zeta^{(-1)}, \nabla_{\delta'}\zeta^{(-1)})=J((N\delta)*\zeta^{(0)}, \delta'*\zeta^{(0)})$$

(equivalently $(N\delta)*\zeta^{(0)}=\tilde{\nabla}_{w(\delta)}\zeta^{(0)}$ cf. (1.10.1)).

(4.8) *Duality and a homogeneity of N.* From (2.1) iv) follows the following a duality of N

(4.8.1) $$N+N^*=(n+1)id.$$

Here N^* signifies the adjoint endomorphism of $\mathcal{G}$ w.r.t. the inner product $J(\delta*\zeta^{(0)}, \delta'*\zeta^{(0)})$ for $\delta, \delta'\in\mathcal{G}$.

From the commutation relation $[\delta, t_1]=\delta t_1$ follows the following homogeneity relation

(4.8.2) $$\nabla_{\delta}(t_1*\delta')+\delta*(n+1)\delta'=N(\delta*\delta')+t_1*\nabla_{\delta}\delta'+(\delta t_1)\delta'.$$

Note. Using the endomorphism N above, the homogeneity condition (4.3.1) is expressed as $N\delta_1=r\delta_1$. Then using the duality (4.8.1) one computes easily $sJ(\zeta^{(0)}, \zeta^{(0)})=0$. Hence unless $s=0$, which is the case only when F is a family of ordinary double points, $\delta_1 z=J(\zeta^{(0)}, \zeta^{(0)})=0$. Therefore z is a function on T.

(4.9) If $\zeta^{(0)}$ is a primitive form (enough only (3.2. iv)). Then $\nabla N=0$.

(4.9.1) $$\nabla_{\delta}(N\delta')=N(\nabla_{\delta}\delta') \qquad \forall\delta, \delta'\in\mathcal{G}.$$

§ 5. Flat embedding of S into Ω_f.

(5.1) *Flat embedding.* Let us fix a primitive form $\zeta^{(0)}$. We construct an embedding depending on $\zeta^{(0)}$ of S into a vector space.

Remember that connection ∇ is torsion free (4.5.3) and integrable (4.5.1). Thus there exists a coordinate system $t_1, \cdots, t_\mu$ of S, up to linear transformations, such that $\ker \nabla = \bigoplus_{i=1}^{\mu} \boldsymbol{C}(\partial/\partial t_i)$. Then the embedding

$$t \in S \longrightarrow \sum_{i=1}^{\mu} t_i(t) \frac{\partial}{\partial t_i} \in \ker \nabla \tag{5.1.1}$$

does not depend on the choice of $t_1, \cdots, t_\mu$.

DEFINITION. The embedding (5.1.1) will be called a flat embedding and $t_1, \cdots, t_\mu$ will be called flat coordinates.

One identifies $\ker \nabla$ with $\Omega_f := q_* \Omega_F / m_T q_* \Omega_F$ (where m_T is the maximal ideal of $\mathcal{O}_{T,0}$), through the isomorphism (4.4.1).

(5.2) *Hessian direction coordinate z of Ω_f.* The vector space Ω_f is decomposed into a direct sum,

$$\Omega_f = \boldsymbol{C}\delta_1 \oplus \Omega \oplus \boldsymbol{C} \frac{\partial}{\partial z}. \tag{5.2.1}$$

Explanation of the decomposition. First, notice that $\delta_1 \in \ker \nabla$ and put $\delta_1^\perp := \{\delta \in \ker \nabla : J(\delta_1, \delta) = 0\}$. Second, denote by δ_μ the element of $\ker \nabla$ which corresponds to the Hessian of f by the identification $\ker \nabla \cong \mathcal{O}_{q^{-1}(0)}/(\partial f/\partial x_0, \cdots, \partial f/\partial x_n)$ and put $\delta_\mu^\perp = \{\delta \in \ker \nabla : J(\delta_\mu, \delta) = 0\} = \ker \nabla \cap m_T \mathcal{G}$.

On the other hand, the flat function z is identified with a linear form on Ω_f by the flat embedding (5.1.1), so that z vanishes identically on the factor $\delta_1^\perp$. ($\because$ $\delta z = J(\delta * \zeta^{(0)}, \zeta^{(0)})$ for any $\delta \in \mathcal{G}$.)

Noting that $J(\delta_1, \delta_\mu) \neq 0$, we get a decomposition $\Omega_f = \boldsymbol{C}\delta_1 \cap (\delta_1^\perp \cap \delta_\mu^\perp) \cap \boldsymbol{C}(\partial/\partial z)$.

(5.3) *Example.* The base space S of a universal unfolding of a simple singularity is identified with Spec $\mathcal{R}$, where $\mathcal{R}$ is the subring of the polynomial ring of the Cartan algebra invariant under the Weyl group action ([1]). Then the flat structure of S is identified with the linear structure of Spec $\mathcal{R}$ introduced in [3]. Explicit calculation for types A_k, B_k, C_k, D_k, E_6, F_4, H_3, H_4, G_2, $I_2(p)$ are given in [5] and for the type E_7 in [8] and for the type E_8 in [2].

§ 6. Exponents.

(6.1) *Exponents.* As in (5.1) let us fix a primitive form $\zeta^{(0)}$. The property (4.9) means that N is a $\boldsymbol{C}$-linear endomorphism of $\ker\nabla=\Omega_f$. Thus we define

DEFINITION. The eigenvalues $\alpha_1, \cdots, \alpha_\mu$ of N are called exponents. It is obvious from the duality (4.8.1) that the exponents have the following duality property,

(6.1.1) $$\{\alpha_1, \cdots, \alpha_\mu\}=\{n+1-\alpha_1, \cdots, n+1-\alpha_\mu\}.$$

The distribution of exponents seems to have a strong concentration near to the center $(n+1)/2$. One conjectures that the following inequarity holds:

$$\sharp\{\alpha_i : \alpha_i\leqq r\}<\mu\int_0^r P_{n+1}(x)dx \qquad \text{for } 0<r<\frac{(n+1)}{2},$$

where $P_n(x)=\int_{-\infty}^{\infty}(2\sin(\pi t)/t)^n \exp(\sqrt{-1}\,\pi t(n-2x))dt$ (cf. [6]). As a consequence one conjectures that the geometric genus of the singular point $f^{-1}(0)\leqq\mu/(n+1)!$.

To describe the distribution of exponents, we introduce a characteristic function $\chi(t)$ by

(6.1.2) $$\chi(t):=\sum_{i=1}^{\mu}\exp(2\pi\sqrt{-1}\,\alpha_i t).$$

(6.2) *Example* 1. If the singularity $f(z)$ is weighted homogeneous of type $(1: r_0, \cdots, r_n)$, the N is semi-simple and the characteristic function χ is given by,

$$\chi(t)=\prod_{i=0}^{n}(T-T^{r_i})/\prod_{i=0}^{n}(T^{r_i}-1), \qquad T=\exp 2\pi\sqrt{-1}\,t.$$

Example 2. If $f(z)$ is a simple singularity, then the exponents are $m_j/h+n/2$, $j=1, \cdots, \mu$, where m_j, $j=1, \cdots, \mu$ are the Coxeter exponents.

(6.3) A duality among the degree of the flat coordinates and the degrees of the logarithmic fields. Let $\Omega_f^\vee$ be the space of $\boldsymbol{C}$-linear forms on Ω_f and let $w(\Omega_f)$ be a space of logarithmic fields s.t. $\mathrm{Der}_S(\log\Delta)=\mathcal{O}_S\underset{C}{\otimes}w(\Omega_f)$ (1.6.2). Then we have

Assertion i) The derivation by the Euler field E induces an affine transformation $E:\Omega_f^\vee\to\Omega_f^\vee$, whose linear part is the transpose of $-(N-(r+1))$: $\Omega_f\to\Omega_f$.

ii) The bracket product by E induces a linear map

$$E: w(\Omega_f) \longrightarrow w(\Omega_f),\ [E, w(\delta)] = w((N-r)\delta).$$

As a consequence of the assertion, one get a duality. Namely, by suitably pairing eigenvalues of $E: \Omega_f^{\vee} \rightarrow \Omega_f^{\vee}$ with those of $E: w(\Omega_f) \rightarrow w(\Omega_f)$, the sum of each pair is equal to 1=degree of $F(x, t)$. (cf. [3], [4]) This partially explains the duality observed by Orlik-Solomon [9], cf. also [7].

§ 7. The intersection form of vanishing homology classes.

(7.1) In this paragraph we give an algebraic presentation of the intersection form of vanishing homology of the family $X \rightarrow S$.

Let us consider $\mathcal{O}_T$-linear maps $v^{(k)}$, $w^{(k)}$, for $k \in \boldsymbol{Z}$,

(7.1.1) $$v^{(k)}: \mathcal{G} \longrightarrow \pi_* \mathcal{H}_F^{(k)}\ (= \nabla_{\delta_1}^k \mathcal{H}^{(0)}), \qquad \delta \mapsto \nabla_\delta \zeta^{(k-1)},$$

(7.1.2) $$w^{(k)}: \mathcal{G} \longrightarrow \pi_* \mathcal{H}_F^{(k-1)},$$

$$w^{(k)}(\delta) := \nabla_{w(\delta)} \zeta^{(k-1)} = t_1 v^{(k)}(\delta) - v^{(k)}(t_1 * \delta) = v^{(k-1)}((N-k)\delta).$$

For any $p \in \boldsymbol{Z}$, let us define

(7.1.3) $$I_p := \sum_{i=1}^{\mu} v^{(p)}(e_i) \underset{\mathcal{O}_S}{\otimes} w^{(n-p)}(e^{i*}) \in \mathcal{H}_F^{(p)} \underset{\mathcal{O}_S}{\otimes} \mathcal{H}_F^{(n-p-1)},$$

where $e_1, \cdots, e_\mu$ and $e^{1*}, \cdots, e^{\mu*}$ are $\mathcal{O}_T$-dual bases of w.r.t. the bilinear form J.

Using the duality of N (cf. (4.8)) it is now a rather easy task to show the following

PROPOSITION i) *$I = (-1)^p I_p$ does not depend on $p \in \boldsymbol{Z}$.*

ii) *I is (skew-) symmetric for even (odd) n respectively.*

iii) *$\nabla I = 0$. Here ∇ is the connection on the tensor $\mathcal{H}_F \otimes \mathcal{H}_F$ induced from the Gauß-Manin connection.*

COROLLARY. *I defines a bilinear form on the local system defined by the Gauß-Manin connection, which is invariant under the action of the monodromy group.*

(7.2) On the other hand, using the Picard-Lefschetz formula, one gets a lemma.

PROPOSITION. *A bilinear form defined on the vanishing homology group of a hypersurface isolated singular point is a constant multiple of the intersection form if it is invariant under the total monodromy group action, except in case A_1 with odd n.*

(7.3) *An explicit formula.* Let δ_i, δ^{i*}, $i=1, \cdots, \mu$ be basis and dual basis of $\mathcal{G}$ and let $N=(N_i^j)$ and $A=(A_i^j)$ be matrix presentations of the endomorphisms N and t_1* with respect to the basis. Then we have

$$I=(-1)^n \sum (N-n)(t_1-A)^{-1}(N-(n-1))(t_1-A)^{-1} \cdots$$

$$\cdots (N-1) \operatorname{Res}\begin{bmatrix} \delta_i F dx \\ F(x, t) \end{bmatrix} \otimes \operatorname{Res}\begin{bmatrix} \delta^{i*} F dx \\ F(x, t) \end{bmatrix}.$$

§ 8. Periods of a primitive form.

(8.1) *Vanishing cycle.* Let $X_A \to S_A$ be a family of ordinary double points of the fiber dimension n defined by a Hamiltonian $A(x, t)$ and let t_{1A} be the standard energy function (i.e. By a suitable coordinates $A(x, t)=t_{1A}-(x_0^2+\cdots+x_n^2)$.) There exists a unique primitive form ζ_A (up to sign) such as

(8.1.1) $$K^{(k)}(\zeta_A^{(0)}, \zeta_A^{(0)})=\left(\frac{1}{2\pi}\right)^{n+1}(-1)^{n(n+1)/2} \quad \text{for } k=0, \quad =0 \quad \text{for } k>0$$

$$t_{1A}\zeta_A^{(0)}=\frac{n+1}{2}\zeta_A^{(-1)}.$$

Since $\zeta_A^{(0)}$ is an $\mathcal{O}_{S_A}$ free base of $\mathcal{H}_A^{(0)}$, we define

(8.1.2) $$\gamma_A : g(t)\zeta_A^{(0)} \in \mathcal{H}_F^{(0)} \longmapsto g(t)t_{1A}^{(n-1)/2}/\Gamma\left(\frac{n+1}{2}\right) \in \mathcal{O}_{S_A}[t_{1A}^{1/2}]$$

(where Γ is the gamma function), which form a $\boldsymbol{C}$-base of the local system $\mathcal{H}om_{\mathrm{Der}_{S_A}}(\mathcal{H}_F^{(0)}, \mathcal{O}_{S_A})|_{S-D_A}$.

DEFINITION. γ_A will be called the vanishing homology class associated to the ordinary double points of A.

(8.2) *Period map.* For a given Hamiltonian $F(x, t)$, in the local system $\mathcal{H}om_{\mathrm{Der}_S}(\mathcal{H}_F^{(0)}, \mathcal{O}_S)|_{S-D}$, there exists an integral lattice L_F of rank μ generated by μ vanishing cycles. The Lattice L_F is stable under the isomorphism (1.8.5). By restricting the integral operator $\mathcal{H}_F^{(0)} \times \mathcal{H}om(\mathcal{H}_F^{(0)}, \mathcal{O}_S) \to \mathcal{O}_S$ on $\zeta^{(0)} \times L_F$ for $\zeta^{(0)} \in \Gamma(S, \mathcal{H}_F^{(0)})$, one gets periods of $\zeta^{(0)}$, $t \in S-D \mapsto \mathrm{Hom}(L_{F,t}, \boldsymbol{C})$.

Let us choose and fix a base point $t_0 \in S-D$ and trivialize the local system L_F on the monodromy covering space $\widetilde{S-D}$. Thus we obtain a period map which we shall denote by

(8.2.1) $$\int_{\gamma(t)} \zeta^{(0)} : \widetilde{S-D} \longrightarrow \mathrm{Hom}(L_{F,t_0}, \boldsymbol{C}) \cong \mathrm{Hom}(Z^{\mu}, \boldsymbol{C}),$$

where the covering transformation of $\widetilde{S-D}$ and the monodromy action on $\mathrm{Hom}(\boldsymbol{Z}^{\mu}, \boldsymbol{C})$ are equivariant. For even n, one may construct a ramified covering $\tilde{S}\rightarrow S$, by adding divisors to $\widetilde{S-D}$, so that the map (8.2.1) extends to $\tilde{S}$.

(8.3) Let $\zeta^{(0)}$ be a primitive form. Let $t_1, \cdots, t_{\mu}$ be flat coordinates of S and $\gamma_1, \cdots, \gamma_{\mu}$ be basis of L_F. Then using the duality (4.8.1), one computes the following Jacobian determinant

$$\det\left(\frac{\partial}{\partial t_i}\int_{\gamma_j}\zeta^{(k-2)}\right)_{ij}=\mathrm{const}\,\Delta^{(n+1)/2-k}.$$

For even n, this implies that the period map

$$\text{(8.3.1)}\qquad \int_{\gamma}\zeta^{(n/2-1)}=\left(\frac{\partial}{\partial t_1}\right)^{n/2-1}\int_{\gamma}\zeta^{(0)}:\tilde{S}\longrightarrow \mathrm{Hom}(L_{F,t_0}, \boldsymbol{C})$$

is everywhere bi-regular. If F is a universal unfolding of a simple singularity, (8.3.1) is bijective.

(8.4) *Example.* Suppose $n=1$. Put $\Omega_{f,\alpha}=\{e\in\Omega_f:(N-\alpha)^2e=0\}$ and $\Omega_{f,1>}:=\bigoplus_{1>\alpha}\Omega_{f,\alpha}$, $\Omega_{f,1<}:=\bigoplus_{1<\alpha}\Omega_{f,\alpha}$ so that $\Omega_f=\Omega_{f,1>}\oplus\Omega_{f,1}\oplus\Omega_{f,1<}$. One calls $v^{(0)}(e)\in v^{(0)}(\Omega_{f,1>})$ ($v^{(0)}(\Omega_{f,1<})$, $v^{(0)}(\Omega_{f,1})$) differential of the first (second, third, respectively) kind.

Assume that $\Omega_{f,1}=0$. Let $\partial/\partial t_1, \cdots, \partial/\partial t_{\mu/2}$ be basis of $\Omega_{f,1>}$. Then the $\mu/2\times\mu$ matrix $\left(\frac{\partial}{\partial t_i}\int_{\gamma_j}\zeta^{(-1)}\right)=\left(\int_{\gamma}v^{(0)}\left(\frac{\partial}{\partial t_i}\right)\right)$ satisfies Rieman's conditions on periods so that one gets a holomorphic family of Abelian variety over $\Omega_{f,\gamma+1\geq}$.

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