Quotients of polynomials and a theorem of Pisot and Cantor

By D. J. LEWIS and Patrick MORTON

To the memory of Professor Takuro Shintani

1. Introduction.

It is an elementary exercise using the division algorithm to prove that the quotient of two polynomials $A(x)$ and $B(x)$ with rational coefficients is again a polynomial, whenever $A(n)/B(n)\in Z$ for an infinite set of integers n. It is natural to ask if a similar result holds for polynomials in several variables. In particular, what subsets S of the k-dimensional lattice $\boldsymbol{Z}^k$ have the following property?

Property D. If $A(\underline{x})$ and $B(\underline{x})\neq 0$ are any two polynomials over Q in k variables $\underline{x}=(x_1, \cdots, x_k)$, and if $A(\underline{n})/B(\underline{n})\in \boldsymbol{Z}$ for all those $\underline{n}\in S$ for which $B(\underline{n})\neq 0$, then $B(\underline{x})$ divides $A(\underline{x})$ in $Q[\underline{x}]$.

In this note we exhibit a class of sets S having the property D, which are composed of lattice points on certain exponential curves. These are the sets

$$S=\{(m_1^n, \cdots, m_k^n)\,;\, n\geqq 0\}, \tag{1}$$

where the m_i are fixed integers $\geqq 2$ and are relatively prime in pairs.

The fact that these sets satisfy D rests on the following theorem of Pisot [1], p. 233 (see also Cantor [4]).

THEOREM. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ ($b_n\neq 0$ *for* $n\geqq 0$) *be two power series representing rational functions, and assume that* $\sum_{n=0}^{\infty} b_n x^n$ *has exactly one (simple) pole on its radius of convergence. Then, if* $a_n/b_n\in \boldsymbol{Z}$ *for all* $n\geqq 0$, *it follows that* $\sum_{n=0}^{\infty} \frac{a_n}{b_n} x^n$ *is also the power series of a rational function.*

A result of Cantor [3] shows that the assumption on the simplicity of the pole may be discarded. However, we shall only use the theorem in the above form.

In § 2 we state this theorem in terms of linear recurrences, and give a new proof, based on a division algorithm for exponential polynomials which is due to Ritt [8]. Then in §3 we deduce that the sets S in (1) satisfy property D. We remark that a similar result may be proved for polynomials over a *real* algebraic number field, using a result of Cantor [4]. However this involves no new ideas, so for simplicity we shall restrict ourselves to the rational field $\boldsymbol{Q}$.

Finally, in §4 we prove the following result, valid for an arbitrary number field K: if $\mathcal{O}$ is the ring of integers in K, and $A(\underline{\mu})/B(\underline{\mu})\in\mathcal{O}$ for all $\underline{\mu}\in\mathcal{O}^k$ for which $B(\underline{\mu})P(\underline{\mu})\neq 0$ ($P(\underline{x})$ an arbitrary non-zero polynomial over K), then $A(\underline{x})/B(\underline{x})\in K[\underline{x}]$.

2. The Pisot-Cantor theorem.

We first recall the following well-known facts concerning the power series of rational functions.

(A) $\sum_{n=0}^{\infty} a_n x^n$ represents a rational function if and only if $\{a_n\}$ is a linear recurring sequence, i.e. if and only if there are complex numbers $c_1, \cdots, c_r$ ($c_r\neq 0$, $r\geqq 0$), for which

$$a_{n+r}=\sum_{k=1}^{r} c_k a_{n+r-k}\,, \qquad \text{for} \quad n\geqq n_0\,, \tag{2}$$

where n_0 is some sufficiently large integer.

(B) If $\alpha_1, \cdots, \alpha_s$ are the distinct roots of the polynomial

$$x^r-c_1x^{r-1}-\cdots-c_r=0\,, \tag{3}$$

with the c_k as in (2), and if the multiplicity of α_k as a root of (3) is e_k, then the sequences

$$\{n^j\alpha_k^n\}\,, \qquad 1\leqq k\leqq s,\ 0\leqq j\leqq e_k-1\,,$$

are independent (over $\boldsymbol{C}$) and form a basis for the solution space of (2). In particular, any solution $\{a_n\}$ of (2) has a unique representation of the form

$$a_n=\sum_{k=1}^{s} p_k(n)\alpha_k^n\,, \qquad n\geqq n_0\,, \tag{4}$$

where $p_k(x)\in\boldsymbol{C}[x]$ and $\deg p_k(x)\leqq e_k-1$.

(C) If $\{a_n\}$ satisfies (4), where the α_k are distinct and no $p_k(x)$ is zero, then the rational function $\sum_{n=0}^{\infty} a_n x^n$ has a pole of order $1+\deg p_k$ at $\frac{1}{\alpha_k}$, for $1\leqq k\leqq s$, and no other poles.

We shall refer to these facts simply as (A), (B), (C).

In order to prove the Pisot-Cantor theorem (see the introduction), we shall work with the coefficients a_n and b_n in the form (4). It is then convenient to introduce the "exponential polynomial"

$$a(x)=\sum_{k=1}^{s} p_k(x)e^{x\log \alpha_k}, \tag{5}$$

corresponding to (4), where the logarithms are chosen arbitrarily. We also define

$$\mu(a)=\mu(a(x))=\max_{1\le k\le s}\operatorname{Re}\log\alpha_k=\max_{1\le k\le s}\log|\alpha_k|.$$

The following lemma is due to Ritt, and forms the basis for his discussion in [8] of the arithmetic in the ring of exponential polynomials. (Note that Lemma 1 is more general than Ritt's lemma, but the proof is exactly the same.)

LEMMA 1. *Let $a(x)$ and $b(x)$ be exponential polynomials, and assume $b(x)$ has the special form*

$$b(x)=\sum_{j=1}^{r-1} q_j(x)e^{\lambda_j x}+q_r e^{\lambda_r x}, \tag{6}$$

where q_r is a non-zero constant and $\operatorname{Re}\lambda_r>\operatorname{Re}\lambda_j$ for $1\leqq j\leqq r-1$. Then there are exponential polynomials $\kappa(x)$ and $\rho(x)$ which satisfy

$$a(x)=\kappa(x)b(x)+\rho(x), \tag{7}$$

and

$$\text{either}\quad \rho(x)=0 \quad\text{or}\quad \mu(\rho)<\mu(b). \tag{8}$$

We shall also require the following result of Pisot. (See [5], page 138.)

LEMMA 2. *Let $\{z_n\}$ be a sequence of real numbers satisfying a linear recurrence of the form (2), and let $\{A_n\}$ be a sequence of rational integers. If*

$$\sum_{n=0}^{\infty}|z_n-A_n|^2<\infty,$$

then $\{A_n\}$ also satisfies a linear recurrence of the form (2).

We now prove the theorem of Pisot and Cantor.

THEOREM 1. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers which satisfy linear recurrences. Assume that $b_n\neq 0$ for $n\geqq 0$, and that the minimal recurrence satisfied by b_n is of the form (2), where the corresponding equation (3) has a unique largest root, of multiplicity one. Then the hypothesis $\frac{a_n}{b_n}\in \boldsymbol{Z}$ for $n\geqq 0$ implies that $\left\{\frac{a_n}{b_n}\right\}$ also satisfies a linear recurrence.*

PROOF. By the remarks in (B) we may set

$$a_n=\sum_{k=1}^{s} p_k(n)\alpha_k^n\,, \qquad n\geqq n_0\,,$$

$$b_n=\sum_{j=1}^{r-1} q_j(n)\beta_j^n+q_r\beta_r^n\,, \qquad n\geqq n_0\,,$$

where $p_k, q_j\in \boldsymbol{C}[x]$, $\alpha_k, \beta_j, q_r\in \boldsymbol{C}$, $q_r\neq 0$, and

$$|\beta_r|>|\beta_j| \qquad \text{for} \quad 1\leqq j\leqq r-1\,. \tag{9}$$

We now define $a(x)$ by (5) and $b(x)$ by (6), where $\lambda_j=\log \beta_j$ is chosen arbitrarily. By (9) and by Lemma 1 there exist exponential polynomials $\kappa(x)$ and $\rho(x)$ satisfying (7) and (8). Setting $x=n$ in (7) gives that

$$\frac{a_n}{b_n}=\frac{a(n)}{b(n)}=\kappa(n)+\frac{\rho(n)}{b(n)}\,, \qquad \text{for} \quad n\geqq n_0\,.$$

Now $\{\kappa(n)\}$ clearly satisfies a linear recurrence, by (A) and (C). Moreover $\mu(\rho)<\mu(b)$ implies that

$$\left|\frac{\rho(n)}{b(n)}\right|\leqq ce^{-\delta n}\,, \qquad n\geqq n_0\,,$$

for some positive constants c and δ. Since

$$\sum_{n=0}^{\infty} e^{-2\delta n}<\infty\,,$$

the assumptions of Lemma 2 are fulfilled with

$$A_n=\frac{a_n}{b_n}\,, \qquad z_n=\kappa(n)\,.$$

Therefore $\left\{\frac{a_n}{b_n}\right\}$ does satisfy a linear recurrence. Q.E.D.

REMARKS. 1. The equivalence of Theorem 1 and the theorem stated in § 1 follows from (A), (B) and (C).

2. It is clear that we need only assume

$$b_n\neq 0 \quad \text{and} \quad \frac{a_n}{b_n}\in \boldsymbol{Z} \qquad \text{for} \quad n\geqq n_0\,,$$

for some fixed $n_0\geqq 0$, in order to guarantee the conclusion of Theorem 1.

3. The sets S.

For the proof of our main result we need two more lemmas, the first of which deals with a special case of Theorem 1.

LEMMA 3. *Let*

$$(10) \qquad a_n=\sum_{i=1}^{r} c_i\alpha_i^n\,, \qquad b_n=\sum_{j=1}^{s} d_j\beta_j^n\,,$$

where the c_i and d_j are non-zero and real, and where the α_i and β_j are positive and respectively pairwise distinct. If $\frac{a_n}{b_n}\in \boldsymbol{Z}$ for those n for which $b_n\neq 0$, then

$$(11) \qquad a_n=b_n\sum_{k=1}^{t} u_k\gamma_k^n\,, \qquad for \quad n\geqq 0\,,$$

where the u_k and γ_k are real and $\gamma_k>0$. Moreover γ_k lies in the multiplicative group G generated by the α_i and β_j.

PROOF. Since $b_n=0$ for at most finitely many n, and since some β_j must dominate the β_i with $i\neq j$, Theorem 1 and (B) imply that

$$\frac{a_n}{b_n}=\sum_{k=1}^{t} u_k(n)\gamma_k^n\,, \qquad \text{for} \quad n\geqq n_0\,,$$

for some non-zero polynomials $u_k(x)\in \boldsymbol{C}[x]$ and distinct γ_k in $\boldsymbol{C}$. Thus

$$(12) \qquad \sum_{i=1}^{r} c_i\alpha_i^n=\Big(\sum_{j=1}^{s} d_j\beta_j^n\Big)\Big(\sum_{k=1}^{t} u_k(n)\gamma_k^n\Big)\,, \qquad \text{for} \quad n\geqq n_0\,.$$

Assume that the assertion of the lemma is false, and let γ_k be the γ of least absolute value and smallest argument which is not equal to any $\frac{\alpha_i}{\beta_j}$, or for which $u_k(x)$ is not a real constant. If β_1 is the smallest of the β's, then the term $d_1u_k(n)(\beta_1\gamma_k)^n$ in the product on the right side of (12) does not combine with any other term in the product. Hence (B) shows that $d_1u_k(n)(\beta_1\gamma_k)^n$ must equal some term $c_i\alpha_i^n$, for $n\geqq n_0$. But this can only happen if

$$d_1u_k(n)=c_i \quad \text{and} \quad \beta_1\gamma_k=\alpha_i\,;$$

i.e. $u_k(x)=u_k$ is a real constant and $\gamma_k=\alpha_i/\beta_1$. Hence the γ's all lie in the multiplicative group G. Now (12) becomes

$$(13) \qquad \sum_{i=1}^{r} c_i\alpha_i^n=\Big(\sum_{j=1}^{s} d_j\beta_j^n\Big)\Big(\sum_{k=1}^{t} u_k\gamma_k^n\Big)\,, \qquad n\geqq n_0\,,$$

which shows that the real exponential polynomial

$$\sum_{i=1}^{r} c_ie^{x\log\alpha_i}-\Big(\sum_{j=1}^{s} d_je^{x\log\beta_j}\Big)\Big(\sum_{k=1}^{t} u_ke^{x\log\gamma_k}\Big)$$

has infinitely many real zeros. Thus it must be identically zero, so that (13) holds for $n\geqq 0$. This completes the proof of the lemma.

Our final lemma is a special case of Fatou's lemma [6], [7].

LEMMA 4. *Let*

$$a_n=\sum_{i=1}^{r} c_i q_i^n\,, \qquad \text{for} \quad n\geqq n_0\,,$$

where c_i, $q_i\in Q$, $c_iq_i\neq 0$ *and the* q_i *are distinct. If* $a_n\in \boldsymbol{Z}$ *for* $n\geqq n_0$, *then* $q_i\in \boldsymbol{Z}$ *for* $1\leqq i\leqq s$.

We give the following simple inductive proof.

PROOF. If $r=1$ the assertion is obvious. Assume its truth for r, and let

$$a_n=\sum_{i=1}^{r} c_i q_i^n + c_{r+1}q_{r+1}^n\,,$$

where $q_{r+1}=\dfrac{u}{v}$ with u, $v\in \boldsymbol{Z}$. Form the sequence

$$\begin{aligned} b_n &= va_{n+1}-ua_n \\ &= \sum_{i=1}^{r} c_i(vq_i-u)q_i^n\,, \qquad n\geqq n_0\,. \end{aligned}$$

Since $b_n\in \boldsymbol{Z}$ it follows by induction that $q_1, \cdots, q_r$ are integers. Applying the same argument with q_1 replacing q_{r+1} shows that $q_{r+1}\in \boldsymbol{Z}$ also, and this completes the proof.

A similar proof works for the most general case of Fatou's lemma.

We are now ready to prove

THEOREM 2. *Let* $m_1, \cdots, m_k$ *be* k *integers* $\geqq 2$ *which are relatively prime in pairs. Let* $A(x_1, \cdots, x_k)$ *and* $B(x_1, \cdots, x_n)$ *be non-zero polynomials with rational coefficients, and assume that*

$$\frac{A(m_1^n, \cdots, m_k^n)}{B(m_1^n, \cdots, m_k^n)}\in \boldsymbol{Z}$$

for those n *for which the denominator is not zero. Then* $B(x_1, \cdots, x_k)$ *divides* $A(x_1, \cdots, x_n)$ *in* $Q[x_0, \cdots, x_n]$.

PROOF. Let

$$a_n=A(m_1^n, \cdots, m_k^n)$$

$$b_n=B(m_1^n, \cdots, m_k^n)$$

for $n\geqq 0$. It is clear that a_n and b_n have the form (10), where the α_i and β_j are equal to distinct monomials in the m_i. Thus our assumptions, together with Lemma 3, imply that

$$a_n=b_n\sum_{i=1}^{t} u_i\gamma_i^n\,, \qquad \text{for} \quad n\geqq 0\,,$$

where $u_i\in Q$ (this follows from the proof of Lemma 3 or from a simple determinant argument), and where the γ_i lie in the multiplicative subgroup of Q generated by $m_1, \cdots, m_k$. However Lemma 4 shows that each γ_i lies in $\boldsymbol{Z}$, and since the m_i are pairwise relatively prime it follows that

$$\gamma_i = m_1^{e_{i1}} \cdots m_k^{e_{ik}} \qquad \text{with} \quad e_{ik} \geqq 0 .$$

Therefore

$$a_n = b_n \sum_{i=1}^{t} u_i (m_1^{e_{i1}} \cdots m_k^{e_{ik}})^n = b_n P(m_1^n, \cdots, m_k^n),$$

for $n \geqq 0$, where $P(\underline{x}) \in Q(\underline{x})$. Thus, as in the proof of Lemma 3, the real exponential polynomial

$$A(e^{x \log m_1}, \cdots, e^{x \log m_k}) - B(e^{x \log m_1}, \cdots, e^{x \log m_k}) P(e^{x \log m_1}, \cdots, e^{x \log m_k})$$

is identically zero. But $e^{x \log m_1}, \cdots, e^{x \log m_k}$ are algebraically independent over Q, and so we have that

$$A(x_1, \cdots, x_k) = B(x_1, \cdots, x_k) P(x_1, \cdots, x_k) . \qquad \text{Q.E.D.}$$

We note that the same arguments can be used to prove:

THEOREM 3. *Let K be a real algebraic number field, let $A, B \in K[x_1, \cdots, x_k]$ and let $\mu_1, \cdots, \mu_k$ be k positive algebraic integers of K which are not units and are relatively prime in pairs. If*

$$\frac{A(\mu_1^n, \cdots, \mu_k^n)}{B(\mu_1^n, \cdots, \mu_k^n)} = \frac{a_n}{b_n}$$

is an algebraic integer for all those $n \geqq 0$ for which the denominator is non-zero, then $A/B \in K[x_1, \cdots, x_k]$.

The only change in the proof is at the first step. To deduce that $\frac{a_n}{b_n}$ satisfies a linear recurrence, we must appeal to a result of Cantor [4] (Lemma 2, applied to the valuation which is the ordinary absolute value on K). The proof then proceeds in exactly the same manner using Lemma 3 and Lemma 4 (i.e., its generalization to number fields).

We also make the following remark. If A and B have integer coefficients in Theorem 2 and B is primitive (the greatest common divisor of its coefficients is 1), then A/B will have integer coefficients as well. If B is not primitive then this need not be the case. For example, take $B(x)=2$ and $A(x)=x(x-1)$. Then $A(x)/B(x) = \frac{x(x-1)}{2} = \binom{x}{2}$ is an integer for all integral values of x.

4. A result for number fields.

Let $\mathcal{O}$ be the ring of integers in an algebraic number field K, and let $\underline{x}=(x_1, \cdots, x_k)$. In this section we prove

THEOREM 4. *Let* $A(\underline{x}), B(\underline{x})\in K[\underline{x}]$, *where* $B(\underline{x})\neq 0$. *If* $A(\underline{\mu})/B(\underline{\mu})\in\mathcal{O}$ *for all* $\underline{\mu}\in\mathcal{O}^k$ *for which* $B(\underline{\mu})P(\underline{\mu})\neq 0$, *where* $P(\underline{x})$ *is any fixed non-zero polynomial in* $K[\underline{x}]$, *then* $A(\underline{x})/B(\underline{x})\in K[\underline{x}]$.

PROOF. First we consider the case $k=1$, $B(x)$ irreducible over K. Write

$$\rho A(x)=Q(x)B(x)+R(x),$$

where $\rho\in\mathcal{O}$, $Q(x)$, $R(x)\in\mathcal{O}[x]$, and $\deg R<\deg B$. Then for every $\mu\in\mathcal{O}$ for which $P(\mu)B(\mu)\neq 0$, we have

$$\frac{R(\mu)}{B(\mu)}=\rho\frac{A(\mu)}{B(\mu)}-Q(\mu)\in\mathcal{O}.$$

Assume $R(x)\neq 0$, and let $\mathcal{P}$ be a prime ideal of $\mathcal{O}$ with the property that $B(x)$ splits completely into distinct linear factors modulo $\mathcal{P}$. (The existence of such a $\mathcal{P}$ follows easily from standard results in algebraic number theory. See also [2], p. 258.) Then for some $\mu_0\in\mathcal{O}$ we have

$$B(\mu_0)\equiv 0 \pmod{\mathcal{P}}, \qquad R(\mu_0)\not\equiv 0 \pmod{\mathcal{P}},$$

since the congruence

$$B(x)\equiv 0 \pmod{\mathcal{P}}$$

has $\deg B>\deg R$ roots. Without loss of generality we may also assume $B(\mu_0)P(\mu_0)\neq 0$. But in that case $R(\mu_0)/B(\mu_0)\notin\mathcal{O}$; this contradiction shows that $R(x)=0$, i.e. $B(x)\mid A(x)$ in $K[x]$.

If $k=1$ and $B(x)$ is reducible, write

$$B(x)=B_1(x)\cdots B_m(x)$$

with irreducible polynomials $B_i(x)$ and apply the above reasoning successively to the polynomials

$$A_i(x)=\frac{\rho^{m-i}A(x)}{B_1(x)\cdots B_{i-1}(x)} \quad\text{and}\quad B_i(x), \qquad 1\leq i\leq m.$$

Here ρ is a non-zero integer of K with the property that $\rho B_i(x)\in\mathcal{O}[x]$ for $i=1, \cdots, m$. We then get successively

$$\frac{A}{B_1}, \frac{A}{B_1B_2}, \cdots, \frac{A}{B}\in K[x],$$

since

$$\frac{A_i(\mu)}{B_i(\mu)}=\frac{A(\mu)}{B(\mu)}\rho B_{i+1}(\mu)\cdots\rho B_m(\mu)\in\mathcal{O}$$

for all $\mu\in\mathcal{O}$ for which $B(\mu)P(\mu)\neq 0$.

Now we consider the case $k>1$. Assume that

$$A(\underline{y}, x_k),\ B(\underline{y}, x_k),\ P(\underline{y}, x_k)\in K[\underline{y}, x_k],\quad \text{where } \underline{y}=(x_1, \cdots, x_{k-1}),$$

that B involves the variable x_k, and that

$$\frac{A(\underline{\xi}, \eta)}{B(\underline{\xi}, \eta)}\in\mathcal{O}\quad\text{for all } (\underline{\xi}, \eta)\in\mathcal{O}^k \quad\text{for which } P(\underline{\xi}, \eta)B(\underline{\xi}, \eta)\neq 0.$$

For fixed $\underline{\xi}\in\mathcal{O}^{k-1}$, the first part of the proof shows that

$$A(\underline{\xi}, x_k)=f_{\underline{\xi}}(x_k)B(\underline{\xi}, x_k),\qquad f_{\underline{\xi}}(x_k)\in K[x_k], \tag{14}$$

for all $\underline{\xi}\in\mathcal{O}^{k-1}$ for which $P(\underline{\xi}, x_k)B(\underline{\xi}, x_k)$ is not identically zero. we divide $A(\underline{y}, x_k)$ by $B(\underline{y}, x_k)$ with respect to the variable x_k and obtain

$$D(\underline{y})A(\underline{y}, x_k)=B(\underline{y}, x_k)Q(\underline{y}, x_k)+R(\underline{y}, x_k)$$

where $D(\underline{y})\in K[\underline{y}]$, Q, $R\in K[\underline{y}, x_k]$ and $\deg_{x_k}R<\deg_{x_k}B$. If $B_0(\underline{y})$ denotes the leading coefficient of B with respect to x_k, then for all $\underline{\xi}\in\mathcal{O}^{k-1}$ satisfying

$$D(\underline{\xi})P(\underline{\xi}, x_k)B_0(\underline{\xi})\neq 0$$

we have $R(\underline{\xi}, x_k)=0$ by (14). But if $R(\underline{y}, x_k)\neq 0$ there is certainly a $\underline{\xi}\in\mathcal{O}^{k-1}$ for which

$$R(\underline{\xi}, x_k)D(\underline{\xi})P(\underline{\xi}, x_k)B_0(\underline{\xi})\neq 0.$$

Thus $R(\underline{y}, x_k)$ is identically zero and

$$D(\underline{y})A(\underline{y}, x_k)=Q(\underline{y}, x_k)B(\underline{y}, x_k).$$

Applying the same argument to x_i in place of x_k, we see that for every variable x_i appearing in $B(\underline{x})$,

$$D_i(\underline{x}^{(i)})A(\underline{x})=Q_i(\underline{x})B(\underline{x}), \tag{15}$$

where $\underline{x}^{(i)}$ contains all variables $x_1, \cdots, x_n$ except x_i, and D_i, Q_i are polynomials. If $B(\underline{x})$ involves only one variable, then (15) shows that $A(\underline{x})/B(\underline{x})\in K[\underline{x}]$. Otherwise

$$\frac{A(\underline{x})}{B(\underline{x})}=\frac{Q_i(\underline{x})}{D_i(\underline{x}^{(i)})}=\frac{Q_j(\underline{x})}{D_j(\underline{x}^{(j)})}$$

for all i, j, $i\neq j$, such that x_i and x_j appear in $B(\underline{x})$. Hence

$$D_j(\underline{x}^{(j)})Q_i(\underline{x})=D_i(\underline{x}^{(i)})Q_j(\underline{x}),$$

which implies that any factor of $D_j(\underline{x}^{(j)})$ involving x_i divides $Q_i(\underline{x})$. But clearly

$D_j(\underline{x}^{(j)})$ involves only variables appearing in $B(\underline{x})$, and thus $D_j(\underline{x}^{(j)})|Q_j(\underline{x})$ in $K[\underline{x}]$, i.e. $A(\underline{x})/B(\underline{x}) \in K[\underline{x}]$. Q.E.D.

References

[1] Benzaghou, B., Algèbres de Hadamard, Bull. Soc. Math. France **98** (1970), 209-252.
[2] Brillhart, J. and I. Gerst, On the prime divisors of polynomials, Amer. Math. Monthly **78** (1971), 250-266.
[3] Cantor, D., On arithmetic properties of coefficients of rational functions, Pacific J. Math. **15** (1965), 55-58.
[4] Cantor, D., On arithmetic properties of the Taylor series of rational functions II, Pacific J. Math. **41** (1972), 329-334.
[5] Cassels, J. W. S., An Introduction to Diophantine Approximation, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, 1957.
[6] Fatou, P., Séries trigonométriques et séries de Taylor, Acta Math. **30** (1906), 335-400.
[7] Pólya, G. and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Band, 2. Berlin, 1964, p. 142, Problem 156.
[8] Ritt, J., On the zeros of exponential polynomials, Trans. Amer. Math. Soc. **31** (1929), 680-686.

(Received June 15, 1981)

D. J. Lewis
University of Michigan
Ann Arbor, MI 48109
U.S.A.

Patrick Morton
California Institute of Technology
Pasadena, CA 91125
U.S.A.