On self-dual, completely reducible finite subgroups of $GL(2, k)$

By Nagayoshi IWAHORI and Takeo YOKONUMA

Dedicated to the late Takuro Shintani

In order to explain J. McKay's marvelous observation [2] concerning the mysterious relation between the irreducible complex representations of the binary polyhedral groups and the Dynkin diagrams of Euclidean type, Happel-Preiser-Ringel [3] gave the classification of the finite groups which admit a faithful self-dual two-dimensional representation over a field whose characteristic does not divide the order of the groups. However the conditions given in [3] on the fields for a given group do not seem to be precise. It seems that they did not consider the necessity of the conditions.

In this note, we give the above classification by different method and show the necessary and sufficient conditions on the fields. Then using the Perron-Frobenius theory on non-negative matrices, we determine all the diagrams which can be realized as the representation graphs of the above representations.

Notations. (1) We denote by $\mathfrak{C}_n$ and $\mathfrak{D}_{2n}$ the cyclic group of order n and the dihedral group of order $2n$ respectively.

(2) The polyhedral groups (l, m, n) $(1<l\leqq m\leqq n)$ are the groups generated by three elements P, Q, R with the defining relations: $P^l=Q^m=R^n=PQR=1$. It is known that the group (l, m, n) is finite if and only if $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}>1$, i.e. one of the following cases $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ ([1]). These groups are respectively $\mathfrak{D}_{2n}$, $\mathfrak{A}_4$, $\mathfrak{S}_4$, $\mathfrak{A}_5$.

(3) The binary polyhedral groups $\langle l, m, n\rangle$ $(1<l\leqq m\leqq n)$ are the groups generated by three elements P, Q, R with the defining relations: $P^l=Q^m=R^n=PQR$. The group $\langle l, m, n\rangle$ is finite if and only if the corresponding polyhedral group (l, m, n) is finite. When this is the case, $PQR=Z$ is a central element of order 2 and the kernel of the canonical homomorphism is $\langle Z\rangle$.

1. The determination of the finite groups admitting a faithful, completely reducible, self-dual two-dimensional representation.

First of all, let us introduce the definition of the field for representation of a given group.

DEFINITION. *Let G be a finite group. A field k is called a field for representation (of degree 2) of G if k satisfies both of the following conditions* (1) *and* (2). (We denote by char(k) the characteristic of k.)

(1) *The group algebra kG of G over k is semi-simple (i.e. the* char(k) *does not divide the order of G).*

(2) *G has a faithful self-dual two dimensional representation over k.*

We now determine the finite groups which have fields for representation.

THEOREM 1. *For a finite group G, the following three conditions* (I), (II) *and* (III) *are equivalent.*

(I) *G has a field for representation.*

(II) *The complex number field $\boldsymbol{C}$ is a field for representation of G.*

(III) *G is isomorphic to one of the following groups: $\mathfrak{C}_n$, $\mathfrak{D}_{2n}$, the finite binary polyhedral groups $\langle l, m, n\rangle$.*

PROOF. (III)$\Rightarrow$(II) is well-known (e.g. [2]) and (II)$\Rightarrow$(I) is trivial. Let us show (I)$\Rightarrow$(III). Let k be a field for representation of G and $\rho: G\rightarrow GL(2, k)$ be a faithful self-dual two-dimensional representation over k. Identifying G and $\rho(G)$, we can assume that $G\subset GL(2, k)$. Since ρ is self-dual, the eigenvalues α, β of $\sigma\in G$ in the algebraic closure $\bar{k}$ of k coincide with those of ${}^t\sigma^{-1}$. Then we have either $\alpha=\alpha^{-1}$, $\beta=\beta^{-1}$ or $\alpha=\beta^{-1}$, $\beta=\alpha^{-1}$. So $\alpha\beta=\det(\sigma)=\pm 1$ and G is contained in the group $SL^{\pm}(2, k)=\{\tau\in GL(2, k) \mid \det(\tau)=\pm 1\}$.

Case 1. $G\not\subset SL(2, k)$ (then char(k)$\neq 2$).

Let $H=G\cap SL(2, k)$, then $[G:H]=2$. For $\sigma\in G-H$, we have $\sigma^2=1$. In fact, the eigenvalues α, β of σ satisfy $\alpha\beta=-1$. From above remark, we have $\alpha^2=\beta^2=1$. Then, σ being diagonalizable, $\sigma^2=1$. Then we have $\sigma h\sigma^{-1}=h^{-1}$ for $\sigma\in G-H$, $h\in H$ and this implies that H is abelian. H is isomorphic to a finite subgroup of the multiplicative group $\bar{k}-\{0\}$. So H is cyclic and $G\cong\mathfrak{D}_{2n}$ if H is of order n.

Case 2. $G\subset SL(2, k)$.

Since $PSL(2, k)=SL(2, k)/\mathfrak{z}$ operates naturally on the projective line $\boldsymbol{P}_1$ over $\bar{k}$, $\bar{G}=G/\mathfrak{z}_0$ also operates on $\boldsymbol{P}_1$ where $\mathfrak{z}=\{\pm 1\}$ and $\mathfrak{z}_0=G\cap\mathfrak{z}$.

Case 2.1. *There exists a common fixed point on $\boldsymbol{P}_1$ for all elements in $\bar{G}$.*

By our assumption, we see that $\bar{k}^2$ is the direct sum of two one-dimensional G-invariant subspaces. So G is isomorphic to a subgroup consisting of diagonal matrices in $SL(2, k)$, i.e. G is cyclic.

Case 2.2. *There does not exist any common fixed point.*

Since every $\sigma\in G$ is diagonalizable over $\bar{k}$, we see that each element $\bar{\sigma}$ in $\bar{G}-\{1\}$ has just two fixed points in $\boldsymbol{P}_1$. Let $\bar{G}^{\#}=\bar{G}-\{1\}$, $\boldsymbol{P}_1^{\bar{\tau}}$ be the set of the fixed points on $\boldsymbol{P}_1$ of $\bar{\sigma}$, and $\Omega=\bigcup_{\bar{\sigma}\in\bar{G}^{\#}}\boldsymbol{P}_1^{\bar{\sigma}}$. Then by [4] Ω is the disjoint union of three $\bar{G}$-orbits Ω_1, Ω_2 and Ω_3. Let $p_i\in\Omega_i$ and $\bar{G}_i$ be the stabilizer of p_i in $\bar{G}$. Let ν_i be the order of $\bar{G}_i$, where we can assume $\nu_1\leqq\nu_2\leqq\nu_3$. We have the following possibilities [4]:

ν_1	ν_2	ν_3	$\lvert\bar{G}\rvert$	$\bar{G}$	$\bar{G}$
2	2	n	$2n$	$(2, 2, n)$	$\mathfrak{D}_{2n}$
2	3	3	12	$(2, 3, 3)$	$\mathfrak{A}_4$
2	3	4	24	$(2, 3, 4)$	$\mathfrak{S}_4$
2	3	5	60	$(2, 3, 5)$	$\mathfrak{A}_5$

In all cases, $\lvert\bar{G}\rvert$ is even. So $\lvert G\rvert$ is even and G has an element a of order 2. We then have $a=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, $\mathfrak{z}=\{1, a\}\subseteqq G$ and $\mathfrak{z}_0=\mathfrak{z}$. Therefore we can conclude easily that G is isomorphic to one of the finite binary polyhedral groups.

2. The necessary and sufficient conditions for a field for representation.

THEOREM 2. *Let k be a field. The necessary and sufficient conditions for k to be a field for representation of the finite groups $\mathfrak{C}_n$, $\mathfrak{D}_{2n}$, $\langle 2, 2, n\rangle$, $\langle 2, 3, n\rangle$ are given by the following table.*

G	The conditions for the field k
$\mathfrak{C}_n$	$n=1$ none $n=2$ $\mathrm{char}(k)\neq 2$ $n\geqq 3$ $\mathrm{char}(k)\nmid n$ and $\gamma_n\in k$
$\mathfrak{D}_{2n}=(2, 2, n)$	$n=2$ $\mathrm{char}(k)\neq 2$ $n\geqq 3$ $\mathrm{char}(k)\nmid 2n$ and $\gamma_n\in k$
$\langle 2, 2, n\rangle$	$\mathrm{char}(k)\nmid 2n$, $\gamma_{2n}\in k$ and there exist $x, y\in k$ such that $x^2+y^2=\gamma_{2n}^2-4$
$\langle 2, 3, n\rangle$ $(3\leqq n\leqq 5)$	$\mathrm{char}(k)\nmid 6n$, $\gamma_{2n}\in k$ and there exist $x, y\in k$ such that $x^2+y^2=\gamma_{2n}^2-3$

where $\zeta_d\in\bar{k}$ is a primitive d-th root of unity and $\gamma_d=\zeta_d+\zeta_d^{-1}$. We note that if $\gamma_d\in k$ for some ζ_d, then it is so for every primitive d-th root of unity in $\bar{k}$.

PROOF. 1) $G=\mathfrak{C}_n=\langle\sigma\rangle$. The cases $n=1$, $n=2$ and the necessity of the condition when $n\geqq 3$ follow easily. To show that the condition is sufficient, consider the representation ρ given by $\rho(\sigma)=\begin{pmatrix} 0 & 1 \\ -1 & \gamma_n \end{pmatrix}$.

2) $G=(2, 2, n)=\langle P, Q, R \mid P^2=Q^2=R^n=PQR=1\rangle$. Similarly to the case of $G=\mathfrak{C}_n$, define ρ by $\rho(R)=\begin{pmatrix} 0 & 1 \\ -1 & \gamma_n \end{pmatrix}$, $\rho(Q)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho(P)=\rho(Q)\rho(R)$.

3) $G=\langle 2, 2, n\rangle=\langle P_0, Q_0, R_0 \mid P_0^2=Q_0^2=R_0^n=P_0Q_0R_0\rangle$. Let $\rho: G\to GL(2, k)$ be a faithful self-dual representation. By the proof of Theorem 1, $\rho(G)$ is contained in $SL(2, k)$. Let $\rho(P_0)=P$, $\rho(Q_0)=Q$, $\rho(R_0)=R$. Then,

$$(*) \qquad P^2=Q^2=R^n=PQR=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can assume $P=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $Q=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $ad-bc=1$, $a+d=\zeta_4+\zeta_4^{-1}$ $=0$ and $R=-(PQ)^{-1}=\begin{pmatrix} b & d \\ -a & -c \end{pmatrix}$. It follows that $b-c=\zeta_{2n}+\zeta_{2n}^{-1}=\gamma_{2n}\in k$. Since $b=c+\gamma_{2n}$, $d=-a$ and $ad-bc=1$, we have $a^2+\left(c+\frac{\gamma_{2n}}{2}\right)^2=\frac{1}{4}(\gamma_{2n}^2-4)$. The elements $x=2a$, $y=2\left(c+\frac{\gamma_{2n}}{2}\right)$ of k satisfy $x^2+y^2=\gamma_{2n}^2-4$. Since $\mathrm{char}(k)\nmid |G|$, we have $\mathrm{char}(k)\nmid 2n$.

Now we show that the condition is sufficient. Taking $x, y\in k$ such that $x^2+y^2=\gamma_{2n}^2-4$, we define $a=\frac{x}{2}$, $c=\frac{1}{2}(y-\gamma_{2n})$, $d=-a$, $b=c+\gamma_{2n}$. Then, from the above calculation, $P=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Q=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $R=\begin{pmatrix} b & d \\ -a & -c \end{pmatrix}$ are contained in $SL(2, k)$ and satisfy $(*)$. So these elements determine a self-dual representation ρ of G. Let us show that ρ is faithful. Since the order of R is $2n$ and $|G|$ $=4n$, the order of $\mathfrak{N}=\mathrm{Ker}\,\rho$ is at most 2. Suppose that $|\mathfrak{N}|=2$, then $\mathfrak{N}=\langle Z_0\rangle$, where $Z_0=P_0Q_0R_0$, so $G/\mathfrak{N}\cong\mathfrak{D}_{2n}$. On the other hand $G/\mathfrak{N}\cong\langle R\rangle\cong\mathfrak{C}_{2n}$. This is a contradiction.

4) $G=\langle 2, 3, n\rangle=\langle P_0, Q_0, R_0 \mid P_0^2=Q_0^3=R_0^n=P_0Q_0R_0\rangle$. Similarly to the case 3), we have the relations

$$(**) \qquad P^2=Q^3=R^n=PQR=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, $a+d=\zeta_6+\zeta_6^{-1}=1$ for $Q=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we have $\left(a-\frac{1}{2}\right)^2+\left(c+\frac{\gamma_{2n}}{2}\right)^2=\frac{1}{4}(\gamma_{2n}^2-3)$. To show the converse, we prove that the representation ρ constructed as above is faithful. The subgroup $G_1=\langle P, Q, R\rangle$ of $SL(2, k)$

has k as a field for representation and we see that G_1 is not abelian, considering the order of $P=QR$. Suppose Ker $\rho=\mathfrak{N}\neq\{1\}$, then G_1 is a quotient group of $G/\langle Z_0\rangle=(2, 3, n)$, where $Z_0=P_0Q_0R_0$. But G_1 has the central element $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ of order 2. We can conclude that $\mathfrak{N}=\{1\}$.

COROLLARY. *Let k be a finite field $\boldsymbol{F}_q$ of $q=p^e$ elements where p is a prime number. Then we can restate the conditions in Theorem 2 as follows.*

G	The conditions for the field k	
$\mathfrak{C}_n$	$q\equiv\pm1$	(mod n)
$\mathfrak{D}_{2n}$	$q\equiv\pm1$	(mod n), $p>2$
$\langle 2, 2, n\rangle$	$q\equiv\pm1$	(mod $2n$)
$\langle 2, 3, n\rangle$	$q\equiv\pm1$	(mod $2n$), $p>3$

PROOF. It is well-known that any element α in a finite field k can be expressed in the form $\alpha=x^2+y^2$ ($x, y\in k$). So the last conditions in the cases of $\langle 2, 2, n\rangle$, $\langle 2, 3, n\rangle$ hold always. $\gamma_d=\zeta_d+\zeta_d^{-1}\in k$ is equivalent to $(\zeta_d+\zeta_d^{-1})^q=\zeta_d^q+\zeta_d^{-q}=\zeta_d+\zeta_d^{-1}$. This means that $\{\zeta_d^q, \zeta_d^{-q}\}$ and $\{\zeta_d, \zeta_d^{-1}\}$ satisfy the quadratic equation $t^2-\gamma_d t+1=0$. So we obtain that $\gamma_d\in k\Leftrightarrow\zeta_d^q=\zeta_d$ or $\zeta_d^q=\zeta_d^{-1}\Leftrightarrow q-1\equiv0$ (mod d) or $q+1\equiv0$ (mod d). And then q and d are relatively prime.

REMARK 1. Some of the conditions on k mentioned in Theorem 1 of [3] are not the necessary conditions. For example, in the case of $G=\langle 2, 2, m\rangle$, Theorem 1 of [3] says that $\zeta_{2m}\in k$ for even m and $\zeta_{4m}\in k$ for odd m. But we know from Corollary that $\boldsymbol{F}_7$ is a field for representation of G when $m=3$. Then $\zeta_4\notin\boldsymbol{F}_7$ and $\zeta_{12}\notin\boldsymbol{F}_7$. When $m=4$, though $\zeta_8\notin\boldsymbol{F}_7$, $\boldsymbol{F}_7$ is a field for representation. In the case of $G=\langle 2, 3, 3\rangle$, Theorem 1 of [3] says that $\zeta_4\in k$ for a field of representation. But in this case also, $k=\boldsymbol{F}_7$ is a field for representation. Let us show an example of characteristic 0 case. From Theorem 2, $k=\boldsymbol{Q}(\zeta_3)$ is a field for representation of $\langle 2, 3, 3\rangle$, because $\zeta_6+\zeta_6^{-1}=1\in k$ and $1^2+(\sqrt{3}i)^2=-2$. But $\zeta_4\notin\boldsymbol{Q}(\zeta_3)$. In the cases of $\langle 2, 3, 4\rangle$, $\langle 2, 3, 5\rangle$ also, the conditions on k in [3] should be adjusted.

REMARK 2. It is well-known that $\langle 2,3,3\rangle\cong SL(2,\boldsymbol{F}_3)$ and $\langle 2,3,5\rangle\cong SL(2,\boldsymbol{F}_5)$. One can prove these by constructing the matrices which satisfy the defining relations of $\langle 2, 3, 3\rangle$ or $\langle 2, 3, 5\rangle$ by the method of Theorem 2, taking the existence of the unipotent matrices into account. Therefore, from Corollary, we have the following well-known inclusion relations, where p is a prime number.

For $p=3$ or $p>3$, $SL(2, \boldsymbol{F}_3)\hookrightarrow SL(2, \boldsymbol{F}_p)$.
For $p=5$ or $p>5$, $p\equiv\pm 1 \pmod 5$, $SL(2, \boldsymbol{F}_5)\hookrightarrow SL(2, \boldsymbol{F}_p)$.

3. The representation graphs and generalized Euclidean diagrams.

To begin with, we recall the definition of the representation graphs for the convenience of readers. Let k be a field for representation of a finite group G. Let $kG=\mathfrak{A}_1\oplus\cdots\oplus\mathfrak{A}_n$ be the decomposition of kG into the simple components $\mathfrak{A}_i$. We can assume that $\mathfrak{A}_i$ is the total matric algebra $M_{d_i}(E_i)$ of degree d_i over a skew field E_i. Then we have $|G|=\sum_{i=1}^{n} d_i^2 e_i$, where $e_i=\dim_k E_i$. Let (ρ_i, V_i) be the irreducible representation corresponding to $\mathfrak{A}_i$. Then the dimension $\tilde{d}_i$ of V_i is equal to $d_i e_i$. Let (ρ, V) be a faithful self-dual two-dimensional representation of G over k. Then the multiplicity c_{ij} of ρ_i in $\rho\otimes\rho_j$ determines a non-negative matrix $C=(c_{ij})$. We can express C in the form of a graph, which is called the representation graph of ρ: Consider the graph Γ with n vertices $\rho_1, \cdots, \rho_n$. If $c_{ii}>0$, then we write c_{ii} loops around the vertex ρ_i. If $c_{ij}+c_{ji}>0$ for $i\neq j$, we write $\overset{\rho_i}{\circ}\underset{(c_{ij},\, c_{ji})}{\text{———}}\overset{\rho_j}{\circ}$. If $c_{ij}=c_{ji}=0$ for $i\neq j$, there is no edge between the vertices ρ_i and ρ_j. We use the following convention: If $c_{ij}=c_{ji}=1$, we write $\overset{\rho_i}{\circ}\text{——}\overset{\rho_j}{\circ}$ and if $c_{ij}>1$ and $c_{ji}=1$, we write $\overset{\rho_i}{\circ}\Longrightarrow\overset{\rho_j}{\circ}$ (c_{ij} arrows).

Since ρ is faithful, the matrix C is indecomposable (i.e. for every (i, j), $i\neq j$, there exists a sequence $(i_1, \cdots, i_\nu)$ such that $c_{ii_1}c_{i_1i_2}\cdots c_{i_\nu j}\neq 0$) ([2]) and we can apply the Perron-Frobenius theory on non-negative matrices. We quote the following relation from [3]: $e_i c_{ij}=e_j c_{ji}$ for $1\leqq i, j\leqq n$.

PROPOSITION 1. *The matrix $C=(c_{ij})$ has the following properties.*

(α) *Every component c_{ij} is a non-negative integer.*

(β) *$c_{ij}>0$ if and only if $c_{ji}>0$.*

(γ) *The matrix C is indecomposable and the Frobenius root of C equals 2.*

(δ) *There exist row vectors whose components are all positive integers $\tilde{\boldsymbol{x}}=(\tilde{d}_1, \cdots, \tilde{d}_n)$, $\boldsymbol{x}=(d_1, \cdots, d_n)$ such that*

$$\begin{cases} 2\tilde{\boldsymbol{x}}=\tilde{\boldsymbol{x}}C, \quad 2\boldsymbol{x}=\boldsymbol{x}\,{}^tC, \\ \operatorname{Min}\{\tilde{d}_1, \cdots, \tilde{d}_n\}=\operatorname{Min}\{d_1, \cdots, d_n\}=1. \end{cases}$$

(ε) *For all i, $d_i | \tilde{d}_i$.*

PROOF. Comparing the degrees of $\rho\otimes\rho_j=\sum_i c_{ij}\rho_i$, we have $2\tilde{\boldsymbol{x}}=\tilde{\boldsymbol{x}}C$. Then, since $\tilde{d}_i=e_i d_i$ and $e_i c_{ij}=e_j c_{ji}$, we have $2\boldsymbol{x}=\boldsymbol{x}\,{}^tC$.

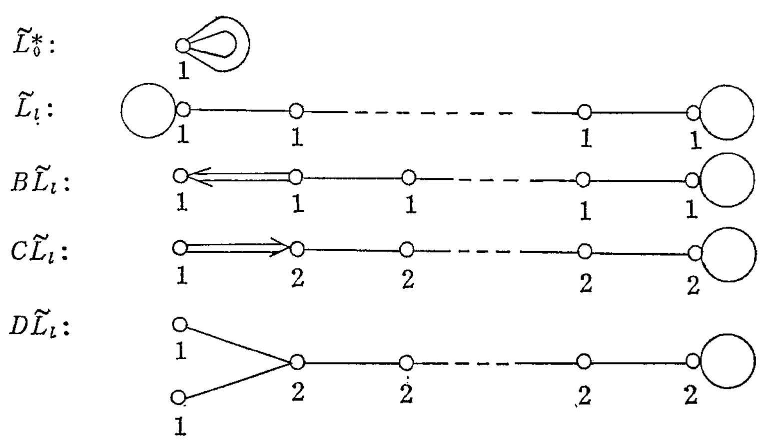
Remark that $\tilde{x}$ and x in (δ) are uniquely determined.

DEFINITION. *The matrices C and their graphs satisfying* (α), (β) *and* (γ) *in Proposition* 1 *are called of generalized Euclidean type, and those which moreover satisfy* $c_{ii}=0$ *for all i are called of Euclidean type.*

Since the classification of the matrices of Euclidean type is well known, we show here the classification of the matrices of generalized Euclidean type which are not of Euclidean type.

Although the following result is given in [3], we shall prove it here by using the Perron-Frobenius theory on non-negative matrices (cf. e.g. [5]).

THEOREM 3. *The matrices of generalized Euclidean type which are not of Euclidean type are given by the following graphs.*



PROOF. First of all, we can verify that the numbers under the vertices give the components of the vector $\tilde{x}$ satisfying $\tilde{x}C=2\tilde{x}$, and can conclude that these are of generalized Euclidean type.

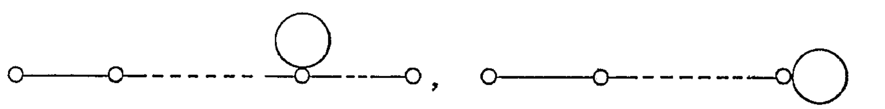
We remark that, by the Perron-Frobenius theory, the Frobenius root of the graph decreases if some of vertices or edges are taken off from the original graph. Therefore, the Frobenius root of the graph which contains properly one of the graphs in Theorem 3 is larger than 2, and the Frobenius root of the proper subgraph is less than 2.

Now, we show the generalized Euclidean graph with loops coincides with one of those in Theorem 3.

Case 1. *For some i*, $c_{ii}\geqq 2$. The graph contains $\tilde{L}_0^*$. Therefore it coincides with $\tilde{L}_0^*$.

Case 2. *For all* i, $c_{ii}\leqq 1$, *and* $\sum_i c_{ii}\geqq 2$. The graph contains $\tilde{L}_l$. Therefore it coincides with $\tilde{L}_l$.

Case 3. $\sum_i c_{ii}=1$. Let $m=\operatorname{Max}_{i\neq j} c_{ij}$. Suppose $m\geqq 2$, then it contains $\widetilde{BL}_l$ or $\widetilde{CL}_l$. Therefore $m=2$ and it coincides with $\widetilde{BL}_l$ or $\widetilde{CL}_l$. Suppose $m=1$. If the graph has a junction (i.e. the vertex i which has at least three j such that $c_{ij}>0$), it contains $\widetilde{DL}_l$ and is equal to $\widetilde{DL}_l$. If there is no junction, the graph is one of the followings:

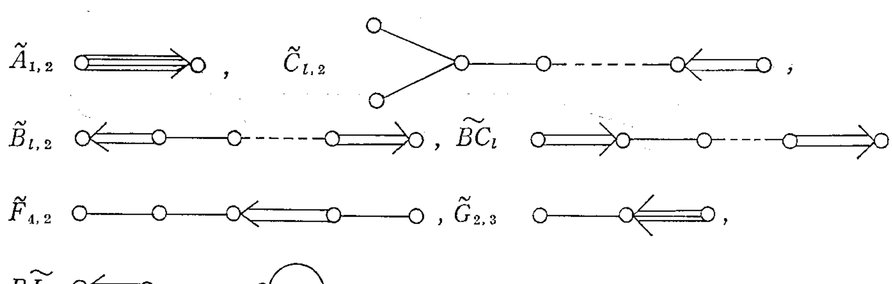


If the former case occurs, the graph coincides with $D\tilde{L}_2$. The latter case is impossible because of $\tilde{L}_l$.

4. The determination of representation graphs.

In this section we determine the graphs of generalized Euclidean type which can be realized as the representation graphs of a finite group admitting a field for representation.

From the condition (ε) of Proposition 1, we can conclude that the following graphs are impossible:



Calculating $\tilde{d}_i$, d_i, $e_i=\tilde{d}_i/d_i$ for the remaining graphs, we verify that $e_i\leqq 3$. (This is given in [3].)

PROPOSITION 2. *If a field k is a field for representation of a finite group G, for the simple component $\mathfrak{A}_i=M_{d_i}(E_i)$ of the group algebra kG, we have $\dim_k E_i\leqq 3$. In particular, the skew fields E_i are all commutative.* (In fact, we show later that $\dim_k E_i\leqq 2$.)

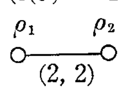
COROLLARY. *Let G' be the commutator subgroup of G. Then $[G:G']=\sum_{d_i=1} e_i$.*

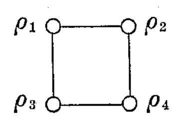
In particular, G is abelian if and only if all d_i are equal to 1.

PROOF. The irreducible representations of G/G' over k are those ρ_i such that $\mathrm{Ker}\,\rho_i \supset G'$. Then $\rho_i(k[G/G']) = \rho_i(kG) \cong \mathfrak{A}_i \cong M_{d_i}(E_i)$. Therefore, $\mathrm{Ker}\,\rho_i \supset G' \Leftrightarrow \rho_i(G)$ is abelian $\Leftrightarrow \rho_i(kG) \cong M_{d_i}(E_i)$ is commutative $\Leftrightarrow d_i = 1$ (because E_i is commutative). So $k[G/G'] = \bigoplus_{d_i=1} \mathfrak{A}_i$ and $[G : G'] = \sum_{d_i=1} \dim_k E_i = \sum_{d_i=1} e_i$.

Now, we determine the representation graphs of faithful self-dual two-dimensional representations over k of the groups mentioned in Theorem 1, (III). In the following, these representations are called the realization of G.

Case 1. *G is abelian,* i.e. $G = \mathfrak{C}_n$ or $\mathfrak{C}_2 \times \mathfrak{C}_2$.

If $G = \mathfrak{C}_1$, the realization is $\rho = 1_G \oplus 1_G$ (1_G is the unit representation) and the representation graph is $\tilde{L}_0^*$. If $G = \mathfrak{C}_2 = \langle \sigma \rangle$, the irreducible representations of G are $\rho_1 = 1_G$ and $\rho_2 = \varepsilon$ ($\varepsilon(\sigma) = -1$). The realization are $\rho_1 \oplus \rho_2$ and $2\rho_2$, their graphs being $\tilde{L}_1$ and 

respectively. For $G = \mathfrak{C}_2 \times \mathfrak{C}_2$, the realization is essentially unique and the graph is $\tilde{A}_3$ (

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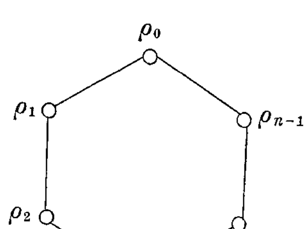
Let G be $\mathfrak{C}_n = \langle \sigma \rangle$ ($n \geqq 3$), and ρ be a realization of G. Then the eigenvalues α, β of $\rho(G)$ are the primitive n-th roots of unity and $\alpha\beta = 1$ because ρ is self-dual. Put $\alpha = \theta = \zeta_n$ and $\beta = \theta^{-1}$.

Case 1.1. $\theta \in k$.

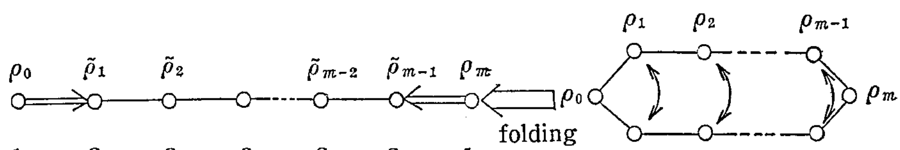
The irreducible representations of G over k are ρ_i ($0 \leqq i \leqq n-1$) where $\rho_i(\sigma) = \theta^i$ and $\rho = \rho_1 \oplus \rho_{n-1}$. Then $\rho \otimes \rho_j = \rho_{j+1} \oplus \rho_{j-1}$ ($\rho_n = \rho_0$) and the representation graph is $\tilde{A}_{n-1}$,



Case 1.2. $\theta \notin k$, $n = \text{even} = 2m$.

From $\theta + \theta^{-1} \in k$, we have $\theta^j + \theta^{-j} \in k$. Then $\tilde{\rho}_j(\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & \theta^j + \theta^{-j} \end{pmatrix}$ ($1 \leqq j \leqq m-1$) is the irreducible representation over k and equivalent to $\rho_j \oplus \rho_{n-j}$ over $k(\theta)$. These $(m-1)$ representations and ρ_0, ρ_m (defined in Case 1.1) are the complete representative system of the irreducible representations over k of G.

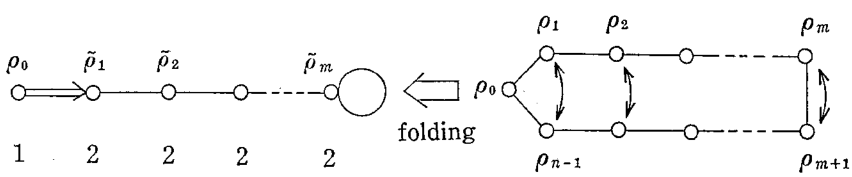
Then we have the representation graph which is obtained by "folding" of that of Case 1.1.



$\tilde{d}_i$	1	2	2	2 …	2	2	1
d_i	1	1	1	1	1	1	1
e_i	1	2	2	2	2	2	1

Case 1.3. $\theta \notin k$, n=odd$=2m+1$.

Similarly, $\tilde{\rho}_j$ $(1 \leqq j \leqq m)$ and ρ_0 give the complete representative system. The representation graph is as follows.



$\tilde{d}_i$	1	2	2	2	2
d_i	1	1	1	1	1
e_i	1	2	2	2	2

REMARK. Now we conclude that $\tilde{L}_l$ $(l \geqq 2)$ cannot be realized as representation graph because all d_i of $\tilde{L}_l$ are 1.

Case 2. $G=\mathfrak{D}_{2n}=(2, 2, n)$ $(n \geqq 3)$.

Case 2.1. n=even$=2m$.

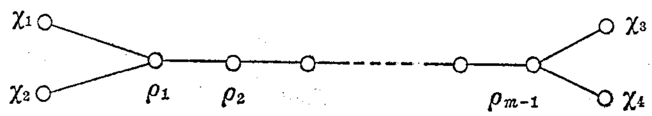
From the defining relations $G=\langle P, Q, R \mid P^2=Q^2=R^n=PQR=1\rangle=\langle Q, R \mid Q^2=R^n=1, QRQ^{-1}=R^{-1}\rangle$, the one-dimensional representations of G are the following χ_i $(1 \leqq i \leqq 4)$: $\chi_1(Q)=\chi_1(R)=1$; $\chi_2(Q)=-1$, $\chi_2(R)=1$; $\chi_3(Q)=1$, $\chi_3(R)=-1$; $\chi_4(Q)=\chi_4(R)=-1$.

Define $(m-1)$ representations ρ_j $(1 \leqq j \leqq m-1)$ of G over k by $\rho_j(Q)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho_j(R)=\begin{pmatrix} 0 & 1 \\ -1 & \theta^j+\theta^{-j} \end{pmatrix}$ where $\theta=\zeta_n$. Then we can verify;

1) $\rho_1, \cdots, \rho_{m-1}$ are absolutely irreducible and inequivalent.

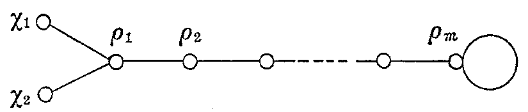
2) $\{\chi_1, \chi_2, \chi_3, \chi_4, \rho_1, \cdots, \rho_{m-1}\}$ is the complete representative system of the irreducible representations.

We can assume that the realization ρ equals ρ_1. The representation graph is as follows.



Case 2.2. $n=\text{odd}=2m+1$.

Similarly, $\{\chi_1, \chi_2, \rho_1, \cdots, \rho_m\}$ is the complete representative system of the irreducible representations.



Case 3. $G=\langle 2, 2, n\rangle$ $(n\geqq 2)$.

Case 3.1. $n=\text{even}=2m$.

From the defining relations $G=\langle P, Q, R \mid P^2=Q^2=R^n=PQR\rangle=\langle Q, R \mid Q^2=R^n, QPQ^{-1}=R^{-1}\rangle$, the one-dimensional representations of G are the following χ_i $(1\leqq i\leqq 4)$: $\chi_1(Q)=\chi_1(R)=1$; $\chi_2(Q)=-1$, $\chi_2(R)=1$; $\chi_3(Q)=1$, $\chi_3(R)=-1$; $\chi_4(Q)=\chi_4(R)=-1$.

Let $\theta=\zeta_{2n}$, $s=\theta+\theta^{-1}$, $s_j=\theta^j+\theta^{-j}$ $(j=1, 2, \cdots)$. The field k for representation of G satisfies $s\in k$. We can assume that $\rho(Q)=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $\rho(R)=\begin{pmatrix}0 & 1\\ -1 & s\end{pmatrix}$. Then $\alpha\delta-\beta\gamma=1$, $\alpha+\delta=0$ and the defining relations give

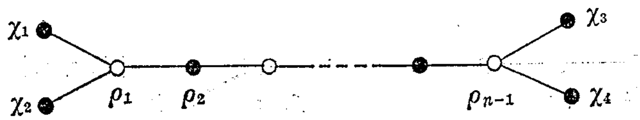
$$(\text{☆}) \qquad \alpha^2+s\alpha\beta+\beta^2=-1\,.$$

Then for each j, there exist $x, y\in k$ which satisfy

$$(\text{☆☆}) \qquad x^2+s_jxy+y^2=(-1)^j\,.$$

In fact, if $\theta\in k$, then put $x=\theta^{mj}$, $y=0$. If $\theta\notin k$, then $k(\theta)/k$ is a Galois extension of degree 2, and $\theta^j\notin k$ for $1\leqq j\leqq n-1$. Therefore $k(\theta)=k(\theta^j)=K$. Consider the norm map N from K to k. Then (☆☆) means $(-1)^j\in N(K)$. On the other hand (☆) means $(-1)\in N(K)$ and we have the result.

Using x and y, we can find a_j, b_j, c_j and $d_j\in k$ such that $\rho_j(Q)=\begin{pmatrix}a_j & b_j\\ c_j & d_j\end{pmatrix}$ and $\rho_j(R)=\begin{pmatrix}0 & 1\\ -1 & s_j\end{pmatrix}$ satisfy the defining relations and give the representation ρ_j of G. ρ_j $(1\leqq j\leqq n-1)$ are absolutely irreducible and non-equivalent. And $\{\chi_1, \chi_2, \chi_3, \chi_4, \rho_1, \cdots, \rho_{n-1}\}$ gives the complete representative system of the irreducible representations of G and the representation graph is as follows for $\rho=\rho_1$:

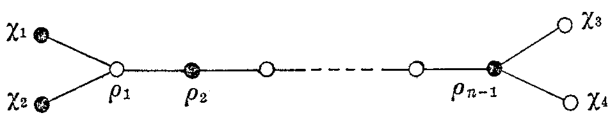


(all $e_i=1$)

The representations corresponding to black vertices are the representations of $G/\langle PQR\rangle=(2, 2, n)=\mathfrak{D}_{2n}$.

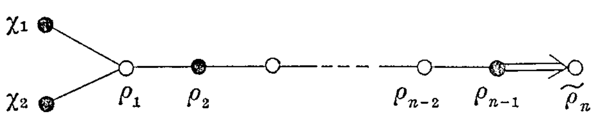
Case 3.2. $n=\text{odd}=2m+1$, $\zeta_4\in k$.

G has the following four one-dimensional representations: $\chi_1(Q)=\chi_1(R)=1$; $\chi_2(Q)=-1$, $\chi_2(R)=1$; $\chi_3(Q)=\zeta_4$, $\chi_3(R)=-1$; $\chi_4(Q)=-\zeta_4$, $\chi_4(R)=-1$. The absolutely irreducible representations $\rho_1, \cdots, \rho_{n-1}$ of degree 2 are determined as Case 3.1 and we have the following representation graph.

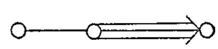


Case 3.3. $n=\text{odd}=2m+1$, $\zeta_4\notin k$.

G has two one-dimensional representations χ_1 and χ_2. Let $\tilde{\rho}_n=\chi_3\oplus\chi_4$. Then $\tilde{\rho}_n$ is an irreducible representation over k and we have the following representation graph.



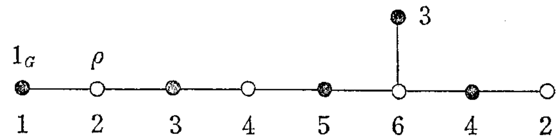
$\begin{pmatrix} e_i=1 \text{ except } \tilde{\rho}_n \\ e_i=2 \text{ for } \tilde{\rho}_n \end{pmatrix}$

REMARK. We verify that $\tilde{G}_{2,1}$  cannot be realized as representation graph. Suppose that a finite group G has $\tilde{G}_{2,1}$ as a representation graph. Then $|G|=8$, $[G:G']=4$ and $G=\langle 2, 2, 2\rangle$ or $(2, 2, 4)$, which is impossible. Since $\tilde{G}_{2,1}$ is the only possible case where $e_i=3$ appears, we know that $\dim_k E_i\leqq 2$.

Case 4. $G=\langle 2, 3, n\rangle$ $(3\leqq n\leqq 5)$.

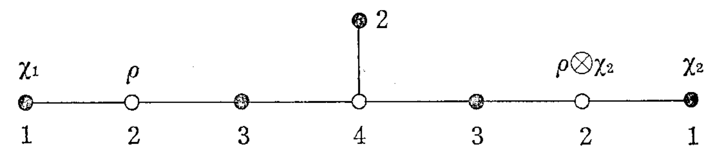
These groups have the irreducible representations of degree >2 over the algebraic closure $\bar{k}$ of k. So the maximum of the degrees of the irreducible representations over k is also at least 3. On the other hand, the possible graphs of generalized Euclidean types which have $\operatorname{Max}\tilde{d}_j>2$ are of type $\tilde{F}_{4,1}$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$. Comparing the orders of G, we know that the representation graphs of $\langle 2, 3, 3\rangle$ are of type $\tilde{F}_{4,1}$ or $\tilde{E}_6$; those of $\langle 2, 3, 4\rangle$ are of type $\tilde{E}_7$; those of $\langle 2, 3, 5\rangle$ are of type $\tilde{E}_8$.

$G=\langle 2, 3, 5\rangle$



The numbers mean $\tilde{d}_i=d_i$; the representations corresponding to black vertices are the representations of $(2, 3, 5)=\mathfrak{A}_5$.

$G=\langle 2, 3, 4\rangle$

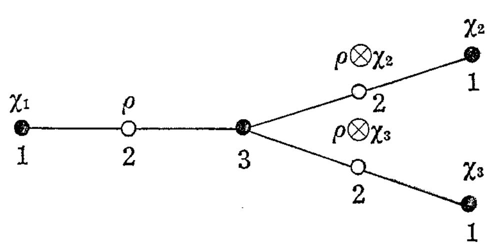


The representations corresponding to black vertices are the representations of $(2, 3, 4)=\mathfrak{S}_4$; $\chi_1=1_G$, χ_2 is the one-dimensional representation given by

$$\chi_2(P)=\chi_2(R)=-1\,, \qquad \chi_2(Q)=1\,.$$

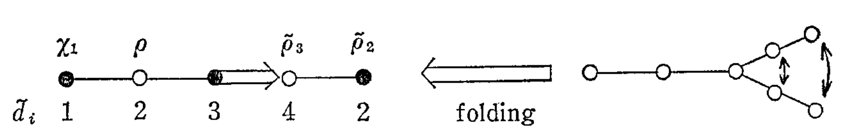
Case 4.1. $G=\langle 2, 3, 3\rangle$.

When $\omega=\zeta_3\in k$, G has three one-dimensional representations χ_1, χ_2 and χ_3 where $\chi_1=1_G$; $\chi_2(P)=1$, $\chi_2(Q)=\omega$, $\chi_2(R)=\omega^2$; $\chi_3(P)=1$, $\chi_3(Q)=\omega^2$, $\chi_3(R)=\omega$. The representation graph is of type $\tilde{E}_6$:



The representations corresponding to black vertices are the representations of $(2, 3, 3)=\mathfrak{A}_4$.

When $\omega=\zeta_3\notin k$, $\tilde{\rho}_2=\chi_2\oplus\chi_3$, $\tilde{\rho}_3=(\rho\otimes\chi_2)\oplus(\rho\otimes\chi_3)$ are the irreducible representations over k. The representation graph is of type $\tilde{F}_{4,1}$:



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(Received June 12, 1981)

Nagayoshi Iwahori
Department of Mathematics
Faculty of Sciences
University of Tokyo
Hongo, Tokyo
113 Japan

Takeo Yokonuma
Department of Mathematics
Faculty of Science and Technology
Sophia University
Chiyoda-ku, Tokyo
102 Japan