On Springer's representations

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To the memory of Takuro Shintani

To the Weyl group representations constructed by Springer in [10, 11], various approaches have been exemplified in these years ([9], [7], [8]). But identifications among them have not been realized until very recently and now they have been carried out after Lusztig's new construction [8] using the Deligne-Goresky-MacPherson cohomology [12]. In § 1 of this paper, identifications among the main three constructions, i.e., Springer's [10], Slodowy's [9] and Lusztig's [8], will be reported. We note that just recently in [1], the identification between Springer's [10] and Lusztig's [8] has been announced. We shall show that Springer's representation in [10] coincides with Lusztig's one in [8] up to the sign representation and that Lusztig's one coincides with Slodowy's one in [9].

In § 2, we try to give an expression of the action of a simple reflexion with respect to the basis of the top homology given by the fundamental cycles. This has been announced in essence at Oberwolfach (April 1979) and at Sapporo (August 1979) but there was some incorrect point (in particular, if an irreducible component of $\mathcal{B}^A$ is singular in codimension 1, the formula in [4] is incorrect). In this occasion, we shall give a precise form. The author understands that a similar formula has been observed also by Lusztig in his letter to Springer dated March, 1978. As a merit of our method in § 2, it can be shown directly that Springer's representation [10] of the top cohomology has a $\boldsymbol{Z}$-basis.

The author thanks Slodowy for valuable discussions in several occasions and for having informed the author of the above mentioned letter of Lusztig.

After writing up the manuscript, the author has been informed by Slodowy and Springer that, besides Lusztig, Springer and MacPherson have also got the same results as in § 1. The author particularly thanks Springer who has written him a successive development and has pointed out that the idea as in Lemma 3 in § 1 was known to him earlier.

§ 1. Identifications.

1.1. Let G be a connected reductive algebraic group over an algebraically

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closed field k, and $\mathscr{B}$ be the set of all Borel subgroups in G. Fix a Borel subgroup B once and for all; hence $\mathscr{B}\cong G/B$ as varieties. Fix a maximal torus T in B and let $W=N_G(T)/T$ be the Weyl group. Denote by $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$ the Lie algebras of G, B and T respectively. Let $\mathfrak{g}'$, $\mathfrak{t}'$ be the k-linear dual spaces of $\mathfrak{g}$, $\mathfrak{t}$ respectively and $\mathfrak{t}'\hookrightarrow\mathfrak{g}'$ be the embedding given by the root space decomposition for (G, T).

Let $\tilde{\mathfrak{g}}=G\times^B\mathfrak{b}$ be the vector bundle over $\mathscr{B}=G/B$ associated to the principal B-bundle $G\rightarrow G/B$ with the adjoint action of B on $\mathfrak{b}$. Then $\tilde{\mathfrak{g}}\ni(g, A)\mapsto(gBg^{-1}, gA)\in\mathscr{B}\times\mathfrak{g}$ gives an isomorphism onto the closed subvariety $\{(B', A)\in\mathscr{B}\times\mathfrak{g};\ A\in\operatorname{Lie}B'\}$. Taking the projection to $\mathfrak{g}$, one has a surjective proper map $\rho:\tilde{\mathfrak{g}}\rightarrow\mathfrak{g}$ with fibers $\rho^{-1}(A)=\mathscr{B}^A=\{B'\in\mathscr{B};\ A\in\operatorname{Lie}B'\}$. The direct sum decomposition $\mathfrak{b}=\mathfrak{t}\oplus\mathfrak{u}$ ($\mathfrak{u}$ is the nilpotent radical of $\mathfrak{b}$) gives the projection $\mathfrak{b}\ni A\mapsto A_{\mathfrak{t}}\in\mathfrak{t}$, which gives rise to the smooth projection $\tilde{\mathfrak{g}}\rightarrow\mathfrak{t}$ ($(g, A)\mapsto A_{\mathfrak{t}}$). Note that $(bA)_{\mathfrak{t}}=A_{\mathfrak{t}}$ for $b\in B$. Thus one has Grothendieck's resolution

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\ \rho\ } & \mathfrak{g} \\ \downarrow & & \downarrow \\ \mathfrak{t} & \longrightarrow & W\backslash\mathfrak{t} \end{array}$$

where $\mathfrak{g}\rightarrow W\backslash\mathfrak{t}$ is Chevalley's invariant map (see [9]). (We assume that the characteristic of k is 0 or large enough.)

1.2. Assume char $k=p>0$, q is a power of p, $\boldsymbol{F}_q$ is the finite field with q elements. The affine line $\boldsymbol{A}^1$ over k has an $\boldsymbol{F}_q$-Galois covering $\boldsymbol{A}^1\rightarrow\boldsymbol{A}^1$ $(x\mapsto x^q-x)$. Fix a non-trivial additive character $\psi:\boldsymbol{F}_q\rightarrow\overline{\boldsymbol{Q}}_l^\times$ ($\overline{\boldsymbol{Q}}_l^\times$ is the multiplicative group of the algebraic closure $\overline{\boldsymbol{Q}}_l$ of the l-adic number field, $l\neq p$), which gives rise to the locally constant sheaf $\mathcal{A}_\psi$ on $\boldsymbol{A}^1$ (a "smooth" constructible $\overline{\boldsymbol{Q}}_l$-sheaf [2]).

Now consider the map $\alpha:\mathfrak{t}'\times\tilde{\mathfrak{g}}\rightarrow\mathfrak{t}'\times\mathfrak{t}\rightarrow\boldsymbol{A}^1$, where $\tilde{\mathfrak{g}}\rightarrow\mathfrak{t}$ is the projection in Grothendieck's resolution and the second map is the natural pairing. Let $\mathcal{E}=\mathcal{E}_\psi=\alpha^*\mathcal{A}_\psi$ be the locally constant sheaf pulled back on $\mathfrak{t}'\times\tilde{\mathfrak{g}}$. Then by definition, $\mathcal{E}$ is constant on $0\times\tilde{\mathfrak{g}}$ ($0\in\mathfrak{t}'$) and on $\mathfrak{t}'\times(G\times^B\mathfrak{u})=\mathfrak{t}'\times\rho^{-1}(\mathfrak{N})$ ($\mathfrak{N}$ is the nilpotent variety of $\mathfrak{g}$).

1.3. In general, for a group W and a W-variety X, we call a sheaf on X a W-sheaf if there exists an isomorphism $w^*\mathcal{S}\xrightarrow{\sim}\mathcal{S}$ on X ($w\in W$) satisfying the usual compatibility condition with the group structure of W; hence, for example, $\Gamma(X, \mathcal{S})$ then turns out to be a W-module by $\Gamma(X, \mathcal{S})\xrightarrow{\sim}\Gamma(X, w^{*-1}\mathcal{S})\xrightarrow{\sim}\Gamma(X, \mathcal{S})$, where the first isomorphism is the natural one.

Consider $\mathfrak{t}'\times\mathfrak{g}$ as a W-variety letting W act dually on $\mathfrak{t}'$ and trivially on $\mathfrak{g}$. Put $\tilde{\rho}=\mathrm{Id}_{\mathfrak{t}'}\times\rho:\mathfrak{t}'\times\tilde{\mathfrak{g}}\rightarrow\mathfrak{t}'\times\mathfrak{g}$. We shall regard $R^*\tilde{\rho}_*\mathcal{E}$ as W-sheaves using the

Deligne-Goresky-MacPherson extension as in [8]. Let $\mathfrak{g}^0$ be the set of strongly regular semisimple elements (i.e. whose centralizers are tori) in $\mathfrak{g}$, and put $\tilde{\mathfrak{g}}^0 = \rho^{-1}(\mathfrak{g}^0)$. Note that $G/T \times \mathfrak{t}^0 \simeq \tilde{\mathfrak{g}}^0$ by $(gT, H) \mapsto (g, H) \in G \times^B \mathfrak{b}$ where $\mathfrak{t}^0 = \mathfrak{t} \cap \mathfrak{g}^0$, $H \in \mathfrak{t}^0$. If ρ^0 is the restriction of ρ to $\tilde{\mathfrak{g}}^0$, then under the above identification, $\rho^0(gT, H) = gH$ for $(gT, H) \in G/T \times \mathfrak{t}^0$. By the W-action $(gT, H) \mapsto (g\dot{w}T, w^{-1}H)$ ($\dot{w} \in N_G(T)$ represents $w \in W$), $\rho^0 : \tilde{\mathfrak{g}}^0 \cong G/T \times \mathfrak{t}^0 \to \mathfrak{g}^0$ is a W-principal bundle. Let $\mathcal{E}^0$ be the restriction of $\mathcal{E}$ to $\mathfrak{t}' \times \tilde{\mathfrak{g}}^0$. Then under the above W-action on $\mathfrak{t}' \times \tilde{\mathfrak{g}}^0$, $\mathcal{E}^0$ is a W-sheaf since $\mathcal{E}^0$ is induced by a W-invariant map $\mathfrak{t}' \times G/T \times \mathfrak{t}^0 \to \mathfrak{t}' \times \mathfrak{t}^0 \to \boldsymbol{A}^1$ where the first map is $\mathrm{Id}_{\mathfrak{t}'} \times \mathrm{pr}_2$. Hence under a W-equivariant map $\tilde{\rho}^0 : \mathfrak{t}' \times \tilde{\mathfrak{g}}^0 \to \mathfrak{t}' \times \mathfrak{g}^0$, the locally constant sheaf $\tilde{\rho}^0_* \mathcal{E}^0$ is a W-sheaf.

1.4. Consider the Deligne-Goresky-MacPherson extension ${}^{\pi}(\tilde{\rho}^0_* \mathcal{E}^0)$ on the whole $\mathfrak{t}' \times \mathfrak{g}$ as in [8]. Then by functoriality, ${}^{\pi}(\tilde{\rho}^0_* \mathcal{E}^0)$ turns out to be a complex of W-sheaves on $\mathfrak{t}' \times \mathfrak{g}$.

LEMMA 1. *${}^{\pi}(\tilde{\rho}^0_* \mathcal{E}^0) \simeq \boldsymbol{R} \tilde{\rho}_* \mathcal{E}$ in the derived category generated by constructible sheaves on $\mathfrak{t}' \times \mathfrak{g}$.*

PROOF. Essentially by the same reason in [8] (see [12]). The difference is that in our case, the sheaf $\mathcal{E}$ is not constant. However, one can reduce our $\mathcal{E}$ in constant case by considering the $\boldsymbol{F}_q$-covering on $\mathfrak{t}' \times \tilde{\mathfrak{g}}$ given by α. Then on this covering, the corresponding isomorphism holds and the ψ-isotypic part is the above isomorphism.

1.5. Recall that in [8] Lusztig defines a W-sheaf structure by considering the isomorphism ${}^{\pi}(\rho^0_* \boldsymbol{Q}_l) \cong \boldsymbol{R} \rho_* \boldsymbol{Q}_l$ on $\mathfrak{g}$. Relation with our construction is simply as follows.

LEMMA 2. *${}^{\pi}(\tilde{\rho}^0_* \mathcal{E}^0)|_{0 \times \mathfrak{g}} \cong {}^{\pi}(\rho^0_* \overline{\boldsymbol{Q}}_l) \cong \boldsymbol{R} \rho_* \overline{\boldsymbol{Q}}_l$ on $0 \times \mathfrak{g}$ ($0 \in \mathfrak{t}'$) coincide with Lusztig's construction.*

PROOF. Immediately by checking the definition of the DGM extension combined with the base change theorem since $\mathcal{E}|_{0 \times \tilde{\mathfrak{g}}} \simeq \overline{\boldsymbol{Q}}_l$.

1.6. We are assuming that char $k = p$ is sufficiently large as in [10]. Let $\mathfrak{t}'_0$ be the set of strongly regular elements in $\mathfrak{t}'$ ((i.e. whose centralizers are tori). By the base change theorem, $\boldsymbol{R} \tilde{\rho}_* \mathcal{E}|_{\mathfrak{t}' \times \mathfrak{N}} \cong \boldsymbol{R} \tilde{\rho}_* \overline{\boldsymbol{Q}}_l|_{\mathfrak{t}' \times \mathfrak{N}}$ since $\mathcal{E}$ is constant on $\mathfrak{t}' \times \rho^{-1}(\mathfrak{N})$. We claim that the given W-sheaf structures on $\mathfrak{t}'_0 \times \mathfrak{N}$ then coincides essentially with Springer's construction [10] (up to the sign representation).

We first review Springer's paper [10]. Consider a sequence of the maps

$$\mathfrak{t}_0'\times G/T\times\mathfrak{g}\xrightarrow{f}\mathfrak{t}_0'\times\mathcal{B}\times\mathfrak{g}\xrightarrow{p}\mathfrak{t}_0'\times\mathfrak{g}$$

where $f(\xi, gT, A)=(\xi, gB, A)$, $p=pr_{13}$ (f depends on the choice of B). The W-action on $\mathfrak{t}_0'\times G/T\times\mathfrak{g}$ is defined by $w.(\xi, gT, A)=(w\xi, g\dot{w}^{-1}T, A)$. The composite $\pi=p\circ f$ is then W-equivariant. Let $\tilde{\alpha}: \mathfrak{t}_0'\times G/T\times\mathfrak{g}\to\boldsymbol{A}^1$ be such that $\tilde{\alpha}(\xi, gT, A)=\langle g\xi, A\rangle$, where $\langle\,,\rangle$ is the natural pairing $\mathfrak{g}'\times\mathfrak{g}\to\boldsymbol{A}^1$. Put $\mathcal{F}=\tilde{\alpha}^*\mathcal{A}_\psi$ for $\mathcal{A}_\psi$ as in 1.2. Then by [10; 3.5], $R^{2d}f_!\mathcal{F}$ is supported on the subvariety $\mathfrak{t}_0'\times\tilde{\mathfrak{g}}$ in $\mathfrak{t}_0'\times\mathcal{B}\times\mathfrak{g}$, and $R^if_!\mathcal{F}=0$ $(i\neq 2d)$ $(d=\dim\mathcal{B})$.

LEMMA 3. *$R^{2d}f_!\mathcal{F}|_{\mathfrak{t}_0'\times\tilde{\mathfrak{g}}}\xrightarrow{\sim}\mathcal{E}|_{\mathfrak{t}_0'\times\tilde{\mathfrak{g}}}$ (the Tate twist disregarded) and hence $R^{i+2d}\pi_!\mathcal{F}\xrightarrow{\sim}R^i\tilde{\rho}_*\mathcal{E}|_{\mathfrak{t}_0'\times\mathfrak{g}}$.*

PROOF. The following diagram

$$\begin{array}{ccc}
f^{-1}(\mathfrak{t}_0'\times\tilde{\mathfrak{g}})\cong\mathfrak{t}_0'\times(G\times^T\mathfrak{h}) & \xrightarrow{\tilde{\alpha}} & \boldsymbol{A}^1\\
f\downarrow \qquad\qquad \downarrow & & \|\\
\mathfrak{t}_0'\times\tilde{\mathfrak{g}}\quad\cong\mathfrak{t}_0'\times(G\times^B\mathfrak{b}) & \xrightarrow{\alpha} & \boldsymbol{A}^1
\end{array}$$

commutes (hence $f^*\mathcal{E}\cong\mathcal{F}$). Taking the $\boldsymbol{F}_q$-coverings both over $\mathfrak{t}_0'\times\tilde{\mathfrak{g}}$ and $f^{-1}(\mathfrak{t}_0'\times\tilde{\mathfrak{g}})$, by α and $\tilde{\alpha}=\alpha\circ f$, one has $\tilde{f}$: $f^{-1}(\mathfrak{t}_0'\times\tilde{\mathfrak{g}})\times_{A^1}\boldsymbol{A}^1\to(\mathfrak{t}_0'\times\tilde{\mathfrak{g}})\times_{A^1}\boldsymbol{A}^1$. Then $R^{2d}\tilde{f}_!\overline{\boldsymbol{Q}}_l$ is constant since it is locally constant and constant on some open dense subset (a big cell!). Taking the ψ-isotypic part, one has the required isomorphism.

1.7. By the W-action on $\mathfrak{t}_0'\times G/T\times\mathfrak{g}$ in 1.6, $R^*\pi_!\mathcal{F}$ acquires the W-sheaf structure on $\mathfrak{t}_0'\times\mathfrak{g}$. On the other hand, $R^i\tilde{\rho}^*\mathcal{E}|_{\mathfrak{t}_0'\times\mathfrak{g}}$ has the W-sheaf structure given by Lemma 1.

LEMMA 4. *$R^{i+2d}\pi_!\mathcal{F}\xrightarrow{\sim}\mathrm{sgn}\otimes R^i\tilde{\rho}_*\mathcal{E}|_{\mathfrak{t}_0'\times\mathfrak{g}}$ as W-sheaves, where* sgn *is the sign representation.*

PROOF. We first see that the isomorphism with W-actions holds on $\mathfrak{t}_0'\times\mathfrak{g}^0$. It suffices to compare at geometric stalks. Let $(\xi, A)\in\mathfrak{t}_0'\times\mathfrak{g}^0$, and one may assume $A\in\mathfrak{t}^0$. Then in $\xi\times G/T\times A\xrightarrow{f}\xi\times\mathcal{B}\times A\xrightarrow{p}\xi\times A$, $p^{-1}(\xi, A)\cap\xi\times\tilde{\mathfrak{g}}=\{(\xi, sB, A);\ s\in W\}$ and $f^{-1}(\xi, sB, A)\cong\{(\xi, suT, A);\ u\in U=\text{unipotent radical of } B\}\cong\boldsymbol{A}^d$. Thus our isomorphism turns out to be

$$\phi_B;\ R^{2d}\pi_!\mathcal{F}_{(\xi,A)}\xrightarrow{\sim}\bigoplus_{s\in W}H_c^{2d}(\boldsymbol{A}^d)_{(\xi,sB,A)}\xrightarrow{\sim}\bigoplus_{s\in W}\mathcal{E}_{(\xi,sB,A)}(\cong\overline{\boldsymbol{Q}}_l[W]).$$

Making the commutative diagram

$$\begin{array}{ccc} R^{2d}\pi_!\mathcal{F}_{(\xi,A)} & \xrightarrow[\sim]{\phi_B} & \bigoplus_{s\in W}\mathcal{E}_{(\xi,sB,A)} \\ {\scriptstyle w^*}\uparrow & & \uparrow{\scriptstyle \phi(w)} \\ R^{2d}\pi_!\mathcal{F}_{(w\xi,A)} & \xrightarrow[\phi_B]{\sim} & \bigoplus_{s\in W}\mathcal{E}_{(w\xi,sB,A)} \end{array},$$

one sees that $\phi(w)=\phi_B\circ\phi_{\bar{w}_B}^{-1}\circ\bar{w}^*$ where ${}^wB=wBw^{-1}$ and $\bar{w}^*$ denotes the W-action on $\tilde{\rho}^0_*\mathcal{E}$. But then the map $\phi_B\circ\phi_{\bar{w}_B}^{-1}$ is computed in [10; 3.12] and it equals the sign of w, $(-1)^{l(w)}=\mathrm{sgn}(w)$. This implies the isomorphism with W-action on $\mathfrak{t}_0'\times\mathfrak{g}^0$.

Next consider the following diagram.

$$\begin{array}{ccccccc} \mathfrak{t}_0'\times G/T\times\mathfrak{g} & & \xleftarrow{\quad\quad\quad\tilde{j}\quad\quad\quad} & & \mathfrak{t}_0'\times G/T\times\mathfrak{g}^0 \\ {\scriptstyle f}\downarrow & & & & \\ \mathfrak{t}_0'\times\mathcal{B}\times\mathfrak{g} & \longleftarrow & \mathfrak{t}_0'\times\tilde{\mathfrak{g}} & \xleftarrow{\ j\ } & \mathfrak{t}_0'\times\tilde{\mathfrak{g}}^0 & & \swarrow{\scriptstyle \pi^0} \\ {\scriptstyle p}\downarrow & & \downarrow & & {\scriptstyle \tilde{\rho}^0}\downarrow & & \\ \mathfrak{t}_0'\times\mathfrak{g} & = & \mathfrak{t}_0'\times\mathfrak{g} & \longleftarrow & \mathfrak{t}_0'\times\mathfrak{g}^0 & & \end{array}$$

By what has been just proved, one has

$$\mathrm{sgn}\otimes \boldsymbol{R}\pi^0_!\tilde{j}^*\mathcal{F}\cong\boldsymbol{R}\tilde{\rho}^0_*j^*\mathcal{E}[-2d].$$

Applying the DGM extension to $\mathfrak{t}_0'\times\mathfrak{g}$, one has by Lemma 1

$$\mathrm{sgn}\otimes{}^{\pi}(\boldsymbol{R}\pi^0_!\tilde{j}^*\mathcal{F})\cong{}^{\pi}(\boldsymbol{R}\tilde{\rho}^0_*j^*\mathcal{E})[-2d]\cong\boldsymbol{R}\tilde{\rho}_*\mathcal{E}[-2d]$$

on $\mathfrak{t}_0'\times\mathfrak{g}$. But by the functoriality of the constructions, one has a quasi-isomorphism as complexes of W-sheaves ${}^{\pi}(\boldsymbol{R}\pi^0_!\tilde{j}^*\mathcal{F})\cong\boldsymbol{R}\pi_!\tilde{j}_*\tilde{j}^*\mathcal{F}$ and $\tilde{j}_*\tilde{j}^*\mathcal{F}\cong\mathcal{F}$, which implies $\mathrm{sgn}\otimes\boldsymbol{R}\pi_!\mathcal{F}\cong\boldsymbol{R}\tilde{\rho}_*\mathcal{E}[-2d]$ on $\mathfrak{t}_0'\times\mathfrak{g}$. Hence Lemma 4.

1.8. We are now ready to identify Springer's representation with Lusztig's one. Since $R^*\tilde{\rho}_*\mathcal{E}$ is constant on $\mathfrak{t}'\times A$, for a nilpotent $A\in\mathfrak{N}$,

$$H^i(\mathcal{B}^A,\bar{\boldsymbol{Q}}_l)\cong\Gamma(\mathfrak{t}_0'\times A,\ R^{i+2d}\pi_!\mathcal{F})\cong\Gamma(\mathfrak{t}_0'\times A,\ \mathrm{sgn}\otimes R^i\tilde{\rho}_*\mathcal{E})$$

acquires the W-module structure, which is Springer's representation. On the other hand, $\mathcal{E}|_{0\times\mathfrak{g}}$ is constant and, by Lemma 2, $H^i(\mathcal{B}^A,\bar{\boldsymbol{Q}}_l)\cong R^i\tilde{\rho}_*\mathcal{E}_{(0,A)}$ is the W-module defined by Lusztig in [8]. Since $R^*\tilde{\rho}_*\mathcal{E}$ is constant on $\mathfrak{t}'\times A$, one has isomorphisms as W-modules, by Lemma 4,

$$\begin{aligned}\Gamma(\mathfrak{t}_0'\times A,\ \mathrm{sgn}\otimes R^{i+2d}\pi_!\mathcal{F})&\cong\Gamma(\mathfrak{t}_0'\times A,\ R^i\tilde{\rho}_*\mathcal{E})\\ &\cong\Gamma(\mathfrak{t}'\times A,\ R^i\tilde{\rho}_*\mathcal{E})\\ &\cong R^i\tilde{\rho}_*\mathcal{E}_{(0,A)}.\end{aligned}$$

Thus Springer's W-action on $H^i(\mathcal{B}^A, \bar{Q}_l)$ coincides with Lusztig's one multiplied with sgn.

1.9. The coincidence of Lusztig's representation with Slodowy's one [9] can be seen rather easily. Here the base field k is assumed to be the complex number field $\boldsymbol{C}$. For a nilpotent $A\in\mathcal{N}$, let A' be an opposite nilpotent in Jacobson-Morosov's triplet. Then $S=A+\mathfrak{z}_{\mathfrak{g}}(A')$ in $\mathfrak{g}$ is Slodowy's transversal slice for the orbit of A, where $\mathfrak{z}_{\mathfrak{g}}(A')$ is the centralizer of A'. Put $\tilde{S}=\rho^{-1}(S)\subset\tilde{\mathfrak{g}}$. Then in Grothendieck's resolution in 1.1, $\tilde{S}\to\mathfrak{t}$ is a simultaneous resolution of fibers of $S\to W\backslash\mathfrak{t}$. By Slodowy [9], $\tilde{S}\to\mathfrak{t}$ is a topologically trivial fiber bundle and the special fiber $\tilde{S}_0=\rho^{-1}(S_{\bar{0}})$ $(0\in\mathfrak{t}, \bar{0}\in W\backslash\mathfrak{t})$ contracts to $\mathcal{B}^A$ in $\tilde{S}_0$. Hence the monodromy representation of $\pi_1(W\backslash\mathfrak{t}^0, \bar{t})$ on $H^*(S_{\bar{t}}, Q)$ $(\bar{t}\in W\backslash\mathfrak{t}^0)$ gives rise to the representation of W on $H^*(\tilde{S}_0, Q)\cong H^*(\mathcal{B}^A, \boldsymbol{Q})$.

Let $t\in\mathfrak{t}^0$ be a regular element with $\bar{t}\in W\backslash\mathfrak{t}^0$ its image. The monodromy representation of W is defined on $H^*(S_{\bar{t}}, Q)$ as above. Let ${}^{\pi}(\rho_*^0\boldsymbol{Q})\cong\boldsymbol{R}\rho_*\boldsymbol{Q}$ be Lusztig's construction [8]. Then

$$\boldsymbol{H}^*(S_{\bar{0}}, {}^{\pi}(\rho_*^0\boldsymbol{Q}))\cong\boldsymbol{H}^*(S_{\bar{0}}, \boldsymbol{R}\rho_*\boldsymbol{Q})\cong H^*(\tilde{S}_0, \boldsymbol{Q})$$

where $\boldsymbol{H}^*$ denotes the hypercohomology. By these isomorphisms, $H^*(\tilde{S}_0, \boldsymbol{Q})$ $(\cong H^*(\mathcal{B}^A, \boldsymbol{Q}))$ aquires a W-action via ${}^{\pi}(\rho_*^0\boldsymbol{Q})$. It suffices to show that this action coincides with Slodowy's monodromy action.

Now for $\bar{t}\in W\backslash\mathfrak{t}^0$, let $\boldsymbol{H}^*(S_{\bar{0}}, {}^{\pi}(\rho_*^0\boldsymbol{Q}))\to\boldsymbol{H}^*(S_{\bar{t}}, {}^{\pi}(\rho_*^0\boldsymbol{Q}))$ be the cospecialization map. Since ${}^{\pi}(\rho_*^0\boldsymbol{Q})|_{S_{\bar{t}}}\cong\rho_*^0\boldsymbol{Q}|_{S_{\bar{t}}}$ $(S_{\bar{t}}\subset\mathfrak{g}^0)$, $\boldsymbol{H}^*(S_{\bar{t}}, {}^{\pi}(\rho_*^0\boldsymbol{Q}))\cong H^*(S_{\bar{t}}, \rho_*^0\boldsymbol{Q})$. Taking a suitable C^∞-trivialization of $\tilde{S}\to\mathfrak{t}$, one has as C^∞-manifolds

$$\rho^{-1}(S_{\bar{t}})=\coprod_{w\in W}\tilde{S}_{wt}\xrightarrow{\sim}\tilde{S}_0\times Wt\longrightarrow S_{\bar{t}}$$

where $\tilde{S}_0$ has a suitable topological W-action such that the last map is the quotient by W. Under the first map, one has $a: H^*(S_{\bar{t}}, \rho_*^0\boldsymbol{Q})\xrightarrow{\sim} H^*(\tilde{S}_0, \boldsymbol{Q})\otimes_{\boldsymbol{Q}}\boldsymbol{Q}[W]$, where $H^*(\tilde{S}_0, \boldsymbol{Q})$ is a W-module obtained by the above W-action on $\tilde{S}_0$, which gives Slodowy's representation. One also see the following commutative diagram of W-maps

$$\begin{array}{ccccc} H^*(\tilde{S}_0, \boldsymbol{Q}) & & \xhookrightarrow{\quad\quad b\quad\quad} & & H^*(\tilde{S}_0\times Wt, \boldsymbol{Q}) \\ \wr\| & & & & \wr\| \\ \boldsymbol{H}^*(S_{\bar{0}}, {}^{\pi}(\rho_*^0\boldsymbol{Q})) & \xrightarrow[\text{cosp}]{} & H^*(S_{\bar{t}}, \rho_*^0\boldsymbol{Q}) & \xrightarrow[a]{} & H^*(\tilde{S}_0, \boldsymbol{Q})\otimes_{\boldsymbol{Q}}\boldsymbol{Q}[W] \end{array}$$

where b is the injection given by the projection $\tilde{S}_0\times Wt\to\tilde{S}_0$. In the above diagram, $H^*(\tilde{S}_0, \boldsymbol{Q})$ in the left top corner is equipped with Lusztig's W-action (through $\boldsymbol{H}^*(S_{\bar{0}}, {}^{\pi}(\rho_*^0\boldsymbol{Q}))$) and that in the right bottom corner with Slodowy's one tensored with the group ring $\boldsymbol{Q}[W]$. The image of b in $H^*(\tilde{S}_0, \boldsymbol{Q})\otimes\boldsymbol{Q}[W]$

is by definition $H^*(\tilde{S}_0, \boldsymbol{Q})\otimes(\boldsymbol{Q}[W]^W)\cong H^*(\tilde{S}_0, \boldsymbol{Q})$. One can thus identify both the representations.

THEOREM 1. *On $H^*(\mathcal{B}^A)$ for a nilpotent A, Springer's representation* [10] *coincides with Lusztig's one* [8] *multiplied with the sign representation. Lusztig's representation coincides with Slodowy's one* [9].

§2. Local formulas.

2.1. Basic notations are as in §1. We fix a nilpotent A once and for all and consider Springer's representation of the Weyl group W given by the top cohomology $H^{2n}(\mathcal{B}^A)=H^{2n}(\mathcal{B}^A, \overline{\boldsymbol{Q}}_l)$, where $n=\dim \mathcal{B}^A$. It is known by Steinberg and Spaltenstein that $\mathcal{B}^A$ is connected and of pure dimension $n=(1/2)(\dim Z_G(A) - \operatorname{rank} G)$ ($Z_G(A)$ denotes the centralizer of A), at least if the characteristic of the base field k is 0 or large enough. Let I be the set of all irreducible components of $\mathcal{B}^A$. Put

$$H_{2n}(\mathcal{B}^A)=H^{2n}(\mathcal{B}^A)^\vee=\text{the dual space of } H^{2n}(\mathcal{B}^A)\,.$$

Then each $C\in I$ defines the fundamental cycle $[C]\in H_{2n}(\mathcal{B}^A)$ given by

$$[C]:\ H^{2n}(\mathcal{B}^A)\xrightarrow{\text{rest}} H^{2n}(C)\xrightarrow{\mathrm{Tr}_C}\overline{\boldsymbol{Q}}_l$$

(the Tate twist being disregarded), and $\{[C];\ C\in I\}$ forms a basis of $H_{2n}(\mathcal{B}^A)$.

2.2. Springer's representation depends on the choice of Borel subgroups. Fix a Borel subgroup B once and for all and consider the W-action on $H^*(\mathcal{B}^A)$ and hence on $H_{2n}(\mathcal{B}^A)$. The Borel subgroup B determines a system of generators of W consisting of simple reflexion $s\in W$ so that $P_s=B\cup BsB$ is a (parabolic) subgroup of G. For such s, put $\mathcal{P}_s=G/P_s$ and let ϕ_s: $\mathcal{B}=G/B\to\mathcal{P}_s$ be the natural projection. Then ϕ_s is a $\boldsymbol{P}^1$-bundle locally trivial in the Zariski topology and hence $\dim\phi_s(C)$ for $C\in I$ is equal to n or $n-1$. We try to describe the s-action on $H_{2n}(\mathcal{B}^A)$ with respect to the basis $\{[C];\ C\in I\}$.

2.3. It was already shown in [5] that $s[C]=[C]$ if and only if $\dim\phi_s(C) =n-1$. We shall first show the following formula:

If $\dim\phi_s(C)=n$, then there exist numbers $n_C^{C'}(s)\in\overline{\boldsymbol{Q}}_l$ for $C'\in I$ with $\phi_s(C') \subsetneqq\phi_s(C)$ such that

$$s[C]=-[C]-\sum_{C'} n_C^{C'}(s)[C']\,.$$

For simplicity, we write $P=P_s$, $\phi=\phi_s$, and $\mathcal{P}=\mathcal{P}_s$ since we fix s henceforth. Let $\mathcal{P}^A$ be the closed subvariety of $\mathcal{P}=G/P$ consisting of gP ($g\in G$) such that

A belongs to the Lie algebra Lie gPg^{-1} of gPg^{-1}. Then $\phi(\mathcal{B}^A)=\mathcal{P}^A$. Let $\bar{\phi}=\phi|_{\mathcal{B}^A}$ be the restriction of ϕ to $\mathcal{B}^A\subset\mathcal{B}$; hence $\bar{\phi}: \mathcal{B}^A\to\mathcal{P}^A$. Then fibers of $\bar{\phi}$ are connected and hence they are isomorphic to $\boldsymbol{P}^0$ or $\boldsymbol{P}^1$. Hence for the Leray sheaves $R^*\bar{\phi}_*\overline{\boldsymbol{Q}}_l$, one has $R^0\bar{\phi}_*\overline{\boldsymbol{Q}}_l\cong\overline{\boldsymbol{Q}}_l$ and $R^i\bar{\phi}_*\overline{\boldsymbol{Q}}_l=0$ unless $i=0, 2$. It is shown in [5] that the exact sequence

$$0\longrightarrow H^{2n}(\mathcal{P}^A)\longrightarrow H^{2n}(\mathcal{B}^A)\longrightarrow H^{2n-2}(\mathcal{P}^A, R^2\bar{\phi}_*\overline{\boldsymbol{Q}}_l)\longrightarrow 0$$

is that as s-modules, where s acts on $H^{2n}(\mathcal{P}^A)$ as -1 and on $H^{2n-2}(\mathcal{P}^A, R^2\bar{\phi}_*\overline{\boldsymbol{Q}}_l)$ as 1. Taking the duals, one has a surjection $H_{2n}(\mathcal{B}^A)\to H_{2n}(\mathcal{P}^A)$ which maps $[C]$ to $[\phi(C)]$ if $\dim\phi(C)=n$ and whose kernel is the linear span $\langle[C']; C'\in I, \dim\phi(C')=n-1\rangle$.

Fix $C\in I$ such that $\dim\phi(C)=n$. Then $\phi(C)$ is an irreducible component of $\mathcal{P}^A$ and the n-dimensional irreducible components of $\bar{\phi}^{-1}(\phi(C))$ in $\mathcal{B}^A$ consist of $C'\in I$ with $\phi(C')\subset\phi(C)$. Note that $\dim\phi(C')=n-1$ if $C'\neq C$ and $\phi(C')\subset\phi(C)$. In fact, if $\phi(C')=\phi(C)$, then on some non-empty open $U\subset\phi(C)$, $\bar{\phi}^{-1}(u)\,(u\in U)$ is a point. Then $\bar{\phi}^{-1}(U)$ is contained in $C\cup C'$ which implies $C=C'$ (actually one can show that C is birational to $\phi(C)$; see later). Restricting the above exact sequence to $\phi(C)\subset\mathcal{P}^A$, one has

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2n}(\phi(C)) & \longrightarrow & H^{2n}(\bar{\phi}^{-1}(\phi(C))) & \longrightarrow & H^{2n-2}(\phi(C), R^2\bar{\phi}_*\overline{\boldsymbol{Q}}_l) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^{2n}(\mathcal{P}^A) & \longrightarrow & H^{2n}(\mathcal{B}^A) & \longrightarrow & H^{2n-2}(\mathcal{P}^A, R^2\bar{\phi}_*\overline{\boldsymbol{Q}}_l) & \longrightarrow & 0\,,
\end{array}$$

where all maps are s-homomorphisms ([5]). Dualizing the above diagram, from what one has seen, one has

$$\begin{array}{ccccccccc}
0 & \leftarrow & \langle[\phi(C)]\rangle & \leftarrow & \langle[C']; \phi(C')\subset\phi(C)\rangle & \leftarrow & \langle[C']; \phi(C')\subsetneqq\phi(C)\rangle & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & H_{2n}(\mathcal{P}^A) & \longleftarrow & H_{2n}(\mathcal{B}^A) & \longleftarrow & \langle[C']; \dim\phi(C')=n-1\rangle & \leftarrow & 0
\end{array}$$

Hence $s[C]\equiv-[C] \bmod \langle[C']; \phi(C')\subsetneqq\phi(C)\rangle$, which implies our assertion:

$$s[C]=-[C]-\sum_{C'} n_C^{C'}(s)[C']$$

with certain numbers $n_C^{C'}(s)\in\overline{\boldsymbol{Q}}_l$ for $\phi(C')\subsetneqq\phi(C)$.

2.4. We want to describe the numbers $n_C^{C'}(s)$ in geometric terms of (C, C'). They turn out to be natural numbers. Let $\nu: \widetilde{\phi(C)}\to\phi(C)$ be the normalization of $\phi(C)$. Consider the diagram:

$$\begin{array}{ccccccc}
\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A & \xrightarrow{\tilde{\nu}} & \bar{\phi}^{-1}(\phi(C)) & \subset & \mathcal{B}^A & \subset & \mathcal{B} \\
\tilde{\phi}\downarrow & & \downarrow & & \downarrow\bar{\phi} & & \downarrow\phi \\
\widetilde{\phi(C)} & \xrightarrow{\nu} & \phi(C) & \subset & \mathcal{P}^A & \subset & \mathcal{P}
\end{array}$$

Fix $C' \in I$ with $\phi(C') \subsetneqq \phi(C)$. If the number of irreducible components of $\tilde{\nu}^{-1}(C')$ is $r(C')$ and

$$\tilde{\nu}^{-1}(C') = \bigcup_{i=1}^{r(C')} \tilde{C}'_i$$

is the irreducible decomposition, then so is

$$\nu^{-1}(\phi(C')) = \bigcup_{i=1}^{r(C')} \tilde{\phi}(\tilde{C}'_i) \qquad \text{in} \quad \widetilde{\phi(C)}.$$

LEMMA 1. *Let* $\nu_i(C', C) = [\tilde{\phi}(\tilde{C}'_i)_{\mathrm{red}} : \phi(C')]_{\mathrm{sep}}$ *be the separable degree of* $\tilde{\phi}(\tilde{C}'_i)_{\mathrm{red}} \to \phi(C')$ *where* $\tilde{\phi}(\tilde{C}'_i)_{\mathrm{red}}$ *is the reduced scheme attached to* $\tilde{\phi}(\tilde{C}'_i)$. *Then in the map*

$$\tilde{\nu}_* : H_{2n}(\widetilde{\phi(C)} \times_{\mathcal{P}} \mathcal{B}^A) \longrightarrow H_{2n}(\bar{\phi}^{-1}(\phi(C))),$$

one has

$$\tilde{\nu}_*([\tilde{C}'_i]) = \nu_i(C', C)[C'],$$

where $[\tilde{C}'_i]$ *is the fundamental cycle corresponding to* $(\tilde{C}'_i)_{\mathrm{red}}$.

PROOF. Since $\bar{\phi}^{-1}(\phi(C')) = \phi^{-1}(\phi(C')) \to \phi(C')$ is a $\boldsymbol{P}^1$-bundle locally trivial in the Zariski topology, the separable degree of $(\tilde{C}'_i)_{\mathrm{red}} \to C'$ coincides with $\nu_i(C', C)$ which is a mapping degree in the étale cohomology.

2.5. If one defines an s-module structure on $H_{2n}(\widetilde{\phi(C)} \times_{\mathcal{P}} \mathcal{B}^A)$ such that $\tilde{\nu}_*$ is an s-homomorphism and if one has a formula:

$$s[\tilde{C}] = -[\tilde{C}] - \sum_{C'} \sum_{i=1}^{r(C')} d_i(C', C)[\tilde{C}'_i]$$

for some numbers $d_i(C', C)$ where $\tilde{C} = \tilde{\nu}^{-1}(C)$, then one has

$$s[C] = -[C] - \sum_{C'} \left(\sum_{i=1}^{r(C')} d_i(C', C)\nu_i(C', C) \right)[C']$$

by taking the image under $\tilde{\nu}_*$. (Note that $\tilde{\nu}_*([\tilde{C}]) = [C]$ since $\widetilde{\phi(C)}$ is the normalization of $\phi(C)$.) Thus under our notations,

$$n_C^{C'}(s) = \sum_{i=1}^{r(C')} d_i(C', C)\nu_i(C', C).$$

For this purpose, we shall recall Springer's first construction [10] and "localize" it near a divisor $\tilde{C}'_i$.

2.6. Let $T \subset B$, $\mathfrak{t}'_0$ be as in § 1. Fix $\xi \in \mathfrak{t}'_0$ and defiine $\alpha_\xi : G/T \to \boldsymbol{A}^1$ by $\alpha_\xi(gT) = \langle g\xi, A \rangle = \langle \xi, g^{-1}A \rangle$. Let $\mathcal{A}_\psi$ be the Artin-Schreier sheaf over $\boldsymbol{A}^1$ as in § 1, and $\mathcal{F} = \mathcal{F}_\xi = \alpha_\xi^* \mathcal{A}_\psi$ over G/T. If $f : G/T \to G/B = \mathcal{B}$ is the projection, then

$R^{2d}f_!\mathcal{F}\cong i_*\bar{Q}_l$ where $i: \mathcal{B}^A\hookrightarrow\mathcal{B}$ is the embedding, and $H_c^{2d+i}(G/T, \mathcal{F})\cong H^i(\mathcal{B}^A)$ has a W-module structure defined by the W-action on $\{\mathcal{F}_\xi\}_{\xi\in\Gamma_0'}$. Under the map $\widetilde{\phi(C)}\xrightarrow{\nu}\phi(C)\hookrightarrow\mathcal{P}^A\hookrightarrow\mathcal{P}$, let

$$\widetilde{\phi(C)}\times_{\mathcal{P}}G/T \longrightarrow \widetilde{\phi(C)}$$

be the pull-back of $\phi\circ f: G/T\to\mathcal{P}$, and $\tilde{\mathcal{F}}$ be the sheaf over $\widetilde{\phi(C)}\times_{\mathcal{P}}G/T$, the pull-back of $\mathcal{F}$. Then by the base change theorem,

$$H_c^{i+2d}(\widetilde{\phi(C)}\times_{\mathcal{P}}G/T, \tilde{\mathcal{F}}) \xrightarrow{\sim} H^i(\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A),$$

and the right hand side aquires the s-module structure under which $\tilde{\nu}_*$ in Lemma 1 is an s-module homomorphism. Dualizing the exact sequence

$$0 \longrightarrow H^{2n}(\widetilde{\phi(C)}) \longrightarrow H^{2n}(\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A) \longrightarrow H^{2n-2}(\widetilde{\phi(C)}, R^2\tilde{\phi}_*\bar{Q}_l) \longrightarrow 0$$

one has

$$0 \longleftarrow H_{2n}(\widetilde{\phi(C)}) \longleftarrow H_{2n}(\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A) \longleftarrow \langle[\tilde{C}_i']\,;\,\phi(C')\subsetneqq\phi(C)\rangle \longleftarrow 0\,.$$

Fix a component $\tilde{C}_i'\subset\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A$ for $C'\in I$ such that $\phi(C')\subsetneqq\phi(C)$. Take $U\subset\widetilde{\phi(C)}$ be a non-empty open set such that $U\cap\tilde{\phi}(\tilde{C}_i')\neq\varnothing$ but $U\cap\tilde{\phi}(\tilde{C}_j')=U\cap\nu^{-1}(\phi(C''))=\varnothing$ for $j\neq i$ and $C''\neq C'$ nor C. ($\tilde{\phi}:\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A\to\phi(C)$.) Since $\widetilde{\phi(C)}$ is normal, one may assume that U and $U'=U\cap\tilde{\phi}(\tilde{C}_i')$ are smooth. Similarly as above, pulling back all the data under the map $U\hookrightarrow\widetilde{\phi(C)}\to\mathcal{P}$, one can define also the s-module structure on $H_c^{2n}(U\times_{\mathcal{P}}\mathcal{B}^A)$ and the s-homomorphism diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{2n}(\widetilde{\phi(C)}) & \longrightarrow & H^{2n}(\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A) & \longrightarrow & H^{2n-2}(\widetilde{\phi(C)}, R^2\tilde{\phi}_*\bar{Q}_l) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_c^{2n}(U) & \longrightarrow & H_c^{2n}(U\times_{\mathcal{P}}\mathcal{B}^A) & \longrightarrow & H^{2n-2}(U, R^2\tilde{\phi}_*\bar{Q}_l) & \longrightarrow & 0\,. \end{array}$$

Dualizing the above diagram and putting $H_{2n}(\ )=H_c^{2n}(\ )^\vee$, one has

$$0 \longleftarrow H_{2n}(U) \longleftarrow H_{2n}(U\times_{\mathcal{P}}\mathcal{B}^A) \longleftarrow \langle[\tilde{\phi}^{-1}(U')]\rangle \longleftarrow 0$$

where the map $H_{2n}(\widetilde{\phi(C)}\times_{\mathcal{P}}\mathcal{B}^A)\to H_{2n}(U\times_{\mathcal{P}}\mathcal{B}^A)$ is a surjection which maps $[\tilde{C}]$ to $[U\times_{\phi(C)}C]$ and $[\tilde{C}_i']$ to $[\tilde{\phi}^{-1}(U')]$ and the other elements in the basis to 0. Thus if $d_i(C', C)$ is as in 2.5, then

$$s[U\times_{\phi(C)}C]=-[U\times_{\phi(C)}C]-d_i(C', C)[\tilde{\phi}^{-1}(U')]$$

in $H_{2n}(U\times_{\mathcal{P}}\mathcal{B}^A)$.

2.7. Since $\phi\circ f: G/T\to G/P=\mathcal{P}$ is locally trivial in the Zariski topology, one may assume that

$$U\times_{\mathcal{P}}G/T\cong U\times P/T\longrightarrow G/T.$$

Take a map $\eta: U\to G$ such that $U\times P/T\ni(x, gT)\mapsto\eta(x)gT\in G/T$ is the above map. Then the sheaf $\tilde{\mathfrak{F}}$ on $U\times P/T$ is given by

$$U\times P/T\ni(x, gT)\longmapsto\langle\xi, g^{-1}\eta(x)^{-1}A\rangle\in A^1$$

where $\eta(x)^{-1}A\in\mathfrak{p}=\mathrm{Lie}\,P$ since $U\to\mathcal{P}^A$. Let $\mathfrak{u}_{\mathfrak{p}}$ be the nilpotent radical of $\mathfrak{p}$ and define $\beta: U\to\mathfrak{p}/\mathfrak{u}_{\mathfrak{p}}$ by $\beta(x)=\eta(x)^{-1}A \bmod \mathfrak{u}_{\mathfrak{p}}$. Then $\beta(x)\in\mathfrak{p}/\mathfrak{u}_{\mathfrak{p}}$ is nilpotent and hence β define the map

$$\beta: U\longrightarrow\mathfrak{N},$$

where $\mathfrak{N}$ is the set of nilpotent 2×2-matrices. Since $\tilde{\phi}^{-1}(U')\to U'$ is a $\boldsymbol{P}^1$-bundle, $\beta(x)=0$ for $x\in U'$. Shrinking U, if necessary, one may assume $\beta(x)\neq 0$ for $x\notin U'$. Since $\langle\xi, g^{-1}\eta(x)^{-1}A\rangle$ $(g\in P)$ depends only on $P \bmod Ru(P)$ ($Ru(P)$ is the unipotent radical of P), one may replace $\tilde{\mathfrak{F}}$ on $U\times P/T$ by $\tilde{\mathfrak{F}}$ (the same letter) on $U\times GL_2/T$ (T is a maximal torus of GL_2) given by the map

$$U\times GL_2/T\ni(x, gT)\longmapsto\langle\xi, g^{-1}\beta(x)\rangle\in\boldsymbol{A}^1,$$

and one obtains the same s-module $H_{2n}(U\times_{\mathcal{P}}\mathcal{B}^A)$. Here $U\times_{\mathcal{P}}\mathcal{B}^A\hookrightarrow U\times\boldsymbol{P}^1$ in the above trivialization and the irreducible components are $U'\times\boldsymbol{P}^1=\tilde{\phi}^{-1}(U')$ and the graph of $U\ni x\mapsto(\boldsymbol{P}^1)^{\beta(x)}\in\boldsymbol{P}^1$ $(=U\times_{\phi(C)}C\cong U)$ by the assumption.

2.8. For $\beta: U\to\mathfrak{N}=\{$nilpotent 2×2-matrices$\}$ defined in 2.7, $U'=\beta^{-1}(0)_{\mathrm{red}}$ is a smooth divisor. We define the order of β along U' as follows. For the generic point u_0 of U', $\mathcal{O}_{U,u_0}$ is a discrete valuation ring with valuation $\mathrm{ord}_{U'}:\mathcal{O}_{U,u_0}\to\boldsymbol{Z}$. The matrix entry β_{ij} of β defines the comorphism $\beta_{ij}^*: k[t]\to\mathcal{O}_{U,u_0}$.

DEFINITION. *The order of β along U'* is defined by

$$\mathrm{ord}_{U'}\beta=\mathrm{Min}\{\mathrm{ord}_{U'}\beta_{ij}^*(t); 1\leqq i, j\leqq 2\}.$$

Note that $\mathrm{ord}_{U'}\beta$ is uniquely determined by the components $\tilde{C}_i'$ and $\tilde{C}$. Our claim is:

LEMMA 2. *$d_i(C', C)=\mathrm{ord}_{U'}\beta$ where the left hand side is as in* 2.6.

2.9. We shall above Lemma 2 in the remaining part. Take a (general) closed point $x_0\in U'$ such that there exists a smooth curve $E\ni x_0$ intersecting U' transversally for which the order of $\beta|_E: E\to\mathfrak{N}$ along the divisor $\{x_0\}\subset E$ equals $\mathrm{ord}_{U'}\beta$ (such x_0 exists). First we note that in the étale topology, near $x_0\in U'\subset U$, $\tilde{\mathfrak{F}}$ over $U\times GL_2/T$ is, roughly speaking, equivalent to $\tilde{\mathfrak{F}}|_{E\times GL_2/T}$. Let $D=\bar{E}_{x_0}$ be the strict localization of E at x_0. In general, if $\bar{X}=\{X_i\to X\}$ is a projective system of étale morphisms and F is a smooth sheaf on X, then define the cohomology with proper support by $H_c^*(\bar{X}, F)=\varprojlim_i H_c^*(X_i, F)$. (See [2; page 71], or rather, through the Poincaré duality as below). In our case, this projective limit behaves rather well (Poincaré duality, trace morphisms etc. ⋯). Since $U\times GL_2/T$ and $\tilde{\mathfrak{F}}$ are smooth, by the Poincaré duality

$$H_c^{2n+2}(U\times GL_2/T,\ \tilde{\mathfrak{F}})\cong H^2(U\times GL_2/T,\ \tilde{\mathfrak{F}}^\vee)^\vee$$

where $(\ )^\vee$ is the dual object ($\dim U\times GL_2/T=n+2$). By restricting $D\times GL_2/T\to U\times GL_2/T$,

$$H^2(U\times GL_2/T,\ \tilde{\mathfrak{F}}^\vee)\cong H^2(D\times GL_2/T,\ \tilde{\mathfrak{F}}^\vee)$$

and by the Poincaré duality for $D\times GL_2/T$,

$$H^2(D\times GL_2/T,\ \tilde{\mathfrak{F}}^\vee)^\vee\cong H_c^4(D\times GL_2/T,\ \tilde{\mathfrak{F}})$$

($\dim D\times GL_2/T=3$). Thus one has

$$\begin{aligned}H_c^{2n}(U\times_{\mathcal{P}}\mathcal{B}^A)&\cong H_c^{2n+2}(U\times GL_2/T,\ \tilde{\mathfrak{F}})\\&\cong H_c^4(D\times GL_2/T,\ \tilde{\mathfrak{F}})\\&\cong H_c^2(D\times_{\mathcal{P}}\mathcal{B}^A)\,.\end{aligned}$$

But then it is seen easily that

$$D\times_{\mathcal{P}}\mathcal{B}^A\cong D\times\{\infty\}\cup\{x_0\}\times\boldsymbol{P}^1\subset D\times\boldsymbol{P}^1$$

for some $\infty\in\boldsymbol{P}^1$ (components intersect transversally), and in the dualized isomorphism

$$H_{2n}(U\times_{\mathcal{P}}\mathcal{B}^A)\xrightarrow{\ \sim\ }H_2(D\times_{\mathcal{P}}\mathcal{B}^A)\,,$$

one has

$$[U\times_{\phi(C)}C]\longmapsto[D\times\{\infty\}]\,,\qquad[U'\times\boldsymbol{P}^1]\longmapsto[\{x_0\}\times\boldsymbol{P}^1]\,.$$

Hence it suffices to show that

$$s[D\times\{\infty\}]=-[D\times\{\infty\}]-(\mathrm{ord}_{x_0}\beta|_D)[\{x_0\}\times\boldsymbol{P}^1]$$

in $H_2(D\times_{\mathcal{P}}\mathcal{B}^A)$.

2.10. We thus have to look at the local model on $D\times GL_2/T\to D$ with a given $\beta: D\to\mathfrak{N}$, $\beta^{-1}(0)=\{x_0\}$ where D is the strict localization of $\boldsymbol{A}^1$ at x_0. Now we shall check the above formula in 2.9 for the "homology"

$$H_2(X^\beta)\cong H_2(D\times_{\mathcal{P}}\mathcal{B}^A)\,,$$

where $X^\beta=D\times\{\infty\}\cup\{x_0\}\times\boldsymbol{P}^1\subset D\times\boldsymbol{P}^1$. First one can factorize $\beta: D\to\mathfrak{N}$ as $\beta=\beta_1\circ\gamma\,;\,D\xrightarrow{\gamma}D\xrightarrow{\beta_1}\mathfrak{N}$ such that $\mathrm{ord}_{x_0}\beta_1=1$ and the degree of γ at x_0 is $\mathrm{ord}_{x_0}\beta$. Then in the s-isomorphism

$$\gamma_* : H_2(X^\beta) \xrightarrow{\sim} H_2(X^{\beta_1}),$$

one has

$$\gamma_*([D\times\{\infty\}])=(\mathrm{ord}_{x_0}\beta)[D\times\{\infty\}]$$

$$\gamma_*([\{x_0\}]\times \boldsymbol{P}^1])=[\{x_0\}\times \boldsymbol{P}^1].$$

Hence if the formula is shown in the case $\mathrm{ord}_{x_0}\beta=1$, it holds in all cases.

It can be checked by a direct computation that for $G=GL_3$, A a subregular nilpotent, $\mathcal{B}^A$ a Dynkin curve $\boldsymbol{P}^1\cup\boldsymbol{P}^1$ intersecting transversally at a point x_0, a localization of one component at x_0 gives the β with $\mathrm{ord}_{x_0}\beta=1$. Thus for $\beta: D\to\mathfrak{N}$ with $\mathrm{ord}_{x_0}\beta=1$, our model is isomorphic to the Dynkin curve for GL_3. On the other hand, in this case, $H_2(\mathcal{B}^A)\hookrightarrow H_2(\mathcal{B})$ and the fundamental cycles are given by the Schubert cycles ([6]). Therefore, by using the formula for $H_2(\mathcal{B})$ given in [3], our formula can be checked in this case. Hence the lemma.

2.11. REMARK. Our method goes well also for Kazhdan-Lusztig's construction [7] rather easilier. In particular, the formula in 2.9 can be deduced directly from their definition (i.e., without using GL_3-model), which is an amusing exercise. As a result, one can identify their representation with Springer's one for the top cohomology.

THEOREM 2. *For a nilpotent A, let $n=\dim\mathcal{B}^A$. For a fixed Borel subgroup B, consider Springer's representation of the Weyl group W on the homology group $H_{2n}(\mathcal{B}^A)$. Let I be the set of irreducible components of $\mathcal{B}^A$ and $\{[C];C\in I\}$ be the basis of $H_{2n}(\mathcal{B}^A)$ consisting of fundamental cycles. Let $s\in W$ be a simple reflexion for B and $\phi:\mathcal{B}=G/B\to G/P$ be the corresponding $\boldsymbol{P}^1$-bundle for the parabolic subgroup $P=B\cup BsB$. Then the action of s on $\{[C];C\in I\}$ is described as follows:*

(i) *$s[C]=[C]$ if and only if* $\dim\phi(C)=n-1$.

(ii) *Assume* $\dim\phi(C)=n$. *For $C'\in I$ such that $\phi(C')\subsetneqq\phi(C)$, put*

$$n_C^{C'}(s)=\sum_i d_i(C',C)\nu_i(C',C)$$

where $\nu_i(C',C)$ and $d_i(C',C)$ are the natural numbers given in Lemma 1 (2.4) *and in Lemma* 2 (2.8) *respectively. Then*

$$s[C]=-[C]-\sum_{C'}n_C^{C'}(s)[C'].$$

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