On unitarizable highest weight modules of Hermitian pairs

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Dedicated to the memory of Takuro Shintani

§ 0. Introduction.

We let $\mathfrak{g}$ denote a real, simple Lie algebra with a fixed Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, and we assume $\mathfrak{k}$ has a one-dimensional center, so that $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair. We assume $\pi: \mathfrak{g}_C \rightarrow \mathrm{End}\, V$ is a highest weight representation in the sense of the first paragraph of § 2, (2.1), below. In this paper, we first give a necessary and sufficient condition, in terms of the weights of π, that π be unitarizable (see Theorem 3.1), below. Then we apply this criterion, using a result obtained independently by R. Parthasarathy and Zuckerman (Theorem 4.2, below) to obtain a sufficient condition on the highest weight of π, which guarantees π is unitarizable. This last result is Theorem 4.5, below.

We emphasize that our proofs are *entirely* algebraic. This claim depends, in particular, on the existence of an algebraic proof of unitarity for irreducible finite-dimensional representations of compact, semi-simple Lie algebras. Such an argument is given in [4], Theorem 12.1, for loop algebras and that argument translates directly to the compact, semi-simple case.

In part, the argument in [4] inspired the unitarity result given here for Hermitian pairs. A second inspiration was the paper [10] of R. Parthasarathy. In [10], a necessary and sufficient condition for unitarizability is given for a certain family of highest weight modules. Our initial idea was to see if we could prove the sufficiency (Theorem B of [10]) without involving the spin representation, as in [10], but rather, by relying on an argument similar to that in [4]. This strategy eventually succeeded, but we had to find a substitute for Cartan-Weyl theory, used in [4]. This led us to use Theorem 4.2 in order to prove the sufficiency condition in Theorem 4.5, below. Our final result, Theorem 4.5, below, is somewhat stronger than Parthasarathy's, in that we need not assume we have a nondegenerate infinitesimal character and in that we can relax his integrality condition.

There are several algebraic treatments, in special cases, for the unitarity of highest weight modules of Hermitian symmetric pairs (see, e.g. [7], [8],

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[13]). Also, after completing this work, we learned that Theorem 3.1 was also obtained by Enright, Howe, and Wallach, and used by them in their classification of unitarizable highest weight modules [1].

§ 1. Preliminaries concerning Hermitian pairs.

We let $\mathfrak{g}$ denote a real, simple Lie algebra and we fix a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Thus, if $(\,,)$ denotes the Cartan-Killing form of $\mathfrak{g}$, then $(\,,)$ restricted to $\mathfrak{k}$ is negative-definite, and restricted to $\mathfrak{p}$ is positive-definite. Moreover, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to $(\,,)$. In this paper, *we assume throughout, that* $\mathfrak{k}$ *has a one-dimensional center.* Thus $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair (see [6]).

Whenever $\mathfrak{a}\subseteq\mathfrak{g}$ is a subspace, we let $\mathfrak{a}_C\subseteq\mathfrak{g}_C$ denote the complexification of $\mathfrak{a}$. We fix a maximal torus $\mathfrak{t}\subseteq\mathfrak{k}$ (so $\mathfrak{t}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$), and we let Δ denote the set of roots in $\mathfrak{t}_C'$, the dual space of $\mathfrak{t}_C$. Then we have the root space decomposition

$$\mathfrak{g}_C=\mathfrak{t}_C\oplus\amalg_{\alpha\in\Delta}\mathfrak{g}^{\alpha}\,,$$

where for $\alpha\in\Delta$, $\mathfrak{g}^\alpha$ is the *root space*

$$\mathfrak{g}^\alpha=\{\xi\in\mathfrak{g}_C\,|\,[t,\xi]=\alpha(t)\xi,\ t\in\mathfrak{t}_C\}\,.$$

Now, $[\mathfrak{k}_C, \mathfrak{p}_C]\subseteq\mathfrak{p}_C$, and so, á fortiori $[\mathfrak{t}_C, \mathfrak{p}_C]\subseteq\mathfrak{p}_C$, and of course $[\mathfrak{t}_C, \mathfrak{k}_C]\subseteq\mathfrak{k}_C$, since $\mathfrak{k}_C$ is a subalgebra of $\mathfrak{g}_C$. It then follows that each $\mathfrak{g}^\alpha$ is contained in either $\mathfrak{k}_C$ or in $\mathfrak{p}_C$. In the former case we say α is a *compact root*, and in the latter case, we say α is a *non-compact root*. We let $\Delta_\mathfrak{k}$ denote the compact roots, and $\Delta_\mathfrak{p}$ the non-compact roots. We then have

$$\Delta=\Delta_\mathfrak{k}\cup\Delta_\mathfrak{p}\quad\text{(disjoint union)}.$$

We let $\mathfrak{q}$ denote the *real subspace* $\mathfrak{q}=i\mathfrak{g}$ in $\mathfrak{g}_C$. Then, of course, $\mathfrak{g}_C$ is the direct sum of real subspaces

$$\mathfrak{g}_C=\mathfrak{q}+i\mathfrak{q}\,,$$

and we let * denote conjugation with respect to this direct sum decomposition. Thus, if $\xi=q_1+iq_2$, $q_1, q_2\in\mathfrak{q}$, then we set $\xi^*=q_1-iq_2$. It is easily checked that * is conjugate linear, and is a Lie algebra anti-automorphism in the sense that

$$[\xi_1^*, \xi_2^*]=[\xi_2, \xi_1]^*,\quad \xi_1, \xi_2\in\mathfrak{g}_C\,. \tag{1.1}$$

Incidentally, the fact that * is bijective is obvious from the definition. In fact, it is also obvious from the definition that * is involutive in the sense that $(\xi^*)^*=\xi$, for all $\xi\in\mathfrak{g}_C$.

If $t\in\mathfrak{t}\subseteq i\mathfrak{q}$, then $t^*=-t$, and it follows from this and from (1.1) that

(1.2) $$(\mathfrak{g}^{\alpha})^* = \mathfrak{g}^{-\alpha}, \quad \alpha \in \Delta.$$

Indeed, if $e \in \mathfrak{g}^{\alpha}$, we have for $t \in \mathfrak{t}$,

$$\begin{aligned}[t, e^*] &= [e, t^*]^* \\ &= [e, -t]^* = [t, e]^* \\ &= \overline{\alpha(t)} e^* = -\alpha(t) e^*,\end{aligned}$$

since $\alpha(t)$ is pure imaginary for $t \in \mathfrak{t}$. We can further refine (1.2).

LEMMA 1.3. *We may choose a set of nonzero root vectors* $\{e_{\alpha}\}_{\alpha \in \Delta}$, $e_{\alpha} \in \mathfrak{g}^{\alpha}$ *for each* $\alpha \in \Delta$, *so that*

$$e_{\alpha}^* = e_{-\alpha}, \ (e_{\alpha}, e_{-\alpha}) = 1, \quad \alpha \in \Delta_{\mathfrak{k}}.$$

$$e_{\alpha}^* = -e_{-\alpha}, \ (e_{\alpha}, e_{-\alpha}) = 1, \quad \alpha \in \Delta_{\mathfrak{p}}.$$

PROOF. It follows from (1.2) that if $e_{\alpha} \in \mathfrak{g}^{\alpha}$, then $e_{\alpha}^* \in \mathfrak{g}^{-\alpha}$, and hence it suffices to prove that we can choose the family $\{e_{\alpha}\}_{\alpha \in \Delta}$ so that

$$(e_{\alpha}, e_{\alpha}^*) = 1, \quad \alpha \in \Delta_{\mathfrak{k}}; \quad (e_{\alpha}, e_{\alpha}^*) = -1, \quad \alpha \in \Delta_{\mathfrak{p}}.$$

If $x_{\alpha} \in \mathfrak{g}^{\alpha}$, then for $c \in \boldsymbol{C}$,

$$(c x_{\alpha}, (c x_{\alpha})^*) = |c|^2 (x_{\alpha}, x_{\alpha}^*),$$

and so it suffices to show that if $x_{\alpha} \in \mathfrak{g}^{\alpha}$, $x_{\alpha} \neq 0$, then

(1.4) $$\begin{aligned}(x_{\alpha}, x_{\alpha}^*) &> 0, \quad \alpha \in \Delta_{\mathfrak{k}}, \\ (x_{\alpha}, x_{\alpha}^*) &< 0, \quad \alpha \in \Delta_{\mathfrak{p}}.\end{aligned}$$

However,

$$x_{\alpha} + x_{\alpha}^* \in \mathfrak{q}, \quad i(x_{\alpha} - x_{\alpha}^*) \in \mathfrak{q},$$

and so

$$\begin{aligned}i(x_{\alpha} + x_{\alpha}^*) &\in \mathfrak{k} \quad \text{if} \quad \alpha \in \Delta_{\mathfrak{k}}, \\ i(x_{\alpha} + x_{\alpha}^*) &\in \mathfrak{p} \quad \text{if} \quad \alpha \in \Delta_{\mathfrak{p}}.\end{aligned}$$

Thus, if $\alpha \in \Delta_{\mathfrak{k}}$,

$$\begin{aligned}0 &> (i(x_{\alpha} + x_{\alpha}^*), \ i(x_{\alpha} + x_{\alpha}^*)) \\ &= -2(x_{\alpha}, x_{\alpha}^*).\end{aligned}$$

If $\alpha \in \Delta_{\mathfrak{p}}$, we have similarly,

$$0 < -2(x_{\alpha}, x_{\alpha}^*).$$

Thus, we obtain (1.4), and hence, as we have noted, the lemma. ∎

We introduce an ordering on the roots Δ, such that $\alpha\in\Delta$ is positive if $\alpha(t_c)>0$, where $t_c\in\mathfrak{t}$ is some fixed nonzero element of $\mathfrak{i}$ (=center of $\mathfrak{k}$). We let $\Delta_\pm$ denote the corresponding set of $\pm$ roots, and we let

$$\mathfrak{p}_\pm=\coprod_{\alpha\in\Delta_{\mathfrak{p},\pm}}\mathfrak{g}^\alpha,$$

where

$$\Delta_{\mathfrak{p},\pm}=\Delta_\mathfrak{p}\cap\Delta_\pm,$$

$$\Delta_{\mathfrak{k},\pm}=\Delta_\mathfrak{k}\cap\Delta_\pm,$$

by definition. Then

$$[\mathfrak{p}_+,\mathfrak{p}_+]=[\mathfrak{p}_-,\mathfrak{p}_-]=0.$$

We let $\mathcal{U}(\mathfrak{g}_C)$ denote the universal enveloping algebra of $\mathfrak{g}_C$. We fix a set of root vectors $\{e_\alpha\}_{\alpha\in\Delta}$ as in Lemma 1.3, and we fix an orthonormal basis $h_1,\cdots,h_l$ ($l=\dim_R\mathfrak{t}$) of $\mathfrak{t}_C$. Then we can write the Casimir element $C\in\mathcal{U}(\mathfrak{g}_C)$, corresponding to the Cartan-Killing form, as

$$C=\sum_{i=1}^l h_i^2+\sum_{\alpha\in\Delta}e_{-\alpha}e_\alpha. \tag{1.5}$$

We let

$$\mathfrak{b}=\mathfrak{t}_C\oplus\coprod_{\alpha\in\Delta_+}\mathfrak{g}^\alpha,$$

so $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}_C$, and has the property that

$$\begin{aligned}\mathfrak{b}&=(\mathfrak{b}\cap\mathfrak{k}_C)\oplus(\mathfrak{b}\cap\mathfrak{p}_C)\\&=\mathfrak{b}_\mathfrak{k}+\mathfrak{p}_+,\end{aligned}$$

where $\mathfrak{b}_\mathfrak{k}=\mathfrak{b}\cap\mathfrak{k}_C$, by definition. We say more generally that if $\mathfrak{s}$ is any subspace of $\mathfrak{g}_C$, then $\mathfrak{s}$ is θ-stable if $\theta\mathfrak{s}=\mathfrak{s}$, where θ is the automorphism of $\mathfrak{g}_C$ which is $+1$ on $\mathfrak{k}_C$ and -1 on $\mathfrak{p}_C$. Then, $\mathfrak{s}$ is θ-stable if and only if

$$\mathfrak{s}=(\mathfrak{s}\cap\mathfrak{k}_C)\oplus(\mathfrak{s}\cap\mathfrak{p}_C).$$

§ 2. Highest weight modules.

Let V denote a (possibly infinite-dimensional) vector space over C, and let

$$\pi:\mathfrak{g}_C\to\operatorname{End}V$$

be a Lie algebra representation. We will *always assume* that: (See also [5].)

(2.1) (i) π is irreducible.

(ii) V is a direct sum of finite-dimensional $\mathfrak{k}_C$-modules.

(iii) π is a highest weight representation; i.e., there exists $v_0\in V$, $v_0\neq0$, and there exists $\mu\in\mathfrak{t}_C'$ such that

$$\pi(t)\cdot v_0=\mu(t)v_0\,, \qquad t\in\mathfrak{t}_C\,,$$

$$\mu(\mathfrak{t}_c) \text{ is real, and}$$

$$\pi(e_\alpha)\cdot v_0=0 \qquad \text{if} \quad \alpha\in\Delta_{\mathfrak{k},+} \quad \text{or} \quad \alpha\in\Delta_{\mathfrak{p},-}\,.$$

We note that $*: \mathfrak{g}_C\to\mathfrak{g}_C$ extends to a conjugate-linear involution (still denoted by $*$) of $\mathcal{U}(\mathfrak{g}_C)$, such that

$$(u_1u_2)^*=u_2^*u_1^*, \quad u_1, u_2\in\mathcal{U}(\mathfrak{g}_C)\,.$$

We also extend π to a representation of $\mathcal{U}(\mathfrak{g}_C)$ (still denoted by π). It is then known (see [13]) that V admits a non-trivial, Hermitian inner product $\{,\}$, (not necessarily positive-definite), such that

$$\{\pi(u)v_1, v_2\}=\{v_1, \pi(u^*)v_2\}, \quad v_1, v_2\in V,\ u\in\mathcal{U}(\mathfrak{g}_C)\,. \tag{2.2}$$

We may reformulate (2.2) as asserting that $\pi(u)^*=\pi(u^*)$, $u\in\mathcal{U}(\mathfrak{g}_C)$, where $\pi(u)^*$ denotes the Hermitian-adjoint of $\pi(u)$, with respect to $\{,\}$ (in particular, (2.2) asserts that $\pi(u)^*$ is well-defined).

If $\nu\in\mathfrak{t}_C'$, then we set

$$V_\nu=\{v\in V\,|\,\pi(t)\cdot v=\nu(t)v,\ t\in\mathfrak{t}_C\}\,.$$

If $V_\nu\neq 0$, then we call ν a weight of V or of π, and V_ν the corresponding weight space. The following facts are easily deduced from (2.1).

(2.3) (i) $V=\amalg_i W_i$ (direct sum) where i ranges over some index set, and each W_i is an irreducible $\mathfrak{k}_C$-module.

(ii) The space V is a direct sum

$$V=\amalg_\nu V_\nu$$

of weight spaces, and each weight space V_ν is finite dimensional, and is a direct sum of its intersections with the subspaces W_i.

(iii) Each weight ν is of the form $\nu=\mu-\beta_{\mathfrak{k}}+\beta_{\mathfrak{p}}$, where $\beta_{\mathfrak{k}}$ is a sum of positive compact roots, and $\beta_{\mathfrak{p}}$ is a sum of positive, noncompact roots.

We may take $\Delta_{\mathfrak{k},-}\cup\Delta_{\mathfrak{p},+}$ as a new set of positive roots, and in this case, take $\alpha_1, \cdots, \alpha_l$ to be the corresponding simple roots.

DEFINITION 2.3. For a weight ν of π, we let $\mathrm{ht}(\nu)$ (the "height" of ν) be defined by

$$\mathrm{ht}(\nu)=n\,, \quad n=\textstyle\sum_{i=1}^l n_i\,,$$

where

$$\nu=\mu+\textstyle\sum_{i=1}^l n_i\alpha_i\,, \quad \text{each } n_i \text{ a nonnegative integer} \tag{2.4}$$

(note that every weight ν of π is of the form (2.4), thanks to (2.3), (iii), and our choice of simple roots).

LEMMA 2.5. (i) *If ν, ν' are two distinct weights of π, then V_ν and $V_{\nu'}$ are perpendicular with respect to* $\{,\}$.

(ii) *If W_1 and W_2 are two inequivalent, irreducible $\mathfrak{k}_C$-submodules of V, then W_1 and W_2 are perpendicular with respect to* $\{,\}$.

PROOF. If $t\in i\mathfrak{t}$ is chosen so that $\nu(t)\neq\nu'(t)$, then $t^*=t$, and for $v\in V_\nu$, $v'\in V_{\nu'}$, we have

$$\nu(t)\{v, v'\}=\{\pi(t)v, v'\}$$

$$=\{v, \pi(t)v'\}=\nu'(t)\{v, v'\},$$

and since $\nu(t)\neq\nu'(t)$, we conclude

$$\{v, v'\}=0.$$

This proves (i).

If W_i has highest weight $\nu_i\in\mathfrak{t}_C'$, and if $v_i\in W_i$ is a highest weight vector ($i=1, 2$), then (i) implies

$$\{v_1, v_2\}=0.$$

If $w_1\in W_1$ is any vector, however, then if $\mathfrak{u}_\mathfrak{k}^-=\coprod_{\alpha\in\Delta_{\mathfrak{k},-}}\mathfrak{g}^\alpha$ (so $\mathfrak{u}_\mathfrak{k}^-$ is a maximal nilpotent subalgebra of $\mathfrak{k}_C$), then there exists $\xi\in\mathcal{U}(\mathfrak{u}_\mathfrak{k}^-)$ so that

$$w_1=\pi(\xi)\cdot v_1.$$

Moreover $(\mathfrak{u}_\mathfrak{k}^-)^*=\mathfrak{u}_\mathfrak{k}^+$, and thus

$$\{w_1, v_2\}=\{v_1, \pi(\xi^*)v_2\}$$

$$=c\{v_1, v_2\}, \qquad c\in \boldsymbol{C},$$

since v_2 is a highest weight vector of W_2. However $\{v_1, v_2\}=0$, as we noted earlier. Hence, v_2 is orthogonal to W_1. But then, since $W_1^\perp\cap W_2$ is a $\mathfrak{k}_C$-submodule of the irreducible $\mathfrak{k}_C$-module W_2, we have $W_1^\perp\supseteq W_2$, and this proves (ii). ∎

If $\nu\in\mathfrak{t}_C'$, and

$$\frac{2(\nu, \alpha)}{(\alpha, \alpha)}\geqq 0, \quad \frac{2(\nu, \alpha)}{(\alpha, \alpha)}\in \boldsymbol{Z},$$

for all $\alpha\in\Delta_{\mathfrak{k},+}$ (in this case we say that ν is $\mathfrak{k}$-dominant), then by the $\mathfrak{k}$-type of V corresponding to ν, we mean the sum of all irreducible $\mathfrak{k}$-submodules, with highest weight ν. When we speak of such a $\mathfrak{k}$-type, it will be understood that it is non-zero.

§3. A necessary and sufficient condition for unitarity.

We let ρ (resp., $\rho_{\mathfrak{k}}$; resp., $\rho_{\mathfrak{p}}$) denote one-half the sum of the roots in Δ_+ (resp., in $\Delta_{\mathfrak{k},+}$; resp., in $\Delta_{\mathfrak{p},+}$). We let $(\,,\,)$ denote the symmetric bilinear form on $\mathfrak{t}'_C$, corresponding to the Cartan-Killing form, and note that $(\,,\,)$ is then positive-definite on the $\boldsymbol{R}$-span of the roots. We let $\|\;\|$ denote the corresponding norm. We then have:

THEOREM 3.1. *If* (π, V) *is a highest weight module as in* (2.1), *with highest weight* μ *and highest weight vector* v_0, *then a nontrivial Hermitian inner product* $\{,\}$, *such that* $\{v_0, v_0\}>0$, *and such that* (2.2) *holds for* $\{,\}$, *is positive-definite if and only if for every weight* $\nu\neq\mu$ *which is the highest weight of a* $\mathfrak{k}$*-type, we have*

$$\|\nu+\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2>\|\mu+\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2 . \tag{3.2}$$

PROOF. Assume that (3.2) holds for every weight $\nu\neq\mu$ which is the highest weight of a $\mathfrak{k}$-type. Then, in order to prove $\{,\}$ is positive definite, it suffices to prove $\{,\}$ restricted to each $\mathfrak{k}$-type of V is positive definite by (ii) of Lemma 2.5. But then we argue by induction on the height of the highest weight ν of the $\mathfrak{k}$-type. Indeed, for ν of height zero, we have $\nu=\mu$, and the corresponding $\mathfrak{k}$-type has multiplicity one. Since $\{v_0, v_0\}>0$, by assumption, we deduce by the $\mathfrak{k}$-invariance of $\{,\}$ (which follows from 2.2) that $\{,\}$ is positive-definite on the $\mathfrak{k}_C$-module generated by v_0. (Incidentally, this can be proved algebraically, using the same type of argument we are developing now for Hermitian pairs, combined with Cartan-Weyl theory: see [4], where precisely this argument is used in the case of loop algebras.)

Now assume $\{,\}$ is positive-definite on all $\mathfrak{k}$-types with highest weight having height $<m$. Assume ν is the highest weight of a $\mathfrak{k}$-type, and $\mathrm{ht}(\nu)=m$. To complete the induction, it will suffice to prove $\{w, w\}>0$, whenever w is a linear combination of highest weight vectors of the $\mathfrak{k}$-type corresponding to ν. (Exactly as for the case $\nu=\mu$, above, it suffices to restrict attention to such w.)

So, let w be such an element of V_ν. Then

$$\pi(e_\alpha)\cdot w=0\,,\qquad \alpha\in\Delta_{\mathfrak{k},+}\,. \tag{3.3}$$

From the expression (1.5) for C, we have

$$C=C_1+C_2\,, \tag{3.4}$$

where

$$C_1=2\sum\nolimits_{\alpha\in\Delta_{\mathfrak{k},+}}e_{-\alpha}e_\alpha+2\sum\nolimits_{\alpha\in\Delta_{\mathfrak{p},+}}e_\alpha e_{-\alpha}\,, \tag{3.5}$$

and where

$$C_2=\sum_{i=1}^{l} h_i^2+2h_{\rho_{\mathfrak{k}}}-2h_{\rho_{\mathfrak{p}}}. \tag{3.6}$$

In (3.6), $h_{\rho_{\mathfrak{k}}}$ (resp., $h_{\rho_{\mathfrak{p}}}$) is the element of $\mathfrak{t}_C$ corresponding to $\rho_{\mathfrak{k}}$ (resp., to $\rho_{\mathfrak{p}}$), when we identify $\mathfrak{t}_C$ with $\mathfrak{t}_C'$ by means of the Killing form.

We then consider

$$\{\pi(C)w,\ w\},$$

and evaluate this expression in two different ways, using (3.4)-(3.6), and (3.3).

First, since π is a highest weight module, $\pi(C)$ is a scalar multiple of the identity operator, and to find this scalar, it suffices to compute $\pi(C)\cdot v_0$ for the highest weight vector v_0. But

$$\pi(C_1)\cdot v_0=0,$$

and thus

$$\pi(C)\cdot v_0=\pi(C_2)v_0=(\|\mu+\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2-\|\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2)v_0.$$

Thus

$$\{\pi(C)w,\ w\}=(\|\mu+\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2-\|\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2)\{w,\ w\}. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}\{\pi(C_1)w,\ w\}&=2\sum_{\alpha\in\Delta_{\mathfrak{k},+}}\{\pi(e_{-\alpha})\pi(e_\alpha)w,\ w\}+2\sum_{\alpha\in\Delta_{\mathfrak{p},+}}\{\pi(e_\alpha)\pi(e_{-\alpha})w,\ w\}\\&=-2\sum_{\alpha\in\Delta_{\mathfrak{p},+}}\{\pi(e_{-\alpha})w,\ \pi(e_{-\alpha})w\},\end{aligned}$$

by Lemma 1.3, by (2.2), and by the assumption that w is a linear combination of highest weight vectors of a $\mathfrak{k}$-type. By induction, this last expression is negative, i.e. we have

$$\{\pi(C_1)w,\ w\}<0. \tag{3.8}$$

(At least one of the above summands $\{\pi(e_{-\alpha})w,\ \pi(e_{-\alpha})w\}$, $\alpha\in\Delta_{\mathfrak{p},+}$, is strictly positive, since w is not a highest weight vector.) On the other hand,

$$\{\pi(C_2)w,\ w\}=(\|\nu+\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2-\|\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2)\{w,\ w\},$$

and using (3.8), we thus obtain

$$\{\pi(C)w,\ w\}<(\|\nu+\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2-\|\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}\|^2)\{w,\ w\}. \tag{3.9}$$

But it follows from this, from (3.7), and from (3.2), that

$$\{w,\ w\}>0.$$

Finally, we note that, if conversely, we assume $\{,\}$ is positive-definite, then we obtain (3.8), and hence (3.9). Since (3.7) holds in any case, we then obtain (3.2), for every weight $\nu\neq\mu$ which is the highest weight of a $\mathfrak{k}$-type, from the fact that $\{w,\ w\}>0$. ■

§4. A sufficient condition for unitarity.

Let $\mathfrak{r}\supseteqq\mathfrak{b}$ be any proper parabolic subalgebra of $\mathfrak{g}_C$, containing $\mathfrak{b}$. Let $\mathfrak{u}_\mathfrak{r}$ denote the unipotent radical of $\mathfrak{r}$, and let $\Delta_\mathfrak{r}$ denote the set of all roots $\alpha\in\Delta_+$, such that $\mathfrak{g}^\alpha\subset\mathfrak{u}_\mathfrak{r}$. We let

$$\Delta_{\mathfrak{r},\mathfrak{p}}=\Delta_\mathfrak{r}\cap\Delta_\mathfrak{p},$$

and note that $\Delta_{\mathfrak{r},\mathfrak{p}}=\Delta_\mathfrak{r}\cap\Delta_{\mathfrak{p},+}\subseteqq\Delta_{\mathfrak{p},+}$.

DEFINITION 4.1. We say that the highest weight module V, with highest weight μ, satisfies the $\mathfrak{r}$-cone condition, if for every $\mathfrak{k}$-type of V, the highest weight ν of that $\mathfrak{k}$-type is of the form

$$\nu=\mu+\beta,$$

where β is a sum of roots in $\Delta_{\mathfrak{r},\mathfrak{p}}$.

We let $\rho_{\mathfrak{r},\mathfrak{k}}$ denote one-half the sum of the roots in $\Delta_{\mathfrak{r},\mathfrak{k}}=\Delta_\mathfrak{r}-\Delta_{\mathfrak{r},\mathfrak{p}}$.

THEOREM 4.2 (Parthasarathy-Zuckerman). *If V is a highest weight module in the sense of* 2.1, *with highest weight μ, and if for all $\alpha\in\Delta_{\mathfrak{r},\mathfrak{p}}$, one has*

$$(\mu+\rho_\mathfrak{k}-\rho_\mathfrak{p},\ \alpha)\geqq 0, \tag{4.3}$$

and if for all $\alpha\in\Delta_+-\Delta_\mathfrak{r}$, one has

$$(\mu+2\rho_{\mathfrak{r},\mathfrak{k}},\ \alpha)=0, \tag{4.4}$$

then V satisfies the $\mathfrak{r}$-cone condition.

THEOREM 4.5 (See also [10].) *If V is a highest weight module with highest weight μ, μ satisfying* (4.3) *and* (4.4), *then V is unitarizable* (*as a $\mathfrak{g}$-module*).

PROOF. Let $\{,\}$ be a non-trivial, Hermitian inner product on V which satisfies (2.2), and such that $\{v, v\}>0$ for some $v\in V$. By Theorem 4.2, we then know that if ν is the highest weight of a $\mathfrak{k}$-type, then

$$\nu=\mu+\beta,$$

where β is a sum of roots in $\Delta_{\mathfrak{r},\mathfrak{p}}$. But then,

$$\begin{aligned}&\|\nu+\rho_\mathfrak{k}-\rho_\mathfrak{p}\|^2\\ &=\|\mu+\rho_\mathfrak{k}-\rho_\mathfrak{p}\|^2+2(\mu+\rho_\mathfrak{k}-\rho_\mathfrak{p},\ \beta)+(\beta,\ \beta)\\ &\geqq\|\mu+\rho_\mathfrak{k}-\rho_\mathfrak{p}\|^2+(\beta,\ \beta),\quad \text{by (4.3)}\\ &>\|\mu+\rho_\mathfrak{k}-\rho_\mathfrak{p}\|^2,\qquad \text{if } \mu\neq\nu, \text{ so that } \beta\neq 0.\end{aligned}$$

Hence, (3.2) holds, and thus by Theorem 3.1, $\{,\}$ is positive-definite. ∎

§ 5. Proof of Theorem 4.2.

We first observe

LEMMA 5.1. *Suppose there exists a* $\mathcal{U}(\mathfrak{g}_C)$*-module* A *such that*

(i) *A is a direct sum of finite dimensional irreducible* $\mathfrak{k}_C$*-modules.*

(ii) *The* $\mathfrak{k}$*-type with highest weight* μ *occurs with multiplicity one in* A.

(iii) *For every* $\mathfrak{k}$*-type of* A*, the highest weight* ν *is of the form* $\nu=\mu+\beta$*, where* β *is a sum of roots in* $\Delta_{\mathfrak{r},\mathfrak{p}}$.

Then the highest weight module V *with highest weight* μ *is a subquotient of* A*, and hence* V *satisfies the* $\mathfrak{r}$*-cone condition.*

PROOF. Let $A(\mu)$ be the $\mathfrak{k}$-type of A corresponding to the irreducible $\mathfrak{k}$-module W_μ with highest weight μ. (The definition of $\mathfrak{k}$-type is given after (3.2)). The $\mathfrak{g}_C$-submodule $\mathcal{U}(\mathfrak{g}_C)A(\mu)$ in A contains W_μ with multiplicity one. Hence, every proper $\mathfrak{g}_C$-submodule of $\mathcal{U}(\mathfrak{g}_C)A(\mu)$ does not contain W_μ. Let B be the sum of all the proper $\mathfrak{g}_C$-submodules of $\mathcal{U}(\mathfrak{g}_C)A(\mu)$. Then B itself is proper and maximal, and the quotient $C=(\mathcal{U}(\mathfrak{g}_C)A(\mu))/B$ is nontrivial and irreducible over $\mathcal{U}(\mathfrak{g}_C)$.

By (iii), A, and hence C has no $\mathfrak{k}$-type with highest weight $\mu-\alpha$, $\alpha\in\Delta_{\mathfrak{p}}^{+}$. It follows that for the highest weight vector v_0 of C_μ, $\pi(e_\alpha)v_0=0$, for $\alpha\in\Delta_{\mathfrak{p},-}$. Thus, C is a highest weight module in the sense of (2.1) and C has highest weight μ. By a standard argument, we must have $C\cong V$. Finally, property (iii) implies the $\mathfrak{r}$-cone condition for $C\cong V$.

THEOREM 5.2. *Suppose* μ *satisfies conditions* (4.3) *and* (4.4) *of Theorem* 4.2. *Then there exists a module* A *satisfying conditions* (i), (ii), *and* (iii) *of Lemma* 5.1.

REMARKS. There are two proofs of Theorem 5.2: the first, found by R. Parthasarathy, is published in [9] and uses a construction generalizing the work of Enright and Varadarajan [3]; the second, found by Zuckerman, is unpublished [14], (but was presented in spring of 1978 to a seminar at the Institute for Advanced Study in Princeton, N. J.). This second proof uses an algebraic version of a construction in the (unpublished) Berkeley thesis of Wilfried Schmid (see [11]). Cone conditions were first discussed by R. Blattner and independently by Schmid, in his thesis (see also [12]).

Here is a sketch of Zuckerman's construction:

Write $\mathfrak{r}=\mathfrak{l}+\mathfrak{u}$, where $\mathfrak{l}$ is reductive and θ-stable, and $\mathfrak{u}=\mathfrak{u}_{\mathfrak{r}}$ is the nilradical ($\mathfrak{u}$ is also θ-stable). Then $\mathfrak{l}\cap\mathfrak{k}_C$ is a reductive subalgebra of $\mathfrak{k}_C$. Let $\mathcal{M}(\mathfrak{g}_C, \mathfrak{k}_C)$ be the category of $\mathfrak{g}_C$-modules which are direct sums of irreducible finite dimensional $\mathfrak{k}_C$-modules. Similarly, define the category $\mathcal{M}(\mathfrak{g}_C, \mathfrak{l}\cap\mathfrak{k}_C)$. For any $\mathfrak{g}_C$-module X, let $X^{(\mathfrak{k}_C)}$ be the sum in X of all irreducible, finite-dimensional $\mathfrak{k}_C$-submodules. Then $X^{(\mathfrak{k}_C)}$ is a $\mathfrak{g}_C$-submodule of X, and the rule $X\rightsquigarrow X^{(\mathfrak{k}_C)}$ defines a left exact functor S from $\mathcal{M}(\mathfrak{g}_C, \mathfrak{k}_C\cap\mathfrak{l})$ to $\mathcal{M}(\mathfrak{g}_C, \mathfrak{k}_C)$.

LEMMA 5.3. *Any object in $\mathcal{M}(\mathfrak{g}_C, \mathfrak{k}_C\cap\mathfrak{l})$ has a resolution by injective objects in $\mathcal{M}(\mathfrak{g}_C, \mathfrak{k}_C\cap\mathfrak{l})$.*

We can thus introduce the right derived functors R^iS, $i\geqq 0$ of the functor S: for any object X in $\mathcal{M}(\mathfrak{g}_C, \mathfrak{k}_C\cap\mathfrak{l})$, let I^* be an injective resolution. Then SI^* is a complex of objects in $\mathcal{M}(\mathfrak{g}_C, \mathfrak{k}_C)$. Define

$$R^iSX=H^i(SI^*)\,. \tag{5.4}$$

For our purposes, the key general fact about the "derived functor" modules R^iSX is the following:

PROPOSITION 5.5. *For each irreducible $\mathfrak{k}$-type W_ν, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{k}_C}(W_\nu, R^iSX)\cong H^i(\mathfrak{k}_C, \mathfrak{k}_C\cap\mathfrak{l}, W_\nu'\otimes X)\,, \tag{5.6}$$

where W_ν' is the module dual to W_ν, and $H^*(\mathfrak{k}_C, \mathfrak{k}_C\cap\mathfrak{l}, -)$ is the relative Lie algebra cohomology functor.

Now let μ satisfy condition (4.4) of Theorem 4.2.

LEMMA 5.7. *There exists a unique one-dimensional $\mathfrak{r}$-module E such that $\mathfrak{t}_C$ acts through the linear functional*

$$\mu+2\rho_{\mathfrak{r},\mathfrak{k}}\,. \tag{5.8}$$

We now form the $\mathfrak{g}_C$-module

$$\tilde{X}=\mathrm{Hom}_{\mathcal{U}(\mathfrak{r})}(\mathcal{U}(\mathfrak{g}_C), E) \tag{5.9}$$

where the left action of $\mathfrak{g}_C$ on $\tilde{X}$ comes from the action of $\mathfrak{g}_C$ on $\mathcal{U}(\mathfrak{g}_C)$ via right multiplication.

Next, we pass to the $\mathfrak{g}_C$-module

$$X=\tilde{X}^{(\mathfrak{k}_C\cap\mathfrak{l})}\,. \tag{5.10}$$

Finally, let $d=\frac{1}{2}(\dim \mathfrak{k}_C-\dim(\mathfrak{k}_C\cap\mathfrak{l}))$.

Theorem 5.2 follows now from the statement:

(5.11) If μ satisfies both conditions (4.3) and (4.4) of Theorem 4.2, then the module $A=R^dSX$, for X as in (5.10), satisfies conditions (i), (ii), and (iii) of Lemma 5.1.

In fact, condition (i) of Lemma 5.1 follows from the definition of the derived functor module R^dSX. Conditions (ii) and (iii) of Lemma 5.1 follow from a technical argument using the isomorphism (5.6), as applied to the module X in (5.10). Here, only the structure of X as a $\mathfrak{k}_C$-module plays a role.

REMARKS. 1) A deeper argument using the $\mathfrak{g}_C$-module structure of X and a duality theorem for the derived functors R^*S (see [2]) leads to the sharper statement:

(5.12) If μ satisfies both conditions (4.3) and (4.4) of Theorem 4.2, then for X as in (5.10),

$$R^iSX=0 \quad \text{for} \quad i\neq d, \tag{5.13}$$

and R^dSX is irreducible as a $\mathfrak{g}_C$-module.

2) Parthasarathy [10] has proven the following result, which complements Theorem 4.5:

THEOREM 5.14. *If V is a unitary highest weight module with highest weight μ, and for every root α of $\mathfrak{t}_C$ in $\mathfrak{g}_C$,*

$$\frac{2(\mu+\rho_{\mathfrak{k}}-\rho_{\mathfrak{p}}, \alpha)}{(\alpha|\alpha)} \tag{5.15}$$

is a non-zero integer, then there exists a parabolic subalgebra $\mathfrak{r}\supseteqq\mathfrak{b}$ such that μ satisfies conditions (4.3) *and* (4.4) *relative to $\mathfrak{r}$.*

3) Combining 1) and 2) we see that if a highest weight module V is unitary and μ satisfies (5.15), then for some $\mathfrak{r}\supseteqq\mathfrak{b}$, V is a derived functor module R^dSX for X as in (5.10).

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