Analytic representations of SL_2 over a $\mathfrak{p}$-adic number field

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In memory of Professor Shintani, who was our friend and teacher

Introduction.

Let p be a prime number, and let $\boldsymbol{Q}_p$ be the p-adic number field. Let k be an algebraically closed field containing $\boldsymbol{Q}_p$ such that the standard p-adic valuation of $\boldsymbol{Q}_p$ can be extended to a valuation $|\ |$ of k and k is complete with respect to this valuation $|\ |$.

Let $\boldsymbol{P}^1(k)=k\cup\{\infty\}$ be the one dimensional projective space over k. Then the linear fractional group $PSL(2, k)$ acts on $\boldsymbol{P}^1(k)$ by

$$PSL(2, k)\ni\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \boldsymbol{P}^1(k)\ni z\longmapsto (az+c)/(bz+d)\in\boldsymbol{P}^1(k).$$

Hence subgroups of $PSL(2, k)$ can be regarded as transformation groups on $\boldsymbol{P}^1(k)$.

Let L be a finite extension of $\boldsymbol{Q}_p$ contained in k, and let $\mathfrak{o}$ be the integer ring of L. Let $G=SL(2, L)$ and $K=SL(2, \mathfrak{o})$. Then G is a locally compact group and K is a maximal compact open subgroup of G. Let $\mathfrak{G}=\{X\in M_2(L);\ \mathrm{tr}(X)=0\}$. Then $\mathfrak{G}$ can be regarded as the Lie algebra of G and K (cf. § 1). Let D be the complement in $\boldsymbol{P}^1(k)$ of $\boldsymbol{P}^1(L)=L\cup\{\infty\}$, and let V be the space of k-valued analytic functions on D (cf. [6]). Then V contains the space U of all rational functions $f(z)\in k(z)$ such that all poles of $f(z)$ belong to $\boldsymbol{P}^1(L)$. Further V can be regarded as the completion of U with respect to a countable number of semi-norms (cf. § 2).

For each negative integer s, put

$$\pi_s(g)f(z)=(bz+d)^s f((az+c)/(bz+d))$$

for any $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$ and $f(z)\in V$. Then π_s is a continuous representation of G on V, and U is a G-invariant subspace of V. We may say that these representations are the p-adic analogue of the holomorphic discrete series of $SL(2, \boldsymbol{R})$. Also, these representations seem to be closely related with the Schottky uniformization of degenerating curves (cf. [8], [3], [4], [1]).

Let U_s be the subspace of U consisting of all $f(z)\in U$ such that f has a partial fractional expansion of the form

$$f(z)=\sum_{m=0}^{\infty} d_m^{(\infty)} z^m+\sum_{j=1}^{l} \sum_{m=s}^{-\infty} d_m^{(j)}(z-b_j)^m \qquad \text{(a finite sum)}$$

with $b_j \in L$ and $d_*^{(*)} \in k$. Then we see that U_s is a closed G-invariant subspace of U. Let V_s be the closure of U_s in V.

The object of this paper is to study the continuous representation $\pi_s: G \to \mathrm{Aut}_k V$. We conjecture that (1) V_s and V/V_s are topologically irreducible G-modules, and that (2) no two of them for various s are topologically equivalent.

On the other hand, π_s induces two more important representations: One is the continuous representation of K on V, and the other is the algebraic representation of the pair $(\mathfrak{G}, K)$ on U. For these representations, we can prove the following results (cf. § 3):

(i) V and V_s are topologically indecomposable K-modules;

(ii) U_s and U/U_s are algebraically irreducible $(\mathfrak{G}, K)$-modules.

In particular, U_s and U/U_s are topologically irreducible K-modules. We also study the equivalence between the U_s and the U/U_s.

As for generalization of the representations π_s, we can construct certain infinite dimensional representations T_χ parametrized by locally analytic characters $\chi: L^\times \to k^\times$ in spaces of locally analytic functions on L. Our representation π_s can be obtained as the dual representation of one of such representations T_χ. The details about the representations T_χ will be published in a following paper.

§ 1. The p-adic Lie algebra $\mathfrak{G}$.

Let $\boldsymbol{Q}_p$ be the p-adic number field, and let k be an algebraically closed field containing $\boldsymbol{Q}_p$. We assume that the standard p-adic valuation of $\boldsymbol{Q}_p$ can be extended to a valuation $|\ |$ of k, and that k is complete with respect to $|\ |$. Let L be a finite extension of $\boldsymbol{Q}_p$ contained in k, let $\mathfrak{o}$ be the integer ring of L, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. Let $G=SL(2, L)$ and let $K=SL(2, \mathfrak{o})$. Then K is an open compact subgroup of G.

Let

$$\mathfrak{G}=\{X \in M_2(L)\,;\ \mathrm{tr}(X)=0\}.$$

Then $\mathfrak{G}$ becomes a Lie algebra with

$$\mathfrak{G} \times \mathfrak{G} \ni (X, Y) \longmapsto [X, Y]=XY-YX \in \mathfrak{G}.$$

We consider $\mathfrak{G}$ as the Lie algebra of G. Since K is an open subgroup of G, $\mathfrak{G}$ can be regarded also as the Lie algebra of K.

Let X be an element of $\mathfrak{G}$. Since k is an algebraically closed field containing L, there is $P\in SL(2, k)$ such that $P^{-1}XP$ has the form

$$\text{(i)}\quad \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad (\lambda\in k) \quad \text{or} \quad \text{(ii)} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let t be an element of L, and let

$$\exp(tX)=\sum_{n=0}^{\infty}\frac{(tX)^n}{n!} \qquad \text{in} \quad M_2(L).$$

Since

$$P^{-1}(tX)P=\begin{pmatrix} t\lambda & 0 \\ 0 & -t\lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix},$$

this series converges for $|t\lambda|<|p^{1/(p-1)}|$ in case (i), and converges for any t in case (ii). Furthermore, if this condition is satisfied, $\exp(tX)$ is an element of $M_2(L)\cap SL(2, k)=SL(2, L)=G$. Since

$$P^{-1}\exp(tX)P=\begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{-t\lambda} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

where e^z is defined by $e^z=\sum_{n=0}^{\infty} z^n/n!$, $\exp(tX)$ satisfies

$$|\mathrm{tr}(\exp(tX))-2|=|e^{t\lambda}+e^{-t\lambda}-2|=|(e^{t\lambda}-1)(e^{-t\lambda}-1)|<|p^{2/(p-1)}| \qquad \text{in case (i)}$$

and $|\mathrm{tr}(\exp(tX))-2|=0$ in case (ii). We observe that any element $g\in G$ satisfying $|\mathrm{tr}(g)-2|<|p^{2/(p-1)}|$ can be written as $g=\exp(Y)$ with $Y=\sum_{n=1}^{\infty}(g-1)^n(-1)^{n-1}/n\in\mathfrak{G}$. In particular, the image of the exponential map contains any sufficiently small principal congruence subgroup $K_n=\left\{g\in SL(2, \mathfrak{o});\ g\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ modulo } \mathfrak{p}^n\right\}$ of K.

Example. $\mathfrak{G}$ is generated by $X_+=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_-=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $Y=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For these elements of $\mathfrak{G}$, $\exp(tX)$ is given by: $\exp(tX_+)=\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\exp(tX_-)=\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ for any $t\in L$, and $\exp(tY)=\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ for $|t|<|p^{1/(p-1)}|$.

§ 2. The space V of analytic functions on $\boldsymbol{P}^1(k)\backslash\boldsymbol{P}^1(L)$.

Let k and L be as in § 1, and let $\boldsymbol{P}^1(k)$ and $\boldsymbol{P}^1(L)$ be the one dimensional projective space over k and over L, respectively. Let $D=\boldsymbol{P}^1(k)\backslash\boldsymbol{P}^1(L)$. Since $\boldsymbol{P}^1(L)$ is a compact subset of $\boldsymbol{P}^1(k)$, D is an open subset of $\boldsymbol{P}^1(k)$. Let $V=\mathcal{O}(\boldsymbol{P}^1(k)\backslash\boldsymbol{P}^1(L))$ be the space of k-valued analytic functions on D (see [6] for the definition and the proof of the following assertions).

Let $\{r_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers such that each r_n belongs to $|k^{\times}|$ and $\lim r_n=0$. Since $\boldsymbol{P}^1(L)$ is a compact subset of $\boldsymbol{P}^1(k)$, $\boldsymbol{P}^1(L)$ is covered by $\{z\in\boldsymbol{P}^1(k);\ |z|>r_n^{-1}\}$ and a finite number of open balls of the form $\{z\in k;\ |z-a|<r_n\}$ $(a\in L)$. We denote by E_n the complement in $\boldsymbol{P}^1(k)$ of this covering. Then

$$E_1\subset E_2\subset\cdots\subset E_n\subset\cdots\subset D \quad\text{and}\quad D=\cup E_n\,.$$

By our definition, each E_n has the form

$$\{z\in\boldsymbol{P}^1(k);\ |z|\leqq r_n^{-1},\ |z-a_i|\geqq r_n \quad (i=1,\cdots,l_n)\},$$

where $l_n\in\boldsymbol{N}$, $a_i\in L$, $|a_i|\leqq r_n^{-1}$ and $|a_i-a_j|>r_n$ $(i\neq j)$. Let $\mathcal{O}(E_n)$ be the space of k-valued analytic functions on E_n. Then $\mathcal{O}(E_n)$ is the set consisting of $f\colon E_n\to k$ such that f can be expanded on E_n in the form

$$(\mathrm{C}_n)\qquad f(z)=\sum_{m=0}^{\infty}c_m z^m+\sum_{i=1}^{l_n}\sum_{m=-1}^{-\infty}c_m^{(i)}(z-a_i)^m$$

with $c_m,\ c_m^{(i)}\in k$. Here we assume that this series converges on E_n. Hence $|c_m|r_n^{-m}\to 0$ $(m\to\infty)$ and $|c_m^{(i)}|r_n^m\to 0$ $(m\to-\infty)$. It is known that the coefficients c_m and $c_m^{(i)}$ are uniquely determined by f. For such an element f of $\mathcal{O}(E_n)$, let

$$\|f\|_n=\mathrm{Max}\Big(\mathop{\mathrm{Max}}_{0\leq m<\infty}|c_m|r_n^{-m},\ \mathop{\mathrm{Max}}_{\substack{1\leq i\leq l_n\\ -\infty<m\leq -1}}|c_m^{(i)}|r_n^m\Big).$$

Then $\|\ \|_n$ is a norm of the k-vector space $\mathcal{O}(E_n)$, and $\mathcal{O}(E_n)$ is complete with respect to $\|\ \|_n$. It is known that

$$\|f\|_n=\mathop{\mathrm{Max}}_{z\in E_n}|f(z)|$$

holds. Since $\mathcal{O}(D)$ is the set consisting of $f\colon D\to k$ such that the restriction of f to each E_n belongs to $\mathcal{O}(E_n)$, the semi-norms $\|\ \|_n$ $(n=1,2,\cdots)$ give on $\mathcal{O}(D)$ a structure of a Fréchet k-vector space. In particular, $\mathcal{O}(D)$ is complete with respect to this topology.

Let U be the set consisting of all rational functions $f(z)$ of z such that the coefficients of $f(z)$ belong to k and $f(z)$ has no pole in D. Since k is algebraically closed, each element f of U has a partial fractional expansion:

$$f(z)=\sum_{m=0}^{\infty} d_m z^m+\sum_{j=1}^{l}\sum_{m=-1}^{-\infty} d_m^{(j)}(z-b_j)^m \qquad \text{(a finite sum)},$$

where d_m, $d_m^{(j)}\in k$ and $b_j\in L$. It is easy to see that f can be expanded in the form (C_n). Hence U is a k-subspace of $V=\mathcal{O}(D)$. Furthermore, any finite sum of the form (C_n) belongs to U. Hence U is dense in each $\mathcal{O}(E_n)$. Since $\mathcal{O}(E_n)\subset\mathcal{O}(E_{n-1})\subset\cdots\subset\mathcal{O}(E_1)$, U is dense in $V=\varprojlim\mathcal{O}(E_n)$.

Let $G=SL(2, L)$. For any $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$ and $z\in \boldsymbol{P}^1(k)$, we write $g(z)=(az+c)/(bz+d)$. By (6) of Proposition 3.4 of [6], $f(g(z))$ is an analytic function on $g(D)=g(\boldsymbol{P}^1(k))\backslash g(\boldsymbol{P}^1(L))=D$. It is easy to see that this action

$$SL(2, L)\times V\ni(g, f(z))\longrightarrow f(g(z))\in V$$

is continuous.

§ 3. Discrete series.

3-1. The representation π_s. Let k, L, V, U, G, K, $\cdots$ be as in § 2. We fix a negative integer s. For any element f of $V=\mathcal{O}(\boldsymbol{P}^1(k)\backslash\boldsymbol{P}^1(L))$ and for any $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G=SL(2, L)$, put

$$\pi_s(g)f(z)=(bz+d)^s f\left(\frac{az+c}{bz+d}\right).$$

Since $(bz+d)^s$ is analytic on $D=\boldsymbol{P}^1(k)\backslash\boldsymbol{P}^1(L)$, $\pi_s(g)f(z)$ is an analytic function on D. Hence $\pi_s(g)$ is a k-linear endomorphism of V. It is easy to see

PROPOSITION 1. *π_s is a continuous representation of G on V, and U is a dense G-invariant subspace of V.*

It is well-known that G is generated by $A(a)=\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $(a\in L^\times)$, $C(c)=\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ $(c\in L)$ and $I=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For such an element g, $\pi_s(g)f(z)$ is given by the following formula:

$$\pi_s(A(a))f(z)=a^{-s}f(a^2z);$$

$$\pi_s(C(c))f(z)=f(z+c);$$

$$\pi_s(I)f(z)=(-z)^s f(-1/z).$$

Let $\mathfrak{G}$, $\exp(tX)$, X_+, X_-, Y, $\cdots$ be as in § 1. For any $X\in\mathfrak{G}$ and $f(z)\in V$, we observe that

$$(d\pi_s)(X)f(z)=\lim_{t\to 0}\frac{1}{t}\{\pi_s(\exp(tX))f(z)-f(z)\}$$

is well-defined. Explicitly,

$$(d\pi_s)(X_+)f(z)=\lim_t \frac{1}{t}\left\{(tz+1)^s f\left(\frac{z}{tz+1}\right)-f(z)\right\}$$
$$=-z^2 f'(z)+szf(z);$$
$$(d\pi_s)(X_-)f(z)=\lim_t \frac{1}{t}\{f(z+t)-f(z)\}=f'(z);$$
$$(d\pi_s)(Y)f(z)=\lim_t \frac{1}{t}\{e^{-st}f(e^{2t}z)-f(z)\}=2zf'(z)-sf(z).$$

We note that, if W is a closed K-invariant k-subspace of V, then $(d\pi_s)(\mathfrak{G})W\subset W$.

3-2. Indecomposability. For each negative integer s, let U_s be the subspace of U consisting of all rational functions f whose partial fractional expansions have the form

$$f(z)=\sum_{m=0}^{\infty} d_m z^m+\sum_{j=1}^{l}\sum_{m=s}^{-\infty} d_m^{(j)}(z-b_j)^m$$

(d_m, $d_m^{(j)}\in k$, $b_j\in L$). We observe that U_s is closed in U, and that the closure of U_s in V is the set V_s consisting of all functions $f\in\mathcal{O}(D)$ such that for any positive integer n, the restriction of f to E_n has an expansion of the form

$$f(z)=\sum_{m=0}^{\infty} c_m z^m+\sum_{i=1}^{l_n}\sum_{m=s}^{-\infty} c_m^{(i)}(z-a_i)^m \qquad (c_m,\ c_m^{(i)}\in k).$$

Let

$$f(z)=\sum_{m=0}^{\infty} d_m z^m+\sum_{j=1}^{l}\sum_{m=s}^{-\infty} d_m^{(j)}(z-b_j)^m \qquad (d_m,\ d_m^{(j)}\in k,\ l\in\boldsymbol{N},\ b_j\in L)$$

be the partial fractional expansion of $f\in U_s$. Then the $\pi_s(A(a))f(z)$ ($a\in L^\times$) and the $\pi_s(C(c))f(z)$ ($c\in L$) have partial fractional expansions of the same type. Further, since

$$\pi_s(I)z^m=(-z)^{s-m}\in U_s \qquad (m\geqq 0 \text{ or } m\leqq s)$$

and since

$$\pi_s(I)(z-b)^m=(-z)^{s-m}(bz+1)^m=b^m\sum_{i=0}^{s-m}\binom{s-m}{i}(-1)^i(z+b^{-1})^{i+m}b^{m-s+i}\in U_s$$
$$(b\in L^\times,\ m\leqq s),$$

$\pi_s(I)f(z)$ is an element of U_s. Hence U_s is a closed G-invariant subspace of U. Since V_s is the closure of U_s in V, V_s is a closed G-invariant subspace of V.

Let $\mathfrak{o}$ be the integer ring of L, and let $K=SL(2, \mathfrak{o})$. Then K acts transitively on $\boldsymbol{P}^1(L)$. Let $\mathfrak{G}$ be the Lie algebra of K (cf. § 1). Then we have

THEOREM 1. *Let s be a negative integer, and let $\pi_s: G \to \mathrm{Aut}_k V$ be as before. Then the smallest $(\mathfrak{G}, K)$-invariant k-subspace of U containing $1/z$ is U.*

PROOF. Let W be the smallest $(\mathfrak{G}, K)$-invariant k-subspace of U containing $1/z$. Since $W \ni 1/z$, W contains $\pi_s(C(-c))z^{-1}=(z-c)^{-1}$ for any $c \in \mathfrak{o}$. Since W is a $\mathfrak{G}$-invariant k-subspace, W contains

$$-(d\pi_s)(X_-)z^{-1}=z^{-2}.$$

Repeating the same argument, we observe that W contains

$$z^{-h} \text{ and } (z-c)^{-h} \quad (c \in \mathfrak{o})$$

for any positive integer h. Since $\pi_s(I)z^{s-h}=(-z)^s(-z^{-1})^{s-h}=(-z)^h$, W contains z^h for any non-negative integer h. Since W is a k-subspace of U, W contains all rational functions f of the form

$$f(z)=\sum_{m=0}^{\infty} d_m z^m + \sum_{j=1}^{l} \sum_{m=-1}^{-\infty} d_m^{(j)}(z-b_j)^m$$

($d_m, d_m^{(j)} \in k$, $b_j \in \mathfrak{o}$, $l \in N$). Let c be an element of $L \backslash \mathfrak{o}$. Then $|-c|>1$. Hence $-c^{-1} \in \mathfrak{o}$. Since

$$\pi_s(I)(z-c)^{-1}=(-z)^s(-z^{-1}-c)^{-1}=(-z)^{s+1}c^{-1}(z+c^{-1})^{-1}$$

is contained in W, $(z-c)^{-1}$ belongs to W. Hence, repeating the above argument, we see that $(z-c)^{-h}$ belongs to W for any positive integer h. Since W is a k-subspace of U, W contains all rational functions f of the form

$$f(z)=\sum_{m=0}^{\infty} d_m z^m + \sum_{j=1}^{l} \sum_{m=-1}^{-\infty} d_m^{(j)}(z-b_j)^m$$

($d_m, d_m^{(j)} \in k$, $b_j \in L$, $l \in N$). This shows $U \subset W$. Since U is a $(\mathfrak{G}, K)$-invariant k-subspace containing $1/z$, $W \subset U$. Hence $W=U$ and the theorem is proved.

Since U is dense in V, and since any closed K-invariant k-subspace is invariant under $\mathfrak{G}$, the following corollary follows from Theorem 1.

COROLLARY. *Let $\pi_s: G \to \mathrm{Aut}_k V$ be as in Theorem 1. Then the smallest closed K-invariant k-subspace of V containing $1/z$ is V.*

Since $(-1)^s \pi_s(I)1=z^s$, repeating similar arguments as in the proof of Theorem 1, we can prove the following theorem.

THEOREM 2. *Let $\pi_s: G \to \mathrm{Aut}_k V$ be as before. Then* (i) *the smallest $(\mathfrak{G}, K)$-invariant k-subspace of U_s containing* 1 *is U_s; and* (ii) *the smallest closed K-invariant k-subspace of V_s containing* 1 *is V_s.*

Since $\boldsymbol{P}^1(L)$ is compact, $D=\boldsymbol{P}^1(k)\backslash\boldsymbol{P}^1(L)$ is a completely regular quasi-connected set (cf. [6], Proposition 2.3). Hence the theorem of identity holds for functions in $\mathcal{O}(D)$ (cf. [6], Theorem 3.7). Namely, let f and g be analytic functions on D, and let Δ be a subset of D having at least one accumulation point in D. We assume that $f(z)=g(z)$ for any $z\in\Delta$. Then $f(z)=g(z)$ for any $z\in D$.

By making use of this fact, we can prove

LEMMA 1. *Let π_s be as before, and let f be an element of V. Then*:

(1) *$f(z)$ is a constant iff $(d\pi_s)(X_-)f(z)=0$.*

(2) *$f(z)=\alpha/z$ $(\alpha\in k)$ iff $(d\pi_s)(Y)f(z)=(-s-2)f(z)$.*

PROOF. Since $(d\pi_s)(X_-)f(z)=f'(z)$ and $(d\pi_s)(Y)f(z)=2zf'(z)-sf(z)$, the necessity is obvious in each case. Hence we shall prove that it is sufficient.

Let $f(z)$ be an element of $V=\mathcal{O}(D)$ satisfying $f'(z)=0$. Since $f(z)$ can be expanded into a convergent power series at each point $z_0\in D$, f is locally a constant at each point $z_0\in D$. Then, by the theorem of identity, f is globally a constant on D. Hence (1) is proved.

Let $f(z)$ be an element of V satisfying $2zf'(z)-sf(z)=(-s-2)f(z)$. Then $zf'(z)=-f(z)$. Let $F(z)=zf(z)$. Then $F(z)\in\mathcal{O}(D)$ and

$$F'(z)=f(z)+zf'(z)=0\,.$$

Hence $F(z)$ is a constant. Therefore $f(z)=F(z)/z$ has the form $f(z)=\alpha/z$ $(\alpha\in k)$. Therefore the lemma is proved.

Now we can prove the indecomposability of π_s.

THEOREM 3. *Let $\pi_s: G\to\mathrm{Aut}_k V$ be as before. Then U is an indecomposable $(\mathfrak{G}, K)$-module, and V is a topologically indecomposable K-module.*

PROOF. Let $U=H_1\oplus H_2$ be the direct sum of two $(\mathfrak{G}, K)$-invariant k-subspaces. Then $z^{-1}\in U$ is a sum of $h_i\in H_i$ $(i=1, 2)$: $z^{-1}=h_1+h_2$. By Lemma 1, $(d\pi_s)(Y)z^{-1}=(-s-2)z^{-1}$ holds. Hence

$$(d\pi_s)(Y)h_1+(d\pi_s)(Y)h_2=(d\pi_s)(Y)z^{-1}=(-s-2)z^{-1}=(-s-2)h_1+(-s-2)h_2\,.$$

Since the H_i $(i=1, 2)$ are $\mathfrak{G}$-invariant, this equality shows $(d\pi_s)(Y)h_i=(-s-2)h_i$ $(i=1, 2)$. By Lemma 1, this shows $h_i(z)=\alpha_i/z$ $(\alpha_i\in k)$. Since $\alpha_1/z+\alpha_2/z=h_1+h_2=1/z$, either α_1 or α_2 is not zero. Hence H_1 or H_2 contains $1/z$. It follows from Theorem 1 that $H_1=U$ or $H_2=U$. Therefore U is an indecomposable $(\mathfrak{G}, K)$-module.

If V is a direct sum of two closed K-invariant k-subspaces, then, repeating the same argument as in the above case, we can show that one of the subspaces contains $1/z$. It follows from Corollary to Theorem 1 that this subspace coincides with V. Hence V is topologically indecomposable.

Since $\mathfrak{G}$ is the Lie algebra of K, the following corollary follows from Theorem 3.

COROLLARY. *Let π_s be as before. Then U is a topologically indecomposable K-module.*

REMARK. By repeating similar arguments, we can show that V_s is a topologically indecomposable K-module.

3-3. Irreducibility. We need the following lemma to prove the irreducibility of U_s and U/U_s.

LEMMA 2. *Let $\pi_s: G \to \mathrm{Aut}_k V$ be as before, and let*

$$f(z)=\sum_{m=-\infty}^{+\infty} d_m z^m + \sum_{j=1}^{l} \sum_{m=-1}^{-\infty} d_m^{(j)}(z-b_j)^m$$

$(d_m,\ d_m^{(j)} \in k,\ b_j \in L^\times)$ be the partial fractional expansion of $f \in U$. For any two different integers q and r, let

$$T_{q,r}f(z)=\frac{1}{(q-r)}\left\{\frac{1}{2}(d\pi_s)(Y)f(z)+\left(\frac{s}{2}-r\right)f(z)\right\}.$$

Then

$$T_{q,r}f(z)=\sum_{m=-\infty}^{+\infty} \frac{m-r}{q-r} d_m z^m + \sum_{j=1}^{l} \sum_{m=-1}^{-\infty} \left(\frac{m-r}{q-r} d_m^{(j)} + \frac{m+1}{q-r} b_j d_{m+1}^{(j)}\right)(z-b_j)^m .$$

PROOF. Since

$$\frac{1}{2}(d\pi_s)(Y)f(z)+\left(\frac{s}{2}-r\right)f(z)=zf'(z)-rf(z),$$

we have

$$(q-r)T_{q,r}z^m=(m-r)z^m$$

and

$$(q-r)T_{q,r}(z-b)^m=(m-r)(z-b)^m+mb(z-b)^{m-1}.$$

Since $T_{q,r}$ is k-linear, we obtain the lemma.

Now we can prove that the $(\mathfrak{G}, K)$-module U_s is algebraically irreducible. Namely, we have

THEOREM 4 *Let $\pi_s: G \to \mathrm{Aut}_k V$, U_s, K, $\mathfrak{G}$ and $d\pi_s$ be as before. We consider U_s as a $(\mathfrak{G}, K)$-bimodule. Then U_s has no $(\mathfrak{G}, K)$-invariant k-subspace W satisfying $\{0\} \subsetneqq W \subsetneqq U_s$.*

PROOF. Let W be a $(\mathfrak{G}, K)$-invariant k-subspace of U_s different from $\{0\}$.

Then W contains a non-zero element f. Let

$$f(z)=\sum_{m=-\infty}^{+\infty} d_m z^m+\sum_{j=1}^{l}\sum_{m=s}^{-\infty} d_m^{(j)}(z-b_j)^m$$

$(d_m,\ d_m^{(j)}\in k,\ d_{-1}=d_{-2}=\cdots=d_{s+1}=0,\ b_j\in L^\times,\ b_i\neq b_j\ (i\neq j))$ be the partial fractional expansion of f. Then one of the d_m and the $d_m^{(j)}$ is not zero. Since

$$\pi_s(I)(z-b)^{s-h}=(-1)^s(-b)^{s-h}\sum_{i=0}^{h}\binom{h}{i}b^{-h+i}(z+b^{-1})^{s-h+i}(-1)^{h-i}$$

$(h\geqq 0,\ b\neq 0)$, we may assume that either $d_m\neq 0$, or $d_m^{(j)}\neq 0$ with $b_j\in\mathfrak{o}$. Since $\pi_s(C(b))(z-b)^m=z^m$ $(m\leqq s)$, we may assume that $d_m\neq 0$. Since $\pi_s(I)z^m=(-1)^{s-m}z^{s-m}$, we may assume that $d_m\neq 0$ with $m\geqq 0$. Since W is $\mathfrak{G}$-invariant and since $(d\pi_s)(X_-)f(z)=f'(z)$, we may assume that $d_0\neq 0$. Hence W contains an element f of the form

$$f(z)=1+\sum_{m=1}^{\infty} d_m z^m+\sum_{m=s}^{-\infty} d_m z^m+\sum_{j=1}^{l}\sum_{m=s}^{-\infty} d_m^{(j)}(z-b_j)^m .$$

Since this is a finite sum, we use a finite number of the operators $T_{0,r}$ $(r\in\boldsymbol{Z},\ r\geqq 1$ or $r\leqq s)$ and construct an element f of W of the form

$$f(z)=1+\sum_{j=1}^{l}\sum_{m=s}^{-\infty} d_m^{(j)}(z-b_j)^m \qquad (d_m^{(j)}\in k,\ b_j\in L^\times).$$

Then

$$\pi_s(C(b_l))f(z)=1+\sum_{m=s}^{-\infty} d_m^{(l)} z^m+\sum_{j=1}^{l-1}\sum_{m=s}^{-\infty} d_m^{(j)}(z-b_j+b_l)^m .$$

Hence we have constructed an element f of W of the same form and with a strictly less number l. Repeating this process, it follows that W contains 1. Since W is a $(\mathfrak{G}, K)$-invariant k-subspace of U_s, it follows from Theorem 2 that $W=U_s$. Therefore we have proved Theorem 4.

As for the quotient $(\mathfrak{G}, K)$-module U/U_s, we can prove the following result.

THEOREM 5. *Let $\pi_s: G\to \mathrm{Aut}_k V$, U, U_s, K, G and $d\pi_s$ be as before. Let f be an element of U which is not contained in U_s. Then the smallest $(\mathfrak{G}, K)$-module containing f is U. In particular, U/U_s is an algebraically irreducible $(\mathfrak{G}, K)$-module.*

PROOF. Let W be the smallest $(\mathfrak{G}, K)$-invariant k-subspace of U containing f, and let

$$f(z)=\sum_{m=-\infty}^{+\infty} d_m z^m+\sum_{j=1}^{l}\sum_{m=-1}^{-\infty} d_m^{(j)}(z-b_j)^m$$

(d_m, $d_m^{(j)}\in k$, $b_j\in L^\times$, $b_i\neq b_j$ $(i\neq j)$) be the partial fractional expansion of f. Then one of the d_m and the $d_m^{(j)}$ $(0>m>s)$ is not zero. If $d_{m_0}^{(j)}\neq 0$ with $b_j\notin \mathfrak{o}$ and $0>m_0>s$, we may assume that m_0 is the smallest one with this property so that $d_m^{(j)}=0$ for any m with $m_0>m>s$. Since

$$\pi_s(I)(z-b)^m=(-z)^s(-z^{-1}-b)^m=(-z)^{s-m}b^m(z+b^{-1})^m$$

$(b\in L^\times,\ 0>m>s)$ has poles only at $z=0$ and $z=-b^{-1}$, $\pi_s(I)(z-b)^m$ has a partial fractional expansion of the form

$$\sum_{i=0}^{s-m} e_i^* z^i+\sum_{i=-1}^{m} e_i(z+b^{-1})^i$$

$(e_i^*,\ e_i\in k,\ e_m\neq 0)$. Since $\pi_s(I)z^m=(-z)^{s-m}$ and $\pi_s(I)U_s\subset U_s$, we see that the coefficient of $(z+b_j^{-1})^{m_0}$ of the partial fractional expansion of $\pi_s(I)f\in W$ is not zero. Hence we may assume that one of the d_m $(0>m>s)$ and the $d_m^{(j)}$ $(0>m>s,\ b_j\in\mathfrak{o},\ b_j\neq 0)$ is not zero. If $d_m^{(j)}$ is not zero, then the coefficient of z^m of the partial fractional expansion of $\pi_s(C(b_j))f$ is not zero. Hence we may assume

$$f(z)=z^{m_0}+\sum_{m\neq m_0} d_m z^m+\sum_{j=1}^{l}\sum_{m=-1}^{-\infty} d_m^{(j)}(z-b_j)^m \qquad (0>m_0>s).$$

Now we apply the procedure in the proof of Theorem 4 and erase the second and the third sum in the right hand side of the above equation (use $T_{m_0,q}$ $(q\neq m_0)$ instead of $T_{0,q}$). It follows that W contains z^{m_0}. Since

$$\frac{1}{m_0}(d\pi_s)(X_-)z^{m_0}=z^{m_0-1}\in W\,,$$

we repeat this procedure and see that W contains z^{s+1}. Since

$$\pi_s(I)z^{s+1}=(-z)^s(-z^{-1})^{s+1}=-z^{-1}\in W\,,$$

W contains $1/z$. Since the smallest $(\mathfrak{G}, K)$-invariant k-subspace of U containing $1/z$ is U, W contains U. Since W is a k-subspace of U, U contains W. Hence $W=U$ and hence the theorem is proved.

REMARK. It follows from Theorems 4 and 5 that U_s and U/U_s have no non-trivial closed K-invariant k-subspaces. It is likely that V_s and V/V_s have no non-trivial closed K-invariant k-subspaces.

We note here that the transitivity of K on $\boldsymbol{P}^1(k)$ is essential for the irreducibility. For example, if we replace K by the principal congruence subgroup K_n of level p^n $(n\geqq 1)$, then the irreducibility fails.

3-4. Equivalence of representations. As for equivalence of the representations, we have the following result:

THEOREM 6. *Let s, s' and s'' be negative integers. Then*

(1) *U_s is not $\mathfrak{G}$-equivalent to any $\mathfrak{G}$-submodule of $V_{s'}$ $(s'\neq s)$ nor $V/V_{s''}$.*

(2) *If s is not -1, then U/U_s is not $\mathfrak{G}$-equivalent to any $\mathfrak{G}$-submodule of $V_{s'}$ nor $V/V_{s''}$ $(s''\neq s)$.*

PROOF. Let s and s' be negative integers, and let π_s and $\pi_{s'}$ be as before. Then $1\in U_s$ is a non-zero solution of

$$\begin{cases} (d\pi_s)(X_-)f=0 \\ (d\pi_s)(Y)f=-sf\,. \end{cases}$$

If (U_s, π_s) is equivalent as a $\mathfrak{G}$-module to a $\mathfrak{G}$-submodule W of $(V, \pi_{s'})$, then

$$\begin{cases} (d\pi_{s'})(X_-)f=0 \\ (d\pi_{s'})(Y)f=-sf \end{cases}$$

has a non-zero solution f in W. By Lemma 1, $(d\pi_{s'})(X_-)f=0$ implies that f is a constant. Hence

$$(d\pi_{s'})(Y)f=2zf'(z)-s'f(z)=-s'f(z)\,.$$

Since $f\neq 0$, $s=s'$. Further, since $1\in W$, it follows from Theorem 2 that $U_s\subset W$. Therefore $s=s'$ and $U_s\subset W$.

Let $\tilde{\pi}_{s''}\colon G\rightarrow \mathrm{Aut}_k(V/V_{s''})$ be the quotient representation of $\pi_{s''}$. If (U_s, π_s) is equivalent as a $\mathfrak{G}$-module to a $\mathfrak{G}$-submodule W of $(V/V_{s''}, \tilde{\pi}_{s''})$, then

$$\begin{cases} (d\pi_{s''})(X_-)f\equiv 0 & \text{modulo } V_{s''} \\ (d\pi_{s''})(Y)f\equiv -sf & \text{modulo } V_{s''} \end{cases}$$

has a non-zero solution in W. For each n, let E_n, a_i, $\cdots$ be as in § 2. Then

$$f(z)\equiv \sum_{m=0}^{\infty} c_m z^m+\sum_{i=1}^{l_n}\sum_{m=-1}^{-\infty} c_m^{(i)}(z-a_i)^m \qquad \text{modulo } V_{s''}$$

with c_m, $c_m^{(i)}\in k$. Since $V_{s''}$ is $\mathfrak{G}$-invariant and since

$$(d\pi_{s''})(X_-)f(z)=f'(z)\equiv \sum_{m=0}^{\infty} mc_m z^{m-1}+\sum_{i=1}^{l_n}\sum_{m=-1}^{-\infty} mc_m^{(i)}(x-a_i)^m \qquad \text{modulo } V_{s''},$$

it follows from $(d\pi_{s''})(X_-)f(z)\equiv 0$ modulo $V_{s''}$ that $c_m^{(i)}=0$ for $0>m>s''+1$. Hence

$$f(z)\equiv \sum_{m=0}^{\infty} c_m z^m+\sum_{i=1}^{l_n}\sum_{m=s''+1}^{-\infty} c_m^{(i)}(z-a_i)^m \qquad \text{modulo } V_{s''}\,.$$

Since $f(z) \notin V_{s''}$, we choose n such that at least one of the $c_{s''+1}^{(i)}$ $(1 \leqq i \leqq l_n)$ is not zero. Since $V_{s''}$ is $\mathfrak{G}$-invariant and since

$$(d\pi_{s''})(Y)f(z)=2zf'(z)-s''f(z)$$

$$\equiv \sum_{m=0}^{\infty}(2m-s'')c_m z^m+\sum_{i=1}^{l_n}\sum_{m=s''+1}^{-\infty}c_m^{(i)}\{2zm(z-a_i)^{m-1}-s''(z-a_i)^m\} \quad \text{modulo } V_{s''}$$

$$\equiv \sum_{m=0}^{\infty}(2m-s'')c_m z^m+\sum_{i=1}^{l_n}(s''+2)c_{s''+1}^{(i)}(z-a_i)^{s''+1}$$

$$+\sum_{i=1}^{l_n}\sum_{m=s''}^{-\infty}\{(2m-s'')c_m^{(i)}+2(m+1)a_i c_{m+1}^{(i)}\}(z-a_i)^m \quad \text{modulo } V_{s''},$$

it follows from $(d\pi_{s''})(Y)f(z)\equiv -sf(z)$ modulo $V_{s''}$ that $s''+2=-s$. Then $s+s''=-2$ and $s,\ s''\leqq -1$. Hence $s=s''=-1$. Then $V=V_s$ and $f\in V_s$. Since this is a contradiction, $(U,\ \pi_s)$ is not equivalent as a $\mathfrak{G}$-module to any $\mathfrak{G}$-submodule of $(V/V_{s''},\ \tilde{\pi}_{s''})$.

We assume that $s\neq -1$. Then $\tilde{f}=z^{s+1}$ modulo U_s is a non-zero solution of

$$\begin{cases}(d\tilde{\pi}_s)(X_-)\tilde{f}=0\\ (d\tilde{\pi}_s)(Y)\tilde{f}=(s+2)\tilde{f}\,.\end{cases}$$

If $(U/U_s,\ \tilde{\pi}_s)$ is equivalent as a $\mathfrak{G}$-module to a $\mathfrak{G}$-submodule W of $(V,\ \pi_{s'})$, then

$$\begin{cases}(d\pi_{s'})(X_-)f=0\\ (d\pi_{s'})(Y)f=(s+2)f\end{cases}$$

has a non-zero solution $f\in W$. By Lemma 1, $(d\pi_{s'})(X_-)f=0$ implies $f=\alpha$ $(\alpha\in k)$. Then

$$(d\pi_{s'})(Y)f(z)=2zf'(z)-s'f(z)=-s'f(z)\,.$$

Hence $s+2=-s'$. Since $s,\ s'\leqq -1$, this shows that $s=s'=-1$. Since this contradicts the assumption $s\neq -1$, $(U/U_s,\ \tilde{\pi}_s)$ is not equivalent as a $\mathfrak{G}$-module to any $\mathfrak{G}$-submodule of $(V,\ \pi_{s'})$.

If $(U/U_s,\ \tilde{\pi}_s)$ is equivalent as a $\mathfrak{G}$-module to a $\mathfrak{G}$-submodule of $(V/V_{s''},\ \tilde{\pi}_{s''})$, then

$$\begin{cases}(d\pi_{s''})(X_-)f(z)\equiv 0 & \text{modulo } V_{s''}\\ (d\pi_{s''})(Y)f(z)\equiv(s+2)f(z) & \text{modulo } V_{s''}\end{cases}$$

has a solution $f\in V\backslash V_{s''}$. For each $n\in \boldsymbol{N}$, let $E_n,\ a_i,\ \cdots$ be as in §2. Then

$$f(z)=\sum_{m=0}^{\infty}c_m z^m+\sum_{i=1}^{l_n}\sum_{m=-1}^{-\infty}c_m^{(i)}(z-a_i)^m \quad \text{on } E_n$$

with c_m, $c_m^{(i)} \in k$. Since $(d\pi_{s''})(X_-)f(z) \equiv 0$ modulo $V_{s''}$, we have $c_m^{(i)}=0$ for $1 \leqq i \leqq l_n$, $s''+1<m \leqq -1$. Since $f \notin V_{s''}$, we choose n such that at least one of the $c_{s''+1}^{(i)}$ is not zero. Since $V_{s''}$ is $\mathfrak{G}$-invariant,

$$(d\pi_{s''})(Y)f(z)=2zf'(z)-s''f(z)$$

$$\equiv \sum_{m=0}^{\infty}(2m-s'')c_m z^m + \sum_{i=1}^{l_n}(s''+2)c_{s''+1}^{(i)}(z-a_i)^{s''+1}$$

$$+\sum_{i=1}^{l_n}\sum_{m=s''}^{-\infty}\{(2m-s'')c_m^{(i)}+2(m+1)a_i c_{m+1}^{(i)}\}(z-a_i)^m \qquad \text{modulo } V_{s''}.$$

Since one of the $c_{s''+1}^{(i)}$ is not zero, it follows from $(d\pi_{s''})(Y)f(z) \equiv (s+2)f(z)$ modulo $V_{s''}$ that $s=s''$.

REMARK. If V_s and V/V_s are topologically irreducible K- (or G-) modules, then Theorem 6 implies that no two of them for various s are equivalent.

REMARK. It follows from the proof of Theorem 6 that $\dim_k \mathrm{Hom}_{\mathfrak{G}}(U_s, V_s)=1$ in the category of $\mathfrak{G}$-modules. Since U_s is dense in V_s, it follows that $\dim_k \mathrm{Hom}_K(V_s, V_s)=1$ in the category of continuous K-modules. But this fact does not necessarily imply the irreducibility of V_s as a topological K-module.[1]

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1) We can also show $\dim_k \mathrm{Hom}_K(V, V)=1$, though V is not a topologically irreducible K-module.

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