On Stark's conjectures on the behavior of $L(s, \chi)$ at $s=0$

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To the memory of Takuro Shintani

In this paper, we discuss some conjectures of H. M. Stark [17; II, IV], [19] on the leading coefficient of the Taylor expansion of an Artin L-function $L(s,\chi)$ at $s=0$. Some special cases of these conjectures were found independently by Takuro Shintani, [16], who also made an important contribution by developing new formulas for the coefficient in question, in terms of the double gamma function. His death is a great tragedy; I would like to dedicate this paper to his memory.

Our discussion is mainly a reformulation, with perhaps some slight added precision, generality, and unity, of ideas of Stark and Shintani. I hope it may help make these very interesting ideas more accessible. Our point of view has already helped inspire two separate p-adic analogs of the conjectures: one by B. Gross [8], at $s=0$, and one by J.-P. Serre, at $s=1$, [12], [21]. Gross' ideas in that connection have in turn suggested to us the consideration of the non-archimedean analog of the conjecture "over $\boldsymbol{Z}$" of Stark which we discuss implicitly in §4 and explicitly in §5.

In the course of our discussion we mention a few new results whose proofs will be published in [21]. That reference also contains the details of many arguments which are only sketched here.

§1. The main conjecture.

We begin by fixing some assumptions and notation.

k is a number field (finite extension of the rational field $\boldsymbol{Q}$),

K/k is a finite Galois extension,

$G=\mathrm{Gal}(K/k)$ is the Galois group.

We denote places of k (even archimedean ones) by symbols like $\mathfrak{p}$, $\mathfrak{q}$, $\cdots$ and those of K by $\mathfrak{P}$, $\mathfrak{Q}$, $\cdots$. For each place $\mathfrak{p}$ of k we let $G_{\mathfrak{p}}\subset G$ denote a decomposition group for $\mathfrak{p}$, well determined up to conjugation. If $\mathfrak{p}$ is finite (i.e., non-archimedean), then $I_{\mathfrak{p}}$ denotes the inertia group and $\sigma_{\mathfrak{p}}$ the Frobenius substitution generating $G_{\mathfrak{p}}/I_{\mathfrak{p}}$, well determined modulo $I_{\mathfrak{p}}$.

S is a finite set of places of k, including the archimedean ones.

V is a finite-dimensional $\boldsymbol{C}$-linear representation space for G.
$\chi_V : G \to \boldsymbol{C}$ is the *character* of $V : \chi_V(\sigma) = \mathrm{Tr}(\sigma | V)$.
$V^G = \{x \in V \mid \sigma x = x \ \forall \sigma \in G\}$.
V^* is the contragredient to V; $\chi_{V^*} = \bar{\chi}_V$.
$L(s, V) = L_S(s, \chi_V, K/k)$ is the Artin L-function associated to V (or to χ_V), but with the Euler factors corresponding to primes $\mathfrak{p}$ in S removed. Thus

$$L(s, V) = \prod_{\mathfrak{p} \notin S} \det(1 - \sigma_{\mathfrak{p}} N_{\mathfrak{p}}^{-s} | V^{I_{\mathfrak{p}}})^{-1}, \qquad \text{for } R(s) > 1.$$

It is important for the reader to remember that many of the things in our discussion, like $L(s, V)$, X and U below, etc., depend on the choice of S, even though we usually write just L instead of L_S, etc. It is known that $L(s, V)$ has a pole of order $\dim V^G$, i.e., a zero of order $-\dim V^G$, at $s=1$. From this fact and the functional equation relating $L(s, V)$ and $L(1-s, V^*)$ it follows that $L(s, V)$ has at $s=0$ a zero of order

$$r(V) = -\dim(V^G) + \sum_{\mathfrak{p} \in S} \dim(V^{G_{\mathfrak{p}}}).$$

Let $L(V) \in \boldsymbol{C}^*$ be the first non-zero coefficient of the Taylor expansion of $L(s, V)$ at $s=0$, so that

$$L(s, V) \sim L(V) s^{r(V)}, \qquad \text{as } s \to 0.$$

Stark's basic idea is that $L(V)$ can be expressed as the product of a "regulator" R and an algebraic number, which he denotes by Θ. His regulator depends on choices, but is always the determinant of an $r(V) \times r(V)$ matrix whose entries are linear forms in the logs of the absolute values of S-units of K at places of K above S, with coefficients coming from the representation V. To define the regulators we introduce

$U = U_{S,K} = \{\alpha \in K \mid \|\alpha\|_{\mathfrak{P}} = 1 \ \forall \mathfrak{P} \nmid S\}$, the group of *S-units* of K.
$Y = Y_{S,K} = \bigoplus_{\mathfrak{P} | S} \boldsymbol{Z}\mathfrak{P}$ is the free $\boldsymbol{Z}$-module with basis the set of places $\mathfrak{P}$ of K above S.
$X = X_{S,K}$ is the submodule of Y consisting of the elements $\sum_{\mathfrak{P} | S} n_{\mathfrak{P}} \mathfrak{P}$ with $\sum n_{\mathfrak{P}} = 0$.

For any ring R and any $\boldsymbol{Z}$-module M we shall denote the R-module $R \otimes_{\boldsymbol{Z}} M$ simply by RM. This notation will be used mainly with $R = \boldsymbol{Q}$ (rational field), $\boldsymbol{R}$ (real field), or $\boldsymbol{C}$ (complex field).

Let $\lambda : \boldsymbol{R}U \to \boldsymbol{R}X$ (or $\boldsymbol{C}U \to \boldsymbol{C}X$) be the $\boldsymbol{R}$- (or $\boldsymbol{C}$-) linear map such that

$$\lambda(1 \otimes \varepsilon) = \sum_{\mathfrak{P} | S} \log \|\varepsilon\|_{\mathfrak{P}} \mathfrak{P}, \qquad \text{for } \varepsilon \in U$$

where $\|\varepsilon\|_{\mathfrak{P}}$ denotes the *normed* absolute value of ε at $\mathfrak{P}$ (the ordinary one if $\mathfrak{P}$ is real, its square if $\mathfrak{P}$ is complex, and given by $(N\mathfrak{P})^{-n}$ where $n = \mathrm{ord}_{\mathfrak{P}} \varepsilon$ if $\mathfrak{P}$

is finite). The "S-unit theorem" states that λ is an isomorphism.

The group G acts on U and on X and $\lambda: \boldsymbol{C}U \to \boldsymbol{C}X$ is a $\boldsymbol{C}[G]$-isomorphism. (Our convention is that G acts always on the left, even though we sometimes write α^σ instead of $\sigma(\alpha)$, for $\alpha \in K$ and $\sigma \in G$, so that $\alpha^{(\sigma\tau)} = (\alpha^\tau)^\sigma$. The action $\mathfrak{P}^\sigma = \sigma\mathfrak{P}$ on places is that for which $\|\alpha^\sigma\|_{\sigma\mathfrak{P}} = \|\alpha\|_{\mathfrak{P}}$.)

For a homomorphism $\varphi: T \to T'$ of $\boldsymbol{C}[G]$-modules we denote by

$$\varphi_V: \operatorname{Hom}_G(V^*, T) \longrightarrow \operatorname{Hom}_G(V^*, T')$$

the linear map induced by φ. For a $\boldsymbol{C}[G]$-homomorphism $f: \boldsymbol{C}X \to \boldsymbol{C}U$, the map $(\lambda f)_V = \lambda_V f_V$ is a linear transformation of the space

$$\operatorname{Hom}_G(V^*, \boldsymbol{C}X) = (V \underset{\boldsymbol{C}}{\otimes} \boldsymbol{C}X)^G = (V \underset{\boldsymbol{Z}}{\otimes} X)^G$$

into itself, and we put

$$R(V, f) \overset{\text{defn}}{=} \det (\lambda f)_V .$$

This is an $r(V) \times r(V)$-determinant. Indeed, we have

$$\boldsymbol{C}X \oplus \boldsymbol{C} \approx \boldsymbol{C}Y \approx \bigoplus_{\mathfrak{p} \in S} \operatorname{Ind}_{G_\mathfrak{p}}^G \boldsymbol{C} ,$$

hence

$$\operatorname{Hom}_G(V^*, \boldsymbol{C}X) \oplus \operatorname{Hom}_G(V^*, \boldsymbol{C}) \approx \bigoplus_{\mathfrak{p} \in S} \operatorname{Hom}_{G_\mathfrak{p}}(V^*, \boldsymbol{C}) ,$$

and so $\dim \operatorname{Hom}_G(V^*, \boldsymbol{C}X) = -\dim V^G + \sum_{\mathfrak{p} \in S} \dim V^{G_\mathfrak{p}} = r(V)$, as claimed.

Put

$$A(V, f) = \frac{R(V, f)}{L(V)} = \lim_{s \to 0} \frac{R(V, sf)}{L(s, V)} .$$

For $\alpha \in \operatorname{Aut} \boldsymbol{C}$ we define V^α and f^α by the "base change" $\alpha: \boldsymbol{C} \to \boldsymbol{C}$. Perhaps the simplest form of Stark's conjecture is

(1.1) Main Conjecture (*first form*): *For all* $\alpha \in \operatorname{Aut} \boldsymbol{C}$ *we have*

$$A(V^\alpha, f^\alpha) = A(V, f)^\alpha .$$

Clearly, for given f, $R(V, f)$ and $A(V, f)$ depend only on the isomorphism class of V, hence only on the character χ_V. If χ is a character of G, we sometimes write $R(\chi, f)$ and $A(\chi, f)$ with the obvious meaning. A fancier form of the conjecture is

(1.2) Main Conjecture (*second form*): *Let E be a field of characteristic 0, isomorphic to a subfield of* $\boldsymbol{C}$. *Let $\chi: G \to E$ be the character of a representation of G in E or in an extension field of E. Let $f: EX \to EU$ be an $E[G]$-homomorphism. Then there exists a* (*unique*) *element $A(\chi, f) \in E$ such that, for* *every homomorphism* $\beta: E\to \boldsymbol{C}$, $A(\chi, f)^{\beta}=A(\chi^{\beta}, f^{\beta})$.

While it is sometimes convenient theoretically to consider f's of the above generality, it is not at all necessary. The rational representation spaces $\boldsymbol{Q}X$ and $\boldsymbol{Q}U$ for G become isomorphic (canonically, via λ) after the base extension $\boldsymbol{Q}\to\boldsymbol{R}$. Hence, as Herbrand observed, ([9], see also [1]) they are isomorphic, though not canonically so, i.e., there exist G-isomorphisms $f: \boldsymbol{C}X\simeq\boldsymbol{C}U$ which are defined over $\boldsymbol{Q}$ in the sense that $f(\boldsymbol{Q}X)=\boldsymbol{Q}U$, or equivalently, $f^{\alpha}=f$ for all $\alpha\in\operatorname{Aut}\boldsymbol{C}$.

(1.3) MAIN CONJECTURE (*third form*): *Let* χ *be a complex valued character of* G *and let* $\boldsymbol{Q}(\chi)$ *be the subfield of* $\boldsymbol{C}$ *generated by its values* $\chi(\sigma)$, $\sigma\in G$. *Let* f *be defined over* $\boldsymbol{Q}$ *as above. Then* $A(\chi, f)\in\boldsymbol{Q}(\chi)$, *and* $A(\chi^{\gamma}, f)=A(\chi, f)^{\gamma}$ *for all* $\gamma\in\operatorname{Gal}(\boldsymbol{Q}(\chi)/\boldsymbol{Q})$.
(Note that each $\chi(\sigma)$ is a sum of roots of unity, so that $\boldsymbol{Q}(\chi)$ is an abelian extension of $\boldsymbol{Q}$.)

It is easy to see, for a given V, that if any one of the three forms of the conjecture is true for V and *one iso*morphism f_0, then all three forms are true for V and every homomorphism f. The point is that $\lambda f=\lambda f_0 g$, where $g=f_0^{-1}f$, so $A(V, f)=A(V, f_0)\det g_V$; and $\det(g_{V\alpha}^{\alpha})=(\det g_V)^{\alpha}$.

The third form of the conjecture is essentially that given by Stark. Instead of using a map f to define the regulator $R(\chi)$, Stark [17, II] uses what he calls a system of Artin units $(\varepsilon_{\mathfrak{P}})$. This is a family of elements $\varepsilon_{\mathfrak{P}}\in U$, one for each place $\mathfrak{P}$ of K above S such that $\varepsilon_{\mathfrak{P}}^{\sigma}=\varepsilon_{\sigma\mathfrak{P}}$ for all $\sigma\in G$ and each $\mathfrak{P}|S$, and such that the only relation among the $\varepsilon_{\mathfrak{P}}$ is $\prod_{\mathfrak{P}|\infty}\varepsilon_{\mathfrak{P}}^{n_{\mathfrak{P}}}=1$, where $n_{\mathfrak{P}}=[K_{\mathfrak{P}}:\boldsymbol{R}]$. It is not hard to show that Stark's regulator $R(\chi, (\varepsilon_{\mathfrak{P}}))$ is equal to our $R(\chi, f)$ where $f: \boldsymbol{C}X\to\boldsymbol{C}U$ is the G-isomorphism defined over $\boldsymbol{Q}$ obtained by restricting to X the homomorphism $\mathfrak{P}\mapsto\varepsilon_{\mathfrak{P}}$ of Y into U. Hence Stark's $\Theta(\chi, (\varepsilon_{\mathfrak{P}}))$ is equal to our $A(\chi, f)^{-1}$. While Stark has never published a general statement quite as precise as (1.3) he has formulated its equivalent in many special cases [17, IV], [19], [20], especially in the case $r(\chi)=1$.

§ 2. Functorality; independence of choices.

In this section we discuss various formal properties of the Main Conjecture.

(2.1) PROPOSITION. *If the conjecture* (1.1)-(1.3) *is true for a given character* χ *with one choice of* S, *it is true for* χ *with any other choice.*

Indeed, suppose $S^{*}=S\cup\{\mathfrak{p}\}$ is the set obtained by adding a new place $\mathfrak{p}$ of k to S. Let us indicate quantities associated with the choice S^{*} by adding a star to the symbol. For example, we have

$$r^*(V)=r(V)+r_{\mathfrak{p}}(V)\,, \qquad \text{where } r_{\mathfrak{p}}(V)=\dim V^{G_{\mathfrak{p}}}\,,$$

$$L^*(s, V)=P_V(q^{-s})L(s, V)\,, \qquad \text{where } q=N\mathfrak{p}\,,$$

and

$$P_V(T)=\det(1-\sigma_{\mathfrak{p}}T\,|\,V^{I_{\mathfrak{p}}})=(1-T)^{r_{\mathfrak{p}}}Q_V(T)\,, \qquad \text{say},$$

where $Q_V(1)\neq 0$ satisfies $Q_{V^\alpha}(1)=(Q_V(1))^\alpha$ for $\alpha\in\operatorname{Aut}\boldsymbol{C}$.

For the leading coefficients we have

$$L^*(V)=(\log q)^{r_{\mathfrak{p}}}Q_V(1)L(V)\,.$$

On the other hand, let $U_{\mathfrak{p}}=U^*/U$ and $Y_{\mathfrak{p}}=X^*/X=\oplus_{\mathfrak{P}|\mathfrak{p}}\boldsymbol{Z}\mathfrak{P}$. The map λ^*: $\boldsymbol{C}U^*\to\boldsymbol{C}X^*$ induces an isomorphism $\lambda_{\mathfrak{p}}:\boldsymbol{C}U_{\mathfrak{p}}\to\boldsymbol{C}Y_{\mathfrak{p}}$ such that $(\log q)^{-1}\lambda_{\mathfrak{p}}$ is defined over $\boldsymbol{Q}$, as one sees from the formula $\log\|\varepsilon\|_{\mathfrak{P}}=-f_{\mathfrak{p}}(\log q)\operatorname{ord}_{\mathfrak{P}}\varepsilon$, $f_{\mathfrak{p}}$ denoting the residue degree of K/k at $\mathfrak{p}$. Thus, if we use an $f^*:\boldsymbol{C}X^*\to\boldsymbol{C}U^*$ which is defined over $\boldsymbol{Q}$ and carries $\boldsymbol{C}X$ into $\boldsymbol{C}U$ and induces $\lambda_{\mathfrak{p}}^{-1}\log q$ on $\boldsymbol{C}X_{\mathfrak{p}}$, then $R(V, f^*)=(\log q)^{r_{\mathfrak{p}}}R(V, f)$ and $A(V, f^*)=\boldsymbol{Q}_V(1)^{-1}A(V, f)$. This shows that the conjecture is true for V with S^* if and only if it is true for V with S.

As function of χ the quantity $R(\chi, f)$ obeys the same formalism as Artin's L-functions, and consequently the same is true for $A(\chi, f)$:

(2.2) PROPOSITION. *Let $f:\boldsymbol{C}X\to\boldsymbol{C}U$ be a G-homomorphism.*

(a) *If χ_1 and χ_2 are characters of G, then $A(\chi_1+\chi_2, f)=A(\chi_1, f)A(\chi_2, f)$.*

(b) *If χ is a character of a subgroup H of G, then*

$$A(\operatorname{Ind}_H^G\chi, f)=A(\chi, f)\,.$$

(c) *If χ is a character of a quotient group $G'=G/H$, then*

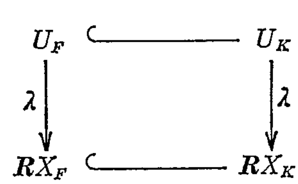
$$A(\operatorname{Infl}_{G'}^G\chi, f)=A(\chi, f\,|\,\boldsymbol{C}X^H)\,.$$

Indeed, (a) is obvious, and (b) follows from the existence of a functorial isomorphism

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G V, \boldsymbol{C}X)\approx\operatorname{Hom}_H(V, \boldsymbol{C}X)$$

if V is a realization of χ.

Let $F=K^H$ be the fixed field of H, and let X_F (resp. U_F) be to F as X (resp. U) is to K. Then (c) is obvious, once we explain how to identify X_F with a subgroup of X_K in such a way that the diagram



is commutative, and $CX_F=(CX)^H$. This is achieved by the imbedding

$$\mathfrak{P}' \longmapsto \sum_{\mathfrak{P}|\mathfrak{P}'} [K_{\mathfrak{P}} : F_{\mathfrak{P}'}]\mathfrak{P}$$

for places $\mathfrak{P}'$ of F above S.

Let 1_G denote the trivial representation of G.

(2.3) PROPOSITION. *Suppose f is induced by an injective G-homomorphism $f_0 : X \to U$. Then*

$$A(1_G, f) = \frac{(U_k : f_0 X_k)}{h},$$

where $h = |\mathrm{Pic}\, O_S|$ is the "S-class number" of k.

By 2.2 (c) we can suppose $K=k$, $G=\{1\}$. Let x_i be a $\boldsymbol{Z}$-base for $X=X_k$ and let u_j be a base for $U=U_k$ modulo torsion. Suppose

$$\lambda \tilde{u}_j = \sum a_{ji} x_i, \quad \text{and} \quad f x_i = \sum b_{ij} \tilde{u}_j,$$

where $\tilde{u}_j$ is the image of u_j in $\boldsymbol{Q}U$. Then

$$R(1_G, f) = \det \lambda f \quad \text{on} \quad \boldsymbol{C}X = \det(a_{ji}) \cdot \det(b_{ij}),$$

and

$$\pm \det(a_{ji}) = R, \quad \text{the "S-regulator" of } k$$

$$\pm \det(b_{ij}) = (\tilde{U} : fX) = \frac{(U : f_0 X)}{w},$$

where $\tilde{U}$ is the image of U in $\boldsymbol{Q}U$ and w is the order of the kernel of $U \to \tilde{U}$. The proposition follows upon combining these formulas with the formula

$$L(1_G) = -\frac{hR}{w}$$

which is derived via the functional equation from the familiar formula for the residue of $L(s, 1_G) = \zeta_k(s)$ at $s=1$.

(2.4) COROLLARY. *The Main Conjecture is true for permutation representations.*

By (2.3) the conjecture is true for the trivial representation 1_H of each subgroup H of G, hence, by (2.2) (a) and (b) it is true for any sum of representations of the form $\mathrm{Ind}_H^G(1_H)$.

If χ is a character with rational values, there is an integer $b>0$ and permutation characters χ_1 and χ_2 such that $b\chi = \chi_1 - \chi_2$. Hence the Main Conjecture is true for $b\chi$. This is the content of Theorem 1 of [17, II]. However, using methods of Ono [11] and Lichtenbaum [10] involving the cohomology of class field theory and results of Swan on integral representations of G, one can prove

(2.5) THEOREM. *The Main Conjecture* (1.1)-(1.3) *is true for rational characters* χ.

For the proof, see [21] and also [4].

(2.6) THEOREM. *The Main Conjecture* (1.1)-(1.3) *is true for characters* χ *such that* $r(\chi)=0$.

Indeed, if $r(\chi)=0$, then $R(\chi, f)=1$ and $A(\chi, f)=1/L(0, \chi)$. The fact that $L(0, \chi^{\alpha})=L(0, \chi)^{\alpha}$ for $\alpha\in\operatorname{Aut}\boldsymbol{C}$ was shown by Siegel [13], and later, by a different method by Shintani [14], in the abelian case. The general case follows by Brauer induction; see [21].

§ 3. **The case** $r(\chi)=1$: In this case Stark's conjecture is especially striking for it is equivalent to the existence of an S-unit ε of K with suitable absolute values at the places above S.

For an irreducible character χ of G, let

$$e_{\chi}=\frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma^{-1})\sigma$$

be the central idempotent in $\boldsymbol{C}[G]$ which acts as identity on a realization of χ and kills the other types of irreducible representations of G.

Let $\mathcal{X}$ be a set of irreducible characters of G such that $r(\chi)=1$ for each $\chi\in\mathcal{X}$ and such that $\chi\in\mathcal{X}\Rightarrow\chi^{\alpha}\in\mathcal{X}$ for all $\alpha\in\operatorname{Aut}\boldsymbol{C}$. Suppose that $a=(a_{\chi})$, $\chi\in\mathcal{X}$, is a family of complex numbers such that $a_{(\chi^{\alpha})}=(a_{\chi})^{\alpha}$ for $\alpha\in\operatorname{Aut}\boldsymbol{C}$. Put

$$\theta_{a}(s)=\sum_{\chi\in\mathcal{X}}a_{\chi}L(s, \chi)e_{\bar{\chi}}\,,$$

a Dirichlet Series with coefficients in $\boldsymbol{Q}[G]$.

(3.1) THEOREM. *Suppose* $y\in\boldsymbol{Q}Y$ *is such that* $e_{\chi}y\in X$ *for each* $\chi\in\mathcal{X}$. *The following statements are equivalent*:

(i) *There exists a (unique)* $u\in\boldsymbol{Q}U$ *such that*

$$\lambda(u)=\theta'_{a}(0)y=\sum_{\chi\in\mathcal{X}}a_{\chi}L'(0, \chi)e_{\bar{\chi}}y\,.$$

(ii) *The Main Conjecture* (1.1)-(1.3) *is true for all* $\chi\in\mathcal{X}$ *such that* $a_{\chi}e_{\bar{\chi}}y\neq 0$.

Let $f:\boldsymbol{Q}X\simeq\boldsymbol{Q}U$ be a G-isomorphism. For $\chi\in\mathcal{X}$, let c_{χ} be the scalar by which λf acts on $e_{\bar{\chi}}\boldsymbol{C}X$. (Note that $e_{\bar{\chi}}\boldsymbol{C}X$ is irreducible because $r(\chi)=1$.) If V

is a realization of χ, then $\mathrm{Hom}_G(V^*, \boldsymbol{C}X)=\mathrm{Hom}_G(V^*, e_{\bar{\chi}}\boldsymbol{C}X)$ is one-dimensional, and $(\lambda f)_V$ acts on this space by the scalar c_χ. Hence $c_\chi=\det(\lambda f)_V=R(\chi, f)=A(\chi, f)L'(0, \chi)$. Let $u=f(x)$, where

$$x=\sum_{\chi\in\mathfrak{X}}\frac{a_\chi}{A(\chi, f)}e_{\bar{\chi}}y\,.$$

Then

$$\lambda u=\lambda fx=\sum_{\chi\in\mathfrak{X}}\frac{c_\chi a_\chi}{A(\chi, f)}e_{\bar{\chi}}y=\sum_{\chi\in\mathfrak{X}}a_\chi L'(0, \chi)e_{\bar{\chi}}y\,.$$

Thus (i) is equivalent to $u\in\boldsymbol{Q}U$ which is the same as $x\in\boldsymbol{Q}X$, i.e., $x=x^\alpha$ for all $\alpha\in\mathrm{Aut}\,\boldsymbol{C}$. Since the non-zero ones among elements $a_\chi e_{\bar{\chi}}y$ lie in distinct irreducible subspaces of $\boldsymbol{C}X$ they are linearly independent, and since $a_{(\chi^\alpha)}=(a_\chi)^\alpha$ and $e_{\chi^\alpha}=(e_\chi)^\alpha$, the condition $x=x^\alpha$ means that $A(\chi^\alpha, f)=A(\chi, f)^\alpha$ for the χ's such that $a_\chi e_{\bar{\chi}}y\neq 0$, and this is (1.1) for those χ's.

If $\mathfrak{X}$ does not contain the trivial character $\chi=1$ we can take y to be a basis element $\mathfrak{P}$ of Y in (3.1). One would like then to conjecture a condition on the coefficient vector $a=(a_\chi)$ which would be sufficient to ensure that $\Theta'_a(0)\mathfrak{P}$ is in λU instead of only in $\lambda\boldsymbol{Q}U$. Such a conjecture might be called a conjecture "over $\boldsymbol{Z}$", as opposed to (1.1)-(1.3) which is only "over $\boldsymbol{Q}$". It would predict the existence of $\varepsilon\in K^*$ such that $\|\varepsilon\|_Q=1$ for all $\boldsymbol{Q}$ not conjugate to $\mathfrak{P}$ over k, and such that, for suitable $b=(b_\chi)=\chi(1)|G|^{-1}(a_\chi)$ and each $\sigma\in G$,

$$\log\|\varepsilon\|_{\sigma\mathfrak{P}}=\sum_{\chi\in\mathfrak{X}}b_\chi L'(0, \chi)\sum_{\tau\in G_{\mathfrak{P}}}\chi(\sigma\tau)\,,$$

where $G_{\mathfrak{P}}$ is the decomposition group of $\mathfrak{P}$. If $\mathfrak{P}$ is archimedean and lies over a *real* place $\mathfrak{p}$ of k, and we identify $k_\mathfrak{p}$ with $\boldsymbol{R}$, such a conjecture would yield an analytic formula for an S-unit $\eta=\|\varepsilon\|_{\mathfrak{P}}\in K\cap\boldsymbol{R}$, namely

$$(*)\qquad \eta=\exp\Big(\sum_{\chi\in\mathfrak{X}}b_\chi L'(0, \chi)\sum_{\tau\in G_{\mathfrak{P}}}\chi(\tau)\Big)\,.$$

In case K/k is abelian, Stark has given what seems to be the "correct" conjecture "over $\boldsymbol{Z}$". Before discussing this in §4, however, we mention that T. Chinburg [3] has examined numerically 5 non-abelian cases, each with $k=\boldsymbol{Q}$, $|G|=48$, and $\mathfrak{X}$ a set of 6 irreducible characters coming from two-dimensional representations $\rho: G\to GL_2(\boldsymbol{C})$ which are "tetrahedral", in the sense that the image of G in $PGL_2(\boldsymbol{C})$ is isomorphic to $\mathcal{A}_4$. In each case Chinburg found 6 independent units η such that $(*)$ holds with an accuracy of 10^{-13} and whose conjugates have suitably small absolute values. On the basis of these data Chinburg has formulated conjectures "over $\boldsymbol{Z}$". The cases treated by him are examples in which $L(s, \chi)$ corresponds to a modular form of weight 1, in accord with the theory of Langlands and Deligne-Serre [7]. Stark [19] has given the modular version of the conjecture "over $\boldsymbol{Q}$" in this case, and views it as giving an explicit construction of the non-abelian extension $K/\boldsymbol{Q}$ corresponding to a new form, whose existence was proved by Deligne and Serre.

§4. **The conjecture** $\mathrm{St}(K/k, S)$. In case G *is abelian,* which we assume from now on, Stark [17, IV] has given an ingenious conjecture "over $\boldsymbol{Z}$" for which there is a great deal of evidence. The conjecture has an extra feature, in that the S-unit ε whose existence is predicted is required to have a special property which we now explain.

Let μ_K be the group of roots of unity in K, let $W=|\mu_K|$ be its order, and let $A_{K/k}$ be the ideal in $\boldsymbol{Z}[G]$ consisting of the elements a such that $\zeta^a=1$ for all $\zeta\in\mu_K$. Let $\varepsilon\to\tilde{\varepsilon}$ denote the canonical map $K^*\to\boldsymbol{Q}K^*$; knowledge of $\tilde{\varepsilon}$ determines ε up to an element $\zeta\in\mu_K$.

(4.1) PROPOSITION. *Let* (σ_i), $i\in I$, *be a system of generators for* G, *and for each* $i\in I$, *let* $n_i\in\boldsymbol{Z}$ *be such that* $\zeta^{\sigma_i}=\zeta^{n_i}$ *for all* $\zeta\in\mu_K$. *Let* $u\in\boldsymbol{Q}K^*$. *The following conditions on* u *are equivalent.*

(i) *There exists* $\varepsilon\in K^*$ *such that* $Wu=\tilde{\varepsilon}$ *and* $K(\varepsilon^{1/W})$ *is abelian over* k.

(ii) *There is a field* $L\supset K$ *such that* L *is abelian over* k, *and an element* $\beta\in L^*$ *such that* $u=\tilde{\beta}$ *in* $\boldsymbol{Q}L^*$.

(iii) *For almost all (i.e., all but a finite number of) primes* $\mathfrak{p}$ *of* k *there is an element* $\varepsilon_{\mathfrak{p}}\in K^*$ *such that* $(\sigma_{\mathfrak{p}}-N\mathfrak{p})u=\tilde{\varepsilon}_{\mathfrak{p}}$, *and such that* $\varepsilon_{\mathfrak{p}}\equiv 1 \pmod{\mathfrak{p}O_K}$.

(iv) *For each* $a\in A$ *there is an* $\varepsilon_a\in K^*$ *such that* $au=\tilde{\varepsilon}_a$, *and for each* $a, b\in A$ *we have* $\varepsilon_a^b=\varepsilon_b^a$.

(v) *There exist* $\varepsilon\in K^*$ *and* $\varepsilon_i\in K^*$, $i\in I$, *such that* $Wu=\tilde{\varepsilon}$, $\varepsilon^{\sigma_i-n_i}=\varepsilon_i^W$ $\forall i\in I$, *and* $\varepsilon_i^{\sigma_j-n_j}=\varepsilon_j^{\sigma_i-n_i}$ *for* $i, j\in I$.

SKETCH OF PROOF. (i) $\Rightarrow$ (ii). Put $\beta=e^{1/W}$, $L=K(\beta)$.

(ii) $\Rightarrow$ (iii). Put $\varepsilon_{\mathfrak{p}}=\beta^{\sigma_{\mathfrak{p}}^L-N\mathfrak{p}}$, where $\sigma_{\mathfrak{p}}^L\in\mathrm{Gal}(L/k)$ is the Frobenius substitution of $\mathfrak{p}$ for L/k.

(iii) $\Rightarrow$ (iv). Use that the elements $\sigma_{\mathfrak{p}}-N\mathfrak{p}$ generate $A_{K/k}$ (indeed, $W=\gcd(1-N\mathfrak{p})$, for $\sigma_{\mathfrak{p}}=1$; [5]), and note that $\varepsilon_{\mathfrak{p}}^{\sigma_{\mathfrak{q}}-N\mathfrak{q}}=\varepsilon_{\mathfrak{q}}^{\sigma_{\mathfrak{p}}-N\mathfrak{p}}$ because both sides are $\equiv 1 \pmod{\mathfrak{p}}$.

(iv) $\Rightarrow$ (v). Put $\varepsilon=\varepsilon_W$ and $\varepsilon_i=\varepsilon_{\sigma_i-n_i}$.

(v) $\Rightarrow$ (i). Exercise in Galois theory.

Let

$$\theta(s)=\theta_{S,K/k}(s)=\sum_{\chi\in G}L_S(s,\chi)e_{\bar{\chi}}=\prod_{\mathfrak{p}\notin S}(1-\sigma_{\mathfrak{p}}^{-1}N\mathfrak{p}^{-s})^{-1}$$

(the last for $\mathrm{Re}\, s>1$) be the $\boldsymbol{C}[G]$-valued function of a complex variable s such that for each $\chi\in\hat{G}$ (the character group of G), we have $\chi(\theta(s))=L(s,\chi^{-1})$. Thus

$$\theta(s)=\sum_{\sigma\in G}\zeta_S(\sigma, s)\sigma^{-1}$$

where $\zeta_S(\sigma, s)$ is the "partial zeta function", defined for $Res>1$ by

$$\zeta_S(\sigma, s)=\sum_{(\mathfrak{a}, S)=1,\, \sigma_{\mathfrak{a}}=\sigma}\sigma_{\mathfrak{a}}^{-1}N\mathfrak{a}^{-s},$$

the sum being over the non-zero integral ideals $\mathfrak{a}$ of O_k prime to S whose image $\sigma_{\mathfrak{a}}$ under the Artin map is σ.

The following remarkable conjecture has been formulated by Stark [17, IV] in case $\mathfrak{p}$ is archimedean. Suppose that S contains a place $\mathfrak{p}$ which splits completely in K. Let $T=S-\{\mathfrak{p}\}$, and suppose that T is not empty and contains the places which are ramified in K. Let $U^T=U^T_{S,K}$ denote the group of elements $\alpha\in U=U_{S,K}$ such that

$$\|\alpha\|_{\mathfrak{Q}}=1 \text{ for } \mathfrak{Q}|T, \qquad \text{if } |T|\geqq 2$$

$$\|\alpha\|_{\mathfrak{Q}} \text{ is constant for } \mathfrak{Q}|\mathfrak{q}, \qquad \text{if } T=\{\mathfrak{q}\}.$$

(4.2) CONJECTURE St$(S, K/k)$. *Suppose* $\mathfrak{p}$, S, *and* T *are as above. Let* $\mathfrak{P}|\mathfrak{p}$. *Then*

(I) *There is an element* $u=u(\mathfrak{P})\in \boldsymbol{Q}U$ *such that*

$$\lambda u=\begin{cases} -\theta'_S(0)\mathfrak{P}, & \text{if } |T|\geqq 2\\ -\theta'_S(0)\left(\mathfrak{P}-\dfrac{1}{|G|}\mathfrak{q}\right), & \text{if } T=\{\mathfrak{q}\}\end{cases}$$

and such that u *satisfies the equivalent conditions* (i)-(v) *of* (4.1).

(II) *There is an element* $\varepsilon=\varepsilon(\mathfrak{P})\in U^T$ *such that*

(a) $\log\|\varepsilon^\sigma\|_{\mathfrak{P}}=-W\zeta'_S(\sigma, 0)$, *for each* $\sigma\in G$

(b) $L'_S(0, \chi)=-\dfrac{1}{W}\sum_{\sigma\in G}\chi(\sigma)\log\|\varepsilon^\sigma\|_{\mathfrak{P}}$, *for each* $\chi\in\hat{G}$,

and such that $K(\varepsilon^{1/W})$ *is abelian over* k.

Note that (IIa) and (IIb) are equivalent. Moreover, if ε satisfies (II) then $u=W^{-1}\bar{\varepsilon}$ satisfies (I) and vice versa; hence (I) and (II) are equivalent. Stark's formulation is (IIb). The operator $\theta'(0)=\sum L'(0, \chi)e_{\bar{\chi}}$ is defined as a sum over all characters χ of G but in (I) we can replace it by the sum over the χ such that $r(\chi)=1$, because $e_{\bar{\chi}}$ kills X if $r(\chi)=0$, and $L'(0, \chi)=0$ if $r(\chi)\geqq 2$. Hence, by (3.1) the existence of $u\in\boldsymbol{Q}U$ satisfying the equations of (I), *but not necessarily the conditions* (i)-(v) of (4.1), is equivalent to the Main Conjecture (1.1)-(1.3) holding for each χ such that $r(\chi)=1$. Conjecture St$(K/k, S)$ is independent of the choice of $\mathfrak{P}$ dividing $\mathfrak{p}$, via $\varepsilon(\mathfrak{P}^\sigma)=\varepsilon(\mathfrak{P})^\sigma$; and it is in fact independent of the choice of splitting place $\mathfrak{p}\in S$ because of

(4.3) PROPOSITION. *Conjecture* St$(K/k, S)$ *is true if* S *contains two places* $\mathfrak{p}$ *and* $\mathfrak{q}$ *which split completely in* K.

If $G_{\mathfrak{p}}=G_{\mathfrak{q}}=1$, then $L'(0, \chi)=0$ for $\chi\neq 1$, and $\xi'(\sigma, 0)=n^{-1}\zeta'(0)$ for each $\sigma\in G$, where $n=|G|=[K: k]$. If $|S|\geqq 3$, then $\zeta'(0)=0$ and (II) is true with $\varepsilon=1$. If $S=\{\mathfrak{p}, \mathfrak{q}\}$, then $\zeta'(0)=-h\log\|\eta\|_{\mathfrak{p}}/wn$, where w is the number of roots of unity in k, $h=|\mathrm{Pic}\, O_S|$ is the "S class-number" of k, and η generates the group $O_S^* \bmod \mu_k$, with $\|\eta\|_{\mathfrak{p}}>1$, so that $\log\|\eta\|_{\mathfrak{p}}$ is the "S-regulator" of k. After cancellation of $\log\|\eta\|_{\mathfrak{p}}$, equation (IIa), with $\varepsilon=\eta^m$, reads $m=Wh/wn$. This in an integer! Indeed, w divides W obviously, and n divides h because K/k is split completely in S and unramified outside S, so that the reciprocity law gives a surjective homomorphism $\mathrm{Pic}\, O_S\to G$. Moreover, $\varepsilon^{1/W}=\varepsilon_0^{1/w}$, where $\varepsilon_0=\eta^{h/n}$, so that $K(\varepsilon^{1/W})$ is indeed abelian over k.

(4.4) COROLLARY. St$(K/k, S)$ *is true if* $K=k$.

(4.5) COROLLARY. St$(K/k, S)$ *is true if* k *has more than one complex place. If* $\mathfrak{p}$ *is non archimedean it is true if* k *is not totally real.*

Suppose $\mathfrak{q}\notin S$. Then

$$\theta_{S\cup\{\mathfrak{q}\}}(s)=(1-\sigma_{\mathfrak{q}}^{-1}N\mathfrak{q}^{-s})\theta_S(s).$$

Differentiating, putting $s=0$ gives

$$\theta'_{S\cup\mathfrak{q}}(0)=(1-\sigma_{\mathfrak{q}}^{-1})\theta'_S(0).$$

Hence, if u satisfies (I) for S, then $(1-\sigma_{\mathfrak{q}}^{-1})u$ satisfies (I) for $S\cup\mathfrak{q}$. In particular,

(4.6) St$(K/k, S)$ *implies* St$(K/k, S')$ *for any* $S'\supset S$.

Suppose $k\subset K'\subset K$. Let $G'=\mathrm{Gal}(K'/k)$. Then $\theta_{S,K'/k}(s)$ is the image of $\theta_{S,K/k}(s)$ under the homomorphism $\boldsymbol{C}[G]\to\boldsymbol{C}[G']$ induced by the natural map $G\to G'$.

(4.7) PROPOSITION. St$(K/k, S)$ *implies* St$(K'/k, S)$ *for* $k\subset K'\subset K$.

If $u\in\boldsymbol{Q}U_{S,K}$ satisfies (I) for K/k and $\mathfrak{P}$, then $u'=N_{K/K'}u\in\boldsymbol{Q}U_{S,K'}$ satisfies (I) for K'/k and $\mathfrak{P}'=N_{K/K'}\mathfrak{P}$. (To verify that u' satisfies 4.1 use (iii), with $\varepsilon'_{\mathfrak{p}}=N_{K/K'}\varepsilon_{\mathfrak{p}}$.)

(4.8) THEOREM. St$(K/k, S)$ *is true if* $k=\boldsymbol{Q}$ *or if* k *is imaginary quadratic.*

This is proved by Stark [17, IV] if $\mathfrak{p}$ is archimedean. For $\mathfrak{p}$ non-archimedean it is true for k imaginary quadratic by (4.5), and for $k=\boldsymbol{Q}$ by Stickelberger's factorization of Gauss and Jacobi sums, see [8].

Shintani's Theorem 2 of [16] is a version "over $\boldsymbol{Q}$" of $\mathrm{St}(K/k, S)$, in case k is real quadratic, the splitting place $\mathfrak{p}$ of k is archimedean, and K is a quadratic extension of an abelian extension of $\boldsymbol{Q}$, but is not itself abelian over $\boldsymbol{Q}$. It would be interesting to investigate whether his methods can be used to prove $\mathrm{St}(K/k, S)$ in that case.

(4.9) PROPOSITION. $\mathrm{St}(K/k, S)$ *is true if* $|S|=2$.

Let S_∞ denote the set of archimedean places of k. By (4.8) we can assume $|S_\infty|\geqq 2$. Since $S_\infty\subset S$ this means $S=S_\infty=\{\mathfrak{p}, \mathfrak{q}\}$, say, with $G_\mathfrak{p}=\{1\}$. Since K/k is unramified outside S, it follows that -1 is a local norm at every place except possibly at $\mathfrak{q}$. Hence -1 is a local norm at $\mathfrak{q}$ also, which means $G_\mathfrak{q}=1$. Hence the conjecture is true by (4.3).

The following seems to be the explanation of Stark's remark near the bottom of p. 199 of [17, IV].

(4.10) PROPOSITION. *Suppose* $[K: k]=2$. *Let* $n=|S|-1=|T|$, *and let* $m=2^{n-2}$. *Then* $\mathrm{St}(K/k, S)$ *is true with an* ε *in condition* (II) *such that* $\varepsilon=\alpha^m$, *where* α *is an element of* K *such that* $K(\alpha^{1/W})$ *is abelian over* k.

We can suppose $n\geqq 2$ by (4.9), and we can suppose $G_\mathfrak{q}=G$ for each $\mathfrak{q}\in T$ by (4.3). Let $G=\{1, \tau\}$. We can suppose k has at least 2 archimedean places by (4.8) and consequently, that τ is a complex conjugation and acts like -1 on roots of 1. With these assumptions one shows

$$\theta'_S(0)=L'(0, \chi)\Big(\frac{1-\tau}{2}\Big)=\frac{|\mathrm{Coker}|\,2^{n-2}(\log\|\eta\|_\mathfrak{P})}{W}(1-\tau)$$

where χ is the non-trivial character of G, Coker denotes the cokernel of the natural homomorphism $\mathrm{Pic}\, O_{S,k}\to\mathrm{Pic}\, O_{S,K}$, and η is the generator of $U^-_{S,K}/\mu_K$ such that $\|\eta\|_\mathfrak{P}>1$ (here $U^-_{S,K}$ means the group of elements $\alpha\in U_{S,K}$ such that $\alpha^\tau=\alpha^{-1}$). For details see [21]. Thus we can take $\alpha=\eta^{-|\mathrm{Coker}|}$. Since $\alpha^{1+\tau}=1$ and $\mu_K^{1+\tau}=1$, $K(\alpha^{1/W})/K$ is abelian (cf. (4.1), (v)).

Let $T_2=\{\mathfrak{q}\in T\,|\,G_\mathfrak{q}$ is of order 2$\}$. As Stark suggests ([17, IV], p. 199), one can use (4.10) to prove that $\mathrm{St}(K/k, S)$ is true whenever G is generated by the $G_\mathfrak{q}$ for $\mathfrak{q}\in T_2$. It might be interesting to consider the general case of G such that $G^2=1$.

(4.11) *The "real" case; numerical confirmation.* Suppose $\mathfrak{p}$ is real. Then we can make the identifications $K_\mathfrak{P}=k_\mathfrak{p}=\boldsymbol{R}$, and $W=2$. Suppose the conjecture is true. Replacing ε by $-\varepsilon$ if necessary we can suppose $\varepsilon>0$, and ε is then unique. Let $P(x)=\prod_{\sigma\in G}(x-\varepsilon^{\sigma})=\sum_{i=0}^{n}(-1)^{i}\alpha_{i}x^{n-i}$, $n=|G|$, be the field equation for ε over k. Since $\sqrt{\varepsilon}$ is abelian over k we have $\varepsilon^{\sigma}>0$ for each $\sigma\in G$, and formula (II a) is equivalent to

$$\varepsilon^{\sigma}=e^{-W\zeta'_{S}(\sigma,0)}, \qquad \text{for } \sigma\in G.$$

Calculating $\zeta'_{S}(\sigma, 0)$ to high accuracy gives approximations $\tilde{\varepsilon}_{\sigma}$ to ε^{σ}. The symmetric functions $\tilde{\alpha}_{i}$ of these $\tilde{\varepsilon}_{\sigma}$ then approximate the α_{i} in $k_{\mathfrak{p}}$. Since $\|\varepsilon\|_{\mathfrak{O}}=1$ for $\mathfrak{O}\nmid\mathfrak{p}$ (assuming $|S|\geqq 3$, which is no harm by (4.9)), the α_{i} are algebraic integers, and satisfy $\|\alpha_{i}\|_{\mathfrak{q}}\leqq\binom{n}{i}^{2}$ for each archimedean place $\mathfrak{q}\neq\mathfrak{p}$, as well as $\|\alpha_{i}-\tilde{\alpha}_{i}\|_{\mathfrak{p}}<10^{-N}$ for a large N. For N sufficiently large, these conditions determine α_{i} uniquely, once $\tilde{\alpha}_{i}$ is given; and if N is somewhat larger than necessary for unicity, then the probability of finding an α_{i}, given a random real number $\tilde{\alpha}_{i}$, is very small. Thus the conjecture can be well-tested by computing the $\tilde{\alpha}_{i}$ very accurately and then miraculously finding the $\alpha_{i}\in O_{k}$. One can also try to check then that the resulting $P(x)$ splits in K, and that the splitting field of $Q(x)=P(x^{2})$ is abelian over k.

In essentially this way the conjecture has been tested by Shintani and Stark in many cases with k real quadratic [16] [17; III, IV] [18] and in one case with k cubic [17, IV].

§ 5. **Conjecture** BS$(K/k, T)$. In case the splitting place $\mathfrak{p}\in S$ is non-archimedean the conjecture St$(K/k, S)$ can be conveniently reformulated, and leads to a refinement of an idea of A. Brumer; hence the name BS (Brumer-Stark).

Let T be a non-empty set of places of k containing the archimedean places and the places ramified in K. Let K^{T} denote the group of $\alpha\in K^{*}$ such that $\|\alpha\|_{\mathfrak{O}}=1$ at each $\mathfrak{O}$ above T, if $|T|\geqq 2$, or such that $\|\alpha\|_{\mathfrak{O}}$ is constant at places $\mathfrak{O}|\mathfrak{q}$, if $T=\{\mathfrak{q}\}$. Suppose $\mathfrak{p}\notin T$ splits completely in K $(G_{\mathfrak{p}}=1)$, and put $S=T\cup\{\mathfrak{p}\}$. Then

$$\theta_{S}(s)=(1-N\mathfrak{p}^{-s})\theta_{T}(s)$$

and consequently

$$\theta'_{S}(0)=(\log N_{\mathfrak{p}})\theta_{T}(0).$$

According to Siegel [13], and also Shintani [14], we have $\theta_{T}(0)\in \boldsymbol{Q}[G]$. Since $\log\|u\|_{\mathfrak{P}}=-(\log N\mathfrak{P})\operatorname{ord}_{\mathfrak{P}}u$, and $N\mathfrak{P}=N\mathfrak{p}$, the condition 4.2 (I) on $u\in\boldsymbol{Q}K^{*}$ is equivalent to $u\in\boldsymbol{Q}K^{T}$ *and* $(u)=\mathfrak{P}^{\theta_{T}(0)}$, where $(u)=\prod\mathfrak{O}^{\operatorname{ord}_{\mathfrak{O}}u}\in\boldsymbol{Q}\mathcal{I}_{K}$ is the "ideal" of u (here $\mathcal{I}_{K}$ denotes the ideal group of K and the homomorphism $u\mapsto(u)$ from $\boldsymbol{Q}K^{*}$ to $\boldsymbol{Q}\mathcal{I}_{K}$ is the unique extension of the one from K^{*} to $\mathcal{I}_{K}$ which associates to each $\alpha\in K^{*}$ its ideal $(\alpha)\in\mathcal{I}_{K}$).

Let $\mathcal{I}_{T,K/k}$ denote the subgroup of $\mathcal{I}_{K}$ consisting of the ideals $\mathfrak{A}$ of K such that $\mathfrak{A}^{\theta_{T}(0)}=(u)$, with $u\in\boldsymbol{Q}K^{T}$ satisfying the equivalent conditions (i)-(v) of

(4.1). As we have just discussed, if $\mathfrak{p}\notin T$, $G_{\mathfrak{p}}=\{1\}$, and $\mathfrak{P}|\mathfrak{p}$, then

(5.1) $\mathrm{St}(K/k, T\cup\{p\})$ *is true* $\Leftrightarrow P\in\mathcal{I}_{T,K/k}$.

On the other hand,

(5.2) *We have* $(\alpha)\in\mathcal{I}_{T,K/k}$ *for each* $\alpha\in K^*$.

Indeed, put $u=\tilde{\alpha}^{\theta_T(0)}\in \boldsymbol{Q}K^*$. Then u satisfies condition 4.1 (iv) with $\varepsilon_a=\alpha^{a\theta_T(0)}$ for each $a\in A$. (Here we use the deep result of Deligne-Ribet [6], and also of Barsky and Cassou-Nogués, [2]), that $a\theta_T(0)\in\boldsymbol{Z}[G]$ for each $a\in A$.) Moreover, $u\in\boldsymbol{Q}K^T$ in view of

(5.3) LEMMA. *For each subgroup* $H\subset G$, *let* $[H]=\sum_{\sigma\in H}\sigma\in\boldsymbol{Z}[G]$. *Then for each* $\mathfrak{q}\in T$

$$[G_{\mathfrak{q}}]\theta_T(0)=\begin{cases} 0, & \text{if } |T|\geqq 2 \\ \text{a multiple of } [G], & \text{if } T=\{\mathfrak{q}\}. \end{cases}$$

Indeed, for each character $\chi\neq 1$ of G we have $\chi([G_{\mathfrak{a}}]\theta_T(0))=\chi([G_{\mathfrak{a}}])L(0,\chi^{-1})=0$, because $L(0,\chi^{-1})=0$ if χ is trivial on $G_{\mathfrak{q}}$. If $|T|\geqq 2$, then $L(0,\chi)=0$ for $\chi=1$ also.

These considerations motivate

(5.4) CONJECTURE $\mathrm{BS}(K/k, T)$. *We have* $\mathcal{I}_{T,K/k}=\mathcal{I}_K$. *In other words, for each ideal* $\mathfrak{A}$ *of* K, *there is an* $\alpha\in K^T$ *such that* $K(\alpha^{1/W})$ *is abelian over* k, *and such that* $A^{W\theta_T(0)}=(\alpha)$.

The idea that the operator $W\theta_T(0)$ kills the ideal class group Pic O_K is due to A. Brumer and generalizes the Stickelberger factorization of Gauss sums (see Coates [5], for example). The idea that $\alpha^{1/W}$ is abelian over k, which generalizes the fact that the Gauss sums lie in cyclotomic fields, is, as we have seen, due to Stark. Hence it seems reasonable to name this conjecture Brumer-Stark. From 5.1 and 5.2 follows

(5.5) PROPOSITION. *Let* P *be a (finite or infinite) set of places of* k *disjoint from* T *such that each* $\mathfrak{p}\in P$ *splits completely in* K *and such that the primes* $\mathfrak{P}$ *of* K *above* P *generate the class group* Pic O_K. *Then* $\mathrm{BS}(K/k, T)$ *is true if and only if* $\mathrm{St}(K/k, T\cup\{p\})$ *is true for each* $\mathfrak{p}\in P$. *In particular,* $\mathrm{BS}(K/k, T)$ *is equivalent to* $\mathrm{St}(K/k, T\cup\{p\})$ *holding for all splitting* $\mathfrak{p}\notin T$.

Thus, "St" for non-archimedean $\mathfrak{p}$ is equivalent to "BS", and the results (4.3)-(4.10) for "St" yield corresponding results for "BS" which we don't bother to list.

As B. Mazur remarked, "BS" has an obvious function-field analog. This has been proven by P. Deligne. He uses "1-motives" to establish property 4.1 (iii) for u, via the theorem of Weil expressing L-series as characteristic polynomials of the Frobenius morphism; see [21] for details.

Taken all together, the evidence for the conjectures seems to me overwhelming.

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