p-adic L-series at $s=0$

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To the memory of Takuro Shintani

The p-adic L-series we will consider in this paper are analogous to the complex L-series of Artin: they are associated to finite dimensional linear representations of the Galois group of a totally real number field. These L-series are known to be meromorphic functions on $\boldsymbol{Z}_p$; we will give a conjectural formula for the leading term in their Taylor expansions at $s=0$. This conjecture (2.12), which expresses the leading term as the product of a p-adic regulator and an algebraic number, was inspired by Tate's formulation of Stark's conjectures for Artin L-series [11, 12, 13].

Following Stark and Tate, we will also present a stronger conjecture (3.13) for the first derivative of abelian L-series at $s=0$. One consequence of this refinement would be the explicit construction of classfields using special values of p-adic analytic functions.

Finally, we will prove that all of our conjectures are true for the p-adic L-series of Kubota and Leopoldt: those associated to 1-dimensional representations of the Galois group of $\boldsymbol{Q}$. The ingredients of the proof are: an analytic formula of Ferrero and Greenberg [3], a transcendence result of Brumer [1], and some results from the p-adic theory of Gauss sums [7].

I would like to thank J. Tate for his generous help and for many stimulating discussions on the subject of Stark's conjectures. Finally, I would like to dedicate this paper to the memory of T. Shintani, whose contributions in the theory of L-series remain as an inspiration to us all.

§0. Notation and conventions.

We will follow the notation of Tate [13] fairly closely. If X is an abelian group and R is a ring we will denote the R-module $R\otimes_Z X$ simply by RX. If $f: X\to Y$ is a group homomorphism we will use the same symbol to denote the induced homomorphism of R-modules $f: RX\to RY$. If τ is an involution of X we let $X^- = \{x\in X: \tau(x)=-x\}$.

We will use the symbols k and K for number fields; usually k will be a subfield of K. We denote places of k (even archimedean ones) by the symbols $\mathfrak{p}, \mathfrak{q}, \cdots$ and places of K by symbols $\mathfrak{P}, \mathfrak{Q}\cdots$. For each place $\mathfrak{P}$ of K we let $K_{\mathfrak{P}}$

denote the completion at $\mathfrak{P}$; if $\mathfrak{P}$ is finite we let $N\mathfrak{P}$ denote the cardinality of the residue field of $K_{\mathfrak{P}}$.

For a finite rational prime p we let $\boldsymbol{C}_p$ denote the completion of an algebraic closure of $\boldsymbol{Q}_p$. We define the p-adic functions $\exp_p(x)$ and $\log_p(x)$ by their convergent power series expansions on $2p\boldsymbol{Z}_p$ and $1+p\boldsymbol{Z}_p$, and extend the latter uniquely to a homomorphism $\log_p: \boldsymbol{Z}_p^* \to \boldsymbol{Z}_p^+$.

§1. A p-adic regulator homomorphism.

To define regulators for p-adic L-series at $s=0$, we need an analog of the homomorphism λ used in the proof of the S-unit theorem. We begin by constructing a theory of p-adic absolute values.

Let K be a number field and let $\boldsymbol{A}_K^*$ denote the idèle group of K. Define the homomorphism

$$\text{(1.1)} \qquad \begin{aligned} \phi: \boldsymbol{A}_K^* &\longrightarrow \boldsymbol{Q}^* \\ (a_{\mathfrak{P}}) &\longmapsto \prod_{\mathfrak{P}\text{ real}} \operatorname{sign}(a_{\mathfrak{P}}) \cdot \prod_{\mathfrak{P}\text{ finite}} (N\mathfrak{P})^{-\operatorname{ord}_{\mathfrak{P}}(a_{\mathfrak{P}})}. \end{aligned}$$

Then $\phi(a)=(N_{K/\boldsymbol{Q}}a)^{-1}$ for all principal idèles $a \in K^*$. Hence ϕ is an algebraic Hecke character of K.

To obtain the usual absolute value map:

$$\text{(1.2)} \qquad \| \ \|: \boldsymbol{A}_K^*/K^* \longrightarrow \boldsymbol{R}_+^*$$

we apply a construction of Serre and Tate [9, §7] to the character ϕ at the infinite place of $\boldsymbol{Q}$. Namely, we define

$$\text{(1.3)} \qquad \|a\| = \phi(a) \cdot N_{(K\otimes\boldsymbol{R})/\boldsymbol{R}}(a_\infty).$$

The restriction of $\| \ \|$ to the subgroup $K_{\mathfrak{P}}^*$ is the normalized local absolute value, for which we have the formulas:

$$\text{(1.4)} \qquad \|\alpha\|_{\mathfrak{P}} = \begin{cases} N_{K_{\mathfrak{P}}/\boldsymbol{R}}(\alpha) & \mathfrak{P} \text{ complex} \\ \operatorname{sign}(\alpha)\cdot\alpha & \mathfrak{P} \text{ real} \\ (N\mathfrak{P})^{-\operatorname{ord}_{\mathfrak{P}}(\alpha)} & \mathfrak{P} \text{ finite}. \end{cases}$$

To obtain p-adic absolute values we simply apply the same construction of Serre and Tate to the character ϕ at a finite place p of $\boldsymbol{Q}$. This gives a continuous homomorphism

$$\text{(1.5)} \qquad \begin{aligned} \| \ \|_p: \boldsymbol{A}_K^*/K^* &\longrightarrow \boldsymbol{Q}_p^* \\ a &\longmapsto \phi(a)\cdot N_{(K\otimes\boldsymbol{Q}_p)/\boldsymbol{Q}_p}(a_p). \end{aligned}$$

In this case however, the image is totally disconnected, so the kernel of $\| \|_p$ contains the connected component of the identity in A_K^*/K^*. This component is precisely the kernel of the Artin homomorphism $r_K: A_K^*/K^* \twoheadrightarrow \mathrm{Gal}(\bar{K}/K)^{ab}$ of global class field theory. Hence the map $\| \|_p$ factors through a Galois character

$$\begin{tikzcd} \| \|_p : A_K^*/K^* \arrow[rr] \arrow[dr, "r_K"'] & & Q_p^* \\ & \mathrm{Gal}(\bar{K}/K)^{ab} \arrow[ur, "\chi"'] & \end{tikzcd} \tag{1.6}$$

Since $\mathrm{Gal}(\bar{K}/K)^{ab}$ is compact, χ takes values in Z_p^*. It is not difficult to show that χ^{-1} gives the Galois action on $T_pG_m = \varprojlim_n \mu_{p^n}$.

If we restrict $\| \|_p$ to the subgroup $K_{\mathfrak{P}}^*$, we obtain a local absolute value

$$\begin{tikzcd} \| \|_{\mathfrak{P},p} : K_{\mathfrak{P}}^* \arrow[rr] \arrow[dr, "r_{\mathfrak{P}}"'] & & Z_p^* \\ & \mathrm{Gal}(\bar{K}_{\mathfrak{P}}/K_{\mathfrak{P}})^{ab} \arrow[ur, "\chi_{\mathfrak{P}}"'] & \end{tikzcd} \tag{1.7}$$

where $r_{\mathfrak{P}}$ is the reciprocity map of local class-field theory and $\chi_{\mathfrak{P}}^{-1}$ gives the local Galois action on T_pG_m. We have the following formulas:

$$\|\alpha\|_{\mathfrak{P},p} = \begin{cases} 1 & \mathfrak{P} \text{ complex}, \\ \mathrm{sign}(\alpha) & \mathfrak{P} \text{ real}, \\ (N\mathfrak{P})^{-\mathrm{ord}_{\mathfrak{P}}(\alpha)} & \mathfrak{P} \text{ finite, not dividing } p, \\ (N\mathfrak{P})^{-\mathrm{ord}_{\mathfrak{P}}(\alpha)} N_{K_{\mathfrak{P}}/Q_p}(\alpha) & \mathfrak{P} \text{ divides } p. \end{cases} \tag{1.8}$$

By (1.5) we have the product formula:

$$\prod_{\mathfrak{P}} \|\alpha\|_{\mathfrak{P},p} = 1 \qquad \text{for } \alpha \in K^*. \tag{1.9}$$

The p-adic absolute values $\| \|_{\mathfrak{P},p}$ are not as sensitive as the real absolute values $\| \|_{\mathfrak{P}}$. For example, if ε is a totally positive unit in the real quadratic field $K = Q(\sqrt{p})$, then $\|\varepsilon\|_{\mathfrak{P},p} = 1$ for all places $\mathfrak{P}$. But the condition $\|\varepsilon\|_{\mathfrak{P}} = 1$ for all places $\mathfrak{P}$ implies that ε is a root of unity.

On the subgroup

$$(K^*)^- \underset{\text{defn.}}{=} \{\varepsilon \in K^* : \|\varepsilon\|_{\mathfrak{P}} = 1 \text{ for all } \mathfrak{P} \text{ dividing } \infty\} \tag{1.10}$$

however, the p-adic absolute values are quite sensitive. We use the symbol $(K^*)^-$ to denote this subgroup as it is the intersection of the minus spaces of all complex conjugations on K^*. The torsion subgroup of $(K^*)^-$ is the group μ_K of roots of unity in K^*.

PROPOSITION 1.11. *Let ε be an element of $(K^*)^-$. Then the following three conditions are equivalent:*

a) *ε is an element of μ_K.*

b) *$\|\varepsilon\|_{\mathfrak{P}}=1$ for all finite places $\mathfrak{P}$.*

c) *$\|\varepsilon\|_{\mathfrak{P},p}$ is a root of unity for all finite places $\mathfrak{P}$.*

PROOF. Since $\|\ \|_{\mathfrak{P}}$ and $\|\ \|_{\mathfrak{P},p}$ are group homomorphisms, they map μ_K into the torsion subgroups of $\boldsymbol{R}_+^*$ and $\boldsymbol{Z}_p^*$ respectively. The former is trivial and the latter consists of the roots of unity in $\boldsymbol{Q}_p^*$. Hence a) implies b) and c).

The fact that b) implies a) is well-known. To show that c) implies b) we use the explicit formulas in (1.4) and (1.8). If $\mathfrak{P}$ does not divide p then $\|\varepsilon\|_{\mathfrak{P},p}=\|\varepsilon\|_{\mathfrak{P}}$ is a positive rational number. If it is also a root of unity in $\boldsymbol{Z}_p^*$ then $\|\varepsilon\|_{\mathfrak{P}}=1$. If $\mathfrak{P}$ divides p and $\|\varepsilon\|_{\mathfrak{P},p}=\zeta$ is a root of unity, we must have $N_{K_{\mathfrak{P}}/\boldsymbol{Q}_p}(\varepsilon)=p^a\zeta$. But $\varepsilon\in(K^*)^-$, so all of the conjugates of ε have complex absolute value 1. Hence $a=0$ and $\|\varepsilon\|_{\mathfrak{P}}=N\mathfrak{P}^{-\mathrm{ord}_{\mathfrak{P}}(\varepsilon)}=1$. $\square$

Let S be a finite set of places of K, which contains all places dividing ∞ and p. Let $U_{S,K}$ denote the S-units of K^* and let $U_{S,K}^-=U_{S,K}\cap(K^*)^-$. Let $Y_{S,K}$ denote the free abelian group on the set S and let $X_{S,K}$ denote the subgroup of elements of degree 0. Since S and K will usually be fixed, we will write U, U^-, X, and Y when the meaning is clear.

Recall the homomorphism λ used in the proof of the S-unit theorem [13, § 1]:

$$\lambda: U \longrightarrow \boldsymbol{R}Y \qquad (1.12)$$
$$\varepsilon \longmapsto \sum_{\mathfrak{P}\in S} \log\|\varepsilon\|_{\mathfrak{P}}\cdot\mathfrak{P}\,.$$

By the product formula, the image of λ lies in $\boldsymbol{R}X$, and Dirichlet's theorem asserts that λ induces an isomorphism $\lambda: \boldsymbol{R}U \xrightarrow{\sim} \boldsymbol{R}X$.

In a similar manner, we may define the p-adic homomorphism

$$\lambda_p: U \longrightarrow \boldsymbol{Q}_pY \qquad (1.13)$$
$$\varepsilon \longmapsto \sum_{\mathfrak{P}\in S} \log_p\|\varepsilon\|_{\mathfrak{P},p}\cdot\mathfrak{P}\,.$$

Again the image lies in $\boldsymbol{Q}_pX$ by the product formula (1.9). Using (1.7) one can show that the image of λ_p is contained in the subgroup $p^n\boldsymbol{Z}_pX$, where p^n exactly divides the number of roots of unity in $K(\mu_{2p})$.

PROPOSITION 1.14. *The induced map* $\lambda_p: \boldsymbol{Q}U^- \to \boldsymbol{Q}_p X$ *is an injection.*

PROOF. We must show that the kernel of λ_p on U^- is equal to the subgroup μ_K. Clearly, μ_K is contained in the kernel. Conversely, if $\lambda_p(\varepsilon)=0$ then $\|\varepsilon\|_{\mathfrak{P},p}$ is in the kernel of $\log_p(x)$ for every $\mathfrak{P}$ in S. This kernel is just the roots of unity in $\boldsymbol{Z}_p^*$. Since ε is also an S-unit, we see that $\|\varepsilon\|_{\mathfrak{P},p}$ is a root of unity for all finite places $\mathfrak{P}$ of K. Hence ε is a root of unity by (1.11). □

In attempting to strengthen (1.14), we are led to our first conjecture:

CONJECTURE 1.15. *The induced map* $\lambda_p: \boldsymbol{Q}_p U^- \to \boldsymbol{Q}_p X$ *is an injection.*

Let $\bar{\boldsymbol{Q}}$ denote the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}_p$. An equivalent form of (1.15) is the conjecture that the map $\lambda_p: \boldsymbol{C}_p U^- \to \boldsymbol{C}_p X$ is an injection. In this direction, we have the following result from transcendence theory.

PROPOSITION 1.16. *The map* $\lambda_p: \bar{\boldsymbol{Q}}U^- \to \boldsymbol{C}_p X$ *is an injection.*

PROOF. Let $u=\sum \alpha_i \otimes \varepsilon_i$ be an element in the kernel of λ_p, where the ε_i are in U^- and the α_i are algebraic numbers. We may further assume that the α_i are linearly independent over $\boldsymbol{Q}$. Since $\lambda_p(u)=0$ we have $\sum \alpha_i \log_p \|\varepsilon_i\|_{\mathfrak{P},p}=0$ for all places $\mathfrak{P}$ in S. By Brumer's p-adic version of Baker's theorem [1], this implies that $\log_p \|\varepsilon_i\|_{\mathfrak{P},p}=0$ for all i. Hence $\|\varepsilon_i\|_{\mathfrak{P},p}$ is a root of unity in $\boldsymbol{Z}_p^*$ for all places $\mathfrak{P}$ in S; since ε_i is an S-unit this implies, by (1.11), that ε_i is a root of unity in K^*. Since this holds for all i, $u=0$ in $\bar{\boldsymbol{Q}}U^-$. □

Although we have stated conjecture (1.15) for an arbitrary number field K, it suffices to prove it in the case where K is a *CM* field: a totally imaginary quadratic extension of a totally real field. In general, K will either contain no *CM*-field or a maximal one K_{CM}. In the first case $(K^*)^- = \langle \pm 1 \rangle$ and conjecture (1.15) is trivially true. In the second, every element ε in $(K^*)^-$ is contained in K_{CM}^*: each complex conjugation of K takes ε to ε^{-1}. Since the diagram

$$(1.17)\qquad \begin{array}{ccc} \boldsymbol{Q}_p U_K^- & \xrightarrow{\lambda_p} & \boldsymbol{Q}_p X_K \\ {\scriptstyle j}\uparrow\wr & & \uparrow{\scriptstyle i} \\ \boldsymbol{Q}_p U_{K_{CM}}^- & \xrightarrow{\lambda_p} & \boldsymbol{Q}_p X_{K_{CM}} \end{array}.$$

is commutative, where j is induced by the inclusion $K_{CM}^* \to K^*$ and $i(\sum_{\mathfrak{Q}} \alpha_{\mathfrak{Q}} \cdot \mathfrak{Q})$

$=\sum_{\mathfrak{Q}}\alpha_{\mathfrak{Q}}(\sum_{\mathfrak{P}|\mathfrak{Q}}[K_{\mathfrak{P}}:K_{CM,\mathfrak{Q}}]\mathfrak{P})$ is injective, we may assume, without loss of generality, that $K=K_{CM}$.

In this case, let τ be the involution of K which induces complex conjugation at every infinite place and enlarge S, if necessary, so that it is stable under τ. Then U^- is just the minus eigenspace of τ on U. Since the map λ_p is τ-equivariant, its restriction to U^- takes values in $\boldsymbol{Z}_pX^-$, the minus eigenspace of $\boldsymbol{Z}_pX$:

$$\lambda_p: U^- \longrightarrow \boldsymbol{Z}_pX^- . \tag{1.18}$$

PROPOSITION 1.19. *Assume K is a CM field. Then the following three statements are all equivalent to conjecture* (1.15).

a) *The map $\lambda_p:\boldsymbol{Z}_pU^-\to\boldsymbol{Z}_pX^-$ has finite kernel and cokernel.*

b) *The map $\lambda_p:\boldsymbol{Q}_pU^-\to\boldsymbol{Q}_pX^-$ is an isomorphism.*

c) *The map $\lambda_p:\boldsymbol{C}_pU^-\to\boldsymbol{C}_pX^-$ is an isomorphism.*

PROOF. Since U^- and X^- are finitely generated abelian groups, the equivalence of a), b), and c) is standard. Clearly b) implies (1.15), the converse follows from the fact that the $\boldsymbol{Q}_p$-vector spaces $\boldsymbol{Q}_pU^-$ and $\boldsymbol{Q}_pX^-$ have the same dimension. To see this, note that the map λ is also τ-equivariant, and gives an isomorphism $\boldsymbol{R}U^-\xrightarrow{\sim}\boldsymbol{R}X^-$. Hence the abelian groups U^- and X^- have the same rank. □

If part a) of (1.19) is true, the kernel of λ_p is the subgroup of p-power roots of unity in K^* and the order of the cokernel of λ_p is an interesting arithmetic invariant. We may refine this invariant as follows. Define the homomorphism

$$\begin{aligned} g: U^- &\longrightarrow X^- \\ \varepsilon &\longmapsto \sum_{\mathfrak{P}\nmid\infty} f_{\mathfrak{P}}\,\mathrm{ord}_{\mathfrak{P}}(\varepsilon)\cdot\mathfrak{P}\,, \end{aligned} \tag{1.20}$$

where $f_{\mathfrak{P}}$ is the degree of the residue field at $\mathfrak{P}$ over the prime field. Comparing g with the map λ, we see it induces an isomorphism $g:\boldsymbol{Q}U^-\xrightarrow{\sim}\boldsymbol{Q}X^-$. Define:

$$R_p=R_{p,K,S}=\det(\lambda_p g^{-1}|\boldsymbol{Q}_pX^-)\,. \tag{1.21}$$

The regulator R_p is non-zero if and only if the conditions in proposition (1.19) are satisfied; its p-adic valuation then determines the order of the cokernel of λ_p.

§ 2. p-adic L-series.

Let k be a totally real number field and let $\bar{k}$ be an algebraic closure of k. Let E be a field of characteristic zero and let V be a finite dimensional vector space over E with a linear action of $\mathrm{Gal}(\bar{k}/k)$. We assume that the representation

$$\rho : \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Aut}_E(V) \tag{2.1}$$

factors through the Galois group of a finite extension K of k.

We say the representation V is totally even if every complex conjugation in the Galois group acts as 1_V; we say the representation V is totally odd if every complex conjugation acts as -1_V. In both cases the field K may be chosen to be a CM field.

Let S denote a finite set of places of k, containing the set S_∞ of places dividing ∞. Given an embedding $\alpha : E \to \boldsymbol{C}$ we let V^α denote the complex representation obtained by change of base. Let $L(V^\alpha, s) = L_S(V^\alpha, s)$ be the Artin L-series of V^α without Euler factors corresponding to primes in S, defined as in Tate [13, § 1].

Let n be a negative integer. Then there is an algebraic number $a_S(V, n)$ in E such that

$$L_S(V^\alpha, n) = a_S(V, n)^\alpha \tag{2.2}$$

for all embeddings $\alpha : E \to \boldsymbol{C}$. When $\dim V = 1$ the existence of $a_S(V, n)$ follows from results of Siegel [10] on the rationality of partial zeta values. In the general case one first reduces to the case $S = S_\infty$; the existence of $a_S(V, n)$ then follows from Siegel's results and Serre's variant of Brauer induction, which takes the parity of V into account [14, Appendix]. For arbitrary S and $n \leqq -1$ we have $a_S(V, n) \neq 0$ if and only if V and n have the opposite parity.

Now assume that the set S also contains all divisors of p. Given an embedding $\beta : E \to \boldsymbol{C}_p$ we may define the p-adic L-series of V^β relative to S. This is a meromorphic function $L_p(V^\beta, s) = L_{p,S}(V^\beta, s)$ on $\boldsymbol{Z}_p$ which is characterized by its values on the dense set of strictly negative integers. Let

$$\omega : \mathrm{Gal}(k(\mu_{2p})/k) \longrightarrow (\boldsymbol{Z}/2p)^* \longrightarrow \boldsymbol{Z}_p^* \tag{2.3}$$

be the Teichmuller character. We then have the formula:

$$L_{p,S}(\omega^{1-n} \otimes V^\beta, n) = a_S(V, n)^\beta \tag{2.4}$$

for all $n \leqq -1$, where $a_S(V, n)$ is the element of E determined by (2.2). When $\dim V = 1$ the existence of a function satisfying (2.4) follows from results of Deligne and Ribet [2], [8]. In the general case one can construct $L_p(V^\beta, s)$

using their results and Serre's variant of Brauer induction [5], [14]. One suspects that the identity (2.4) also holds when $n=0$, but at present this is only known for abelian (or, more generally, monomial) representations V.

By parity considerations, the function $L_p(\omega\otimes V^\beta, s)$ is non-zero if and only if the representation V is *totally odd*. For the rest of this section we assume that this is the case, and factor the representation through the Galois group $G=\mathrm{Gal}(K/k)$, where K is a CM field.

Write the Taylor expansions of $L(V^\alpha, s)$ and $L_p(\omega\otimes V^\beta, s)$ at $s=0$ as follows:

$$(2.5)\qquad \begin{cases} L(V^\alpha, s)\sim L(V^\alpha)s^{r(V^\alpha)} \\ L_p(\omega\otimes V^\beta, s)\sim L_p(V^\beta)s^{r_p(V^\beta)} \end{cases} \quad \text{as } s\to 0.$$

We will give explicit formulas for the integer $r(V^\alpha)$ and the complex number $L(V^\alpha)$ and will conjecture similar expressions for $r_p(V^\beta)$ and $L_p(V^\beta)$.

Let S_K denote the places of K dividing those in S and let $U^-=U^-_{S_K}$ and $X^-=X^-_{S_K}$ be the groups defined in §1. The Galois group G acts on U^- and X^-; the latter representation is simply the minus component of the permutation representation on S_K. The homomorphisms λ, λ_p and g defined in §1 are G-equivariant. Hence we have an isomorphism of $\boldsymbol{Q}[G]$-modules:

$$(2.6)\qquad \boldsymbol{Q}U^-\cong\boldsymbol{Q}X^-\cong\bigoplus_{\mathfrak{p}\in S}(\mathrm{Ind}_{G_\mathfrak{p}}^G 1)^-$$

where $G_\mathfrak{p}$ is a decomposition group for the place $\mathfrak{p}$ in G.

Let V^* denote the contragredient to V and let

$$(2.7)\qquad r(V)=\dim_E(V\otimes EX^-)^G=\dim_E \mathrm{Hom}_G(V^*, EX^-).$$

If V is irreducible, $r(V)$ is the multiplicity of V^* in the G-decomposition of EX^-. By (2.6) and Frobenius reciprocity:

$$(2.8)\qquad r(V)=\sum_{\mathfrak{p}\in S}\dim_E V^{G_\mathfrak{p}}.$$

Notice that we also have the formulas:

$$(2.9)\qquad r(V)=\dim_{\boldsymbol{C}}(V^\alpha\otimes\boldsymbol{C}X^-)^G=\dim_{\boldsymbol{C}_p}(V^\beta\otimes\boldsymbol{C}_pX^-)^G$$

for all $\alpha: E\to\boldsymbol{C}$ and $\beta: E\to\boldsymbol{C}_p$.

Define the regulators

$$(2.10)\qquad \begin{cases} R(V^\alpha)=\det(1\otimes\lambda g^{-1}\mid(V^\alpha\otimes\boldsymbol{C}X^-)^G) \\ R_p(V^\beta)=\det(1\otimes\lambda_p g^{-1}\mid(V^\beta\otimes\boldsymbol{C}_pX^-)^G). \end{cases}$$

The former is a non-zero complex number which Tate denotes $R(V^\alpha, g^{-1})$ in [13, §1]. The latter is non-zero in $\boldsymbol{C}_p$ if and only if λ_p induces an isomorphism on the $(V^*)^\beta$-isotypical components of $\boldsymbol{C}_pU^-$ and $\boldsymbol{C}_pX^-$.

PROPOSITION 2.11. *There is an algebraic number* $A(V)$ *in* E^* *such that for all embeddings* $\alpha: E \to \boldsymbol{C}$ *we have*

a) $r(V^{\alpha})=r(V)$

b) $L(V^{\alpha})=R(V^{\alpha})A(V)^{\alpha}$.

CONJECTURE 2.12. *Let* $A(V)$ *be the algebraic number defined by* (2.11). *Then for all embeddings* $\beta: E \to \boldsymbol{C}_p$ *we have*

a) $r_p(V^{\beta})=r(V)$

b) $L_p(V^{\beta})=R_p(V^{\beta})A(V)^{\beta}$.

A few remarks are in order. Proposition (2.11) is proved in Tate [13, 2.6] when $r(V)=0$; the quantity which we call $A(V)$ is his $A(V, g^{-1})^{-1}$. The general case reduces to this one: we have $r_{S_\infty}(V)=0$ since V is totally odd and the relation between $A(V)$ and $A_{S_\infty}(V)=a_{S_\infty}(V, 0)$ is given by formula (2.16). Following the methods in Tate [13, § 2] one can also show that the statements of (2.12) are compatible with the behavior of p-adic L-series under the operations direct sum, inflation, and induction of representations. Similarly, the conjecture is independent of the choice of S containing the divisors of ∞ and p: there is an obvious relation between the quantities defined for S and those defined for $S^* = S \cup \{\mathfrak{p}^*\}$.

Note that the truth of conjecture 2.12 for *all* odd representations V of G implies the truth of conjecture 1.15 for the map λ_p and the CM field K. For (2.12) implies that $R_p(V^{\beta}) \neq 0$, hence λ_p induces an isomorphism on the $(V^*)^{\beta}$-isotypical components. With this point of view, we can return to deal with some of the questions raised in § 1. Our method follows that of Greenberg [4].

PROPOSITION 2.13. *If* $r(V)=1$ *then* $R_p(V^{\beta}) \neq 0$ *for all embeddings* $\beta: E \to \boldsymbol{C}_p$.

PROOF. By the hypothesis, each irreducible component W of $(V^*)^{\beta}$ occurs with multiplicity less than or equal to one in $\boldsymbol{C}_p U^-$ and $\boldsymbol{C}_p X^-$. To show the regulator is non-zero, it therefore suffices to show λ_p is *not identically zero* on the (W)-component. Since this component may be defined over $\overline{\boldsymbol{Q}}$, this follows from (1.16). □

COROLLARY 2.14. *Let* K *be a CM field with maximal real subfield* K_0. *Let* T *denote the set of places of* K_0 *dividing* p *which split in* K. *Assume that* $\mathrm{Aut}(K_0)$ *contains an abelian subgroup* G_0 *which permutes the places in* T *transitively. Then the map* $\lambda_p: \boldsymbol{C}_p U^- \to \boldsymbol{C}_p X^-$ *is an isomorphism and* $R_{p,K,S} \neq 0$.

PROOF. First let S consist precisely of the divisors of ∞ and p. Under the hypotheses made in (2.14) one can show that the multiplicity of each irreducible representation V of $G=\langle G_0, \tau\rangle$ in $\bar{Q}X^-$ is either zero or one. In the former case, $R_p(V^\beta)=1$, in the latter $R_p(V^\beta)\neq 0$ by (2.13). The result for general S follows easily. □

It may be helpful to unwind the statements in (2.11) and (2.12) a bit further. First note that by the definitions of λ and g we have

$$R(V^\alpha)=\prod_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\nmid\infty}}(-\log p_{\mathfrak{p}})^{\dim V^{G_{\mathfrak{p}}}} \tag{2.15}$$

where $p_{\mathfrak{p}}$ is the characteristic of the residue field at $\mathfrak{p}$. The complex number $L(V^\alpha)$ can be easily calculated from the value $L_{S_\infty}(V^\alpha, 0)=a_{S_\infty}(V, 0)^\alpha\neq 0$. Putting this all together, we obtain the formula

$$A(V)=(-1)^{r(V)}\cdot\prod_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\nmid\infty}}\det(1-\sigma_{\mathfrak{p}}\,|\,V^{I_{\mathfrak{p}}}/V^{G_{\mathfrak{p}}})\cdot f_{\mathfrak{p}}^{\dim V^{G_{\mathfrak{p}}}}\cdot a_{S_\infty}(V, 0)\,. \tag{2.16}$$

For example, suppose V has dimension 1 and corresponds to a totally odd quadratic character χ of $\mathrm{Gal}(\bar{k}/k)$. Then $E=\boldsymbol{Q}$ and K is a quadratic extension with k as its maximal real subfield. By the analytic class-number formula

$$a_{S_\infty}(\chi, 0)=\lim_{s\to 0}\zeta_K(s)/\zeta_k(s)=2^g h^*/WQ\,. \tag{2.17}$$

Here g is the degree of k, $h^*=h_K/h_k$ is the relative class number, W is the order of μ_K, and $Q=\mathrm{Card}(U_{S_\infty, K}/\mu_K U_{S_\infty, k})$ is the unit index. We remark that h^* is an integer, Q is equal to 1 or 2, and $2^g=\prod_{\mathfrak{p}\in S_\infty}(1-\chi(\mathfrak{p}))$. The regulator $R(\chi)$ associated to this representation is just the regulator $R_{p,K,S}$ defined by (1.21). Hence conjecture 2.12 becomes the statements:

a) $\mathrm{ord}_{s=0}L_p(\omega\chi, s)\overset{?}{=}r(\chi)=\mathrm{Card}\{\mathfrak{p}\in S:\chi(\mathfrak{p})=1\}$.

b) $\lim_{s\to 0}L_p(\omega\chi, s)/s^{r(\chi)}\overset{?}{=}(-1)^{r(\chi)}\cdot\prod_{\substack{\mathfrak{p}\in S\\ \chi(\mathfrak{p})\neq 1}}(1-\chi(\mathfrak{p}))\cdot\prod_{\substack{\mathfrak{p}\in S\\ \chi(\mathfrak{p})=1}}f_{\mathfrak{p}}\cdot h^*/WQ\cdot R(\chi)$.

Some theoretical evidence for these identities is provided in [6].

§ 3. The first derivative of abelian L-series.

In this section we assume that the CM field K is an *abelian* extension of the totally real field k. Let W denote the order of μ_K.

Let n be the exponent of $G=\mathrm{Gal}(K/k)$ and assume that E contains the n^{th}-roots of unity. Then all irreducible E-linear representations of G have dimension one and are given by characters $\chi: G\to E^*$. The character χ is totally odd if and only if $\chi(\tau)=-1$.

Let S be a finite set of places of k which contains all divisors of ∞ and p, as well as all places which ramify in K. Let χ be a totally odd character of G; if $\mathfrak{p}$ ramifies in K we put $\chi(\mathfrak{p})=0$, otherwise we let $\chi(\mathfrak{p})=\chi(\sigma_{\mathfrak{p}})$. By (2.8) we have the formula:

$$r(\chi)=\mathrm{Card}\,\{\mathfrak{p}\in S:\chi(\mathfrak{p})=1\}. \tag{3.1}$$

If $r(\chi)=0$ then conjecture (2.12) is true, by formula (2.4). In this case $A(\chi)=a(\chi, 0)$ and $R(\chi^{\beta})=1$ for all $\beta: E\to \boldsymbol{C}_p$. In this section we will examine conjecture (2.12) in the case when $r(\chi)=1$; to insure that $r(\chi)\geqq 1$ for all characters χ of G, we make the further assumption that S contains a (finite) place $\mathfrak{p}$ which splits completely in K.

Instead of using the abelian L-functions, it is convenient to formulate our results with the partial zeta-functions of K/k (all defined relative to the set S). For $\sigma\in G$ we define the complex function:

$$\zeta(\sigma, s)=\zeta_S(\sigma, s)=\sum_{\substack{(\mathfrak{a}, S)=1\\ \sigma_{\mathfrak{a}}=\sigma}} N\mathfrak{a}^{-s}. \tag{3.2}$$

This series converges for $\mathrm{Re}(s)>1$ and has a meromorphic continuation to $\boldsymbol{C}$, regular outside $s=1$. The values of $\zeta(\sigma, s)$ at negative integers are rational numbers and $W\zeta(\sigma, 0)$ is an integer [2], [10].

The p-adic partial zeta functions $\zeta_p(\sigma, s)=\zeta_{p,S}(\sigma, s)$ are also meromorphic on $\boldsymbol{Z}_p$ and regular outside $s=1$. Let $d=[k(\mu_{2p}):k]$; then for all negative integers $n\equiv 0 \pmod d$ we have:

$$\zeta_p(\sigma, n)=\zeta(\sigma, n). \tag{3.3}$$

The existence of such a function follows from results of Deligne and Ribet [2], [8].

Write $S=\{\mathfrak{p}\}\cup T$; then T contains all places dividing ∞ and all places which ramify in K/k. In particular, $\mathrm{Card}(T)\geqq 2$. We may define the complex partial zeta functions relative to T and find $\zeta(\sigma, s)=(1-\sigma_{\mathfrak{p}}N\mathfrak{p}^{-s})\zeta_T(\sigma, s)=(1-N\mathfrak{p}^{-s})\cdot\zeta_T(\sigma, s)$. Hence

$$\left\{\begin{array}{l}\zeta(\sigma, 0)=0\\ \zeta'(\sigma, 0)=\log N\mathfrak{p}\cdot\zeta_T(\sigma, 0)\end{array}\right. \qquad \text{for all } \sigma\in G. \tag{3.4}$$

Using (3.3) we may conclude that

$$\zeta_p(\sigma, 0)=0 \qquad \text{for all } \sigma\in G. \tag{3.5}$$

If the splitting place $\mathfrak{p}$ does *not* divide p, then T also contains all divisors of p and we may define the p-adic partial zeta functions relative to T. In this case we find $\zeta_p(\sigma, s)=(1-\langle N\mathfrak{p}\rangle^{-s})\zeta_{p,T}(\sigma, s)$, so

$$\zeta'_p(\sigma, 0)=\log_p N\mathfrak{p}\,\zeta_{p,T}(\sigma, 0) \qquad \text{if } \mathfrak{p} \text{ does not divide } p. \tag{3.6}$$

Now define the group:

$$U_{\mathfrak{p}}=\{\varepsilon\in K^*: \|\varepsilon\|_{\mathfrak{Q}}=1 \text{ if } \mathfrak{Q} \text{ does not divide } \mathfrak{p}\}. \tag{3.7}$$

Then $U_{\mathfrak{p}}$ is a subgroup of $U_{\bar{S},K}$ of rank $[K:k]/2$; its torsion subgroup is μ_K.

PROPOSITION 3.8. *Let $\mathfrak{P}$ be a divisor of $\mathfrak{p}$ in K. Then there is a unique element $u=u(\mathfrak{P})$ in $\boldsymbol{Q}U_{\mathfrak{p}}$ such that*

$$-\zeta'(\sigma, 0)=\log\|u^{\sigma}\|_{\mathfrak{P}} \qquad \textit{for all } \sigma\in G. \tag{3.9}$$

The following two statements are then equivalent:

a) $-\zeta_p'(\sigma, 0)=\log_p\|u^{\sigma}\|_{\mathfrak{P},p}$ *for all* $\sigma\in G$.

b) *Conjecture* (2.12) *is true for all totally odd characters* $\chi: G\to \boldsymbol{C}_p^*$ *with* $r(\chi)=1$; *for all totally odd characters with* $r(\chi)\geqq 2$ *we have* $r_p(\chi)\geqq 2$.

PROOF. Since $\zeta'(\sigma, 0)=\log N\mathfrak{p}\zeta_T(\sigma, 0)=\log N\mathfrak{P}\zeta_T(\sigma, 0)$ the element $u\in\boldsymbol{Q}U_{\mathfrak{p}}$ satisfying (3.9) is uniquely determined by the condition:

$$\mathrm{ord}_{\mathfrak{P}}(u^{\sigma})=\mathrm{ord}_{\sigma^{-1}(\mathfrak{P})}(u)=\zeta_T(\sigma, 0).$$

If we let $\theta=\sum\zeta_T(\sigma, 0)\sigma^{-1}$ in $\boldsymbol{Q}[G]$, then u generates the rational ideal $\mathfrak{P}^{\theta}=\mathfrak{P}^{(1-\tau)\theta/2}$.

Let $\chi: G\to\boldsymbol{C}_p^*$ be totally odd, and choose m so that the ideal $\mathfrak{P}^{(1-\tau)m}$ is principal with a generator α in U. Let $v=\dfrac{1}{2m}\otimes\alpha$ in $\boldsymbol{Q}U_{\mathfrak{p}}$. When $r(\chi)=1$ we find the formulas:

$$\begin{aligned} A(\chi)&=-f_{\mathfrak{p}}\sum_G\chi(\sigma)\zeta_T(\sigma, 0)\\ R_p(\chi)&=\frac{1}{f_{\mathfrak{p}}}\sum_G\chi(\sigma)\log_p\|v^{\sigma}\|_{\mathfrak{P},p}. \end{aligned} \tag{3.10}$$

The first follows from (2.16). To prove the second, note that the divisor $D=\sum_G\chi(\sigma)\mathfrak{P}^{\sigma}$ generates the one-dimensional space $(\chi\otimes C_pX^-)^G$. Hence $\lambda_p g^{-1}(D)=R_p(\chi)\cdot D$; a short computation then yields (3.10). Note that $R(\chi)\neq 0$ by (2.13).

Since v generates the rational ideal $\mathfrak{P}^{(1-\tau)/2}$ we have $u=v^{\theta}$ in $\boldsymbol{Q}U_{\mathfrak{p}}$. Hence

$$\begin{aligned} A(\chi)R(\chi)&=-\sum_{\sigma'}\chi(\sigma')\zeta_T(\sigma', 0)\cdot\sum_{\sigma}\chi(\sigma)\log_p\|v^{\sigma}\|_{\mathfrak{P},p}\\ &=-\sum_{\sigma}\chi(\sigma)\log_p\|u^{\sigma}\|_{\mathfrak{P},p}. \end{aligned} \tag{3.11}$$

This identity actually holds for all characters χ of G; when χ is not totally odd or when $r(\chi)\geqq 2$, both sides of (3.11) are equal to zero.

But condition a) of (3.8) is equivalent to the statement a′) $$L'_p(\omega\chi,\,0)=-\sum_G \chi(\sigma)\log_p\|u^\sigma\|_{\mathfrak{P},\,p}$$

for all characters χ of G. Combining this with (3.11), we see that a) is equivalent to b). □

To state a refined form of conjecture (2.12) when $r(\chi)=1$ we need a result of Tate [13, 4.1].

PROPOSITION 3.12. *Let* $u\in \boldsymbol{Q}U_{\mathfrak{p}}$. *Then the following three conditions are equivalent*:

a) $Wu=1\otimes\varepsilon$, *where* ε *is in* $U_{\mathfrak{p}}$ *and* $\sqrt[W]{\varepsilon}$ *generates an abelian extension of* k.

b) $u=1\otimes\delta$, *where* δ *is in* $U_{\mathfrak{p},L}$ *and* L *is an abelian extension of* k.

c) *For all prime ideals* $\mathfrak{q}$ *of* k, *relatively prime to* W *and* S, *we have* $(\sigma_{\mathfrak{q}}-N\mathfrak{q})u=1\otimes\varepsilon_{\mathfrak{q}}$ *for a unique* $\varepsilon_{\mathfrak{q}}$ *in* U *with* $\varepsilon_{\mathfrak{q}}\equiv 1 \bmod \mathfrak{q}O_K$.

CONJECTURE 3.13. *The element* $u=u(\mathfrak{P})$ *in* $\boldsymbol{Q}U_{\mathfrak{p}}$ *determined by proposition* 3.8 *satisfies the three equivalent properties of proposition* 3.12 *as well as the identities*: $\zeta'_p(\sigma,\,0)=-\log_p\|u^\sigma\|_{\mathfrak{P},\,p}$ *for all* σ *in* G.

As presented, this conjecture depends on the extension K/k, the finite set $S=\{\mathfrak{p}\}\cup T$, and the choice of prime $\mathfrak{P}$ dividing $\mathfrak{p}$. It is clearly independent of $\mathfrak{P}$: one can take $u(\mathfrak{P}^\sigma)=u(\mathfrak{P})^\sigma$. If it is true for S it is true for the set $S^*=S\cup\{\mathfrak{p}^*\}$. One can take $u^*=(1-\sigma_{\mathfrak{p}^*})u$. Finally, if K' is a subfield of K containing k and conjecture (3.13) is true for $(K/k,\,S)$, it is also true for $(K'/k,\,S)$ [13, 4.7].

The consequences of conjecture (3.13) are particularly striking when the completion $k_{\mathfrak{p}}\cong K_{\mathfrak{P}}$ is isomorphic to $\boldsymbol{Q}_p$. Let W_p be the order of $\mu_{\boldsymbol{Q}_p}$ and write $W_p=m\cdot W$.

PROPOSITION 3.14. *Assume that conjecture* 3.13 *is true and* $K_{\mathfrak{P}}\cong\boldsymbol{Q}_p$; *then*

a) *The ideal* $\mathfrak{P}^{W_p\theta}$ *has a unique generator* α *in* $U_{\mathfrak{p}}$ *which lies in the subgroup* $p^{\boldsymbol{Z}}\cdot(1+2p\boldsymbol{Z}_p)$ *of* $K^*_{\mathfrak{P}}\cong\boldsymbol{Q}^*_p$.

b) *All of the conjugates* α^σ *of* α *also lie in the subgroup of the completion at* $\mathfrak{P}$.

c) *For each* $\sigma\in G$ *we have the analytic formula*

$$\begin{aligned}(3.15)\qquad \alpha^\sigma &= p^{W_p\zeta_T(\sigma,0)}\exp_p(-W_p\zeta'_p(\sigma,\,0))\\ &=\exp(-W_p\zeta'(\sigma,\,0))\cdot\exp_p(-W_p\zeta'_p(\sigma,\,0)) \qquad \text{in } K_{\mathfrak{P}}.\end{aligned}$$

PROOF. By (3.9) and a) of (3.12) we have an element $\varepsilon\in U_{\mathfrak{p}}$ with $(\varepsilon)=\mathfrak{P}^{W\theta}$.

Hence ε^m is a generator of $\mathfrak{P}^{W_p\theta}$. But ε^m lies in the subgroup $p^{\boldsymbol{Z}}\cdot\mu_W\cdot(1+2p\boldsymbol{Z}_p)$ of $\boldsymbol{Q}_p^*$. Hence we may multiply ε^m by a unique root of unity ζ in K^* such that $\alpha=\zeta\varepsilon^m$ lies in $p^{\boldsymbol{Z}}\cdot(1+2p\boldsymbol{Z}_p)$.

To prove b), choose an ideal $\mathfrak{a}$ prime to S with $\sigma_{\mathfrak{a}}=\sigma$. Since $\sqrt[W]{\varepsilon}$ is abelian over k by a) of (3.12), we have $\varepsilon^{\sigma_{\mathfrak{a}}-N\mathfrak{a}}$ in $(K^*)^W$ by Kummer theory. Hence $\alpha^{\sigma_{\mathfrak{a}}-N\mathfrak{a}}=(\zeta)^{\sigma_{\mathfrak{a}}-N\mathfrak{a}}$. $(\varepsilon^m)^{\sigma_{\mathfrak{a}}-N\mathfrak{a}}=(\varepsilon^{\sigma_{\mathfrak{a}}-N\mathfrak{a}})^m$ is in $(K^*)^{W_p}$. Since the latter group clearly injects into $p^{\boldsymbol{Z}}\cdot(1+2p\boldsymbol{Z}_p)$, α^σ is also contained in this subgroup of the completion.

By (3.13) we have the formula:

$$-W_p\zeta_p'(\sigma,0)=\log_p\|\alpha^\sigma\|_{\mathfrak{P},p}=\log_p(\|\alpha^\sigma\|_{\mathfrak{P}}\cdot\alpha^\sigma) \qquad \text{in } K_{\mathfrak{P}}.$$

Since $\|\alpha^\sigma\|_{\mathfrak{P}}\cdot\alpha^\sigma$ lies in $(1+2p\boldsymbol{Z}_p)$, we find after exponentiation that:

$$\exp_p(-W_p\zeta_p'(\sigma,0))=\|\alpha^\sigma\|_{\mathfrak{P}}\cdot\alpha^\sigma \qquad \text{in } K_{\mathfrak{P}}.$$

Part c) then follows from the formula $\|\alpha^\sigma\|_{\mathfrak{P}}=p^{-W_p\zeta_T(\sigma,0)}$. □

§ 4. Gauss sums and the abelian case.

In this section we will prove conjecture (3.13) when $k=\boldsymbol{Q}$. We will use this to show that conjecture (2.12) is true for all *abelian* representations V of $\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})$.

Let K be a complex abelian extension of $\boldsymbol{Q}$ of conductor (m) and let $\mathfrak{p}$ be a finite place of $\boldsymbol{Q}$ which splits completely in K. By the Kronecker-Weber theorem, K is contained in the subfield F of $\boldsymbol{Q}(\mu_m)$ which is fixed by the decomposition group of $\mathfrak{p}$. Let S consist of the places of $\boldsymbol{Q}$ dividing ∞, p, (m), and $\mathfrak{p}$ and write $S=\{\mathfrak{p}\}\cup T$. By the remarks following (3.13) it suffices to prove conjecture (3.13) for the set S. Similarly, we may assume that $K=F$.

Let q be the rational prime which generates the ideal $\mathfrak{p}$. Under the Artin isomorphism $\mathrm{Gal}(\boldsymbol{Q}(\mu_m)/\boldsymbol{Q})\cong(\boldsymbol{Z}/m)^*$ the Frobenius element $\sigma_{\mathfrak{p}}$ corresponds to the class $q \pmod{m}$. Let f be the order of this element in the Galois group; we have the field diagram:

(4.1)
$$\begin{tikzcd}[column sep=small]
\boldsymbol{Q}(\mu_m) \arrow[dr, dash] & & \\
 & K \arrow[dr, dash, "G\cong(\boldsymbol{Z}/m)^*/\langle q\rangle."] & \\
 & & \boldsymbol{Q}
\end{tikzcd}$$

For $a\in(\boldsymbol{Z}/m)^*$ we let σ_a denote the corresponding element of G: this depends only on the coset of a mod $\langle q\rangle$. We let $\langle a/m\rangle$ unique rational number between 0 and 1 which is congruent to $a/m \pmod{\boldsymbol{Z}}$.

The most interesting case of (3.13) is when $q=p$. Here we need two analytic formulas; the first due to Hurwitz and the second to Ferrero and Greenberg [3]:

$$\zeta_T(\sigma_a, 0)=\sum_{i=1}^{f}\left(\frac{1}{2}-\langle p^i a/m\rangle\right)=\frac{f}{2}-\sum_{i=1}^{f}\langle p^i a/m\rangle, \tag{4.2}$$

$$\zeta'_p(\sigma_a, 0)=\sum_{i=1}^{f}\log_p\Gamma_p\langle p^i a/m\rangle=\log_p\left(\prod_{i=1}^{f}\Gamma_p\langle p^i a/m\rangle\right). \tag{4.3}$$

Let $\mathfrak{P}$ be a divisor of $\mathfrak{p}=(p)$ in K. To produce the element $u=u(\mathfrak{P})$ in $\boldsymbol{Q}U_{\mathfrak{p}}$ we appeal to the theory of Gauss sums. Let $\mathcal{O}$ be the ring of integers of $\boldsymbol{Q}(\mu_m)$ and let $\chi:(\mathcal{O}/\mathfrak{P}\mathcal{O})^*\to\mu_m$ be the m^{th} power residue symbol. Let $\psi:\boldsymbol{Z}/p\to\mu_p$ be a nontrivial additive character, and define the Gauss sum:

$$g=-\sum_{\alpha\in(\mathcal{O}/\mathfrak{P}\mathcal{O})^*}\chi^{-1}(\alpha)\psi(\operatorname{Tr}\alpha). \tag{4.4}$$

This element is denoted $g(1/m, \mathfrak{P}\mathcal{O}, \psi\circ\operatorname{Tr})$ in [7].

It is easy to check, using Galois theory, that g lies in the subfield $K(\mu_p)$ of $\boldsymbol{Q}(\mu_{mp})$, and that conjugation by an automorphism of μ_p multiplies g by a root of unity. One also has $g^{1+\tau}=p^f=N(\mathfrak{P}\mathcal{O})$ and Stickelberger's theorem gives the ideal factorization:

$$(g)=\mathfrak{P}^{\sum_G\left(\sum_{i=1}^{f}\langle p^i a/m\rangle\right)\sigma_a^{-1}}. \tag{4.5}$$

Let $h=\sqrt{(-1/p)p}$ if p is odd and $h=(1+i)$ if $p=2$. Then the element $\delta=g/h^f$ lies in the abelian extension $L=K(\mu_{2p})$ of $\boldsymbol{Q}$, and the image $u=1\otimes\delta$ in $\boldsymbol{Q}U_{\mathfrak{p},L}$ actually lies in $\boldsymbol{Q}U_{\mathfrak{p}}$. By (4.2) and (4.5) this is precisely the element $u(\mathfrak{P})$ which satisfies (3.9); by its very construction it satisfies b) of (3.12).

To complete the proof of conjecture (3.13) when $q=p$, we must show that $\zeta'_p(\sigma_a, 0)=\log_p\|u^{\sigma_a}\|_{\mathfrak{P},p}$ for all σ_a in G. By (4.3) and the definition of u, this is equivalent to the p-adic identity:

$$\log_p\left(\prod_{i=1}^{f}\Gamma_p\langle p^i a/m\rangle\right)=\log_p(g^{\sigma_a}) \tag{4.6}$$

which follows from the main result of Gross-Koblitz [7, 1.7].

The case when $\mathfrak{p}=(q)$ with $q\neq p$ is similar, but less difficult as no results from p-adic analysis are needed. We again construct a Gauss sum g using a divisor $\mathfrak{P}$ of $\mathfrak{p}$ in K and let $\delta=g/h^f$ in $\boldsymbol{Q}(\mu_{2q})$. In this case we take $u=1\otimes\delta$ only if (p) ramifies in K, otherwise we take $u=(1-\sigma_p)\,1\otimes\delta$. It is easy to check that this element has the requisite properties.

We now turn to a proof of conjecture (2.12) for all totally odd abelian representations V of $\operatorname{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})$. Since the conjecture is compatible with direct sums, we may assume that V is irreducible. Hence V has dimension one and is given by an odd character $\chi: \mathrm{Gal}(\bar{Q}/Q) \to E^*$. We may also assume that S is minimal, so S consists only of the places ∞ and p.

There are then only two possibilities: $r(\chi)=0$ or $r(\chi)=1$. In the former case the conjecture is true by formula (2.4); in the latter it follows from our proof of conjecture (3.13) and proposition (3.8).

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(Received June 16, 1981)

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