On L-dimension of coherent sheaves

By Takao FUJITA*)

(Communicated by N. Iwahori)

In old days some geometers considered "Riemann-Roch Problem" as the following one: *Let L be a line bundle on a variety V. Then, how is the structure of the graded algebra* $\bigoplus_{t\geq 0} H^0(V, tL)$?

Unfortunately, this turned to be not at all easy. Thus, nowadays, because of the brilliant success of Hirzebruch, "Riemann-Roch" means almost always a result on $\chi(V, tL)$, not on $H^0(V, tL)$ itself.

One of the most fundamental tools to study the original problem in general is the notion of *L-dimension* due to Iitaka together with his fibration theorem for the rational mapping defined by $|tL|$ (see [I]). Generalizing his theory, we consider L-dimension of a coherent sheaf $\mathscr{F}$, which is defined by the asymptotic behaviour of $h^0(\mathscr{F}[tL])$ when $t\to\infty$. This enables us to avoid non-singularity assumption in many cases, and thus our theory works in positive characteristic cases too.

However, our analogue of the fibration theorem is less satisfactory than that of Iitaka. This reflects a real difficulty in the classification theory of algebraic varieties, which takes the form of the existence of quasi-elliptic surfaces in 2-dimensional cases. (See (3.16).)

This paper is organized in the following way. In section 1 we review a couple of theorems of Bertini type in arbitrary characteristic cases. In section 2 we introduce the notion of L-dimension of coherent sheaves. In section 3 we establish a fibration theorem of Iitaka type.

Notation, Convention and Terminology.

Usually we work in the category of K-schemes of finite type, where K is an algebraically closed field of any characteristic. Occasionally we assume K to be *sufficiently big*, that is, every object involved is defined over a subfield K' of K such that $\mathrm{tr.deg}\,(K/K')=\infty$. Some arguments work in other categories too, e. g., that of complex analytic spaces. Any way, an object of our category will be

*) This article was completed when the author was a Miller Fellow at the University of California, Berkeley, during the term 1979-1981.

called a space, and is considered to be a local ringed space.

Basically we employ a similar notation as in [EGA] and [H]. A space is said to be *irreducible* if the underlying topological space is so with respect to its Zariski topology. *Variety* is an irreducible, reduced space. *Point* means a closed point. The meaning of "*generic point*" is similar to that of [W]. Line bundles and invertible sheaves are confused with linear equivalence classes of Cartier divisors, and their tensor products are denoted additively.

Finally we list up our notations which may not be standard.

$\mathrm{Supp}(\mathcal{F})$: The support of a coherent sheaf $\mathcal{F}$ on a space S.

$\mathcal{F}_T$: The pull-back of $\mathcal{F}$ to a space T by a given morphism $T \to S$. Similar notation is used for line bundles, linear systems, etc.

$\mathcal{F}[tL] := \mathcal{F} \otimes L^{\otimes t}$, where L is a line bundle.

ρ_Λ: The rational mapping defined by a linear system Λ.

$Bs\Lambda$: The intersection of all the members of Λ.

Λ is identified with a finite dimensional vector subspace of $H^0(S, [\Lambda])$. A member of Λ is the zero-scheme of a section in this subspace.

A morphism $f: T \to S$ is said to be *birational*, if there is an open dense subset U of S such that $f^{-1}(U) \cong U$ as open subschemes of T and S respectively. Of course, if both T and S are varieties, this is equivalent to say $K(S) \cong K(T)$.

§1. Bertini theorems for coherent sheaves.

In this section we make a review of several tools which are used in the later sections. Most of them are well known to experts, possibly except differences in terminology. But it may not be easy for beginners to find proofs. So we give outlines of them too.

(1.1) THEOREM. *Let S be a space and let $\mathcal{F}$ be a coherent sheaf on S. Then there exists a locally finite family $\{X_\alpha\}$ of irreducible subsets in S such that any irreducible component of the support of any subsheaf of $\mathcal{F}$ is one of X_α.*

PROOF. Any ascending chain of subsheaves of $\mathcal{F}$ is locally of finite stable range. By the standard argument of Noetherian decomposition, we prove the theorem.

(1.2) COROLLARY. *Let S, $\mathcal{F}$ be as above and let Λ be a linear system on S. Then,* $\mathrm{Supp}(\mathrm{Ker}(\delta_{\mathcal{F}})) \subset Bs\,\Lambda$ *for a general member D of Λ, where $\delta_{\mathcal{F}}$ is the natural homomorphism $\mathcal{F}[-D] \to \mathcal{F}$ induced by the defining section of D.*

PROOF. Let $\{X_\alpha\}$ be the associated components of $\mathcal{F}$ as in (1.1). Take D so that $D \not\supset X_\alpha$ for any α with $X_\alpha \not\subset Bs\,\Lambda$. Then D satisfies the desired condition.

In order to visualize the power of this fact, we need a theory of dualizing sheaves.

(1.3) NOTATION. $F(S)$ denotes the category of coherent sheaves on a space S. $\mathcal{F}\in F(S)$ means that $\mathcal{F}$ is an object of this category. If T is a locally closed subspace of S, then $\rho_{T\subset S}$ denotes the restriction functor $F(S)\to F(T)$. $\rho_{T\subset S}(\mathcal{F})$ is often denoted by $\mathcal{F}_T$ for $\mathcal{F}\in F(S)$. If $\iota: S\to U$ is a closed embedding, there is a functor $\iota_*: F(S)\to F(U)$. For $\mathcal{F}\in F(S)$, $\iota_*\mathcal{F}$ is denoted by $\mathcal{F}$ by abuse of notation.

(1.4) THEOREM. *There exists a contravariant functor $\mathcal{D}_S^q: F(S)\to F(S)$ for every space S and every integer q, which satisfies the following conditions.*

a) *If T is an open subspace of S, there is a natural functorial isomorphism $\mathcal{D}_T^q\circ\rho_{T\subset S}=\rho_{T\subset S}\circ\mathcal{D}_S^q$.*

b) *If $\iota: S\to U$ is a closed embedding, there is a natural functorial isomorphism $\mathcal{D}_U^q\circ\iota_*=\iota_*\circ\mathcal{D}_S^q$.*

c) *If S is a manifold of dimension n, there is a natural functorial isomorphism $\mathcal{D}_S^q(*)=\mathcal{E}xt_{\mathcal{O}_S}^{n-q}(*, \omega_S)$, where ω_S is the sheaf of Kaehler n-differentials on S.*

OUTLINE OF PROOF. First we remark that the above conditions determine the functor $\mathcal{D}_S^q$ uniquely. Indeed, take an affine covering $\{T_\alpha\}$ of S such that each T_α is a locally closed subset of A^{N_α}. Take an open neighbourhood U of $T=T_\alpha$ in which T is closed. Then, by the conditions a), b) and c), we have $\mathcal{D}_S^q(\mathcal{F})_T=\mathcal{E}xt_{\mathcal{O}_U}^{N-q}(\mathcal{F}, \omega_U)_T$ for each $\mathcal{F}\in F(S)$, and $\mathcal{D}_S^q(\mathcal{F})$ must be obtained by patching them.

Set $\mathcal{D}_{T,U}^q(\mathcal{F})$ be the right hand side of the above equality. If this is independent of the choice of the embedding $T\subset U\subset A^N$, then $\mathcal{D}_S^q(\mathcal{F})$ is well-defined and we are done. Thus, what we should show is the existence of a natural isomorphism $\mathcal{D}_{T,U}^q(\mathcal{F})\cong\mathcal{D}_{T,V}^q(\mathcal{F})$ for any other embedding $T\subset V\subset A^M$.

Step 1, the case in which U is a closed submanifold of V. The above isomorphism is obtained by the natural isomorphism $\mathcal{E}xt_{\mathcal{O}_V}^{M-N}(\mathcal{O}_U, \omega_V)\cong\omega_U$ and $\mathcal{E}xt_{\mathcal{O}_V}^p(\mathcal{O}_U, \omega_V)=0$ for $p\neq M-N$.

Step 2, the case in which $V=U\times A^{M-N}$ with $\pi\circ\iota_V=\iota_U$, where π is the projection $V\to U$ and ι_V, ι_U are embeddings of T in V and U. The problem is local with respect to U, so we may assume U to be affine. Then we have an embedding $\sigma: U\to V$ such that $\sigma\circ\iota_U=\iota_V$ and $\pi\circ\sigma=id_U$. Now we have $\mathcal{D}_{T,U}^q=\mathcal{D}_{T,\sigma(U)}^q=\mathcal{D}_{T,V}^q$, where the last isomorphism is obtained in Step 1.

Step 3, the general case. We have a closed embedding $T\subset U\times V$ induced by ι_U and ι_V. Moreover, $T\subset U\times V\subset U\times A^M$ is also closed. Hence we have $\mathcal{D}_{T,U}^q=\mathcal{D}_{T,U\times A^M}^q=\mathcal{D}_{T,U\times V}^q$ by Step 2. Similarly we have $\mathcal{D}_{T,V}^q=\mathcal{D}_{T,U\times V}^q$.

Step 4. Finally we check the cocycle condition. Namely, if $T\subset U$, $T\subset V$ and $T\subset W$ are three embeddings, then the isomorphisms among $\mathcal{D}_{T,*}^q$ should make a commutative triangle. This is easily seen in case $U\subset V\subset W$. Examining Step 2, we see this for $T\subset U$, $T\subset U\times V$ and $T\subset U\times V\times W$. In the general case we use the following diagram:

$$\begin{tikzcd}
 & \mathscr{D}_{T,U}{}^q \arrow[rr, equal] \arrow[dl, equal] \arrow[dr, equal] & & \mathscr{D}_{T,U\times V}{}^q \arrow[dl, equal] \arrow[dr, equal] & \\
\mathscr{D}_{T,U\times W}{}^q \arrow[rr, equal] \arrow[dr, equal] & & \mathscr{D}_{T,U\times V\times W}{}^q \arrow[rr, equal] \arrow[dl, equal] \arrow[dr, equal] & & \mathscr{D}_{T,V}{}^q \arrow[dl, equal] \\
 & \mathscr{D}_{T,W}{}^q \arrow[rr, equal] & & \mathscr{D}_{T,V\times W}{}^q . &
\end{tikzcd}$$

REMARK. We sometimes write $\mathscr{D}^q$ in stead of $\mathscr{D}^q_S$ when there is no danger of confusion.

(1.5) COROLLARY. *Let $0\to\mathscr{F}\to\mathscr{G}\to\mathscr{H}\to 0$ be an exact sequence in $F(S)$. Then this induces a natural long exact sequence* $\cdots\to\mathscr{D}^{q+1}(\mathscr{F})\to\mathscr{D}^q(\mathscr{H})\to\mathscr{D}^q(\mathscr{G})\to\mathscr{D}^q(\mathscr{F})\to\mathscr{D}^{q-1}(\mathscr{H})\to\cdots$.

(1.6) COROLLARY. *Suppose S to be a closed subspace of a manifold P. Then* $\mathscr{D}^q(*)=\mathcal{E}xt^{N-q}_{\mathcal{O}_P}(*,\omega_P)$ *where* $N=\dim P$.

(1.7) COROLLARY. *Suppose in addition that P is proper over K. Then, for any $\mathscr{F}\in F(S)$, there is a spectral sequence with $E_2^{p,q}=H^p(S,\mathscr{D}^{-q}(\mathscr{F}))$ converging to the dual of $H^{-p-q}(S,\mathscr{F})$.*

Indeed, this is just the well known *Ext*-spectral sequence in view of the Serre duality on P.

(1.8) THEOREM. $\dim(\mathrm{Supp}(\mathscr{D}^q(\mathscr{F})))\leqq q$. *In particular, $\mathscr{D}^q(\mathscr{F})=0$ for $q<0$.*

PROOF. We may assume S to be an open subset of $\boldsymbol{A}^N$ since the problem is local and because of the property a) in (1.4). We use the induction on q. The assertion is clear for $q<0$ because the homological dimension of $\mathcal{O}_x$ is N for $x\in S$. So, assume that $\mathrm{Supp}(\mathscr{D}^q(\mathscr{F}))$ has a component X with $\dim X>q\geqq 0$. Let x be a generic point on X. Then, by (1.2), we can find a hyperplane H such that $H\ni x$ and the induced homomorphism $h:\mathscr{F}\to\mathscr{F}$ is injective. Let $\mathscr{C}=\mathrm{Coker}(h)$. Then $\mathscr{D}^q(\mathscr{F})\to\mathscr{D}^q(\mathscr{F})\to\mathscr{D}^{q-1}(\mathscr{C})$ is exact, where the first homomorphism is $\mathscr{D}^q(h)$. Hence $\mathrm{Supp}(\mathscr{D}^{q-1}(\mathscr{C}))\supset\mathrm{Supp}(\mathrm{Coker}(\mathscr{D}^q(h)))\supset X\cap H$ where the second inclusion follows from the Nakayama's Lemma. This implies $\dim\mathrm{Supp}(\mathscr{D}^{q-1}(C))\geqq\dim X-1>q-1$, contradicting the induction hypothesis.

(1.9) THEOREM. *The support of $\mathscr{D}^q(\mathscr{F})$ is contained in the union of irre-*

ducible components of Supp$(\mathcal{F})$ *with dimensions* $\geqq q$. *In particular,* $\mathcal{D}^q(\mathcal{F})=0$ *for* $q>\dim \operatorname{Supp}(\mathcal{F})$.

PROOF. It suffices to show the following claim for each point x on S: if d is the maximum of the dimensions of the components of Supp$(\mathcal{F})$ containing x, then $\mathcal{D}^q(\mathcal{F})_x=0$ for $q>d$. This problem is local, and we may assume that S is a locally closed subspace of A^N and $d=\dim \operatorname{Supp}(\mathcal{F})$. We use the induction on d. The assertion is clear for $d=0$. Suppose $d>0$. Let H be a general hyperplane section such that $x\in H$. Let $h: \mathcal{F}\to\mathcal{F}$ be the induced homomorphism and let $\mathcal{K}=\operatorname{Ker}(h)$, $\mathcal{I}=\operatorname{Im}(h)$ and $\mathcal{C}=\operatorname{Coker}(h)$. Then $\operatorname{Supp}(\mathcal{K})\subset\{x\}$ by (1.2) and $\dim(\operatorname{Supp}(\mathcal{C}))=d-1$. So $\mathcal{D}^q(\mathcal{K})=0$ for $q>0$ and this implies $\mathcal{D}^q(\mathcal{I})\cong\mathcal{D}^q(\mathcal{F})$ for $q>0$. We have also the exact sequence $\mathcal{D}^q(\mathcal{F})\to\mathcal{D}^q(\mathcal{I})\to\mathcal{D}^{q-1}(\mathcal{C})$, and the last term vanishes at x for $q>d$ by the induction hypothesis. Hence $\mathcal{D}^q(h)_x: \mathcal{D}^q(\mathcal{F})_x\to\mathcal{D}^q(\mathcal{F})_x$ is surjective. Since $x\in H$, this implies $\mathcal{D}^q(\mathcal{F})_x=0$ by Nakayama's lemma.

(1.10) DEFINITION. Let $\mathcal{F}\in F(S)$ and let X be an irreducible subset of S. We define the *rank* of $\mathcal{F}$ at X, denoted by $rk_X(\mathcal{F})$, as follows.

If $X\not\subset \operatorname{Supp}(\mathcal{F})$, then $rk_X(\mathcal{F})=0$.

Suppose X to be a component of Supp$(\mathcal{F})$. Let $\mathcal{I}=\operatorname{Ker}(\mathcal{O}_S\to\mathcal{E}nd(\mathcal{F}))$. Then $\mathcal{F}$ may be considered to be a sheaf on T, the subspace defined by the ideal $\mathcal{I}$. Of course $\operatorname{Supp}(\mathcal{F})=\operatorname{Supp}(T)$. Let $\mathcal{N}$ be the sheaf of nilpotent functions on T and let $\mathcal{F}_j=\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F}$. Then, in a neighbourhood of a generic point x of X, $\mathcal{F}_j$ looks like a locally free sheaf on X of rank r_j. We define $rk_X(\mathcal{F})=\sum_j r_j$.

In general, we define $rk_X(\mathcal{F})$ to be the maximum of $rk_X(\mathcal{G})$ where $\mathcal{G}$ runs through all the subsheaves of $\mathcal{F}$ such that X is a component of Supp$(\mathcal{G})$ (If no such subsheaf $\mathcal{G}$ exists, then $rk_X(\mathcal{F})=0$.).

(1.11) THEOREM. *Let X be an irreducible subset of a space S with* $\dim X=q$. *Then* $rk_X(\mathcal{D}^q(\mathcal{F}))=rk_X(\mathcal{F})$ *for any* $\mathcal{F}\in F(S)$.

PROOF. This is clear if $X\not\subset \operatorname{Supp}(F)$. Suppose that X is a component of Supp$(\mathcal{F})$. We may consider the problem in a neighbourhood of a generic point x of X. Letting the notations as in (1.10), we infer that $rk_X(\mathcal{F}_j)=rk_X(\mathcal{D}^q(\mathcal{F}_j))$ since $\mathcal{F}_j$ is locally free at x. In view of the exact sequence $0=\mathcal{D}^{q+1}(\mathcal{N}^{j+1}\mathcal{F})\to\mathcal{D}^q(\mathcal{F}_j)\to\mathcal{D}^q(\mathcal{N}^j\mathcal{F})\to\mathcal{D}^q(\mathcal{N}^{j+1}\mathcal{F})\to\mathcal{D}^{q-1}(\mathcal{F}_j)$ and $\mathcal{D}^{q-1}(\mathcal{F}_j)_x=0$ (cf. (1.8)), we infer easily $rk_X(\mathcal{D}^q(\mathcal{N}^j\mathcal{F}))=rk_X(\mathcal{N}^j\mathcal{F})$ by the descending induction on j, proving the assertion as a special case $j=0$.

Now we consider the general case, in which $X\subset \operatorname{Supp}(\mathcal{F})$. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$ such that X is a component of Supp$(\mathcal{G})$. Then $\mathcal{D}^q(\mathcal{F})\to\mathcal{D}^q(\mathcal{G})\to\mathcal{D}^{q-1}(\mathcal{F}/\mathcal{G})$ is exact and $\mathcal{D}^{q-1}(\mathcal{F}/\mathcal{G})_x=0$ for a generic point x on X. Hence $rk_X(\mathcal{D}^q(\mathcal{F}))\geqq rk_X(\mathcal{D}^q(\mathcal{G}))=rk_X(\mathcal{G})$. Thus we see $rk_X(\mathcal{D}^q(\mathcal{F}))\geqq rk_X(\mathcal{F})$.

In order to show the inequality of the converse direction, we let $\mathcal{A}$ be the kernel of the natural homomorphism $\mathcal{O}_S \to \mathcal{E}nd(\mathcal{D}^q(\mathcal{F}))$ and let $\mathcal{H}=\mathrm{Ker}(\mathcal{F}\to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{A}, \mathcal{F}))$. Then $\mathrm{Supp}(\mathcal{D}^q(\mathcal{F}))=\mathrm{Supp}(\mathcal{O}_S/\mathcal{A})\supset \mathrm{Supp}(\mathcal{H})$. So $\dim(\mathrm{Supp}(\mathcal{H}))\leq q$, which implies $rk_X(\mathcal{H})=rk_X(\mathcal{D}^q(\mathcal{H}))$. Hence it suffices to show $\mathcal{D}^q(\mathcal{F})_x\cong \mathcal{D}^q(\mathcal{H})_x$ at a generic point x on X.

Now the problem is local and we may consider everything in a small neighbourhood U of x. Let $a_1, \cdots, a_m$ be sections of $\mathcal{A}$ on U such that $\mathcal{A}$ is generated by them. We define subsheaves $\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_m$ of $\mathcal{F}$ inductively as follows. Set $\mathcal{H}_0=\mathcal{F}$. For $j>0$, set $\mathcal{H}_j=\mathrm{Ker}(\varphi_j)$, where φ_j is the homomorphism $\mathcal{H}_{j-1}\to\mathcal{H}_{j-1}$ induced by a_j. Then $\mathcal{H}_m=\mathcal{H}$ by definition. We prove $\mathcal{D}^q(\mathcal{F})_x\cong\mathcal{D}^q(\mathcal{H}_j)_x$ by induction on j, which is obvious for $j=0$. So suppose $j>0$. Let $\mathcal{I}_j=\mathrm{Im}(\varphi_j)$ and $\mathcal{C}_j=\mathrm{Coker}(\varphi_j)$. Then $\mathcal{D}^q(\mathcal{H}_{j-1})\to\mathcal{D}^q(\mathcal{I}_j)\to\mathcal{D}^{q-1}(\mathcal{C}_j)$ is exact. Hence $\mathcal{D}^q(\mathcal{H}_{j-1})_x \to\mathcal{D}^q(\mathcal{I}_j)_x$ is surjective at x by (1.8). On the other hand, $\mathcal{D}^q(\mathcal{I}_j)\to\mathcal{D}^q(\mathcal{H}_{j-1}) \to\mathcal{D}^q(\mathcal{H}_j)\to\mathcal{D}^{q-1}(\mathcal{I}_j)$ is exact. $\mathcal{D}^q(\varphi_j)_x=0$ by definition of φ_j and by the induction hypothesis. This implies that $\mathcal{D}^q(\mathcal{I}_j)_x\to\mathcal{D}^q(\mathcal{H}_{j-1})_x$ is a zero map. Then, by the above exact sequence and by (1.8) we infer that $\mathcal{D}^q(\mathcal{H}_{j-1})\cong\mathcal{D}^q(\mathcal{H}_j)$ at x, completing the induction. q. e. d.

(1.12) COROLLARY. *X is a component of* $\mathrm{Supp}(\mathcal{D}^q(\mathcal{F}))$ *if and only if there exists a subsheaf $\mathcal{G}$ of $\mathcal{F}$ such that X is a component of* $\mathrm{Supp}(\mathcal{G})$.

(1.13) DEFINITION. $\mathcal{F}\in F(S)$ is said to be *unmixed* if any irreducible component of the support of any subsheaf of $\mathcal{F}$ is a component of $\mathrm{Supp}(\mathcal{F})$. In particular, if $\mathcal{F}$ is a sheaf on a variety V with $\mathrm{Supp}(\mathcal{F})=V$, then $\mathcal{F}$ is unmixed if and only if $\mathcal{F}$ is torsion free.

(1.14) COROLLARY. *$\mathcal{F}$ is unmixed if and only if any q-dimensional component of* $\mathrm{Supp}(\mathcal{D}^q(\mathcal{F}))$ *is a component of* $\mathrm{Supp}(\mathcal{F})$ *for every q. In particular, if $\mathcal{F}$ is a sheaf on a variety V with* $\mathrm{Supp}(\mathcal{F})=V$, *$\mathcal{F}$ is torsion free if and only if* $\dim(\mathrm{Supp}(\mathcal{D}^q(\mathcal{F})))<q$ *for every* $q<\dim V$.

(1.15) THEOREM. *Let $\mathcal{F}$ be a coherent sheaf on S. Then there is a unique subsheaf $\mathcal{T}$ of $\mathcal{F}$ with the following property:*

a) No component of $\mathrm{Supp}(\mathcal{T})$ *is a component of* $\mathrm{Supp}(\mathcal{F})$.

b) The quotient $\mathcal{F}/\mathcal{T}$ is unmixed.

This will be called the unmixed part of $\mathcal{F}$.

PROOF. For an affine open subset U of S, let N_U be $\{\varphi\in H^0(U, \mathcal{F})\,|\,\varphi=0$ at any generic point of each component of $U\cap\mathrm{Supp}(\mathcal{F})\}$. Let $\mathcal{T}_U$ be the subsheaf of $\mathcal{F}_U$ generated by N_U. Letting U run through all the affine open subsets of S and patching $\mathcal{T}_U$ together, we obtain a subsheaf $\mathcal{T}$ of $\mathcal{F}$. It is easy to see that $\mathcal{T}$ has the desired property a) and b).

Now we prove the uniqueness. Let $\mathcal{G}$ be another subsheaf of $\mathcal{F}$ having the property *a*) and *b*). Consider the image $\mathcal{J}$ of the homomorphism $\mathcal{G}\to\mathcal{F}\to\mathcal{F}/\mathcal{T}$. Then $\mathrm{Supp}(\mathcal{J})\subset\mathrm{Supp}(\mathcal{G})$ contains no component of $\mathrm{Supp}(\mathcal{F})=\mathrm{Supp}(\mathcal{F}/\mathcal{T})$. Hence $\mathcal{J}=0$ since $\mathcal{F}/\mathcal{T}$ is unmixed. This implies $\mathcal{G}\subset\mathcal{T}$. Similarly we infer $\mathcal{T}\subset\mathcal{G}$. Thus $\mathcal{G}=\mathcal{T}$.

(1.16) THEOREM. *Let $\mathcal{F}$ be an unmixed sheaf on a space S and let Λ be a linear system on S such that $Bs\,\Lambda=\emptyset$. Then $\mathcal{F}_D$ is unmixed for a generic member D of Λ.*

PROOF. Let $X_1, \cdots, X_r$ be the irreducible components of $\mathrm{Supp}(\mathcal{F})$. Then, by (1.12), any q-dimensional component of $\mathrm{Supp}(\mathcal{D}^q(\mathcal{F}))$ is one of X_j. Let $\delta: \mathcal{F}[-D]\to\mathcal{F}$ be the homomorphism induced by D. Of course $\mathcal{F}_D=\mathrm{Coker}(\delta)$. We take D in such a way that *a*) δ is injective, *b*) $\mathcal{D}^q(\delta): \mathcal{D}^q(\mathcal{F})\to\mathcal{D}^q(\mathcal{F}[-D])=\mathcal{D}^q(\mathcal{F})[D]$ is injective for every q, *c*) any $(q-1)$-dimensional component of $D\cap\mathrm{Supp}(\mathcal{D}^q(\mathcal{F}))$ is a component of one of $\{D\cap X_j\}$ for every q, and *d*) any component of $D\cap X_j$ is a component of $D\cap\mathrm{Supp}(\mathcal{F})$. Then, by *a*) and *b*) we infer that $0\to\mathcal{D}^q(\mathcal{F})\to\mathcal{D}^q(\mathcal{F}[-D])\to\mathcal{D}^{q-1}(\mathcal{F}_D)\to 0$ is exact for any q. So $\mathrm{Supp}(\mathcal{D}^{q-1}(\mathcal{F}_D))=D\cap\mathrm{Supp}(\mathcal{D}^q(\mathcal{F}))$. Let Y be a component of $\mathrm{Supp}(\mathcal{D}^{q-1}(\mathcal{F}_D))$ with $\dim Y=q-1$. Then, by *c*), Y is a component of one of $\{D\cap X_j\}$. Hence, by *d*), Y is a component of $D\cap\mathrm{Supp}(\mathcal{F})=\mathrm{Supp}(\mathcal{F}_D)$. Thus $\mathcal{F}_D$ is unmixed by (1.12).

(1.17) COROLLARY. *Let $f: S\to V$ be a surjective morphism onto a variety V. Let F be a generic fiber of f. Then $\mathcal{F}_F$ is unmixed if $\mathcal{F}\in F(S)$ is so.*

PROOF. We may assume V to be locally closed in A^N. Taking a generic hyperplane and applying (1.16), we prove the assertion by induction on $\dim V$.

(1.18) COROLLARY. *Let f, S, V and F be as above. Suppose that F is irreducible and is non-singular at its generic point. Then F is reduced if S is so.*

PROOF. (1.17) implies that $\mathcal{O}_F$ is unmixed, and has no embedded component. The assertion follows easily from this.

(1.19) DEFINITION. Let $f: V\to W$ be a surjective morphism between varieties. We call f a *fibration* if the rational function field $K(W)$ is algebraically closed in $K(V)$, where we identify $K(W)$ with a subfield of $K(V)$ via f^*.

(1.20) THEOREM. *Let $f: V\to W$ be a surjective morphism between varieties. Then f is a fibration if and only if a generic fiber F of f is a variety. Moreover, if f is proper and V is normal, $rk_W(f_*\mathcal{O}_V)=1$ is also equivalent to them.*

PROOF. Thanks to [N], we may assume f to be proper.

Let V' be the normalization of V and let f' be the induced morphism $V' \to W$. Then f is a fibration if and only if f' is so, and F is birational to a generic fiber F' of f'. By (1.18) we infer that F is a variety if and only if F' is so. Thus we may assume V to be normal.

Suppose that f is not a fibration. Let X be the normalization of W in the algebraic closure of $K(W)$ in $K(V)$, and let $g: V \to X$ and $\pi: X \to W$ be the natural morphisms. By assumption π is not birational. Hence $rk_W(\pi_* \mathcal{O}_X) > 1$. On the other hand, we have a natural injection $\mathcal{O}_X \to g_* \mathcal{O}_V$, and so $\pi_* \mathcal{O}_X \subset \pi_*(g_* \mathcal{O}_V) = f_* \mathcal{O}_V$. Thus we see $rk_W(f_* \mathcal{O}_V) > 1$.

Suppose that $rk_W(f_* \mathcal{O}_V) > 1$. Let $Y = \mathcal{S}pec\,(f_* \mathcal{O}_V)$ and let $\varphi: V \to Y$ and $\alpha: Y \to W$ be the natural morphisms. Then $F = f^{-1}(x) = \varphi^{-1}(\alpha^{-1}(x))$ for a generic point x of W. α is not birational by assumption. So $\alpha^{-1}(x)$, and hence F, cannot be a variety. Moreover, f is not a fibration since $K(Y)$ is a non-trivial algebraic extension of $K(W)$ in $K(V)$.

Thus, it suffices to show that F is a variety assuming f to be a fibration. For this purpose we recall the following.

(1.21) THEOREM. *Let Λ be a linear system on a normal variety V with $N = \dim \Lambda$, $\mathrm{Bs}\,\Lambda = \varnothing$. Let W be the image of the rational mapping ρ_Λ defined by Λ. Then, a generic member of Λ is of the form $p^e Z$, where Z is a sum of different prime divisors, $p = \mathrm{char}\,(K)$ and e is a non-negative integer. (If $p=0$, then $p^e = 1$ always.) If $\dim W \geqq 2$, then Z is irreducible. If $\dim W = 1$, then the number of components of Z is equal to the degree of W in $\boldsymbol{P}^N$.*

For a proof, see [Z], p. 30 or [W], Chap. IX.

(1.22) Proof of (1.20), continued. Assuming f to be a fibration, we prove F to be a variety by induction on $\dim W$. If W is a curve, this follows easily from (1.21). So assume $\dim W \geqq 2$. Thanks to Chow's lemma, we may assume W to be projective. Let H be a generic hyperplane section on W and let D be the corresponding member of $f^*|H|$. Then, by (1.21), $D = p^e Z$ with Z being a prime divisor. We infer $e=0$ since otherwise there is $\phi \in K(V)$ such that $\phi \notin K(W)$ and $\phi^p \in K(W)$. Hence D is a variety by (1.18). To complete the proof by induction, we need only to show that $K(H)$ is algebraically closed in $K(D)$.

Taking a generic Lefschetz pencil belonging to $|H|$ and considering the graph of the associated rational mapping, we obtain a morphism $f^+: V^+ \to W^+$ together with a morphism $\beta: W^+ \to \boldsymbol{P}^1$, such that f^+ is birationally equivalent to $f: V \to W$ and that $\beta^{-1}(x) = H$ and $(\beta \circ f^+)^{-1}(x) = D$ for a generic point x on $\boldsymbol{P}^1$. We may consider all these things to be defined over a subfield K' of K with tr. deg $(K/K') = \infty$, since K is sufficiently large, except the value t of the coordinate on $\boldsymbol{P}^1$ at x, which is transcendental over K'. Denoting by $K''(\ \ )$ the K''-

valued rational function field, we have $K'(t)(H)\cong K'(W^+)$ and $K'(t)(D)\cong K'(V^+)$. Therefore $K'(t)(H)$ is algebraically closed in $K'(t)(D)$. Since $K'(t)\subset K$, this implies that $K(H)$ is algebraically closed in $K(D)$. q.e.d.

(1.23) REMARK. The arguments in this section work in the category of complex analytic spaces too, after an obvious change of the meanings of the terminology. However, the definition (1.19) is not approproiate in this context. We should define f to be a fibration if its general fiber is a variety. If V is smooth, this is equivalent to say that F is connected, because F is smooth.

§ 2. L-dimension of coherent sheaves.

From now on, every space is assumed to be proper over K.

(2.1) DEFINITION. Let L be a line bundle on a space S and let $\mathcal{F}$ be a coherent sheaf on S. We write $\kappa(L, \mathcal{F})\leqq k$ if and only if there exists a polynomial $\varphi(t)$ of degree k such that $h^0(\mathcal{F}[tL])\leqq\varphi(t)$ for any $t\gg 0$. In particular, $\kappa(L, \mathcal{F})=k$ means that, for any polynomial $\psi(t)$ of degree $\leqq k-1$, there are infinitely many positive integers $\{t_j\}$ such that $h^0(\mathcal{F}[t_jL])>\psi(t_j)$. Conventionally we define $\kappa(L, \mathcal{F})=-\infty$ if and only if $h^0(\mathcal{F}[tL])=0$ for any $t\gg 0$. $\kappa(L, \mathcal{O}_S)$ is denoted by $\kappa(L, S)$, or occasionally by $\kappa(L)$.

For a normal variety V defined over a field of characteristic zero, our definition of $\kappa(L, V)$ turns to be equivalent to that of Iitaka [I] (see also [U]).

(2.2) PROPOSITION. *Let* $0\to\mathcal{F}\to\mathcal{G}\to\mathcal{H}\to 0$ *be an exact sequence of coherent sheaves. Then* $\kappa(L, \mathcal{F})\leqq\kappa(L, \mathcal{G})\leqq\mathrm{Max}\,(\kappa(L, \mathcal{F}), \kappa(L, \mathcal{H}))$.

Obvious by definition. The following two results are also obvious.

(2.3) PROPOSITION. *Let L and $\mathcal{F}$ be as in* (2.1) *and let E be a Cartier divisor such that the associated homomorphism* $\mathcal{F}[-E]\to\mathcal{F}$ *is injective. Then* $\kappa(L, \mathcal{F})\leqq\kappa(L+E, \mathcal{F})$.

(2.4) PROPOSITION. $\kappa(L, \mathcal{F})\geqq\kappa(tL, \mathcal{F})$ *for any positive integer t.*

PROBLEM. Does the equality hold always ?

(2.5) THEOREM. $\kappa(L, \mathcal{F})\leqq\dim\,(\mathrm{Supp}\,(\mathcal{F}))$.

PROOF. We use the Noetherian induction on $\mathrm{Supp}\,(\mathcal{F})$. Let T be the subspace of S defined by the ideal $\mathrm{Ker}\,(\mathcal{O}_S\to\mathcal{E}nd\,(\mathcal{F}))$. Then $\mathcal{F}$ may be regarded as a sheaf on T. Of course $\mathrm{Supp}\,(T)=\mathrm{Supp}\,(\mathcal{F})$. Set $n=\dim T$.

Step 1, the case in which T is a variety and $\mathcal{F}=\mathcal{O}_T$. We may assume $H^0(T,mL)\neq 0$ for some $m>0$. So we have a non-zero section $\delta\in\mathcal{H}om_T(\mathcal{O}_T[-mL],\mathcal{O}_T)$. Since T is a variety, δ is injective and $\mathcal{O}_D=\mathrm{Coker}(\delta)$ is supported on a proper closed subset of T. By the Noetherian induction hypothesis we have a polynomial $\psi(t)$ of degree $\leqq n-1$ such that $h^0(\mathcal{O}_D[tL])\leqq\psi(t)$ for any $t\geqq a$, where a is a constant. Then $h^0(\mathcal{O}_T[tL])\leqq h^0(\mathcal{O}_T[(t-m)L])+h^0(\mathcal{O}_D[tL])\leqq h^0(\mathcal{O}_T[t-m)L])+\psi(t)$ for $t\geqq a$. Iterating we obtain $h^0(\mathcal{O}_T[(tm+j)L])\leqq h^0(\mathcal{O}_T[jL])+\sum_{s=1}^{t}\psi(sm+j)$ for $a\leqq j\leqq a+m-1$. The latter term is a polynomial in t of degree $\leqq n$ for each j. Now it is easy to find a polynomial φ of degree $\leqq n$ such that $h^0(\mathcal{O}_T[tL])\leqq\varphi(t)$ for any $t\geqq a$.

Step 2, the case in which T is a variety and $\mathcal{F}$ is general. We use the induction on $r=rk_T\mathcal{F}$. We may assume that there is $\delta\in\mathcal{H}om_T(\mathcal{O}_T[-mL],\mathcal{F})$ for some $m>0$ such that $\delta\neq 0$ at a generic point of T, because otherwise $\kappa(L,\mathcal{F})=\kappa(L,\mathcal{T})<n$ where $\mathcal{T}$ is the torsion subsheaf of $\mathcal{F}$. δ is injective since $\mathcal{O}_T$ is torsion free, and $rk_T\mathcal{C}=r-1$ where $\mathcal{C}=\mathrm{Coker}(\delta)$. If $r=1$, then $\kappa(L,\mathcal{C})<n$ since $\mathrm{Supp}(\mathcal{C})\neq T$. If $r>1$, then $\kappa(L,\mathcal{C})\leqq n$ by the induction hypothesis on r. On the other hand, we have $\kappa(L,\mathcal{O}_T[-mL])\leqq n$ by Step 1. Combining them and using (2.2) we prove $\kappa(L,\mathcal{F})\leqq n$.

Step 3, the case in which T is irreducible. $V=T_{red}$ is a variety. Let $\mathcal{N}$ be the sheaf of nilpotent functions on T. Then $\mathcal{N}^{\mu}=0$ for some $\mu>0$. Set $\mathcal{F}_j=\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F}$. Then $\mathcal{F}_j$'s are sheaves on V. So $\kappa(L,\mathcal{F}_j)\leqq n$ by Step 2. Now, using (2.2), we prove $\kappa(L,\mathcal{N}^j\mathcal{F})\leqq n$ by the descending induction on j. In particular $\kappa(L,\mathcal{F})\leqq n$.

Step 4, the general case. We should consider the case in which T is reducible. Let $\mathrm{Supp}(T)=X\cup Y$ where both X and Y are proper closed subsets of T. Then there is a subsheaf $\mathcal{G}$ of $\mathcal{F}$ such that $\mathrm{Supp}(\mathcal{G})=X$ and $\mathrm{Supp}(\mathcal{F}/\mathcal{G})=Y$. $\kappa(L,\mathcal{G})\leqq n$ and $\kappa(L,\mathcal{F}/\mathcal{G})\leqq n$ by the Noetherian induction hypothesis. So $\kappa(L,\mathcal{F})\leqq n$ by (2.2). q. e. d.

(2.6) In the algebraic category, we have the following more general result.

THEOREM. *Let L be a line bundle on a algebraic space S and let $\mathcal{F}$ be a coherent sheaf on S such that* $\dim(\mathrm{Supp}(\mathcal{F}))=n$. *Then, for every integer p, there exists a polynomial $\varphi(t)$ of degree $\leqq n$ such that $h^p(\mathcal{F}[tL])\leqq\varphi(t)$ for $t\gg 0$.*

OUTLINE OF PROOF. We use the induction on n. Let T be the subspace defined by the ideal $\mathrm{Ker}(\mathcal{O}_S\to\mathcal{E}nd(\mathcal{F}))$ as in (2.5).

Step 1, the case in which T is a projective variety. Take a sufficiently ample line bundle H on T such that both H and $H+L$ are very ample. Let A and B be general members of $|H|$ and $|L+H|$ respectively. Then we have exact sequences $0\to\mathcal{F}[-H]\to\mathcal{F}\to\mathcal{F}_A\to 0$ and $0\to\mathcal{F}[-H-L]\to\mathcal{F}\to\mathcal{F}_B\to 0$. In view of them we infer that $h^p(\mathcal{F}[(t+1)L])\leqq h^p(\mathcal{F}[tL-H])+h^p(\mathcal{F}_B[(t+1)L])\leqq h^p(\mathcal{F}[tL])+h^{p-1}(\mathcal{F}_A[tL])+h^p(\mathcal{F}_B[(t+1)L])$. Applying the induction hypothesis we infer that the last two terms are bounded by a polynomial of degree $\leqq n-1$ for $t\gg 0$. This implies that $h^p(\mathcal{F}[tL])$ is bounded by a polynomial of dgree $\leqq n$ for $t\gg 0$.

Step 2, the case in which T is a variety. By Chow's lemma there is a projective variety V together with a birational morphism $\pi: V\to T$. Let $\alpha:\mathcal{F}\to\pi_*\pi^*\mathcal{F}$ be the natural homomorphism and let $\mathcal{K}=\mathrm{Ker}(\alpha)$, $\mathcal{I}=\mathrm{Im}(\alpha)$, $\mathcal{C}=\mathrm{Coker}(\alpha)$ and $\mathcal{H}_j=R^j\pi_*(\pi^*\mathcal{F})$. Then $\mathcal{K}$, $\mathcal{C}$ and $\{\mathcal{H}_j\}_{j>0}$ are supported on proper subsets of T and we can apply the induction hypothesis to them. In view of the Leray spectral sequence with $E_2^{p,q}=H^p(T, R^q\pi_*(\pi^*\mathcal{F}[tL_V]))=H^p(T, \mathcal{H}_q[tL])$ converging to $H^{p+q}(V, \pi^*\mathcal{F}[tL_V])$, to which Step 1 applies, we infer that $h^p(\pi_*\pi^*\mathcal{F}[tL])$ are bounded by a polynomial of degree $\leqq n$. Using $0\to\mathcal{I}\to\pi_*\pi^*\mathcal{F}\to\mathcal{C}\to 0$ and $0\to\mathcal{K}\to\mathcal{F}\to\mathcal{I}\to 0$ we prove the assertion.

Step 3, the case in which T is irreducible. Same as in Step 3 in (2.5).

Step 4, the general case. Using a similar argument as in Step 4 in (2.5), we prove the assertion by induction on the number of irreducible components of T.

NOTE (added to the first version of this article). The above argument is the same as in Bănică and Ueno [J. Math. Kyoto Univ. **20** (1980), 381-389]. According to them, the result was proved by D. Leistner (Regensburg) in the analytic case too.

(2.7) COROLLARY. *Let $\pi: V\to S$ be a birational morphism between algebraic varieties. Let L be a line bundle on S and let $\mathcal{F}$ be a coherent sheaf on S. Then $\kappa(L, \mathcal{F})=n=\dim S$ if and only if $\kappa(L_V, \pi^*\mathcal{F})=n$.*

PROOF. Let $\alpha:\mathcal{F}\to\pi_*\pi^*\mathcal{F}$ be the natural morphism and let $\mathcal{K}=\mathrm{Ker}(\alpha)$, $\mathcal{I}=\mathrm{Im}(\alpha)$ and $\mathcal{C}=\mathrm{Coker}(\alpha)$. Then $\mathcal{K}$ and $\mathcal{C}$ are supported on proper subsets of S. In view of the exact sequence $0\to\mathcal{I}\to\pi_*\pi^*\mathcal{F}\to\mathcal{C}\to 0$, we infer that $\kappa(L, \mathcal{I})=n$ if and only if $\kappa(L, \pi_*\pi^*\mathcal{F})=\kappa(L_V, \pi^*\mathcal{F})=n$, since $\kappa(L, \mathcal{C})<n$. From the exact sequence $0\to\mathcal{K}\to\mathcal{F}\to\mathcal{I}\to 0$, we infer that $\kappa(L, \mathcal{F})=n$ if and only if $\kappa(L, \mathcal{I})=n$, because both $h^0(\mathcal{K}[tL])$ and $h^1(\mathcal{K}[tL])$ are bounded by a polynomial of degree $<n$. Combining them we obtain the assertion.

(2.8) COROLLARY. *Let L be a line bundle on a projective variety V such that $\kappa(L, V)=n=\dim V$. Then, for any very ample divisor H on V, there is a positive integer m such that $|mL-H|\neq\varnothing$.*

PROOF. We may assume H to be general in $|H|$. So $H^0(V, tL-H)\to H^0(V, tL)\to H^0(H, tL_H)$ is exact. $h^0(V, tL)>h^0(H, tL_H)$ for $t\gg 0$ since $\kappa(L, V)=n>\kappa(L, \mathcal{O}_H)$ by (2.5). Combining these facts we obtain the assertion.

REMARK. The above argument is originally due to Kodaira [K].

(2.9) THEOREM. *Let L be a line bundle on an irreducible space S. Suppose that $\kappa(L, \mathcal{F})=n=\dim S$ for a coherent sheaf $\mathcal{F}$ on S. Then $\kappa(L, \mathcal{G})=n$ for any coherent sheaf $\mathcal{G}$ such that* $\mathrm{Supp}(\mathcal{G})=S$.

PROOF. First we consider the case in which S is reduced. We claim $\kappa(L, S)=n$. Indeed, otherwise, one could prove $\kappa(L, \mathcal{F})<n$ for any coherent sheaf $\mathcal{F}$ on S by a similar argument as in Step 2 in (2.5). Thus, S is algebraic even if we work in the analytic category. (See [U], p. 54.) Thanks to Chow's lemma and (2.7), we may assume S to be projective. For a given $\mathcal{G}$, take a sufficienty very ample line bundle H such that there exists an injection $\mathcal{O}_S[-H] \to \mathcal{G}$ (Note that, any such homomorphism which does not vanish at a generic point of S is necessarily injective.). Then $\kappa(L, \mathcal{G}) \geqq \kappa(L, \mathcal{O}_S[-H])$. By (2.8) we have $m>0$ such that $|mL-H| \neq \emptyset$. So $h^0(S, tmL-H) \geqq h^0(S, (t-1)H)$ for any $t>0$. This implies $\kappa(L, \mathcal{O}_S[-H])=n$. Thus we prove the assertion.

In general case, let $\mathcal{N}$ be the ideal of nilpotent functions on S and set $\mathcal{F}_j=\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F}$ and $\mathcal{G}_j=\mathcal{N}^j\mathcal{G}/\mathcal{N}^{j+1}\mathcal{G}$. $\kappa(L, \mathcal{F}_j)=n$ for some j since otherwise $\kappa(L, \mathcal{F})<n$. Hence, by the above step, $\kappa(L, \mathcal{H})=n$ for any $\mathcal{H} \in F(S_{red})$ with $\mathrm{Supp}(\mathcal{H})=S$. There exists j such that $rk_S\mathcal{N}^j\mathcal{G}>0$ and $rk_S\mathcal{N}^{j+1}\mathcal{G}=0$, since $\mathrm{Supp}(\mathcal{G})=S$. Then $\mathrm{Supp}(\mathcal{G}_j)=S$ and so $\kappa(L, \mathcal{G}_j)=n$. $h^1(\mathcal{N}^{j+1}\mathcal{G}[tL])$ is bounded by a polynomial of degree $<n$ by (2.6). Now, using the exact sequence $0 \to \mathcal{N}^{j+1}\mathcal{G} \to \mathcal{N}^j\mathcal{G} \to \mathcal{G}_j \to 0$, we infer that $\kappa(L, \mathcal{N}^j\mathcal{G})=n$. This implies $\kappa(L, \mathcal{G})=n$, since $\mathcal{N}^j\mathcal{G}$ is a subsheaf of $\mathcal{G}$.

q. e. d.

(2.10) THEOREM. *Let $f: V \to W$ be a surjective morphism from an irreducible algebraic space V onto a variety W. Let F be a generic fiber of f and let L be a line bundle on V and let $\mathcal{F}$ be an unmixed sheaf on V with* $\mathrm{Supp}(\mathcal{F})=V$. *Then* $\kappa(L, \mathcal{F}) \leqq \kappa(L, \mathcal{F}_F)+\dim W$.

To prove this, we need the following

(2.11) LEMMA. *Let $\mathcal{F}$ be an unmixed sheaf on a projective irreducible space V such that* $\mathrm{Supp}(\mathcal{F})=V$. *Then $H^0(\mathcal{F}[-H])=0$ for any sufficiently ample line bundle H on V, unless* $\dim V=0$.

PROOF. In view of (1.7), we infer that $H^0(\mathcal{F}[-H])$ is dual to $H^0(\mathcal{D}^0(\mathcal{F})[H])$ for any sufficiently ample H. On the other hand, $\mathcal{D}^0(\mathcal{F})=0$ by (1.8) and (1.14). Thus we obtain the assertion.

(2.12) LEMMA. *Let V, $\mathcal{F}$ and L be as in* (2.10) *and let $\rho: V \to \boldsymbol{P}^N$ be a morphism such that* $\dim \rho(V)>0$. *Let H be the pull back of the hyperplane section.*

Then, there is a positive constant c such that $H^0(\mathcal{F}[tL-sH])=0$ for any $s, t>0$ with $s>ct$.

PROOF. First we consider the case in which V is projective, using the induction on $\dim V$. When V is a curve, H is ample on V, and we have c_1 such that $H^0(\mathcal{F}[-c_1H])=0$. We have also c_2 such that c_2H-L is very ample on V. Then, it is easy to see that $c=c_1+c_2$ satisfies the condition. So suppose $\dim V\geqq 2$. Let D be a generic hyperplane section on V. Then, by (1.21) and (1.16), D is irreducible and $\mathcal{F}_D$ is unmixed. So, by the induction hypothesis, we have $c>0$ such that $H^0(D, \mathcal{F}_D[tL-sH])=0$ for any $s, t>0$ with $s>ct$. Then $H^0(D, \mathcal{F}_D[tL-sH-uD])=0$ for any $s, t>0, u\geqq 0$ with $s>ct$. In view of the exact sequence $0\to\mathcal{F}[-D]\to\mathcal{F}\to\mathcal{F}_D\to 0$, we infer that $h^0(\mathcal{F}[tL-sH])\leqq h^0(\mathcal{F}[tL-sH-D])\leqq\cdots\leqq h^0(\mathcal{F}[tL-sH-uD])$ for any $s, t>0, u>0$ with $s>ct$. The last term is zero for $u\gg 0$ by (2.11). Thus we see $H^0(\mathcal{F}[tL-sH])=0$.

Now we consider the general case. By Chow's lemma, there is a projective irreducible space V' together with a birational morphism $\pi: V'\to V$ (This means that there is an open dense subset U of V such that $\pi^{-1}(U)\cong U$.). Let $\mathcal{F}'$ be the unmixed part of $\pi^*\mathcal{F}$ (cf. (1.15)). Then we have a natural homomorphism $\alpha: \mathcal{F}\to\pi_*\pi^*\mathcal{F}\to\pi_*\mathcal{F}'$. Clearly α is an isomorphism at a generic point x of V. This implies $\mathrm{Ker}(\alpha)=0$ since $\mathcal{F}$ is unmixed. On the other hand, by the first step, we have $c>0$ such that $H^0(\mathcal{F}'[tL-sH])=0$ for any $s, t>0$ with $s>ct$. Combining them we obtain $H^0(\mathcal{F}[tL-sH])=0$.

(2.13) PROOF OF THEOREM (2.10). $\kappa(L, \mathcal{F})\geqq 0$ implies $\kappa(L, \mathcal{F}_F)\geqq 0$ since F is generic. Therefore we may assume $\kappa(L, \mathcal{F}_F)\geqq 0$.

Step 1, the case in which W is a curve. Let H be a very ample line bundle on W and let D be a generic member of $f^*|H|$. Then D is a union of d generic fibers of f, where $d=\deg H$. Hence $\kappa(L, \mathcal{F}_D)=\kappa(L, \mathcal{F}_F)$, which we set κ. $h^0(\mathcal{F}_D[tL])$ is bounded by a polynomial $\varphi(t)$ of degree κ for any $t\gg 0$. Using the exact sequence $0\to\mathcal{F}[-H]\to\mathcal{F}\to\mathcal{F}_D\to 0$, we obtain $h^0(\mathcal{F}[tL-jH])-h^0(\mathcal{F}[tL-(j+1)H])\leqq h^0(\mathcal{F}_D[tL-jH])\leqq h^0(\mathcal{F}_D[tL])\leqq\varphi(t)$ for any $t>0, j\geqq 0$. Iterating we see $h^0(\mathcal{F}[tL])\leqq h^0(\mathcal{F}[tL-ctH])+ct\varphi(t)=ct\varphi(t)$, where c is a constant as in (2.12). Thus we prove $\kappa(L, \mathcal{F})\leqq\kappa+1$.

Step 2, the case in which W is a projective variety. We use the induction on $\dim W$. We may assume $\dim W\geqq 2$ by Step 1. Similarly as before, let H be a very ample line bundle on W and let D be a generic member of $f^*|H|$. Then, D is irreducible by (1.21) and $\mathcal{F}_D$ is unmixed by (1.16). $f(D)$ is also a variety. Hence we can apply the induction hypothesis to obtain $\kappa(L, \mathcal{F}_D)\leqq\kappa(L, \mathcal{F}_F)+\dim f(D)$. Now, by a similar argument as in Step 1, we infer $\kappa(L, \mathcal{F})\leqq\kappa(L, \mathcal{F}_D)+1\leqq\kappa(L, \mathcal{F}_F)+\dim W$.

Step 3, the general case. By Chow's lemma we have a projective variety

W' together with a birational morphism $\sigma: W' \to W$. Taking a suitable component of $V \times_W W'$, we find an irreducible space V' together with morphisms $f': V' \to W'$ and $\pi: V' \to V$ such that $\sigma \circ f' = f \circ \pi$ and that π is birational (in the sense as in (2.12)). Let $\mathcal{F}'$ be the unmixed part of $\pi^* \mathcal{F}$ (cf. (1.15)). Then, we have a natural injection $\mathcal{F} \to \pi_* \mathcal{F}'$ as in (2.12). On the other hand, we infer $\kappa(L_{V'}, \mathcal{F}') \leqq \kappa(L, \mathcal{F}_F) + \dim W$ by Step 2, since F is isomorphic to a generic fiber of f', the restriction of $\mathcal{F}'$ to which is isomorphic to $\mathcal{F}_F$. Thus we get the assertion because $\kappa(L, \mathcal{F}) \leqq \kappa(L, \pi_* \mathcal{F}') = \kappa(L_{V'}, \mathcal{F}')$.

(2.14) COROLLARY. *Let $f: V \to W$ be a fibration of algebraic varieties. Let F be its generic fiber and let L be a line bundle on V. Then $\kappa(L, V) \leqq \kappa(L, F) + \dim W$.*

(2.15) REMARK. The author suspects that (2.10) is true in the analytic category too. Actually, its corollary (2.14) is true in this context. For a proof, see [U], p. 59. There you will find an additional assumption that V and W are smooth. But the argument found there works without this assumption, provided that we have a fibration theorem of Iitaka type for singular varieties too. This will be done in the next section. (Caution: If V is not normal, our $\kappa(L, V)$ may be different from that of Iitaka and Ueno.)

§ 3. L-dimension of varieties and Iitaka fibration.

Let L be a line bundle on a variety V. In this section we want to study the behaviour of the rational mappings $\rho_{|tL|}$ for $t > 0$. Keeping to be careful about singularities, we follow closely the idea of Iitaka. First we make the following observation.

(3.1) *Let Λ be a linear system on V. Then, there is a normal variety V' together with a birational morphism $\pi: V' \to V$, an effective Cartier divisor E on V' and a linear system Λ' on V' such that $\pi^* \Lambda = E + \Lambda'$ and $Bs\, \Lambda' = \emptyset$. Moreover, the image $\rho_{\Lambda'}(V')$ is independent of the choice of such V'.*

PROOF. Let $\mathcal{L}$ be the invertible sheaf $[\Lambda]$ and let A be the subspace of $H^0(V, \mathcal{L})$ corresponding to Λ. Let $\lambda: \mathcal{O}_V[A] \to \mathcal{L}$ be the natural homomorphism and let $\mathcal{B}$ be the $\mathcal{O}_V$-ideal $\mathrm{Im}(\lambda) \otimes \mathcal{L}^{-1}$. Let V^* be the blowing up of V with center $\mathcal{B}$ and let E^* be the exceptional divisor on V^*, whose defining ideal $\mathcal{J}$ is the pull back of $\mathcal{B}$. $\mathcal{J}$ is invertible and $\mathcal{J} = \mathrm{Im}(\lambda^*) \otimes \mathcal{L}_{V^*}^{-1}$, where $\lambda^*: \mathcal{O}_{V^*}[A] \to \mathcal{L}_{V^*}$ is the pull back of λ. $\lambda^* \otimes \mathcal{J}^{-1}$ defines a linear system Λ^* on V^* such that $Bs\, \Lambda^* = \emptyset$ and $\Lambda_{V^*} = E^* + \Lambda^*$. Let V' be the normalization of V^* with the natural morphism $\pi: V' \to V$ and let E and Λ' be pull backs of E^* and Λ^*

respectively. Then this V', π, E and Λ' satisfies the desired conditions.

The uniqueness of $\rho_{\Lambda'}(V')$ is shown as follows. Let R be the graded subalgebra of $G(V, L)=\bigoplus_{t\geq 0} H^0(V, tL)$ generated by Λ. Then $\rho_{V'}(V')$ is canonically isomorphic to $\mathrm{Proj}(R)$, which is obviously independent of the choice of V'.

DEFINITION. $\rho_{V'}(V')$ is called the *image* of the rational mapping ρ_Λ and is denoted by $\rho_\Lambda(V)$. V' is called a *good graph* of ρ_Λ.

(3.2) Now, let L be the given line bundle on V. Let ρ_t be the rational mapping defined by the linear system $|tL|$, for $t>0$. Let V_t be a good graph of ρ_t and let W_t be the image of ρ_t. Then clearly $K(W_t)\subset K(V_t)=K(V)$, where $K(X)$ denotes the field of rational functions on X. (If $|tL|=\emptyset$, ρ_t is not defined and our statement means nothing.)

(3.3) LEMMA. *If s is a multiple of t, then $K(W_t)\subset K(W_s)$ as subfields of $K(V)$.*

PROOF. Let R_t be the graded subalgebra of $G(V, L)$ generated by $H^0(V, tL)$. Set $s=ct$ and let $\otimes^c H^0(V, tL)\to H^0(V, sL)$ be the natural morphism and let T be its image. Let R' be the subalgebra of $G(V, L)$ generated by T. Then $R'\subset R_s$ and we have a surjective rational mapping $W_s=\mathrm{Proj}(R_s)\to\mathrm{Proj}(R')$. On the other hand, $\mathrm{Proj}(R')$ is the image of $\mathrm{Proj}(R_t)=W_t$ by the natural Veronese embedding $\boldsymbol{P}(H^0(V, tL))\subset\boldsymbol{P}(S^c H^0(V, tL))\supset\boldsymbol{P}(T)$. Hence $\mathrm{Proj}(R')\cong\mathrm{Proj}(R_t)\cong W_t$. Combining them we prove the assertion.

(3.4) LEMMA. *There exists a positive integer m such that $K(W_t)=K(W_m)$ for any positive multiple t of m.*

Indeed, otherwise, we would have an infinite strictly increasing sequence $K(W_{t_1})\subset K(W_{t_2})\subset\cdots$ of subfields of $K(V)$. This is impossible since $K(V)$ is finitely generated over K.

(3.5) Take $m>0$ as in (3.4) and set $\Gamma=V_m$, $W=W_m$ and $\Phi=\rho_m$, and let π be the natural birational morphism $\Gamma\to V$ and let F be a generic fiber of Φ. First we claim that $\dim W\leqq\kappa(L, V)$.

Indeed, we have $h^0(V, tL)\geqq\dim$ (*the degree t part of R_m*)$=h^0(W, m^{-1}tH)$ for any sufficiently large multiple t of m, where H is the hyperplane section on $W\subset\boldsymbol{P}(H^0(V, mL))$. Thus $\kappa(L, V)\geqq\kappa(H, W)=\dim W$.

Actually, we will see the equality holds.

(3.6) LEMMA. *The image of the natural homomorphism $H^0(V, tL)\to H^0(F, tL_F)$ is of dimension one for any $t>0$ with $|tL|\neq\emptyset$.*

PROOF. Suppose to the contrary for some t. Let $s=mt$. Modifying V_s if necessary, we may assume that the natural rational mapping $V_s \to V_m = \Gamma$ is a morphism. By the choice of m, $K(W_s)=K(W_m)$ and we have a birational mapping $W_s \to W_m$. Let y be the generic point on W_s which lies over the generic point $\Phi(F)$ on W. Let F_s be the generic fiber over y. Then $V_s \to V_m$ restricts to a birational morphism $f: F_s \to F$. $\mathcal{O}_F$ is unmixed by (1.17). From this we infer that $\mathcal{O}_F \to f_* \mathcal{O}_{F_s}$ is injective. So $H^0(F, sL_F) \to H^0(F_s, sL)$ is injective. By the choice of s we infer that $\dim(\mathrm{Im}(H^0(V, sL) \to H^0(F, sL))) \geqq \dim(\mathrm{Im}(H^0(V, tL) \to H^0(F, tL))) \geqq 2$. Combining them we obtain $\dim(\mathrm{Im}(H^0(V, sL) \to H^0(F_s, sL))) \geqq 2$. This is impossible by definition of ρ_s.

(3.7) Let $\pi^*|mL|=E+\Lambda'$, where E is the fixed part of $\pi^*|mL|$. Of course $\Phi^*H=[\Lambda']$. Now, we write $E=E_1+E_2$ in such a way that every prime component of E_1 maps onto W while any component of E_2 maps onto a proper subset of W. By (3.6), we infer that tE_1 is a fixed part of $\pi^*|tmL|$ for any $t>0$, since every component of it meets F. On the other hand, for a sufficiently large δ, we have $D \in \Phi^*|\delta H|$ of the form $D=E_2+D'$ with D' being an effective divisor, since $\Phi(E_2)$ is a proper subset of W. Putting things together we infer that $h^0(V, tmL) \leqq h^0(\Gamma, tH+tD)=h^0(W, \Phi_*\mathcal{O}_\Gamma[t(\delta+1)H])$. This implies $\kappa(mL, V) \leqq \kappa(H, \Phi_*\mathcal{O}_\Gamma) \leqq \dim W$ by (2.5). Since $h^0(V, tL) \leqq h^0(V, tmL)$, this proves $\kappa(L, V) \leqq \dim W$.

(3.8) Now, combining the preceding arguments, we obtain the following

THEOREM. *Let L be a line bundle on a variety V. Suppose that $\kappa(L, V) \geqq 0$. Then there is a positive integer m with the following properties:*

a) $\dim W=\kappa(L, V)$, *where W is the image of the rational mapping $\rho_{|mL|}$.*

b) Let $\Phi: \Gamma \to W$ be the morphism from a good graph Γ of $\rho_{|mL|}$ and let F be a generic fiber of Φ. Then $\dim(\mathrm{Im}(H^0(V, tL) \to H^0(F, tL_F))) \leqq 1$ *for any $t>0$.*

Moreover, the triple Γ, W and Φ is independent of the choice of m up to birational equivalence.

DEFINITION. The above W will be called *Iitaka L-model* of V.

(3.9) COROLLARY. $\kappa(L, V)=\underset{t>0}{\mathrm{Max}}(\dim \rho_{|tL|}(V))$. *In particular,* $\kappa(L, V)=0$ *if and only if* $h^0(V, tL) \leqq 1$ *for any* $t>0$ *and* $=1$ *for some* $t>0$.

(3.10) COROLLARY. *Let d be the greatest common divisor of the set* $\{t>0 \mid |tL| \neq \emptyset\}$. *Then, there are positive constants c_1, c_2 and a such that* $c_1 t^\kappa \leqq h^0(V, tdL) \leqq c_2 t^\kappa$ *for any* $t \geqq a$, *where* $\kappa=\kappa(L, V)$.

Indeed, the lower bound is obtained by an argument as in (3.5) and the upper bound is obtained by (3.7).

(3.11) Unlike the classical case (cf. [I] or [U]), we can not guarantee that the morphism Φ is a fibration in the sense (1.19). However we have the following

THEOREM. *Let Σ be the set $\{x\in V \mid V$ is not normal at $x\}$. Suppose that* $\dim\Sigma<\kappa(L, V)$. *Then Φ in* (3.8) *is a fibration. Moreover, a generic fiber F of Φ satisfies the condition $\kappa(L, F)=0$.*

PROOF. Let $\mathcal{A}_t=\Phi_*(\mathcal{O}_\Gamma[tL])$. It suffices to show $rk_W\mathcal{A}_t\leqq 1$ for any $t\geqq 0$. Indeed, $rk_W(\mathcal{A}_0)=1$ implies that Φ is a fibration by (1.20). $rk_W(\mathcal{A}_t)\leqq 1$ implies that $h^0(F_y, tL)\leqq 1$ for any point y on an open dense subset of W, where F_y is the fiber over y. Therefore, $h^0(F_y, tL)\leqq 1$ for any $t\geqq 0$ and for any $y\in W$ off a union of countably many proper closed subsets of W. Hence $\kappa(L, F)=0$ for a generic fiber F.

Assume $rk_W(\mathcal{A}_t)\geqq 2$ for some $t\geqq 0$. Then, $rk_W(\mathcal{A}_{mt})\geqq 2$, where m is as in (3.8) and (3.4). So, replacing L by mL if necessary, we may assume $m=1$ in order to derive a contradiction. Thus we write $\pi^*L=\Phi^*H+E_1+E_2$ as in (3.7), take $\delta\gg 0$, $D\in\Phi^*|\delta H|$ with $D=E_2+D'$ as there, letting the notations as before.

Let X_s' be $\mathrm{Im}\,(H^0(V, tL+sL)\to H^0(\Gamma, tL+sL))$. Since sE_1 is a fixed part of $\pi^*|sL|$ for any $s\geqq 0$, X_s' comes from a subspace X_s of $H^0(\Gamma, tL+sH+sE_2)$ for any $s\geqq 0$. Adding sD', X_s maps onto a subspace X_s'' of $H^0(\Gamma, tL+sH+sD)$ $=H^0(W, \mathcal{A}_t[s(1+\delta)H])$. Thus we get a homomorphism $\alpha_s: \mathcal{O}_W[-s(1+\delta)H]$ $\otimes[X_s'']\to\mathcal{A}_t$, where X_s'' is regarded as a trivial vector bundle on W. Let $\mathcal{G}$ be the subsheaf of $\mathcal{A}_t$ generated by the images of α_s, $s\geqq 0$. Of course $\mathcal{G}$ is coherent on W, and we claim that $rk_W(\mathcal{G})=1$.

To show the claim, let x be a generic point of W and let F be the fiber $\Phi^{-1}(x)$. Then, the germ $(\mathcal{A}_t)_x$ corresponds to $H^0(F, tL_F)\cong H^0(F, tE_1)$. Let ε be the restriction to F of the defining section $\in H^0(\Gamma, E_1)$ of E_1, which is unique up to scalar multiplication. (3.8), b) implies that the image of $H^0(V, sL)\to H^0(F, sL)$ $=H^0(F, sE_1)$ is the one-dimensional subspace generated by $\varepsilon^{\otimes s}$ for any $s\geqq 0$. On the other hand, if I_s be the subspace of $H^0(F, tL)$ corresponding of $\mathrm{Im}\,(\alpha_s)_x$, $I_s\otimes\varepsilon^{\otimes s}$ comes from $H^0(V, tL+sL)$ by definition of α_s. Hence we infer that I_s is independent of s. So $\mathcal{G}_x=I_s$, and $rk_W(\mathcal{G})=\dim\mathcal{G}_x=1$.

Let $\mathcal{B}$ be the unmixed part of the quotient sheaf $\mathcal{A}_t/\mathcal{G}$ (cf. (1.15). $\mathcal{B}$ is torsion free on W, and is a quotient of $\mathcal{A}_t$.). Then we have the following commutative diagram:

$$\begin{tikzcd}
H^0(\Gamma, tL+sH) \arrow[r] \arrow[d] & H^0(\Gamma, tL+sH+sE_2) \arrow[r] & H^0(\Gamma, tL+sH+sD) \arrow[d] \\
H^0(W, \mathcal{A}_t[sH]) \arrow[rr] \arrow[d] & & H^0(W, \mathcal{A}_t[s(1+\delta)H]) \arrow[d] \\
H^0(W, \mathcal{B}[sH]) \arrow[rr] & & H^0(W, \mathcal{B}[s(1+\delta)H]),
\end{tikzcd}$$

where the middle and lower horizontal maps are defined by the multiplication of the section of $H^0(W, s\delta H)$, which corresponds to sD by Φ^*. So the lower one is injective because $\mathcal{B}$ is torsion free. Since H is ample, the vertical mappings are surjective for $s \gg 0$.

Let $Y_s = H^0(\Gamma, tL+sH)$ and let $\sigma: Y_s \to H^0(\Gamma, tL+sH+sE_2)$ be the mapping as in the above diagram, defined by "adding sE_2". By definition of $\mathcal{B}$, we see that $\sigma^{-1}(X_s)$ maps to zero in $H^0(W, \mathcal{B}[s(1+\delta)H])$. Because of the injectivity of the lower horizontal mapping, this implies that $\sigma^{-1}(X_s)$ maps to zero in $H^0(W, \mathcal{B}[sH])$. Since the vertical map is surjective for $s \gg 0$, we obtain $\dim(Y_s/\sigma^{-1}(X_s)) \geqq h^0(W, \mathcal{B}[sH])$. On the other hand, $Y_s/\sigma^{-1}(X_s)$ is isomorphic to a subspace of $H^0(\Gamma, tL+sH+sE_2)/X_s$. Hence $\dim(Y_s/\sigma^{-1}(X_s)) \leqq h^0(\Gamma, tL+sH+sE_2) - \dim X_s \leqq h^0(\Gamma, tL+sL) - h^0(V, tL+sL)$. Thus, we see that $h^0(\Gamma, tL+sL) - h^0(V, tL+sL)$ is bounded from below for $s \gg 0$ by a polynomial of degree $\kappa(L, V)$, since $rk_W(\mathcal{B}) = rk_W(\mathcal{A}_t) - rk_W(\mathcal{G}) > 0$.

On the other hand, consider the natural homomorphism $\mathcal{O}_V \to \pi_*\mathcal{O}_\Gamma$, and let $\mathcal{C}$ be the cokernel of it. Then $h^0(\Gamma, sL) - h^0(V, sL) \leqq h^0(\mathcal{C}[sL])$, and this is bounded by a polynomial of degree $< \kappa(L, V)$, since $\mathrm{Supp}(\mathcal{C}) \subset \Sigma$ (cf. (2.5)). Thus we obtain a contradiction, which proves $rk_W(\mathcal{A}_t) = 1$ for any $t \geqq 0$. q.e.d.

DEFINITION. When the theorem applies, $\Phi: \Gamma \to W$ is called *Iitaka fibration*. This is uniquely determined by L up to birational equivalence.

(3.12) COROLLARY. *If $\kappa(L, V) = \dim V$, then $\rho_{|mL|}$ is a birational mapping for some $m > 0$.*

(3.13) COROLLARY. *If V is normal, then $\Phi: \Gamma \to W$ is a fibration such that $\kappa(L, F) = 0$, where F is a generic fiber of Φ.*

This is essentially the classical fibration theorem of Iitaka (cf. [I]).

(3.14) THEOREM. *Let V and L be as in* (3.11) *and let d be as in* (3.10). *Then there are positive constants c and a such that $h^0(V, tdL) - h^0(V, (t-1)dL) \leqq ct^{\kappa-1}$ for any $t \geqq a$, where $\kappa = \kappa(L, V)$.*

PROOF. We employ the same notation as before. In particular, let m be the integer as in (3.8). $h^0(V, tdL)$ is a monotone increasing function on t. Therefore, replacing L by mL if necessary, it suffices to consider the case in which $m=1$.

By an argument as in the final step of the proof of (3.11), $h^0(\Gamma, tdL) - h^0(V, tdL)$ is bounded by a polynomial in t of degree $\leqq \kappa-1$. So we may assume $\Gamma = V$.

Let A be a generic member of $\Phi^*|(1+\delta)H|$. Then we have $h^0(\Gamma, tL) = h^0(\Gamma, tH+tE_2) \leqq h^0(\Gamma, tH+(t-1)E_2+D) \leqq h^0(\Gamma, (t-1)H+(t-1)E_2) + h^0(A, tH + (t-1)E_2 + D) \leqq h^0(\Gamma, (t-1)L) + h^0(A, tH+tD)$. So $h^0(\Gamma, tL) - h^0(\Gamma, (t-1)L) \leqq h^0(A, tH+tD) = h^0(W, \Phi_* \mathcal{O}_A[t(1+\delta)H])$. Since $\Phi(A)$ is a divisor on W, the last term is bounded by a polynomial of degree $\dim \Phi(A) = \kappa-1$ by (2.5). This proves our assertion.

REMARK. This result, together with (3.10), means that the asymptotic behaviour of $h^0(V, tdL)$ is like a polynomial of degree κ.

QUESTION. Does the estimate (3.14) hold to be true without the assumption $\dim \Sigma < \kappa(L, V)$?

(3.15) REMARK. Perhaps it is not difficult to generalize the preceding results in the case in which V is an irreducible unmixed (: $=\mathcal{O}_V$ is unmixed) space.

(3.16) So far, we haven't seen no great difference depending on $p=\mathrm{char}(K)$. But now it is the time to discuss it.

As a version of the strong Bertini theorem in $p=0$, we can prove that a generic fiber of an Iitaka fibration is normal. Actually, thanks to the theory of resolution of singularities due to Hironaka, we can take Γ to be smooth and then F becomes smooth. However, if $p>0$, F is not necessarily normal even if Γ is taken to be smooth.

In $p=0$, Iitaka fibration is a fundamental tool in the classification theory of algebraic varieties, in which we apply the theory to the case $L=K_V$, the canonical bundle of a smooth variety V, and define the Kodaira dimension $\kappa(V)$ of V as $\kappa(K_V, V)$. In this case F turns to be a smooth variety (taking Γ to be smooth) of Kodaira dimension 0. Thus, we reduce the classification problem to i) the classification of varieties of Kodaira dimension $\dim V$, 0 and $-\infty$, and ii) the study of the structure of fibrations with generic fiber being a manifold of Kodaira dimension zero (Note that, $\dim F < \dim V$ if $0 < \kappa(V) < \dim V$.).

This method encounter several troubles in case $p>0$. First of all, we have no resolution theory of singularities which is as powerful as that of Hironaka in case $p=0$. In particular, there is no guarantee that Γ can be taken to be smooth. There are still other troubles. Indeed, even if Γ is smooth, F need not be normal. We can just show that F is a locally Gorenstein variety with $\kappa(K_F, F)=0$, where K_F is the dualizing sheaf of F. This result may seem not bad as a "general nonsense", but the problem is more delicate than one might expect.

As a simplest example of such phenomena, consider the case in which $\dim V=2$ and $\kappa(V)=1$. Then, by the general theory, we infer that F is either a) a non-singular elliptic curve, or b) a rational curve with one node, or c) a rational curve with one ordinary cusp. However, as a matter of fact, it turns out that only the cases a) and c) are possible, and in case c), p must be 2 or 3 (cf. [M] and [T]). The reason is of subtler nature than our general theory. Moreover, the condition on p in case c) is probably related to the theory of complex multiplications of elliptic curves.

We should expect similar phenomena in higher dimensions too. To determine which candidates among those that have the property required by our general theory do really appear as generic fibers of Iitaka fibrations is surely a very interesting, but perhaps difficult problem even in the next simplest case $\dim V=3$, $\kappa(V)=1$.

(3.17) Finally, we will prove the following

THEOREM. *Let $f: V\to W$ be a surjective morphism between varieties and let L be a line bundle on W. Then $\kappa(L, W)\leqq\kappa(L, V)$. Moreover, if $\dim\Sigma<\kappa(L, V)$ where Σ is the set $\{x\in W \mid W$ is not normal at $x\}$, then the equality holds.*

PROOF. We follow closely the idea of Ueno (cf. [U], p. 61), keeping to be careful about positive characteristic phenomena. The inequality is clear since we have a natural injection $\mathcal{O}_W\to f_*\mathcal{O}_V$. So consider the second assertion.

We may assume W to be normal. Indeed, let V' and W' be normalizations of V and W respectively and let $f': V'\to W'$ be the induced morphism. If $\kappa(L, W')=\kappa(L, V')$, then $\kappa(L, W')\geqq\kappa(L, V)$ by the first inequality, and so $\dim\Sigma<\kappa(L, W')=\kappa(L, \pi_*\mathcal{O}_{W'})$, where π is the morphism $W'\to W$. Let $\mathcal{C}$ be the cokernel of the homomorphism $\mathcal{O}_W\to\pi_*\mathcal{O}_{W'}$ and apply (2.2) to $0\to\mathcal{O}_W\to\pi_*\mathcal{O}_{W'}\to\mathcal{C}\to 0$. Since $\kappa(L, \mathcal{C})<\kappa(L, \pi_*\mathcal{O}_{W'})$ by (2.5), we infer that $\kappa(L, \pi_*\mathcal{O}_{W'})=\kappa(L, \mathcal{O}_W)$. Thus we get $\kappa(L, W)\geqq\kappa(L, V)$.

Clearly we may assume V to be normal. In addition, considering the Stein factorization of f as in [U] if necessary, we may assume f to be a finite morphism. Now, our proof proceeds in several steps.

Step 1, the case in which f is Frobenius. This means the following: Let $F: y=\mathrm{Spec}(K)\to x=\mathrm{Spec}(K)$ be the morphism defined by the p-power map of the field K, where $p=\mathrm{char}(K)$. Then, $f: V\to W$ is equivalent to $W\times_x y\to W$ in the category of x-schemes. In this case we write symbolically $V=F^{-1}(W)$. Note that $K(W)$ is the image of $K(V)$ by the p-power map $\xi\to\xi^p$, where we identify

$K(W)$ with a subfield of $K(V)$ by f^*. In other words, $K(V)$ is the field of the p-power roots of all the elements of $K(W)$.

Let e be any local section of $f_*\mathcal{O}_V$, which is a sheaf of $\mathcal{O}_W$-algebra. Then e^p lies in the image of the natural homomorphism $\mathcal{O}_W \to f_*\mathcal{O}_V$. This gives rise to a mapping $H^0(V, tL) = H^0(W, f_*\mathcal{O}_V[tL]) \to H^0(W, tpL)$. Though this map φ is not K-linear, one can easily verify that $\varphi(\zeta_1), \cdots, \varphi(\zeta_r)$ are linearly independent if $\zeta_1, \cdots, \zeta_r$ are so in $H^0(V, tL)$. Thus we have $h^0(V, tL) \leqq h^0(W, tpL)$ for any $t \geqq 0$. This implies $\kappa(L, V) \leqq \kappa(L, W)$.

Step 2, the case in which $K(V)/K(W)$ is purely inseparable. Defining $F^{-e}(W)$ inductively by $F^{-e-1}(W) = F^{-1}(F^{-e}(W))$, we find an integer s such that $K(F^{-s}(W)) \supset K(V) \supset K(W)$. Since f is finite, we may identify V with the normalization of W in the field $K(V)$. So we have a surjective morphism $F^{-s}(W) \to V$. This implies $\kappa(L, V) \leqq \kappa(L, F^{-s}(W)) = \kappa(L, W)$, the latter equality follows from Step 1.

Step 3, the case in which $K(V)/K(W)$ is separable. The assertion is proved by the same argument as in [U], p. 62.

Step 4, the general case. Clearly we may assume $p = \mathrm{char}(K) > 0$ by Step 3. Let K' be the subfield $\{\alpha \in K(V) \mid \alpha^{p^e} \in K(W) \text{ for some } e > 0\}$ of $K(V)$. Then $K(V)/K'$ is a separable extension and $K'/K(W)$ is purely inseparable. Let W' be the normalization of W in the field K'. Then f factors to the composition of natural surjective morphisms $V \to W' \to W$. In view of Step 2 and 3, we infer that $\kappa(L, V) = \kappa(L, W') = \kappa(L, W)$. q. e. d.

Appendix

As an application, we generalize a result of Zariski (cf. [Z2]).

THEOREM. *Let L be a line bundle on a normal variety V such that $\kappa(L, V) = 1$. Then the graded algebra $\bigoplus_{t \geqq 0} H^0(V, tL)$ is finitely generated.*

PROOF. Let $\Phi : \Gamma \to W$ be an Iitaka fibration associated with L (see (3.8) and (3.11)). To prove the theorem, we may assume that $V = \Gamma$ and W is a smooth curve.

For an effective divisor E on V, we define $\Phi_* E$ to be the maximal effective divisor E' on W such that $E - \Phi^* E'$ is effective on V. Then, using $\kappa(L_F, F) = 0$, we infer that $H^0(V, tL) \cong H^0(W, \Phi_* \Delta)$ for any $t \geqq 0$ and any $\Delta \in |tL|$.

Take a member D of $|aL|$ for some $a > 0$. For each $x \in W$, let $\Phi^*(x) = \sum \mu_i X_i$ be the prime decomposition as a Weil-divisor on V. Let δ_i be the coefficient of X_i in the divisor $D - \Phi^* \Phi_* D$. Set $r(x) = \mathrm{Min}_i(\delta_i/\mu_i)$ and let $b(x)$ be the least positive integer such that $b(x) r(x) \in \boldsymbol{Z}$. Note that $r(x) = 0$ and $b(x) = 1$ except at most finite number of points on W. Let m be the least common multiple of $\{b(x)\}_{x \in W}$. Then we easily see $\Phi_*(E + tmD) = \Phi_* E + t\Phi_*(mD)$ for any

$t \geqq 0$ and any effective divisor E which is proportional to D.

Now, for each $j=0, 1, \cdots, am-1$, let j' be the least integer such that $j' \geqq 0$, $j' \equiv j \pmod{am}$ and $|j'L| \neq \emptyset$. Take $E_j \in |j'L|$. Set $M_j = \bigoplus_{t \geq 0} H^0(V, (j'+tam)L)$ and $N_j = \bigoplus_{t \geq 0} H^0(W, \Phi_* E_j + t\Phi_*(mD))$. Then, by the above observation, we see $M_0 \cong N_0$ as graded algebras. Moreover, $M_j \cong N_j$ as their graded modules for each j. Since $\Phi_*(mD)$ is ample on W, N_0 (resp. each N_j) is a finitely generated algebra (resp. N_0-module). So M_0 (resp. each M_j) is a finitely generated algebra (resp. M_0-module). This implies that $\bigoplus_{t \geq 0} H^0(V, tL) = \bigoplus_j M_j$ is a finitely generated algebra. q. e. d.

COROLLARY. *Let V and L be as above. Then, there exist a positive integer m, d and constants $c_0, c_1, \cdots, c_{m-1}$ such that $h^0(V, (j+tm)L) = dt + c_j$ for each $j=0, \cdots, m-1$ and for any sufficiently large integer t.*

References

[EGA] Grothendieck, A., Eléments de géométrie algébrique, Publ. Math. Inst. H. E. S. **4**, **8**, **11** and **17**, 1960, 1961, 1961 and 1963.

[H] Hartshorne, R., Algebraic Geometry, Graduate Texts in Math., 52, Springer, 1977.

[I] Iitaka, S., On D-dimensions of algebraic varieties, J. Math. Soc. Japan, **23** (1971), 356-373.

[K] Kodaira, K., Holomorphic mappings of polydiscs into compact complex manifolds, J. Differential Geom. **6** (1971), 33-46.

[M] Mumford, D., Enriques' classification of surfaces in char. p, I, Glabal Analysis (Papers in Honor of Kodaira), p. 325-339, Univ. Tokyo Press, 1969.

[N] Nagata, M., Imbedding of an abstract variety in a complete variety, J. Math. Kyoto Univ. **2** (1962), 1-10 and A generalization of the imbedding problem of an abstract variety in a complete variety, the same Journal, **3** (1963), 89-102.

[S] Serre, J. P., Faisceaux algébriques cohérents, Ann. of Math. **61** (1955), 197-278.

[T] Tate, J., Genus change in inseparable extensions of function fields, Proc. Amer. Math. Soc. **3** (1952), 400-406.

[U] Ueno, K., Classification Theory of Algebraic Varieties and Compact Complex Spaces, Lecture Notes in Math., 439, Springer, 1975.

[W] Weil, A., Foundations of Algebraic Geometry, revised edition, Amer. Math. Soc. Colloq. Publ. **29** (1960).

[Z] Zariski, O., Introduction to the Problem of Minimal Models in the Theory of Algebraic Surfaces, Publ. Math. Soc. Japan, WO. 4, 1958.

[Z2] Zariski, O., The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. **76** (1962), 560-615.

(Received October 15, 1980)

College of General Education
University of Tokyo
Komaba, Tokyo
153 Japan

and

University of California
Berkeley, Calif. 94720
U. S. A.