Maximal ideals in the algebra of holomorphic functions

By Hiroji OCHIAI and Niro YANAGIHARA

(Communicated by T. Kimura)

1. Introduction.

Let Ω be a domain of holomorphy in the complex Euclidean space $\mathbb{C}^N$, and let $H(\Omega)$ be the algebra of all holomorphic functions in Ω. $H(\Omega)$ is equipped with the topology of uniform convergence on every compact subset of Ω.

Let M be a maximal ideal in $H(\Omega)$. A well known theorem of Igusa [3] states

THEOREM IG. *The following four conditions* (i)-(iv) *are equivalent for* M:

(i) *M corresponds to a point of Ω.*

(ii) *M is closed.*

(iii) $H(\Omega)/M\cong\mathbb{C}$.

(iv) *M is finitely generated.*

By Theorem IG, we know that, if M does not correspond to any point of Ω, then $H(\Omega)/M$ must be a proper extension of $\mathbb{C}$. In § 3, we will determine the field $H(\Omega)/M$ for this case.

In § 4, we compactify Ω so that M corresponds to a "boundary point" of Ω.

The maximal ideal M can not be finitely generated in this case. We will determine generators of M, for the one-variable case (i.e., for the case $N=1$), in § 5.

2. Preliminaries.

Let M be a maximal ideal in $H(\Omega)$. Take and fix a function $f_0\in M$. Let $\{A_n\}$ be the set of all irreducible components of the analytic set $\{f_0=0\}$.

Suppose there is a finite collection of functions $f_1^1, \cdots, f_{m_1}^1$ in M such that the irreducible components $\{A_{nn_1}\}$, $A_{nn_1}\subset A_n$, of the analytic set $\{f_0=f_1^1=\cdots=f_{m_1}^1=0\}$ have the property

$$0\leqq\dim(A_{nn_1})<\dim(A_n)$$

for every n and n_1 whenever $\dim(A_n)\geqq1$.

Suppose we can continue the procedure further up to the k-th step ($k\geqq0$),

i. e., there are finite collections of functions in M

$$f_0, f_1^1, \cdots, f_{m_1}^1, \cdots, f_1^k, \cdots, f_{m_k}^k \subset M$$

such that, if $\{A_{nn_1\cdots n_j}\}$, $0 \leqq j \leqq k$, be the set of all irreducible components of

$$\{f_0 = f_1^1 = \cdots = f_{m_1}^1 = \cdots = f_1^j = \cdots = f_{m_j}^j = 0\},$$

then for every $(n, n_1, \cdots, n_{j-1}, n_j)$, $j=0, \cdots, k$ $(n_0=n)$,

$$A_{nn_1\cdots n_{j-1}n_j} \subset A_{nn_1\cdots n_{j-1}}$$

and, whenever $\dim(A_{nn_1\cdots n_{j-1}}) \geqq 1$, we have

$$0 \leqq \dim(A_{nn_1\cdots n_{j-1}n_j}) < \dim(A_{nn_1\cdots n_{j-1}}).$$

The procedure must stop after a finite number of steps, say k steps. Then we have the following two possibilities:

Case 1. Every component $A_{nn_1\cdots n_k}$ is of dimension 0, i. e., all irreducible components reduce to points after k steps.

Case 2. There remain some components of dimension not less than one. Then, for any finite collection of functions $g_1, \cdots, g_r$ in M, we have: if $\{A_{nn_1\cdots n_k p}\}$ is the set of all irreducible components of

$$\{f_0 = f_1^1 = \cdots = f_{m_k}^k = g_1 = \cdots = g_r = 0\},$$

then there is a set $(n, n_1, \cdots, n_k, p)$ such that

$$\dim(A_{nn_1\cdots n_k p}) = \dim(A_{nn_1\cdots n_k}) \geqq 1. \tag{2.1}$$

Since $A_{nn_1\cdots n_k}$ is irreducible, we have by [6, p. 76, Theorem 1J],

$$A_{nn_1\cdots n_k p} = A_{nn_1\cdots n_k}. \tag{2.2}$$

We will show that the case 2 does not occur.

Suppose the case 2 would occur. Let $\{A_1', A_2', \cdots\}$ be all irreducible components of dimensions not less than one, of the analytic set

$$\{f_0 = f_1^1 = \cdots = f_{m_1}^1 = \cdots = f_1^k = \cdots = f_{m_k}^k = 0\}.$$

We take two points z_j', z_j'', $z_j' \neq z_j''$, in each A_j', such that

$$z_j' \neq z_i' \quad \text{and} \quad z_j'' \neq z_i'' \quad \text{if} \quad j \neq i, \quad \text{and} \quad z_j' \neq z_i'',$$

and

$$\{z_j', z_j''\,;\ j=1, 2, \cdots\} \text{ does not cluster in } \Omega.$$

Let $F(z)$ be a function in $H(\Omega)$ such that

$$F(z_j')=0, \quad F(z_j'')=1, \qquad j=1, 2, \cdots.$$

Take a finite collection of functions $g_1, \cdots, g_r$ in M arbitrarily. Let $\{A'_{ji}\}$ be irreducible components of

$$\{f_0=f_1^1=\cdots=f_{m_k}^k=g_1=\cdots=g_r=0\}$$

obtained from $\{A'_j\}$. By the assumption in the case 2, there are a number j and a number i such that

$$A'_{ji}=A'_j.$$

Since $F(z)$ has a zero point z'_j in $A'_j=A'_{ji}$, $F(z)$ has common zeros with any finite collection of functions in M, hence F *belongs to* M. But, if $\{A'_{jp}\}$ are all irreducible components of

$$\{f_0=f_1^1=\cdots=f_{m_k}^k=F=0\},$$

obtained from $\{A'_j\}$, then

$$\dim(A'_{jp})<\dim(A'_j) \qquad \text{for every } (j, p),$$

since $F(z)$ does not vanish at $z''_j\in A'_j$. This contradicts the assumption of the case 2.

Therefore, we have the case 1 only. That is, we obtain the following

THEOREM 1. *Let M be a maximal ideal in $H(\Omega)$. Then, there is a finite number of functions $f_0, f_1, \cdots, f_m$, $f_j\in M$, such that all irreducible components of the analytic set*

$$Z=\{f_0=f_1=\cdots=f_m=0\}$$

are of dimension zero, i.e., the analytic set Z is a point sequence (finite or infinite) which does not cluster in Ω: $Z=\{z_n\}$.

REMARK. By [1, p. 44, Satz 2], we can take $(N+1)$ functions in M whose common zeros coincide with the set Z in Theorem 1.

We note that Theorem 1 holds for any maximal ideal in $H(\Omega)$.

3. An extension of the complex number field.

Now we return to the supposition that M does not correspond to any point of Ω.

LEMMA 2. *If the maximal ideal M corresponds to no point of Ω, then the set Z in Theorem 1 is an infinite sequence.*

PROOF. Suppose $Z=\{z_1, \cdots, z_r\}$. By the supposition on M, there are $g_1, \cdots, g_r$, $g_j\in M$, such that $g_j(z_j)\neq 0$ for $j=1, \cdots, r$. Then the finite collection of functions $g_1, \cdots, g_r, f_0, \cdots, f_m$ in M has no common zeros, which is a contradiction, since

M is a proper ideal. Q. E. D.

Now, let I be the set of all positive integers. For a subset $A \subset I$, we denote by Z_A the subsequence of Z corresponding to numbers in A. That is, if $A=\{n_1, n_2, \cdots\}$, then $Z_A=\{z_{n_1}, z_{n_2}, \cdots\}$.

Let Φ be a family of subsets of I defined as follows: $A \subset I$ belongs to Φ if there is a finite collection of functions $g_1, \cdots, g_r$, $g_j \in M$, such that the set of common zeros of $g_1, \cdots, g_r, f_0, \cdots, f_m$ is Z_A.

LEMMA 3. *The family Φ is a ultrafilter on I, i.e.,*

1° $\emptyset \notin \Phi$.

2° *If $A, B \in \Phi$, then $A \cap B \in \Phi$.*

3° *If $A \in \Phi$ and $A \subset B \subset I$, then $B \in \Phi$.*

4° *If $A \subset I$, then A or $I-A$ belongs to Φ.*

PROOF. 1° is obvious since M is a proper ideal. 2°, suppose Z_A and Z_B be the sets of common zeros of $\{g_1, \cdots, g_r, f_0, \cdots, f_m\}$ and $\{h_1, \cdots, h_s, f_0, \cdots, f_m\}$, respectively. Then, the set of common zeros of $\{g_1, \cdots, g_r, h_1, \cdots, h_s, f_0, \cdots, f_m\}$ is $Z_{A \cap B}$. 3°, let $F(z)$ be a function in $H(\Omega)$ such that

$$F(z_j)=0 \quad \text{if } j \in B; \quad F(z_j)=1 \quad \text{if } j \in I-B.$$

Suppose Z_A be the set of common zeros of $\{g_1, \cdots, g_r, f_0, \cdots, f_m\}$. Take a finite collection of functions $h_1, \cdots, h_s$ in M arbitrarily. Then $\{g_1, \cdots, g_r, h_1, \cdots, h_s, f_0, \cdots, f_m\}$ has common zeros contained in Z_A. Since $Z_A \subset Z_B$, $F(z)$ must have common zeros with $h_1, \cdots, h_s$. Because $\{h_j\}$ is chosen arbitrarily in M and M is a maximal ideal, $F(z)$ must belong to M, which shows that $B \in \Phi$. 4°, if the lemma would not hold, there would be a filter $\Phi' \supsetneqq \Phi$. Take a set $A \in \Phi' - \Phi$. There is a function $G \in H(\Omega)$ such that

$$G(z_j)=0 \quad \text{if } j \in A; \quad G(z_j)=1 \quad \text{if } j \in I-A.$$

Since $A \notin \Phi$, G can not belong to M. Thus, there is a finite collection of functions $g_1, \cdots, g_r$, $g_j \in M$, such that $G, g_1, \cdots, g_r$ have no common zeros. Let Z_B be the set of common zeros of $\{g_1, \cdots, g_r, f_0, \cdots, f_m\}$. Then $B \in \Phi$ hence $B \in \Phi'$, therefore $\emptyset = A \cap B \in \Phi'$, which is absurd. Q. E. D.

Let $\mathbb{C}^I$ be the set of all complex sequences. For $a^*=(a_1, a_2, \cdots)$, $b^*=(b_1, b_2, \cdots) \in \mathbb{C}^I$, we define

$$a^* \equiv b^* \text{ means that } \{i;\ a_i=b_i\} \in \Phi.$$

Then, $\equiv$ is an equivalence relation, as easily seen from Lemma 3. We put $\mathbb{C}^* = \mathbb{C}^I/(\equiv)$ and write $[a^*]$ for the equivalence class of a^*.

We put

$$[a^*]+[b^*]=[(a_1+b_1,\ a_2+b_2,\ \cdots)],$$

$$[a^*][b^*]=[(a_1b_1,\ a_2b_2,\ \cdots)],$$

$[a^*]/[b^*]=[(c_1,\ c_2,\ \cdots)]$, where $(c_1,\ c_2,\ \cdots)$ is a sequence such that $\{i\,;\ c_ib_i=a_i\}\in\Phi$, supposing that $[b^*]\neq[0^*]=[(0,\ 0,\ \cdots)]$.

Then, $\mathcal{C}^*$ is a field. If $a\in\mathcal{C}$, we correspond a to the element $[(a,\ a,\ \cdots)]\in\mathcal{C}^*$. Thus, $\mathcal{C}^*$ is an extension of $\mathcal{C}$. We remark that $\mathcal{C}^*$ is a *transcendental* extension of $\mathcal{C}$.

We will show that *$H(\Omega)/M$ is isomorphic to $\mathcal{C}^*$*, constructed above.

Let $[a^*]\in\mathcal{C}^*$, where $a^*=(a_1,\ a_2,\ \cdots)$, and let f^a be a function in $H(\Omega)$ for which $f^a(z_j)=a_j$, $j=1,\ 2,\ \cdots$. We correspond $[a^*]$ to $[f^a]\in H(\Omega)/M$. If $a'^*=(a'_1,\ a'_2,\ \cdots)\in[a^*]$, there is a function $g\in H(\Omega)$ such that $g(z_j)=a'_j$, $j=1,\ 2,\ \cdots$. Then, the set of common zeros of $\{f^a-g,\ f_0,\ \cdots,\ f_m\}$ coincides with $\{i\,;\ a_i=a'_i\}\in\Phi$, hence $f^a-g\in M$, $g\in[f^a]$.

Conversely, let $[g]\in H(\Omega)/M$. We put $a_j=g(z_j)$ and $a^*=(a_1,\ a_2,\ \cdots)$. We correspond $[g]$ to $[a^*]\in\mathcal{C}^*$. If $h\in[g]$ and $a'_j=h(z_j)$, then obviously $a'^*=(a'_1,\ a'_2,\ \cdots)\in[a^*]$. Thus our assertion is proved.

4. A compactification of the domain Ω.

Suppose a maximal ideal M in $H(\Omega)$ does not correspond to any point of Ω. We will show that M corresponds to a boundary point in the sense we now explain.

Let S be the set of all infinite point sequences in Ω which cluster at no point of Ω. For any sequence $Z=\{z_n\}\in S$, let $\mathfrak{U}(Z)$ be the set of all ultrafilters of subsequences of Z, which contains all subsequences F of Z such that $Z-F$ is finite.

Let $Z_1,\ Z_2\in S$ and $\Psi_1\in\mathfrak{U}(Z_1)$, $\Psi_2\in\mathfrak{U}(Z_2)$. We say that $(Z_1,\ \Psi_1)$ and $(Z_2,\ \Psi_2)$ are equivalent, writing as $(Z_1,\ \Psi_1)\sim(Z_2,\ \Psi_2)$, if there is a subsequence $Z'\subset Z_1\cap Z_2$, $Z'\in S$, such that Z' belongs to Ψ_1 as well as to Ψ_2, and $Z'\cap\Psi_1=Z'\cap\Psi_2$, where $Z'\cap\Psi_i=\{Z'\cap E\,;\ E\in\Psi_i\}$, $i=1,\ 2$. We will show the transitivity of this relation.

Suppose $(Z_1,\ \Psi_1)\sim(Z_2,\ \Psi_2)$ and $(Z_2,\ \Psi_2)\sim(Z_3,\ \Psi_3)$. Then, there are $Z'\subset Z_1\cap Z_2$ and $Z''\subset Z_2\cap Z_3$ such that $Z'\in\Psi_1\cap\Psi_2$, $Z''\in\Psi_2\cap\Psi_3$ and

$$Z'\cap\Psi_1=Z'\cap\Psi_2\,,\quad Z''\cap\Psi_2=Z''\cap\Psi_3\,. \tag{4.1}$$

Then, $Z'''=Z'\cap Z''\in Z'\cap\Psi_2=Z'\cap\Psi_1$, hence $Z'''=Z'\cap E'_1$, $E'_1\in\Psi_1$. Since $Z'\in\Psi_1$, we have $Z'''\in\Psi_1$. Similarly we have $Z'''\in\Psi_3$. Of course, we have $Z'''\subset Z_1\cap Z_3$.

From (4.1), we have obviously

$$Z'\cap Z''\cap \Psi_1=Z'\cap Z''\cap \Psi_2=Z'\cap Z''\cap \Psi_3,$$

hence $Z'''\cap \Psi_1=Z'''\cap \Psi_3$, which shows that $(Z_1, \Psi_1)\sim(Z_3, \Psi_3)$.

Sometimes, we denote simply as $\Psi_1\sim\Psi_2$, instead of $(Z_1, \Psi_1)\sim(Z_2, \Psi_2)$. Each equivalence class $[\Psi]$ of a ultrafilter Ψ is said to determine a boundary point b_Ψ of Ω. We write

$$\delta\Omega=\{b_\Psi;\ \Psi\in\mathfrak{U}(Z),\ Z\in S\},\quad \text{and}$$

$$\Omega^*=\Omega\cup\delta\Omega.$$

Arguments in § 3 show that non-closed maximal ideals correspond to boundary points of Ω, defined above. Conversely, let $b_\Psi\in\delta\Omega$, which is determined by (Z, Ψ), $Z\in S$ and $\Psi\in\mathfrak{U}(Z)$. Put

$$J=\{f\in H(\Omega);\ (\text{zero set of } f)\cap Z\in\Psi\}.$$

J is obviously an ideal. If M is a maximal ideal containing J, then M induces an ultrafilter on Z as shown in § 3, which contains Ψ. Since Ψ is an ultrafilter, we conclude that $J=M$, and J is a maximal ideal. It is easy to see that, if $\Psi_1\sim\Psi_2$, then Ψ_1 and Ψ_2 determine the same maximal ideal. Therefore, each boundary point corresponds to a maximal ideal. Let $\mathfrak{M}$ be the maximal ideal space of $H(\Omega)$. By the above, we obtain

THEOREM 4. *Points of Ω^* and $\mathfrak{M}$ correspond to each other in a one-to-one way.*

Now we introduce a topology in Ω^*.

Neighborhoods of points of Ω are defined as usual.

Let $b_\Psi\in\delta\Omega$ be determined by a class $[\Psi]$. A subset N of Ω^* is said *a neighborhood of* b_Ψ if there are $\Psi'\in[\Psi]$ and a sequence $E\in\Psi'$ such that

(4.2) $\begin{cases} N\cap\Omega \text{ is an open set of } \Omega \text{ containing } E, \text{ and} \\ N\cap\delta\Omega \text{ consists of boundary points determined by classes of ultrafilters} \\ \quad \text{on point sequences } (\in S) \text{ contained in } N\cap\Omega. \end{cases}$

By this definition, Ω^* becomes a Hausdorff topological space. We have

THEOREM 5. *The space Ω^*, hence the space $\mathfrak{M}$, is countably compact.*

PROOF. Let $\{p_n\}$ be a sequence in Ω^*. We will show that it has a cluster point in Ω^*. We can assume that it does not cluster at any point of Ω. (For the definition of *countable compactness*, see [4, p. 162].)

Suppose $\{p_n\}$ has an infinite subsequence $\{z_m\}$ contained in Ω. Then $Z=\{z_m\}$ and an ultrafilter Ψ on Z determine a boundary point b_Ψ which is clearly a cluster point of $\{p_n\}$.

Hence we can suppose $\{p_n\}\subset\delta\Omega$. Then, each p_n is determined by a sequence $Z_n=\{z_{n,m}\}$ and an ultrafilter $\Psi_n=\{E_{n,\alpha}\}_{\alpha\in A_n}$. Let $\{K_h\}$ is an increasing sequence of compact subsets of Ω such that

$$K_h\subset\mathrm{int}(K_{h+1}) \quad\text{and}\quad \cup K_h=\Omega .$$

Put (writing the complement of K_h as K_h^C)

$$Z'_n=Z_n\cap K_n^C, \quad E'_{n,\alpha}=E_{n,\alpha}\cap K_n^C, \quad Z=\bigcup_{n=1} Z'_n .$$

We write, for $\beta=(\alpha_1, \alpha_2, \cdots)$, $\alpha_j\in A_j$,

$$\beta_k=(\alpha_k, \alpha_{k+1}, \cdots) \quad (\beta_1=\beta)$$

and

$$E_{\beta_k}=E'_{k,\alpha_k}\cup E'_{k+1,\alpha_{k+1}}\cup\cdots .$$

Then, $\{E_{\beta_k};\ \beta\in\prod_{j=1}^{\infty}A_j,\ k=1, 2, \cdots\}$ forms obviously a basis for filters on Z, since it has the finite intersection property. The sequence Z and an ultrafilter Ψ on Z, containing $\{E_{\beta_k}\}$, determine a boundary point b_Ψ. Let E be a set in Ψ. Suppose that, for some k and each $m\geq k$, there is an α_m such that $E\cap E'_{m,\alpha_m}$ $=$void. Taking $E_{\beta_k}=\bigcup_{j=k} E'_{j,\alpha_j}$, we have $E\cap E_{\beta_k}=$void, which is absurd. Hence, for each k, there is $m\geq k$ such that $E\cap E'_{m,\alpha_m}\neq$void for every $\alpha_m\in A_m$, which shows that p_m belongs to neighborhoods defined by E. Thus, b_Ψ is a cluster point of $\{p_m\}$. Since any sequence in Ω^* has a cluster point, Ω^* is countably compact, and Theorem 5 is proved. Q. E. D.

REMARK. Ω^* does not satisfy the first countability axiom. But for each b_Ψ $\in\delta\Omega$ there is a sequence $Z\in S$ whose closure contains b_Ψ. Compare with the case of one-variable bounded functions [2, p. 85, Corollary]. Since b_Ψ belongs to the closure of Z, then a subnet of Z converges to b_Ψ [4, p. 71]. But any sequence in Ω can not converge to b_Ψ.

THEOREM 6. *Every function in $H(\Omega)$ is continuous on Ω^* as a map from Ω^* into the Riemann sphere.*

PROOF. Let f be a function in $H(\Omega)$. Let $T=\{t_n\}$ be a net in Ω which converges to a boundary point b_Ψ, defined by (Z, Ψ), $Z=\{z_k\}\in S$, $\Psi\in\mathfrak{U}(Z)$. If $\{f(t_n)\}$ would have not a limit, there would be subnets $\{t_{n'}\}$, $\{t_{n''}\}$, such that $f(t_{n'})\to\alpha$, $f(t_{n''})\to\beta$, $\alpha\neq\beta$. Suppose $\alpha\neq\infty$, $\beta\neq\infty$, and $|\alpha-\beta|=3\varepsilon>0$. The case that α or β is ∞ is treated analogously.

There are n'_1 and n''_1 such that if $n'\geq n'_1$ and $n''\geq n''_1$, then

$$|f(t_{n'})-\alpha|<\varepsilon , \quad |f(t_{n''})-\beta|<\varepsilon . \tag{4.3}$$

We put $T_1=\{t_{n'}\}_{n'\geqq n_1'}$, $T_2=\{t_{n''}\}_{n''\geqq n_1''}$, and

$$Z_i=\{z_k\in Z\,;\ \mathrm{dis}(z_k,\ T_i)=0\},\qquad i=1,\ 2.$$

Then we have, from (4.3)

$$|f(z_k)-\alpha|\leqq\varepsilon\ \text{ if } z_k\in Z_1\,;\quad |f(z_h)-\beta|\leqq\varepsilon\ \text{ if } z_h\in Z_2. \tag{4.4}$$

Z_1 belongs to Ψ. For, if not, there is $E\in\Psi$ such that $Z_1\cap E=$void, hence for each point $z_j\in E$, $\delta_j=\mathrm{dis}(z_j,\ T_1)>0$. Put

$$N=\cup\{z\in\Omega\,;\ |z-z_j|<\delta_j,\ z_j\in E\}.$$

N is (intersection with Ω of) a neighborhood of b_Ψ and contains no point of T_1, hence $\{t_{n'}\}$ can not converge to b_Ψ, which is absurd since $\{t_{n'}\}$ is a subnet of $\{t_{n'}\}$. Similarly, $Z_2\in\Psi$. Hence $Z_1\cap Z_2\neq$void, which contradicts (4.4). Hence $f(t_n)$ has a limit when $t_n\to b_\Psi$, $t_n\in\Omega$.

Suppose $b_n\in\delta\Omega$, $b_n\to b_\Psi$. By the above arguments, $f(b_n)\in\mathbb{C}\cup\{\infty\}$. For each n, there is a net $\{z_{n,m}\}$, $z_{n,m}\to b_n$, hence $f(z_{n,m})\to f(b_n)$. Thus we can choose $z_n'=z_{n,m(n)}$ such that $z_n'\to b_\Psi$ [4, p. 69, Theorem 2.4]. Therefore, $\{f(b_n)\}$ has a limit which equals to $\lim_n f(z_n')$.

Thus, f has a limit at each boundary point, and our theorem is proved.

5. Generators for maximal ideals (one-variable case).

Suppose a maximal ideal M does not correspond to any point of Ω. By Theorem IG, M can not be generated by a finite number of its elements. If we restrict ourselves to the case $N=1$, then we are ready to determine a basis for M.

We know that, by the arguments in § 3, there are a sequence $Z\in S$ and an ultrafilter $\Psi\in\mathfrak{U}(Z)$ which correspond to M. For a set $E\in\Psi$, we denote by $g(z\,;E)$ a function in $H(\Omega)$ which has simple zero at each point of E and has no other zeros.

Let $B=\{E_\alpha\}_{\alpha\in A}$ be a subbasis of the filter Ψ, i.e., $B\subset\Psi$ and for each $E\in\Psi$ there are $E_{\alpha_1},\cdots,E_{\alpha_m}\in B$ such that

$$E_{\alpha_1}\cap\cdots\cap E_{\alpha_m}\subset E. \tag{5.1}$$

Write $g_\alpha(z)=g(z\,;E_\alpha)$. Then, the system $G=\{g_\alpha\}_{\alpha\in A}$ is a basis for M. In fact, if $f\in M$, we put

$$E=(\text{zero set of } f)\cap Z\in\Psi.$$

There are $E_{\alpha_1},\cdots,E_{\alpha_m}\in B$ which satisfy (5.1). Then $f(z)/g(z\,;E)\in H(\Omega)$ and $g(z\,;E)$ belongs to the ideal $(g_{\alpha_1},\cdots,g_{\alpha_m})$, hence we have

$$f(z)=h_1(z)g_{\alpha_1}(z)+\cdots+h_m(z)g_{\alpha_m}(z)$$

with $h_j \in H(\Omega)$, and M is generated by G.

6. Hull-kernel topology in the maximal ideal space.

Let $\mathfrak{M}$ be the space of all maximal ideals in $H(\Omega)$. We defined a topology in $\mathfrak{M}$, as stated in (4.2). By theorem 5, $\mathfrak{M}$ is countably compact and, by theorem 6, every function f in $H(\Omega)$ is continuously extended on $\mathfrak{M}$. Further, Ω is dense in $\mathfrak{M}$.

On the other hand, $\mathfrak{M}$ can be topologized as follows: a set $\mathfrak{E} \subset \mathfrak{M}$ is said *closed* if $\mathfrak{E}$ is the *hull* $h(J)$ of some ideal J in $H(\Omega)$, i.e., there is an ideal J in $H(\Omega)$ such that

$$\mathfrak{E} = h(J) = \{M \in \mathfrak{M};\ M \supseteq J\}. \tag{6.1}$$

This topology is called the *hull-kernel topology* or the *Stone topology*. If $\mathfrak{M}$ is endowed with this topology, we write it as $\mathfrak{M}^{HK}$. $\mathfrak{M}^{HK}$ is called as the *strong structure space* of the algebra $H(\Omega)$ [5, p. 78].

By the way, we write in this section as $\mathfrak{M}^*$ if $\mathfrak{M}$ is endowed with the topology defined by (4.2).

For a subset $\mathfrak{E} \subset \mathfrak{M}$, we define the *kernel* $k(\mathfrak{E})$ of $\mathfrak{E}$ as

$$k(\mathfrak{E}) = \bigcap_{M \in \mathfrak{E}} M. \tag{6.2}$$

Thus, for $\mathfrak{E} \subset \mathfrak{M}$,

$$\bar{\mathfrak{E}} = \text{the closure of } \mathfrak{E} \text{ in } \mathfrak{M}^{HK} = h(k(\mathfrak{E})).$$

THEOREM 7. *Topology in $\mathfrak{M}^*$ is stronger than the one in $\mathfrak{M}^{HK}$.*

PROOF. Suppose $\mathfrak{E}$ is closed in $\mathfrak{M}^{HK}$. Then, $\mathfrak{E} = h(J)$ for some ideal J. For a function $f \in H(\Omega)$, we put

$$h(f) = \{M \in \mathfrak{M};\ M \supseteq (f)\}, \tag{6.3}$$

where (f) is the ideal generated by f. Then

$$h(J) = \bigcap_{f \in J} h(f).$$

Thus, we have only to prove that $h(f)$ is closed in $\mathfrak{M}^*$. Take an $M \in \mathfrak{M} - h(f)$. There is a finite collection of functions $f_1, \cdots, f_m \in M$ such that

$$Z = \{f_1 = \cdots = f_m = 0\}$$

is a point sequence and $Z \cap \{f=0\} = \text{void}$. M is written as (Z, Ψ) with a ultrafilter Ψ on Z. Since the analytic set $\{f=0\}$ is closed in Ω, we can take a neighborhood N of $M = (Z, \Psi)$ such that $(N \cap \Omega) \cap \{f=0\} = \text{void}$. Then, $N \cap h(f)$

=void, which shows that $\mathfrak{M}-h(f)$ is open in $\mathfrak{M}^*$.

We denote the maximal ideal corresponding to a point $p\in\Omega$ as $M(p)$. Let F be a closed proper subset of Ω, containing an open set in Ω. Then

$$F^*=\{M(p)\,;\ p\in F\}$$

is a closed set in $\mathfrak{M}^*$, while the closure of F^* in $\mathfrak{M}^{HK}$ is $\mathfrak{M}\supsetneqq F^*$, since

$$k(F^*)=\bigcap_{p\in F} M(p)=(0)\,. \qquad \text{Q. E. D.}$$

But we have

THEOREM 8. *We restrict ourselves to the case one-variable case. Let* $\mathfrak{E}^*$ *be a closed set in* $\mathfrak{M}^*$. *If*

$$J=k(\mathfrak{E}^*)\neq(0)\,,$$

then $\mathfrak{E}^*$ *is closed also in* $\mathfrak{M}^{HK}$.

PROOF. Let $f\in J=k(\mathfrak{E}^*)$, and Z be the zero set of f. Put

$$\mathfrak{E}^*=\{M_\alpha\,;\ \alpha\in A\},\quad\text{and}\quad M_\alpha=(Z,\ \Psi_\alpha)\,.$$

J induces a filter $\mathfrak{F}$ on Z. Let M be a maximal ideal containing J. If $M=(Z,\ \Psi)$, the ultrafilter Ψ contains $\mathfrak{F}$. Take $E\in\Psi$. Then, $E\cap F\neq$void for every $F\in\mathfrak{F}$. Suppose E would not belong to any ultrafilter Ψ_α, $\alpha\in A$. Then, there is an $E_\alpha\in\Psi_\alpha$ such that $E\cap E_\alpha$=void. Put

$$F=\bigcup_{\alpha\in A} E_\alpha\,.$$

Then, $F\cap E$=void. But $E_\alpha\in\Psi_\alpha$, hence $F\in\Psi_\alpha$ for any $\alpha\in A$. Hence F belongs to $\mathfrak{F}=\bigcap_{\alpha\in A}\Psi_\alpha$, which is a contradiction. Therefore, $E\in\Psi_\alpha$ for some $\alpha\in A$. Hence any neighborhood of M contains some $M_\alpha\in\mathfrak{E}^*$, and since $\mathfrak{E}^*$ is closed in $\mathfrak{M}^*$, we obtain that $M\in\mathfrak{E}^*$, which shows that $\mathfrak{E}^*=h(J)$, and $\mathfrak{E}^*$ is closed in $\mathfrak{M}^{HK}$.

In the above, we used the fact that $\mathfrak{F}=\bigcap_{\alpha\in A}\Psi_\alpha$, which is proved as follows: Let $E\in\Psi_\alpha$ for any $\alpha\in A$. Let g be a function whose zero set is E. Then g belongs to any M_α, hence $g\in J$, and we have that $E\in\mathfrak{F}$. Q. E. D.

$\mathfrak{M}^{HK}$ is not a Hausdorff space. To see this, let $p_1,\ p_2\in\Omega$, $p_1\neq p_2$, and U_1, U_2 be open sets in $\mathfrak{M}^{HK}$ such that $p_1\in U_1$, $p_2\in U_2$. There are ideals J_1 and J_2 with

$$U_i=\{M\in\mathfrak{M}\,;\ M\not\supseteq J_i\}\,,\qquad i=1,\ 2.$$

Choose functions $f_1\in J_1$, $f_2\in J_2$ and a point $q\in\Omega$ such that $f_1(q)\neq0$, $f_2(q)\neq0$. Then, the maximal ideal $M(q)$ corresponding to q must belong to both U_1 and U_2, hence $U_1\cap U_2\neq$void. Thus, $\mathfrak{M}^{HK}$ is not Hausdorff, though $\mathfrak{M}^{HK}$ satisfies the T_1-separation axiom, as easily seen.

But we have

THEOREM 9. *We restrict ourselves to the one-variable case. The set $h(f)$ in* (6.3), *with the relative topology as a subset of $\mathfrak{M}^{HK}$, is a Hausdorff space.*

PROOF. Let Z be the zero set of f and let $M_i=(Z, \Psi_i)\in h(f)$, $i=1, 2$. Take $E_1\in \Psi_2-\Psi_1$ and $E_2=Z-E_1\in \Psi_1-\Psi_2$. Put

$$J_i=\{g\in H(\Omega)\,;\ g \text{ vanishes on } E_i\}, \qquad i=1, 2,$$

and

$$O_i=\{M\in h(f)\,;\ M\not\supseteq J_i\}, \qquad i=1, 2.$$

Then, each O_i is open in $h(f)$ and $M_i\in O_i$. Take an $M\in h(f)$. Suppose $M\in O_1$. There is a $g\in J_1$ such that $g\notin M$, thus there is an $h\in M$ such that the zero set of h is contained in E_2. Then, J_2 is contained in M, and $M\notin O_2$. Thus, $h(f)$ is a Hausdorff space. Q. E. D.

COROLLARY 10. *We restrict ourselves to the one-variable case. Each hull $h(J)$ for $J\neq(0)$ is a Hausdorff space with the relative topology as a subset of $\mathfrak{M}^{HK}$.*

THEOREM 9′. *Suppose an ideal J in $H(\Omega)$ contains sufficiently many functions in the sense that there are $f_1, \cdots, f_m\in J$ such that the analytic set $\{f_1= \cdots =f_m=0\}$ is a point sequence Z. Then, the hull $h(J)$ is a Hausdorff space with the relative topology as a subset of $\mathfrak{M}^{HK}$.*

Proof is the same as in the above.

References

[1] Grauert, H., Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik, Math. Ann. **131** (1956), 38-75.

[2] Hoffman, K., Bounded analytic functions and Gleason parts, Ann. of Math. **86** (1967), 79-111.

[3] Igusa, J. I., On a property of the domain of regularity, Mem. Coll. Sci. Univ. Kyôto **27** (1952), 95-97.

[4] Kelley, J. L., General Topology, Van Nostrand Reinhold Co., New York and London, 1955.

[5] Rickart, C. E., General Theory of Banach Algebras, D. Van Nostrand Co., Princeton Toronto-London, 1960.

[6] Whitney, H., Complex Analytic Varieties, Addison-Wesley Publ. Co., 1972.

(Received August 12, 1980)

Department of Mathematics
Faculty of Education
Miyazaki University,
Miyazaki
880 Japan

and

Department of Mathematics
Faculty of Science
Chiba University
1-33 Yayoi-cho, Chiba City
Chiba
260 Japan