Entire solutions of a polynomial difference equation

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Introduction. Consider a difference equation of the form

(E) $$y(x+1)=P(y(x)),$$

where $P(t)$ is a polynomial of degree n ($\geqq 2$):

$$P(t)=a_0+a_1t+\cdots+a_nt^n$$

with complex coefficients.

It will be proved that a solution of (E) meromorphic in $|x|<\infty$, called a meromorphic solution, is always entire. The purpose of this paper is to study entire solutions of (E).

It is clear that if a solution of (E) converges to a certain value α as $\mathrm{Re}\,x$ tends to $-\infty$ or $+\infty$, α is a fixed point of the polynomial map $P: P(\alpha)=\alpha$. If $\alpha\neq\infty$, the transformation $y=z+\alpha$ takes (E) into an equation

(E. α) $$z(x+1)=Q(z(x)),$$

where $Q(t)$ is given by

$$Q(t)=P(t+\alpha)-\alpha .$$

Therefore $Q(t)$ is written as

$$Q(t)=b_1t+b_2t^2+\cdots+b_nt^n ,$$

and the first coefficient b_1 is equal to $P'(\alpha)$.

A transformation

$$z(x)=\phi(u(x))$$

changes (E. α) into

$$u(x+1)=b_1u(x)$$

if and only if ϕ satisfies the equation of Schröder

(E. S) $$\phi(b_1t)=Q(\phi(t)) .$$

It is well known that if $|b_1|\neq 0$, 1, then (E. S) admits a solution holomorphic at $t=0$:

$$\phi(t)=\sum_{j=1}^{\infty} p_j t^j \qquad (p_1=1)$$

and that if $|b_1|>1$, then $\phi(t)$ is continued analytically to an entire function. It follows that, if $|P'(\alpha)|>1$, equation (E) has an entire solution

$$\varphi(x, \alpha)=\phi(b_1^x)+\alpha .$$

In case when $P'(\alpha)=1$, it will be shown that (E) admits an entire solution which is developed asymptotically as

$$\psi(x, \alpha)\cong\alpha+\gamma(x(1+\sum_{i+j\geq 1} q_{ij}x^{-i/m}(x^{-1}\log x)^j))^{-1/m}$$

as $\mathrm{Re}\, x\to-\infty$, where γ and m are determined by $P(t)$ and α.

Section 1 is devoted to a study of properties of solutions of (E). Existence theorems of the solutions $\varphi(x, \alpha)$ and $\psi(x, \alpha)$ will be given in Section 2. In Section 3 we shall study equation (E. S). We shall discuss the orders and Julia's directions of the solutions $\varphi(x, \alpha)$ and $\psi(x, \alpha)$ in the final section.

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REMARK. N. Yanagihara recently proved that any transcendental meromorphic solution of the difference equation

$$y(x+1)=\frac{P(x, y(x))}{Q(x, y(x))},$$

where $P(x, t)$ and $Q(x, t)$ are mutually prime polynomials, is of order ∞ under a certain condition, and studied the equation

$$y(x+1)=\frac{P(y(x))}{Q(y(x))}$$

in a more detail (Cf. [6]).

§1. Properties of solutions of (E).

In this section, we assume the existence of meromorphic solutions of (E) and study their properties.

A fixed point t_0 $(\in \boldsymbol{P}^1(\boldsymbol{C}))$ of $P(t)$ is said to be *of multiplicity* k, if t_0 is a root of multiplicity k of the equation

$$P(t)=t_0 .$$

REMARK 1.1. $t_0=\infty$ is a fixed point of multiplicity n.

Let $\varphi(x)$ be a meromorphic solution of (E). Then the following proposition is easily obtained.

PROPOSITION 1.1. i) *If* α $(\in \boldsymbol{P}^1(\boldsymbol{C}))$ *is a fixed point of* $P(t)$ *and* $\varphi(x_0)=\alpha$, *then*

$$\varphi(x_0+\nu)=\alpha$$

for $\nu=1, 2, \cdots$.

ii) *If* α *is a fixed point of multiplicity* n *of* $P(t)$, *and* $\varphi(x_0)=\alpha$, *then*

$$\varphi(x_0\pm\nu)=\alpha$$

for $\nu=1, 2, \cdots$.

The order of an α-point of a meromorphic solution of (E) is given by the following proposition.

PROPOSITION 1.2. *Assume that* α *is a fixed point of multiplicity* n. *If a meromorphic solution* $\varphi(x)$ *has an* α*-point of order* r *at* $x=x_0$, *then* $x=x_0+\nu$ $(\nu=\pm 1, \pm 2, \cdots)$ *is an* α*-point of order* rn^ν.

PROOF. If $\alpha\neq\infty$, equation (E) can be written in the form

$$y(x+1)-\alpha=a_n(y(x)-\alpha)^n\,. \tag{1.1}$$

By the assumption, for a sufficiently small positive constant δ, $\varphi(x)$ is developed into convergent series

$$\varphi(x)=\alpha+c_0(x-x_0)^r+\cdots$$

if $|x-x_0|<\delta$. Hence by using (1.1), we derive that

$$\varphi(x+\nu)=\alpha+c_0^{(\nu)}(x-x_0)^{rn^\nu}+\cdots\,, \qquad |x-x_0|<\delta,$$

which implies that

$$\varphi(x)=\alpha+c_0^{(\nu)}(x-(x_0+\nu))^{rn^\nu}+\cdots\,,$$

for $|x-(\nu+x_0)|<\delta$. Therefore $\varphi(x)$ has an α-point of order rn^ν at $x=x_0+\nu$.

In case when $\alpha=\infty$, we can prove in a similar way.

COROLLARY 1.3. *If* α *is a fixed point of multiplicity* n, *a meromorphic solution of* (E) *never takes the value* α.

Considering a special fixed point $\alpha=\infty$ of multiplicity n, we obtain the following theorem.

THEOREM 1.4. *Any nontrivial meromorphic solution of* (E) *is transcendental and entire.*

To prove Theorem 1.4, it is sufficient to show that any nontrivial meromorphic solution is not a rational function. This fact is an easy consequence of Proposition 1.2.

Theorem 1.4 leads us to the following theorem concerning Picard exceptional values of an entire solution of (E).

THEOREM 1.5 (Picard exceptional value). *A value* α $(\neq\infty)$ *is a Picard exceptional value of* $\varphi(x)$, *if and only if* α *is a fixed point of multiplicity* n *of* $P(t)$.

PROOF. In view of Corollary 1.3, it is sufficient to show that any value other than the fixed point of multiplicity n is not a Picard exceptional value. To do this, suppose that β $(\neq\infty)$ is not a fixed point of multiplicity n and is a Picard exceptional value:

(1.2) $$\varphi(x)\neq\beta \qquad \text{for } |x|>R.$$

Then there exists a point γ such that

$$P(\gamma)=\beta\,, \quad \gamma\neq\beta,\ \infty\,,$$

and

(1.3) $$\varphi(x)\neq\gamma \qquad \text{for } |x+1|>R.$$

Relations (1.2) and (1.3) contradict the big Picard's theorem. Thus the proof is completed.

§2. Existence of nontrivial entire solutions.

Let us denote by α a fixed point of $P(t)$. As is stated in Introduction, (E) is taken by

$$y(x)=z(x)+\alpha$$

into

(E. α) $$z(x+1)=Q(z(x))$$

where

(2.1) $$Q(t)=b_1t+b_2t^2+\cdots+b_nt^n\,,$$

with

$$b_1=P'(\alpha)\,.$$

Then the following two existence theorems are obtained.

THEOREM 2.1. *If*

$$|P'(\alpha)|>1, \tag{2.2}$$

equation (E) *has a nontrivial entire solution*

$$\varphi(x, \alpha)=\alpha+\Phi(x, \alpha) \tag{2.3}$$

satisfying

$$\varphi(x, \alpha)\to\alpha$$

as Re $x\to-\infty$. *Here* $\Phi(x, \alpha)$ *is developed into convergent series*

$$\Phi(x, \alpha)=\sum_{k\geq 1} c_k \exp(kx\log b_1) \qquad (b_1=P'(\alpha)) \tag{2.4}$$

for $|x|<\infty$.

THEOREM 2.2. *If*

$$P'(\alpha)=1, \tag{2.5}$$

equation (E) *has a nontrivial entire solution*

$$\phi(x, \alpha)=\alpha+\Psi(x, \alpha) \tag{2.6}$$

satisfying

$$\phi(x, \alpha)\to\alpha$$

as Re $x\to-\infty$. *In the right member of* (2.6), $\Psi(x, \alpha)$ *is a holomorphic function of the form*

$$\Psi(x, \alpha)=\gamma(x(1+b(x, x^{-1}\log x)))^{-1/m}, \tag{2.7}$$

for

$$x\in D_l(\varepsilon, R)=\{x \mid |x|>R,\ |\arg x-\pi|<\pi/2-\varepsilon, \text{ or }$$
$$\operatorname{Im}(e^{\sqrt{-1}\varepsilon}x)<-R, \text{ or } \operatorname{Im}(e^{-\sqrt{-1}\varepsilon}x)>R\},$$

ε *being arbitrarily small,* R *being sufficiently large. Here* m *and* γ *are a positive integer and a complex constant, respectively, depending on* $P(t)$, *and* $b(x, \eta)$ *is the convergent series*

$$b(x, \eta)=\sum_{k\geq 0} b_k(x)\eta^k \tag{2.9}$$

for $x\in D_l(\varepsilon, R)$, $|\eta|<r$, *whose coefficient* $b_k(x)$ *admits an asymptotic expansion*

$$b_k(x)\cong\sum_{j\geq 0} q_{jk}x^{-j/m} \qquad q_{00}=0 \tag{2.10}$$

as $x\to\infty$ *through* $D_l(\varepsilon, R)$.

Before going into our proof, let us consider an equation of Schröder of the form

$$\text{(E. S)} \qquad y(b_1x)=Q(y(x))$$

where $Q(t)$ is the polynomial given by (2.1), i. e.

$$Q(t)=b_1t+b_2t^2+\cdots+b_nt^n\,.$$

THEOREM 2.3. *If* $|b_1|>1$, *equation of Schröder* (E. S) *has an entire solution* $\phi(x)$ *satisfying* $\phi(0)=0$, $\phi'(0)=1$.

PROOF. It is well known that, if $|b_1|\neq 1$, equation (E. S) has a holomorphic solution of the form

$$\phi(x)=\sum_{k\geq 1}c_kx^k \qquad c_1=1,$$

at $x=0$, (E. Schröder [4]). Since $|b_1|>1$, $\phi(x)$ can be continued holomorphically into the whole complex plane by utilizing (E. S). Thus we have an entire solution

$$\phi(x)=\sum_{k\geq 1}c_kx^k$$

for $|x|<\infty$.

PROOF OF THEOREM 2.1. Under assumption (2.2), it is sufficient to prove that equation (E. α) has an entire solution $\Phi(x, \alpha)$ which is written as (2.4) and satisfies

$$\Phi(x, \alpha)\to 0 \tag{2.11}$$

as Re $x\to-\infty$.

Let $\phi(x)$ be an entire solution of (E. S) given by Theorem 2.3. Then if we set

$$\Phi(x, \alpha)=\phi(b_1^x)=\sum_{k\geq 1}c_k\exp(kx\log b_1)\,,$$

it is an entire solution of (E. α). In fact, $\Phi(x, \alpha)$ satisfies

$$\Phi(x+1, \alpha)=\phi(b_1^{x+1})=\phi(b_1\cdot b_1^x)=Q(\phi(b_1^x))=Q(\Phi(x, \alpha))\,.$$

It is also verified without difficulty that $\Phi(x, \alpha)$ is a nontrivial entire function and satisfies (2.11).

Next we give the proof of Theorem 2.2. For this purpose, we quote Theorem 14.2 in T. Kimura [2], with slight modification.

THEOREM 2.4. *Consider the difference equation*

$$w(x+1)=F(w(x))$$

with

$$F(z)=z\left(1+\sum_{j=m}^{\infty}b_jz^{-j/m}\right), \qquad b_m=1$$

Then there exists a unique solution $w(x)$ satisfying the following conditions.

i) *$w(x)$ is holomorphic in $D_l(\varepsilon, R)$.*

ii) *$w(x)$ is expressible in the form*

$$w(x)=x(1+b(x, x^{-1}\log x)),$$

where $b(x, \eta)$ is holomorphic for $x\in D_l(\varepsilon, R)$, $|\eta|<r$, and in the expansion

$$b(x, \eta)=\sum_{k=0}^{\infty} b_k(x)\eta^k,$$

$b_k(x)$ is asymptotically developed into

$$\sum_{j=0}^{\infty} q_{jk}x^{-j/m} \qquad q_{00}=0,$$

as $x\to\infty$ through $D_l(\varepsilon, R)$.

PROOF OF THEOREM 2.2. Let us assume that

$$Q(t)=t+b_{m+1}t^{m+1}+\cdots+b_nt^n, \qquad \text{(Cf. (2.1))}$$

with $b_{m+1}\neq 0$, $m+1\leqq n$. Then (E. α) is transformed by

$$-mb_{m+1}(z(x))^m=w(x)^{-1} \tag{2.12}$$

into an equation expressible in the form

$$w(x+1)=w(x)(1+w(x)^{-1}+O(w(x)^{-(m+1)/m})) \tag{2.13}$$

for $|w(x)|>\rho$, ρ being a sufficiently large constant. By applying Theorem 2.4 to equation (2.13), we have a solution

$$\Theta(x)=x(1+b(x, x^{-1}\log x))$$

holomorphic for $x\in D_l(\varepsilon, R)$, where $b(x, \eta)$ satisfies conditions (2.9) and (2.10). Take the constant R sufficiently large so that

$$|\Theta(x)|>\rho$$

for $x\in D_l(\varepsilon, R)$. Then equation (E. α) has a solution

$$\begin{aligned}\Psi(x, \alpha)&=\gamma\Theta(x)^{-1/m}\\ &=\gamma(1+b(x, x^{-1}\log x))^{-1/m}x^{-1/m}\end{aligned}$$

holomorphic for $x\in D_l(\varepsilon, R)$. It is not difficult to verify that the solution $\Psi(x, \alpha)$ can be continued holomorphically into the whole plane and satisfies

$$\Psi(x, \alpha)\to 0$$

as x tends to ∞ through $D_l(\varepsilon, R)$. Thus the theorem is proved.

Now there arises the question whether (E) has a nontrivial entire solution for every polynomial $P(t)$. The following theorem gives an answer to this question.

THEOREM 2.5. *For any polynomial $P(t)$, equation* (E) *has a nontrivial entire solution.*

In view of Theorems 2.1 and 2.2, it is sufficient to prove the following lemma.

LEMMA 2.6. *By $\alpha_1, \cdots, \alpha_n$, we denote the fixed points of $P(t)$. Then, either of the following two cases occurs.*

i) *For some α_l, $P'(\alpha_l)=1$,*

ii) *For some α_l, $|P'(\alpha_l)|>1$.*

This lemma is proved in G. Julia [1]. For the importance of the lemma, we give a more direct proof. For this purpose, we show the following.

CLAIM. *Assume that*

$$\Lambda=\prod_{1\leq i<j\leq n}(\alpha_i-\alpha_j)\neq 0. \tag{2.14}$$

Then, if we put

$$\Lambda_k=\prod_{\substack{\nu=1\\ \nu\neq k}}^{n}(\alpha_k-\alpha_\nu)\neq 0, \qquad k=1, \cdots, n,$$

$$\lambda_k=\Lambda/\Lambda_k\neq 0,$$

we have

$$\sum_{k=1}^{n}\lambda_k=0. \tag{2.15}$$

Verification of Claim. Now, we set

$$F(x_1, \cdots, x_{n-1})=\prod_{1\leq i<j\leq n-1}(x_i-x_j)$$

$$=(-1)^{(n-1)(n-2)/2}\begin{vmatrix} 1 & 1 \\ x_1 & x_{n-1} \\ \cdots & \cdots \\ x_1^{n-2}\cdots & x_{n-1}^{n-2}\end{vmatrix}.$$

Then, a simple computation shows that

$$\lambda_k=\Lambda/\Lambda_k=\prod_{1\leq i<j\leq n}(\alpha_i-\alpha_j)\Big/\prod_{\substack{\nu=1\\ \nu\neq k}}^{n}(\alpha_k-\alpha_\nu)$$

$$=(-1)^{k+1}F(\alpha_1, \cdots, \alpha_{k-1}, \alpha_{k+1}, \cdots, \alpha_n),$$

for $k=1, \cdots, n$. Hence it follows that

$$\sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} (-1)^{k+1} F(\alpha_1, \cdots, \alpha_{k-1}, \alpha_{k+1}, \cdots, \alpha_n)$$

$$= (-1)^{(n-1)(n-2)/2} \sum_{k=1}^{n} (-1)^{k+1} \begin{array}{c} \wedge k \\ \begin{vmatrix} 1 \cdots\cdots 1 \\ \alpha_1 \cdots\cdots \alpha_n \\ \cdots\cdots\cdots \\ \alpha_1^{n-2} \cdots \alpha_n^{n-2} \end{vmatrix} \end{array}$$

$$= (-1)^{(n-1)(n-2)/2} \begin{vmatrix} 1 \cdots\cdots 1 \\ 1 \cdots\cdots 1 \\ \alpha_1 \cdots\cdots \alpha_n \\ \cdots\cdots\cdots \\ \alpha_1^{n-2} \cdots \alpha_n^{n-2} \end{vmatrix}$$

$$= 0,$$

where the notation $\wedge k$ means that the k-th column is dropped out. This completes the proof.

PROOF OF LEMMA 2.6. By the assumption, the polynomial $P(t)$ is written as

$$P(t) = a_n \prod_{k=1}^{n} (t - \alpha_k) + t. \tag{2.16}$$

If (2.16) has a multiple root (for example α_1), it is clear that

$$P'(\alpha_1) = 1,$$

which implies that case i) occurs.

Next we assume that all the roots are simple;

$$\Lambda = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j) \neq 0. \tag{2.17}$$

We shall show that, under assumption (2.17), case ii) occurs. To do this, suppose that case ii) does not occur, namely

$$|P'(\alpha_k)| = |1 + a_n \Lambda_k| = |1 - (-a_n \Lambda / \lambda_k)| \leqq 1 \tag{2.18}$$

for $k=1, \cdots, n$. Then, from (2.17) and (2.18), it follows that

$$-\pi/2 < \arg(-a_n \Lambda / \lambda_k) < \pi/2.$$

Hence, we have

$$\theta - \pi/2 < \arg \lambda_k < \theta + \pi/2, \qquad k=1, \cdots, n,$$

where $\theta = \arg(-a_n \Lambda)$. This implies

$$\sum_{k=1}^{n} \lambda_k \neq 0,$$

which contradicts (2.15). Thus the proof is completed.

§ 3. Equation of Schröder.

As was shown in the preceding section, the equation of Schröder

$$z(b_1 x)=Q(z(x)) \tag{E. S}$$

has a nontrivial entire solution $\phi(x)$, under the condition $|b_1|>1$. The purpose of this section is to study entire solutions of (E. S).

Assume that $|b_1|>1$. Let us denote by $\Phi(x)$ any nontrivial entire solution of (E. S). Then the order of $\Phi(x)$ is given by the following theorem.

THEOREM 3.1. *The order of* $\Phi(x)$ *is* $\log n/\log|b_1|$.

PROOF. By virtue of the Picard's theorem, there exist a point $x=x_0\neq 0$ and a positive constant c so that

$$\Phi(x_0)=c>2,$$

$$|Q(t)|\geqq\frac{1}{2}|t|^n \qquad \text{for } |t|\geqq c.$$

Then if the inequality

$$|\Phi(a)|\geqq c$$

holds, it follows that

$$|\Phi(b_1 a)|=|Q(\Phi(a))|\geqq\frac{1}{2}|\Phi(a)|^n\geqq\frac{1}{2}c^n\geqq c.$$

Hence, if we set

$$M(r,\ \Phi(x))=\max_{|x|=r}|\Phi(x)|,$$

we obtain

$$M(|x_0 b_1^{N+1}|,\ \Phi(x))\geqq\frac{1}{2}M(|x_0 b_1^N|,\ \Phi(x))^n \qquad \text{for } N=0,\ 1,\ 2,\ \cdots.$$

From the relation above, we deduce that

$$M(|x_0 b_1^N|,\ \Phi(x))\geqq\exp(\gamma n^N), \qquad N=0,\ 1,\ 2,\ \cdots,$$

γ being a positive constant. This yields the inequality

$$M(r,\ \Phi(x))\geqq\exp(\gamma_0 n^{\log r/\log|b_1|}). \tag{3.1}$$

Next, we shall obtain the upper estimate of $M(r,\ \Phi(x))$. For some positive constant σ, it holds

$$|\Phi(xb_1)|=|Q(\Phi(x))|\leqq\max(\sigma|\Phi(x)|^n,\ 1)\,. \tag{3.2}$$

Since the function $\Phi(x)$ is bounded for $1\leqq|x|\leqq|b_1|$, we derive from (3.2)

$$M(|xb_1^N|,\ \Phi(x))\leqq\exp(\beta n^N)$$

for $1\leqq|x|\leqq|b_1|$, β being a positive constant. This implies

$$M(r,\ \Phi(x))\leqq\exp(\beta_0 n^{\log r/\log|b_1|}) \tag{3.3}$$

for $r\geqq 1$.

Combining (3.1) and (3.3), we conclude that

$$\varlimsup_{r\to\infty}\frac{\log\log M(r,\ \Phi(x))}{\log r}=\frac{\log n}{\log|b_1|}\,.$$

Let us consider the case when $|b_1|<1$. Suppose that equation (E.S) admits a nontrivial entire solution $\varphi(x)$. Then,

$$z(\xi)=\varphi(b_1^\xi)$$

is a nontrivial entire solution of

$$z(\xi+1)=Q(z(\xi))\,.$$

It is easy to see that

(3.4) $z(\xi)$ is bounded in the domain defined by $\operatorname{Re}\xi\geqq\dfrac{\arg b_1}{\log|b_1|}\operatorname{Im}\xi+c$,

c being a real constant. On the other hand, as will be shown in the next section (Proposition 4.3), there exists a sequence $\{\xi_N\}$ satisfying

$$|z(\xi_N)|\geqq\exp(\gamma n^{\xi_N})\,,\qquad(\gamma>0),$$

and

$$\operatorname{Re}\xi_N\to+\infty\,,$$

which contradicts (3.4). Thus we arrive at the following theorem.

THEOREM 3.2. *If* $|b_1|<1$, *equation* (E.S) *has no nontrivial entire solution.*

§4. Order and Julia's directions of entire solutions.

This section is devoted to the study of complex analytic properties of entire solutions of (E). As an immediate consequence of Theorem 3.1, we have the following.

THEOREM 4.1. *The entire solution* $\Phi(x,\ \alpha)$ *given in Theorem* 2.1 *is of order* ∞.

More generally, we can prove the theorem below.

THEOREM 4.2. *Any nontrivial entire solution* $\varphi(x)$ *of* (E) *is of order* ∞.

To do this, it is sufficient to show the following proposition.

PROPOSITION 4.3. *For any nontrivial entire solution* $\varphi(x)$ *of* (E), *there exists a sequence* $\{x_0+N\}_{N=1,2,\dots}$, *such that*

$$|\varphi(x_0+N)| \geqq \exp(\gamma n^{|x_0+N|}), \tag{4.1}$$

γ *being a positive constant.*

PROOF. The proof is very similar to that of Theorem 3.1. Taking x_0 and c such that

$$\varphi(x_0)=c>2,$$

$$|P(t)| \geqq \frac{1}{2}|t|^n \qquad \text{for } |t| \geqq c,$$

we can deduce that

$$|\varphi(x_0+N+1)| \geqq \frac{1}{2}|\varphi(x_0+N)|^n .$$

From this relation, we can derive (4.1) with no difficulty.

Let $L(b, c)$ denote the line defined by

$$\operatorname{Re} x = b \operatorname{Im} x + c,$$

b and c being real constants. And set

$$M(L(b, c), \varphi(x)) = \sup_{x \in L(b, c)} |\varphi(x)| .$$

Then, the upper estimate on the line is also given by the same reasoning as in the proof of Theorem 3.1.

PROPOSITION 4.4. *If the inequality*

$$\sup_{c \leqq c_0} M(L(b, c), \varphi(x)) \leqq M_0 < \infty,$$

holds, then there exists a positive constant β *such that*

$$M(L(b, c), \varphi(x)) \leqq \exp(\beta n^c) \tag{4.2}$$

for $c \geqq c_0$.

By applying (4.1) and (4.2) to $\varphi(x, \alpha)$, $\psi(x, \alpha)$ given in Theorems 2.1 and 2.2, we obtain

COROLLARY 4.5.

$$\overline{\lim_{r\to\infty}} \frac{\log\log\log M(r, \varphi(x, \alpha))}{\log r} = 1,$$

$$\overline{\lim_{r\to\infty}} \frac{\log\log\log M(r, \psi(x, \alpha))}{\log r} = 1.$$

Finally, we consider Julia's direction of $\varphi(x, \alpha)$ and $\psi(x, \alpha)$. For the solution $\psi(x, \alpha)$, Julia's direction is easily found.

THEOREM 4.6 (Julia's direction of $\psi(x, \alpha)$). *$\psi(x, \alpha)$ has only one Julia's direction*

$$\arg x = 0.$$

PROOF. Recall that $\psi(x, \alpha)$ is bounded in the domain $D_l(\varepsilon, R)$, where ε can be taken arbitrarily small. Therefore, $\arg x = \theta \not\equiv 0 \pmod{2\pi}$ cannot be Julia's direction.

Next, we determine Julia's directions of $\varphi(x, \alpha)$.

LEMMA 4.7. *$\varphi(x, \alpha)$ is bounded in the domain defined by*

$$H_l(\alpha, \lambda) = \left\{ x \mid \operatorname{Re} x < \frac{\arg P'(\alpha)}{\log |P'(\alpha)|} \operatorname{Im} x + \lambda \right\}, \qquad (0 \leqq \arg P'(\alpha) < 2\pi).$$

The verification of this lemma is not difficult. By utilizing this lemma, we obtain a theorem concerning Julia's directions of $\varphi(x, \alpha)$.

THEOREM 4.8 (Julia's directions of $\varphi(x, \alpha)$). *The direction $\arg x = \theta$ is Julia's direction of $\varphi(x, \alpha)$, if and only if the ray $\arg x = \theta$ is contained in the domain*

$$H(\alpha) = \left\{ x \mid \operatorname{Re} x \geqq \frac{\arg P'(\alpha)}{\log |P'(\alpha)|} \operatorname{Im} x \right\}.$$

PROOF. If the ray $\arg x = \theta$ is not contained in $H(\alpha)$, Lemma 4.7 implies that $\varphi(x, \alpha)$ is bounded for

$$|\arg x - \theta| < \varepsilon,$$

ε being a sufficiently small positive constant. Therefore, such a direction is not Julia's direction. On the other hand, assume that the ray $\arg x = \theta$ is contained in $H(\alpha)$. Note that the function

$$\xi = P'(\alpha)^x = \exp(x \log P'(\alpha))$$

maps the region

$$|\arg x-\theta|<\varepsilon$$

into a region D_R such that

$$D_R \supset \{\xi \mid |\xi|>R\},$$

where ε is an arbitrarily small constant, R is a sufficiently large constant. Recalling that

$$\phi(\xi)=\varphi(x, \alpha)$$

is a nontrivial entire solution of (E. S), we infer that $\varphi(x, \alpha)$ takes every value other than Picard's exceptional value, in the region

$$|\arg x-\theta|<\varepsilon,$$

for any positive constant ε. This means that $\arg x=\theta$ is Julia's direction of $\varphi(x, \alpha)$.

References

[1] Julia, G., Memoire sur l'iteration des fonctions rationnelles, J. Math. Pures Appl. **1** (1918), 47-245.
[2] Kimura, T., On the iteration of analytic functions, Funkcial. Ekvac. **14** (1971), 197-238.
[3] Kimura, T., On meromorphic solutions of the difference equation $y(x+1)=y(x)+1+\lambda/y(x)$, Symposium on Ordinary Differential Equations, Lecture Notes in Math., 312, pp. 74-86, Springer-Verlag, Berlin-New York, 1973.
[4] Schröder, E., Über iterierte Funktionen, Math. Ann. **3** (1871), 296-322.
[5] Urabe, M., Equation of Schröder, J. Sci. Hiroshima Univ. **15** (1915), 113-131.
[6] Yanagihara, N., Meromorphic solutions of some difference equations, to appear.

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