On some singular Fourier multipliers

By Akihiko MIYACHI*)

§ 1. Introduction.

In this paper, we shall consider the following Fourier multiplier:

$$m_{a,b}(\xi)=\phi(\xi)|\xi|^{-b}\exp(i|\xi|^{a}),\ \xi\in\boldsymbol{R}^{n},\ a>0,\ b\in\boldsymbol{R},$$

where ϕ is a smooth function which vanishes in a neighborhood of the origin and is equal to 1 outside a compact set. We shall determine the cases where the corresponding operator

(1.1) $$f\mapsto\mathcal{F}^{-1}(m_{a,b}\mathcal{F}f)\qquad(\mathcal{F}=\text{the Fourier transform})$$

defines a bounded operator between the following spaces:

(1.2) $$H^{p}(0<p<\infty),\ L^{1},\ L^{\infty},\ BMO,\ \Lambda_{s}(s\in\boldsymbol{R})\quad\text{and}\quad L_{k}^{\infty}(k\in\boldsymbol{N})$$

(as for these spaces, see § 2.1).

Our basic tool is the application of the theory of H^{p}-spaces, which enables us to obtain sharp results in some cases. For example we obtain the following results: (i) there are critical indices $p_{i}=p_{i}(a, b)$ $(i=0, 1, 2)$, $0<p_{0}\leqq 2$, $1<p_{1}\leqq 2$, $0<p_{2}\leqq 1$, such that (1.1) defines a bounded operator from $H^{p_{0}}$ to $H^{p_{0}}$, from $L^{p_{1}}$ to $L^{p_{1}'}$ $(1/p_{1}'=1-1/p_{1})$, and from $H^{p_{2}}$ to L^{∞} and all the results for the boundedness of the operator (1.1) from H^{p} or BMO to H^{q}, BMO or Λ_{s} can be derived from those results by interpolation and duality; (ii) if a increases, then p_{0} increases and p_{1} and p_{2} decrease, from which it follows that the oscillating factor $\exp(i|\xi|^{a})$ is a bad factor when we consider the operator (1.1) from H^{p} to H^{q} with $0<p\leqq q<2$ or $2<p\leqq q<\infty$ or from H^{p}, $p>2$, to BMO or Λ_{s} but is a good factor when we consider (1.1) from H^{p} to H^{q} with $0<p<2<q<\infty$ or from H^{p}, $0<p<2$, to BMO or Λ_{s}; (iii) p_{i}'s depend continuously on a and b if $0<a<1$ or $a>1$ but they have discontinuities at $a=1$; thus $m_{1,b}$ cannot be considered as a limit of $m_{a,b}$ with $a\neq 1$.

The results of this paper obtained for the operator (1.1) (some of them are obtained for more general operators) explain the typical features of some important operators in analysis. If $0<a<1$, then (1.1) is a typical example of the pseudo-differential operator of the class $S_{1-a,0}^{-b}$. If $a=1$, the operator (1.1) has an

*) Partly supported by the Fûjukai foundation.

intimate connection with the Cauchy problem for the wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=\sum_{j=1}^{n}\dfrac{\partial^2 u}{\partial x_j^2}, & t\in \boldsymbol{R},\ x\in \boldsymbol{R}^n, \\ u(0, x)=f(x), & x\in \boldsymbol{R}^n, \\ \dfrac{\partial u}{\partial t}(0, x)=g(x), & x\in \boldsymbol{R}^n. \end{cases}$$

The solution of this equation can be written as

$$u(t, \cdot)=\mathcal{F}^{-1}(\cos t|\xi|\cdot\mathcal{F}f(\xi))+\mathcal{F}^{-1}(|\xi|^{-1}\sin t|\xi|\cdot\mathcal{F}f(\xi)).$$

Many important properties of the operator $(f, g)\mapsto u(t, \cdot)$ are shared by the operator (1.1) with $a=1$. Similarly the operator (1.1) with $a=2$ has an intimate connection with the Cauchy problem for the Schrödinger equation:

$$\begin{cases} i\dfrac{\partial u}{\partial t}=\sum_{j=1}^{n}\dfrac{\partial^2 u}{\partial x_j^2}, & t\in \boldsymbol{R},\ x\in \boldsymbol{R}^n, \\ u(0, x)=f(x), & x\in \boldsymbol{R}^n. \end{cases}$$

In § 2, we shall recall the definitions of the spaces (1.2) and some properties of them which are used in this paper. In § 3, we study some general properties of Fourier multipliers between the spaces (1.2). In § 4, we state the results for the Fourier multiplier $m_{a,b}$. § 5 is devoted to the proof of the results in § 4. § 6 is an appendix. Most of the contents of § 6 are perhaps well known to many people; they are included in this paper because the present author cannot find appropriate references.

The main results of this paper are stated in Theorems 4.1~4.5, 5.1, 5.2 and Corollary 4.1.

Throughout this paper, we shall use the following

Notation. ϕ denotes a fixed smooth function on $\boldsymbol{R}^n$ such that

$$0\leqq\phi(\xi)\leqq 1,\ \phi(\xi)=0 \quad \text{if} \quad |\xi|\leqq 1 \quad \text{and} \quad \phi(\xi)=1 \quad \text{if} \quad |\xi|\geqq 2.$$

The letters $C, C', C'', \cdots,$ denote positive constants which may have different values in each occasion. Following Schwartz [16], we denote by $\mathcal{D}$ and $\mathcal{S}$ the spaces of the test functions on $\boldsymbol{R}^n$ and by $\mathcal{D}'$ and $\mathcal{S}'$ the spaces of distributions and tempered distributions respectively. The Fourier transform and the inverse Fourier transform are defined by

$$(\mathcal{F}f)(\xi)=\hat{f}(\xi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^n} f(x)e^{-i\xi\cdot x}dx, \qquad \xi\in \boldsymbol{R}^n,$$

and

$$(\mathcal{F}^{-1}f)(\xi)=(\mathcal{F}f)(-\xi).$$

$[s]$ denotes the integer part of a number s; $[s]$ is an integer and $[s]\leqq s<[s]+1$. Differential operators are denoted by D^{α} or $\left(\frac{\partial}{\partial x}\right)^{\alpha}$;

$$(D^{\alpha}f)(x)=\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x)=\frac{\partial^{|\alpha|}f(x)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},$$

$$\alpha=(\alpha_1,\cdots,\alpha_n),\ |\alpha|=\alpha_1+\cdots+\alpha_n .$$

If f is a function or distribution on $\boldsymbol{R}^n$ and $t>0$, we define $f(\cdot|t)$ as follows:

$$f(x|t)=t^{-n}f\left(\frac{x}{t}\right),\qquad x\in\boldsymbol{R}^n.$$

Then it holds that

$$\mathcal{F}(f(\cdot|t))(\xi)=(\mathcal{F}f)(t\xi).$$

$f*g$ denotes the convolution of f and g:

$$(f*g)(x)=\langle f(x-\cdot),\ g\rangle=\langle f,\ g(x-\cdot)\rangle=\int f(x-y)g(y)dy .$$

§ 2. Preliminaries.

2.1. *Spaces of functions and distributions.*

The spaces considered in this paper are $L^p(0<p\leqq\infty)$, $H^p(0<p<\infty)$, BMO, $L_k^{\infty}(k\in\boldsymbol{N})$ and $\Lambda_s(s\in\boldsymbol{R})$. All of them are spaces of functions and/or distributions on $\boldsymbol{R}^n$. We recall the definitions and fundamental properties of those spaces.

DEFINITION 2.1. L^p, $0<p<\infty$, is the class of all measurable functions f such that

$$\|f\|_{L^p}=\left(\int_{\boldsymbol{R}^n}|f(x)|^p dx\right)^{1/p}<\infty .$$

L^{∞} is the class of all measurable functions f such that

$$\|f\|_{L^{\infty}}=\text{ess. sup}\{|f(x)|\}<\infty .$$

If $1\leqq p\leqq\infty$, L^p is a subspace of $\mathcal{S}'$. If $0<p<1$, we consider L^p merely as a Fréchet space of measurable functions.

We shall define H^p following Fefferman and Stein [8]. Let $\varphi\in\mathcal{S}$ be a fixed function such that $\hat{\varphi}(0)\neq 0$; for tempered distribution f, define the maximal function f^+ by

$$f^+(x)=\sup_{0<t<\infty}\{|(\varphi(\cdot|t)*f)(x)|\},\qquad x\in\boldsymbol{R}^n.$$

DEFINITION 2.2. H^p, $0<p<\infty$, is the class of tempered distributions f such that $f^+\in L^p$. The norm in H^p is defined by

$$\|f\|_{H^p}=\|f^+\|_{L^p}.$$

It is known that the above definition does not depend on the choice of the test function φ, *i.e.*, if we replace the function φ, then the class H^p does not change and the norm changes to an equivalent one. It is also well known that, *if* $1<p<\infty$, *then* $H^p=L^p$ *with equivalent norms.* Cf. Fefferman-Stein [8].

We recall some characterizations of H^p. The first characterization uses the Littlewood-Paley function. Let χ be a function in $\mathcal{S}$ with the following properties:

$$\text{(2.1)} \qquad \begin{cases} \text{support } \hat{\chi}\subset\{1/2\leqq|\xi|\leqq 2\} \quad \text{and} \\ \sum_{k=-\infty}^{\infty}\hat{\chi}(2^k\xi)=1 \quad \text{for} \quad \xi\neq 0. \end{cases}$$

For a tempered distribution f, define the *Littlewood-Paley function* $d(f)$ as follows:

$$d(f)(x)=\Big(\sum_{k=-\infty}^{\infty}|(\chi(\cdot|2^k)*f)(x)|^2\Big)^{1/2}, \qquad x\in\boldsymbol{R}^n.$$

Then we have the following

THEOREM A. *Let* $0<p<\infty$. *If* $f\in H^p$, *then* $d(f)\in L^p$ *and*

$$\|d(f)\|_{L^p}\leqq C\|f\|_{H^p}.$$

Conversely, if $f\in\mathcal{S}'$ *and* $d(f)\in L^p$, *then there is a polynomial* P *such that* $f-P\in H^p$ *and*

$$\|f-P\|_{H^p}\leqq C\|d(f)\|_{L^p}.$$

The proof of this theorem can be found in Triebel [21], pp. 167-169.

The second characterization uses the notion of atom. A function f is called a *p-atom* $(0<p\leqq 1)$ if there is a ball $B=B_f$ such that

$$\text{support } f\subset B, \quad \|f\|_{L^\infty}\leqq|B|^{-1/p}$$

($|B|$=Lebesgue measure of B) and

$$\int f(x)x^\alpha\,dx=0 \qquad \text{for} \quad |\alpha|\leqq[n/p-n].$$

THEOREM B (Latter [11]). *Let* $0<p\leqq 1$. *If* λ_j*'s are complex numbers such that* $\sum_{j=1}^{\infty}|\lambda_j|^p<\infty$ *and* f_j*'s are p-atoms, then* $\sum_{j=1}^{\infty}\lambda_j f_j$ *converges in* H^p *and*

$$\Big\|\sum_{j=1}^{\infty}\lambda_j f_j\Big\|_{H^p}\leqq C\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p}.$$

Conversely, if $f\in H^p$, *then there exist complex numbers* $\{\lambda_j\}$ *and p-atoms* $\{f_j\}$ *such that*

$$f=\sum_{j=1}^{\infty}\lambda_j f_j \quad \textit{and} \quad \Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p}\leqq C\|f\|_{H^p}.$$

If $f\in H^p$ *and* f *has a compact support, then it is possible to construct the above decomposition in such a way that* $\bigcup_{j=1}^{\infty}B_j$ *is relatively compact, where* B_j *is the ball corresponding to the* p*-atom* f_j.

Thirdly we refer to the characterization of H^p by means of *Riesz transforms*. For $\alpha=(\alpha_1, \cdots, \alpha_n)$, α_j nonnegative integer, define the operator R_α by

$$R_\alpha f=\mathcal{F}^{-1}\Big(\Big(-i\frac{\xi}{|\xi|}\Big)^\alpha \hat{f}(\xi)\Big), \qquad f\in L^2.$$

We have the following

THEOREM C. *Let* k *be a positive integer and* $p>(n-1)/(n-1+k)$. *Then,* $f\in L^2\cap H^p$ *if and only if* $R_\alpha f\in L^2\cap L^p$ *for all* $|\alpha|\leqq k$, *and*

$$C\|f\|_{H^p}\leqq \sum_{|\alpha|\leqq k}\|R_\alpha f\|_{L^p}\leqq C'\|f\|_{H^p}, \qquad f\in L^2\cap H^p.$$

This theorem is implicit in Fefferman-Stein [8], pp. 167-168.

DEFINITION 2.3. BMO is the class of all locally integrable functions f such that

$$\|f\|_{BMO}=\sup_B\Big[\inf_c\Big\{|B|^{-1}\int_B|f(x)-c|\,dx\Big\}\Big]<\infty,$$

where the infimum is taken over all complex numbers c and the supremum is taken over all balls B.

DEFINITION 2.4. For nonnegative integer k, we set

$$\|f\|_{L_k^\infty}=\sum_{|\alpha|\leqq k}\|D^\alpha f\|_{L^\infty}.$$

L_k^∞ is the space of all $f\in\mathcal{S}'$ such that $\|f\|_{L_k^\infty}<\infty$.

DEFINITION 2.5. Let $s>0$ and $s=k+\varepsilon$ with nonnegative integer k and $0<\varepsilon\leqq 1$. For a function f of class C^k, we set

$$\|f\|_{\tilde{\Lambda}_s}=\begin{cases}\displaystyle\sum_{|\alpha|=k}\sup_{x\neq y}\Big\{\frac{|D^\alpha f(x)-D^\alpha f(y)|}{|x-y|^\varepsilon}\Big\} & \text{if} \quad 0<\varepsilon<1\\ \displaystyle\sum_{|\alpha|=k}\sup_{x\neq y}\Big\{\frac{|D^\alpha f(x)-2D^\alpha f((x+y)/2)+D^\alpha f(y)|}{|x-y|}\Big\} & \text{if} \quad \varepsilon=1,\end{cases}$$

and

$$\|f\|_{\Lambda_s}=\|f\|_{L_k^\infty}+\|f\|_{\tilde{\Lambda}_s}. \tag{2.2}$$

$\tilde{\Lambda}_s$ and Λ_s are the spaces of functions of class C^k such that $\|f\|_{\tilde{\Lambda}_s}<\infty$ or $\|f\|_{\Lambda_s}<\infty$ respectively.

Λ_s, $s>0$, coincides with the *Lipschitz space* $\Lambda(s;\infty,\infty)$ defined by Taibleson [20], I, (cf. Theorem 4, pp. 421-422, and Theorem 10, p. 444, *loc. cit.*) and with the *Besov space* $B^s_{\infty\infty}$ (cf. Bergh-Löfström [1], Chapter 6). $\Lambda(s;\infty,\infty)$ or $B^s_{\infty\infty}$ are defined for all $s\in\boldsymbol{R}$. We define $\Lambda_s=\Lambda(s;\infty,\infty)$ for $s\leqq 0$; as for the definition of $\Lambda(s;\infty,\infty)$, see § 6.1.

There is a simple isomorphism between Λ_s and Λ_t. Set

$$\langle\xi\rangle=(1-\phi(\xi))+\phi(\xi)|\xi|.$$

Then we have the following

THEOREM D. *Let s and t be real numbers. Then the operator*

$$f\longmapsto\mathcal{F}^{-1}(\langle\xi\rangle^{s-t}\hat{f}(\xi))$$

maps Λ_s isomorphically onto Λ_t.

For this theorem, refer, for example, to Taibleson [20], I (Theorem 6, p. 437). The proof of this theorem is easily established if we use the definition of $\Lambda_s=\Lambda(s;\infty,\infty)$ given in § 6.1.

We give a norm in Λ_s which is equivalent to (2.2).

LEMMA 2.1. *If $s>0$, then there exists a function $\varphi\in\mathcal{D}$ such that the inequalities*

$$C\|f\|_{\Lambda_s}\leqq\|f\|_{\tilde{\Lambda}_s}+\|f*\varphi\|_{L^\infty}\leqq C'\|f\|_{\Lambda_s}$$

hold for all functions f of class C^k.

For a proof of this lemma, see § 6.3.

The following lemma is used later.

LEMMA 2.2. (i) *Let Y be one of the following spaces:*

$$H^p(0<p<\infty),\ L^p(1<p\leqq\infty),\ BMO,\ \Lambda_s(s\in\boldsymbol{R}),\ L^\infty_k(k\in\boldsymbol{N}).$$

Then: if $f_n(n=1,2,\cdots)$ and $f\in\mathcal{D}'$ and $f_n\to f$ in $\mathcal{D}'$, then

$$\|f\|_Y\leqq\liminf\|f_n\|_Y. \tag{2.3}$$

(ii) *If $f_n(n=1,2,\cdots)$ and f are measurable functions and $f_n(x)\to f(x)$ almost everywhere, then*

$$\|f\|_{L^p}\leqq\liminf\|f_n\|_{L^p},\qquad 0<p\leqq\infty.$$

PROOF. (ii) is proved by Fatou's lemma. We shall prove (i). If $Y=H^p$, $0<p<\infty$, then (2.3) is obtained by integrating the pointwise inequality

$$f^+(x)\leqq\liminf f_n^+(x).$$

If $Y=L^p$, $1<p\leqq\infty$, (2.3) is proved by using the converse of Hölder's inequality; see (2.4) given below. The case $Y=L_k^\infty$ is reduced to the case $Y=L^\infty$. If $Y=BMO$, we can use, instead of (2.4), the following equality:

$$\|f\|_{BMO}=\sup\{|\langle f, g\rangle|\,\big|\,g\in\mathcal{D},\ g\text{: 1-atom}\},$$

which is valid for all $f\in\mathcal{D}'$ (in particular it holds that, if $f\in\mathcal{D}'$ and the right hand side of the above equality is finite, then $f\in BMO$; cf. §6.2). In the case $Y=\Lambda_s$, $s\leqq 0$, we can easily establish the proof once we see that $\liminf\|f_n\|_{\Lambda_s}<\infty$ implies that $f_{n'}\to f$ in $\mathcal{S}'$ for some subsequence $\{f_{n'}\}$, which, however, can be seen from (i) of Proposition 6.1 (§6.1). Finally consider the case $Y=\Lambda_s$, $s>0$. The Ascoli-Arzela lemma shows that, if $\liminf\|f_n\|_{\Lambda_s}<\infty$, then there is a subsequence $\{f_{n'}\}$ such that $f_{n'}$ and its derivatives of order $<s$ converge uniformly on every compact set. It is easy to see that

$$\|\lim f_{n'}\|_{\Lambda_s}\leqq\liminf\|f_{n'}\|_{\Lambda_s},$$

which implies the desired inequality.

REMARK 2.1. $\|\cdot\|_{L^p}\,(0<p<1)$ and $\|\cdot\|_{H^p}\,(0<p<1)$ are quasi-norms and $\|\cdot\|_{BMO}$ and $\|\cdot\|_{\dot{\Lambda}_s}\,(s>0)$ are semi-norms. When $0<p<1$, the triangular inequality holds for $\|\cdot\|_{L^p}^p$ and $\|\cdot\|_{H^p}^p$:

$$\|f+g\|_{L^p}^p\leqq\|f\|_{L^p}^p+\|g\|_{L^p}^p,$$

$$\|f+g\|_{H^p}^p\leqq\|f\|_{H^p}^p+\|g\|_{H^p}^p.$$

$\|f\|_{BMO}=0$ if and only if f is a constant function. $\|f\|_{\dot{\Lambda}_s}=0$ if and only if f is a polynomial of degree $\leqq[s]$.

2.2. *Inequalities.*

Hölder's inequality and its converse: if $1<p\leqq\infty$ and $1/p+1/p'=1$, then

$$\|f\|_{L^p}=\sup\{|\langle f, g\rangle|\,\big|\,g\in\mathcal{D},\ \|g\|_{L^{p'}}\leqq 1\}\tag{2.4}$$

for $f\in\mathcal{D}'$; if $p=1$ and $p'=\infty$, the above equality holds for locally integrable function f.

Fefferman's inequality: if $f\in H^1$, $g\in BMO$ and $fg\in L^1$, then

$$\left|\int f(x)g(x)dx\right|\leqq C\|f\|_{H^1}\|g\|_{BMO}.$$

Conversely, for $g\in\mathcal{D}'$, we have

$$C\|g\|_{BMO}\leqq\sup\{|\langle f, g\rangle|\,\big|\,f\in\mathcal{D}\cap H^1,\ \|f\|_{H^1}\leqq 1\}\leqq C'\|g\|_{BMO};\tag{2.5}$$

in particular, if $g\in\mathcal{D}'$ and the middle term of (2.5) is finite, then $g\in BMO$. We have another converse to Fefferman's inequality:

(2.6) $$C\|f\|_{H^1}\leqq\sup\{|\langle f, g\rangle|\,|\,g\in\mathcal{D},\ \|g\|_{BMO}\leqq 1\}\leqq C'\|f\|_{H^1}$$

which is also valid for $f\in\mathcal{D}'$. (2.5) explains the duality $(H^1)'=BMO$ and (2.6) explains the duality $(CMO)'=H^1$, where CMO is the closure of $\mathcal{D}$ in BMO.

The following inequalities explain the duality $(H^p)'=\tilde{\Lambda}_{n/p-n}$, $0<p<1$: if $0<p<1$, then, for $g\in\mathcal{D}'$, we have

(2.7) $$C\|g\|_{\tilde{\Lambda}_{n/p-n}}\leqq\sup\{|\langle f, g\rangle|\,|\,f\in\mathcal{D}\cap H^p,\ \|f\|_{H^p}\leqq 1\}\leqq C'\|g\|_{\tilde{\Lambda}_{n/p-n}}.$$

Inequalities (2.5)~(2.7) are proved by using Theorem B. Cf. Coifman-Weiss [7] (Theorem B in p. 593 and Theorem (4.1) in p. 638, *loc. cit.*) and § 6.2 of the present paper.

2.3. *$\mathcal{K}(X, Y)$ and $\mathcal{M}(X, Y)$.*

Let X and Y be spaces of functions or distributions on $\boldsymbol{R}^n$ equipped with norms or quasi-norms or semi-norms. We define $\mathcal{K}(X, Y)$ and $\mathcal{M}(X, Y)$ as follows.

DEFINITION 2.6. For $K\in\mathcal{D}'$, we define

$$\|K\|_{\mathcal{K}(X,Y)}=\sup\left\{\frac{\|K*f\|_Y}{\|f\|_X}\,\middle|\,f\in\mathcal{D}\cap X,\ \|f\|_X\neq 0\right\}.$$

For $m\in\mathcal{S}'$, we define

$$\|m\|_{\mathcal{M}(X,Y)}=\sup\left\{\frac{\|\mathcal{F}^{-1}(m\hat{f})\|_Y}{\|f\|_X}\,\middle|\,f\in\mathcal{D}\cap X,\ \|f\|_X\neq 0\right\},$$

in other words,

$$\|m\|_{\mathcal{M}(X,Y)}=\|\mathcal{F}^{-1}m\|_{\mathcal{K}(X,Y)}.$$

$\mathcal{K}(X, Y)$ is the space of all $K\in\mathcal{D}'$ such that $\|K\|_{\mathcal{K}(X,Y)}<\infty$ and $\mathcal{M}(X, Y)$ is the space of all $m\in\mathcal{S}'$ such that $\|m\|_{\mathcal{M}(X,Y)}<\infty$.

The spaces we shall deal with as X or Y are those in § 2.1.

REMARK 2.2. $\mathcal{D}$ is not contained in H^p if $0<p\leqq 1$; it holds that, when $0<p\leqq 1$,

$$\mathcal{D}\cap H^p=\left\{f\in\mathcal{D}\,\middle|\,\int f(x)x^\alpha dx=0\quad\text{for}\quad|\alpha|\leqq[n/p-n]\right\}.$$

By the way, the following result holds (see § 6.4):

LEMMA 2.3. *Let* $0<p\leqq 1$, $M>n/p$ *and*

$$X_M=\{f\in L^\infty\,|\,(1+|x|)^M f(x)\in L^\infty\}.$$

Then

$$X_M\cap H^p=\left\{f\in X_M\,\middle|\,\int f(x)x^\alpha dx=0\quad\textit{for}\quad|\alpha|\leqq[n/p-n]\right\}$$

and

$$\|f\|_{H^p}\leqq C\|(1+|x|)^M f(x)\|_{L^\infty} \quad for \quad f\in X_M\cap H^p.$$

As a result, $\|\cdot\|_{\mathcal{K}(H^p,Y)}$ and $\|\cdot\|_{\mathcal{M}(H^p,Y)}$ are semi-norms if $0<p\leqq 1$. $\|\cdot\|_{\mathcal{K}(X,BMO)}$ and $\|\cdot\|_{\mathcal{K}(X,\tilde{\Lambda}_s)}$, $s>0$, are also semi-norms. For example, $\|K\|_{\mathcal{K}(H^p,L^1)}=0$, $0<p\leqq 1$, if and only if K is a polynomial of degree $\leqq[n/p-n]$; $\|K\|_{\mathcal{K}(H^p,BMO)}=0$, $0<p\leqq 1$, if and only if K is a polynomial of degree $\leqq[n/p-n]+1$. But we shall call them norms for the sake of convenience.

REMARK 2.3. Let X be one of the following spaces:

$$H^p(0<p<\infty),\ L^p(1\leqq p\leqq\infty),\ BMO,\ \Lambda_s(s>0),\ L_k^\infty(k\in\boldsymbol{N}).$$

Let Y be one of the above spaces too. Suppose that $K\in\mathcal{K}(X, Y)$ and $\hat{K}\in\mathcal{O}_M$ (the space of multipliers for $\mathcal{S}'$; see Schwartz [16], Chapter 7, § 5). Then $K*f$ is well defined for all $f\in\mathcal{S}'$ and *a fortiori* for all $f\in X$. In these circumstances, we can conclude that the inequality

$$\|K*f\|_Y\leqq\|K\|_{\mathcal{K}(X,Y)}\|f\|_X \tag{2.8}$$

holds for *all* $f\in X$. This can be shown by approximating $f\in X$ by functions of $\mathcal{D}\cap X$ and using Lemma 2.2. If $X=H^p(0<p<\infty)$ or $L^p(1\leqq p<\infty)$, there is no difficulty in approximating $f\in X$ since $\mathcal{D}\cap X$ is dense in X (as for the case $X=H^p$, $0<p\leqq 1$, see Calderón-Torchinsky [3], Theorem 1.8, pp. 104-105). If $X=L^\infty$, BMO, $\Lambda_s(s>0)$ or $L_k^\infty(k\in\boldsymbol{N})$, then, for any $f\in X$, we can construct a sequence $\{f_n\}\subset\mathcal{D}$ such that

$$f_n\longrightarrow f \quad \text{in} \quad \mathcal{S}' \quad \text{and} \quad \|f_n\|_X\longrightarrow\|f\|_X\,,$$

which will suffice to prove (2.8). The construction of the above sequence is easy if $X=L^\infty$, $\Lambda_s(s>0)$ or $L_k^\infty(k\in\boldsymbol{N})$. As for the case $X=BMO$, see § 6.5.

REMARK 2.4. If $X=H^p(0<p<\infty)$, $L^p(0<p\leqq\infty)$, BMO, $\Lambda_s(s\in\boldsymbol{R})$ or $L_k^\infty(k\in\boldsymbol{N})$ and $Y=H^q(0<q<\infty)$, $L^q(1\leqq q\leqq\infty)$, $\Lambda_t(t\in\boldsymbol{R})$ or $L_m^\infty(m\in\boldsymbol{N})$, then it can be shown that every bounded linear operator T from X into Y which commutes with translations, *i.e.*

$$T(f(\cdot-y))=(Tf)(\cdot-y) \qquad \text{for all} \quad y\in\boldsymbol{R}^n,$$

is written as

$$Tf=K*f \qquad \text{for all} \quad f\in\mathcal{S}\cap X$$

with some tempered distribution K. In particular, for the above X and Y, the class $\mathcal{K}(X, Y)$ consists only of tempered distributions. For $X=L^p(1\leqq p\leqq\infty)$ and $Y=L^q(1\leqq q\leqq\infty)$, this is shown in [9], pp. 97-98. The same proof can be reproduced for various other X and Y since it is based on Sobolev type inequality

$$\|f\|_{L^\infty} \leqq C \sum_{|\alpha| \leqq k} \|D^\alpha f\|_Y$$

and the fact that the imbedding $\mathcal{S}$ (or $\mathcal{S} \cap X) \subsetneq X$ is continuous.

§ 3. Properties of $\mathcal{K}(X, Y)$ and $\mathcal{M}(X, Y)$.

THEOREM 3.1. *The following classes contain only the zero distribution*:

$$\mathcal{K}(H^p, H^q), \qquad \infty > p > q > 0, \quad p > 1,$$

$$\mathcal{K}(H^p, L^q), \qquad \infty > p > q > 0, \quad p > 1,$$

$$\mathcal{K}(L^p, H^q), \qquad \infty \geqq p > q > 0,$$

$$\mathcal{K}(L^p, L^q), \qquad \infty \geqq p > q > 0,$$

$$\mathcal{K}(BMO, H^p), \qquad \infty > p > 0,$$

$$\mathcal{K}(BMO, L^p), \qquad \infty > p > 0,$$

$$\mathcal{K}(\Lambda_s, H^p), \qquad s \in \boldsymbol{R}, \quad \infty > p > 0,$$

$$\mathcal{K}(\Lambda_s, L^p), \qquad s \in \boldsymbol{R}, \quad \infty > p > 0,$$

$$\mathcal{K}(L_k^\infty, H^p), \qquad k \in \boldsymbol{N}, \quad \infty > p > 0,$$

$$\mathcal{K}(L_k^\infty, L^p), \qquad k \in \boldsymbol{N}, \quad \infty > p > 0.$$

If $1 \geqq p > q > 0$, *then the classes*

$$\mathcal{K}(H^p, H^q) \quad \textit{and} \quad \mathcal{K}(H^p, L^q)$$

contain only the polynomials of degree $\leqq [n/p - n]$.

Hörmander ([9], Theorem 1.1, p. 96) proved that $\mathcal{K}(L^p, L^q) = \{0\}$ if $\infty \geqq p > q \geqq 1$. We shall see that his proof can be applied to the classes in the above theorem. We define τ_h, $h \in \boldsymbol{R}^n$, by

$$(\tau_h f)(x) = f(x-h), \qquad x \in \boldsymbol{R}^n.$$

LEMMA 3.1. (i) *If* $f \in L^p$, $0 < p < \infty$, *then*

$$\|f \pm \tau_h f\|_{L^p} \longrightarrow 2^{1/p} \|f\|_{L^p} \quad \textit{as} \quad |h| \longrightarrow \infty.$$

(ii) *If* $f \in H^p$, $0 < p < \infty$, *then*

$$\|f \pm \tau_h f\|_{H^p} \longrightarrow 2^{1/p} \|f\|_{H^p} \quad \textit{as} \quad |h| \longrightarrow \infty.$$

PROOF. If f has a compact support, then (i) is certainly true since the supports of f and $\tau_h f$ do not meet when $|h|$ is sufficiently large. Since compactly supported functions are dense in L^p and the operators $f \mapsto f \pm \tau_h f$, $h \in \boldsymbol{R}^n$, are equi-continuous in L^p, (i) is true for all $f\in L^p$. As for (ii), observe that

$$(f\pm\tau_h f)^+(x)\leqq f^+(x)+(\tau_h f)^+(x)=f^+(x)+\tau_h(f^+)(x),$$

$$(f\pm\tau_h f)^+(x)\geqq |f^+(x)-(\tau_h f)^+(x)|=|f^+(x)-\tau_h(f^+)(x)|$$

and hence

$$\|f^+-\tau_h(f^+)\|_{L^p}\leqq\|(f\pm\tau_h f)^+\|_{L^p}\leqq\|f^++\tau_h(f^+)\|_{L^p}.$$

Thus, using (i), we have

$$\|(f\pm\tau_h f)^+\|_{L^p}\longrightarrow 2^{1/p}\|f^+\|_{L^p}\quad\text{as}\quad |h|\longrightarrow\infty,$$

which means (ii). This completes the proof of Lemma 3.1.

PROOF OF THEOREM 3.1. Firstly consider the class $\mathcal{K}(H^p, H^q)$, $\infty>p>q>0$. Suppose that K belongs to this class. Since

$$K*(f+\tau_h f)=K*f+\tau_h(K*f),$$

we have

$$\|K*f+\tau_h(K*f)\|_{H^q}\leqq\|K\|_{\mathcal{K}(H^p,H^q)}\|f+\tau_h f\|_{H^p}$$

for all $f\in\mathcal{D}\cap H^p$ and all $h\in\boldsymbol{R}^n$. Let $|h|\to\infty$. By Lemma 3.1, we have

$$\|K*f\|_{H^q}\leqq 2^{1/p-1/q}\|K\|_{\mathcal{K}(H^p,H^q)}\|f\|_{H^p},\qquad f\in\mathcal{D}\cap H^p.$$

Since $2^{1/p-1/q}<1$, the above inequality is possible only if $\|K\|_{\mathcal{K}(H^p,H^q)}=0$. Thus $\mathcal{K}(H^p, H^q)$ contains only trivial elements if $\infty>p>q>0$. Next consider the class $\mathcal{K}(\Lambda_s, H^p)$, $s\in\boldsymbol{R}$, $\infty>p>0$. Suppose that K belongs to this class. Let $\infty>r>p$, $r>1$ and G be any element of the class $\mathcal{K}(H^r, \Lambda_s)$ with compact support. Then $K*G\in\mathcal{K}(H^r, H^p)$ and hence $K*G=0$ by what was shown above. In particular $K*G=0$ for every $G\in\mathcal{D}$. Hence $K=0$. Thus $\mathcal{K}(\Lambda_s, H^p)=\{0\}$. The other classes are treated in a similar way. This completes the proof of Theorem 3.1.

THEOREM 3.2 (Duality relations). *The following equalities hold; in each case, the norms of the corresponding classes are equivalent.*

(i) $\mathcal{K}(H^p, H^1)=\mathcal{K}(BMO, \tilde{\Lambda}_s)$, $\quad 0<p<1$, $\ s=n/p-n$.

(ii) $\mathcal{K}(H^p, L^q)=\mathcal{K}(L^{q'}, \tilde{\Lambda}_s)$, $\quad 0<p<1$, $\ 1\leqq q\leqq\infty$, $\ s=n/p-n$, $\ 1/q+1/q'=1$.

(iii) $\mathcal{K}(H^p, BMO)=\mathcal{K}(H^1, \tilde{\Lambda}_s)$, $\quad 0<p<1$, $\ s=n/p-n$.

(iv) $\mathcal{K}(H^p, \tilde{\Lambda}_t)=\mathcal{K}(H^q, \tilde{\Lambda}_s)$, $\quad 0<p<1$, $\ 0<q<1$, $\ s=n/p-n$, $\ t=n/q-n$.

(v) $\mathcal{K}(H^1, H^1)=\mathcal{K}(BMO, BMO)$.

(vi) $\mathcal{K}(H^1, L^q)=\mathcal{K}(L^{q'}, BMO)$, $\quad 1\leqq q\leqq\infty$, $\ 1/q+1/q'=1$.

(vii) $\mathcal{K}(L^1, H^1)=\mathcal{K}(BMO, L^\infty)$.

(viii) $\mathcal{K}(L^p, L^q)=\mathcal{K}(L^{q'}, L^{p'})$, $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, $1/p+1/p'=1$, $1/q+1/q'=1$.

PROOF. Let $\mathcal{K}(X, Y')=\mathcal{K}(Y, X')$ be one of the equalities in (i)$\sim$(viii). The inequalities in § 1.2 show that

$$C\|h\|_{Y'}\leqq\sup\{|\langle h, g\rangle|\mid g\in\mathcal{D}\cap Y,\ \|g\|_Y\leqq 1\}\leqq C'\|h\|_{Y'}$$

and

$$C\|h\|_{X'}\leqq\sup\{|\langle f, h\rangle|\mid f\in\mathcal{D}\cap X,\ \|f\|_X\leqq 1\}\leqq C'\|h\|_{X'}.$$

From these inequalities we obtain

$$C\|K\|_{\mathcal{K}(X,Y')}\leqq\sup\{|\langle K*f, g\rangle|\}\leqq C'\|K\|_{\mathcal{K}(X,Y')}$$

and

$$C\|G\|_{\mathcal{K}(Y,X')}\leqq\sup\{|\langle f, G*g\rangle|\}\leqq C'\|G\|_{\mathcal{K}(Y,X')},$$

where both the suprema are taken over

$$\{(f, g)\mid f\in\mathcal{D}\cap X,\ g\in\mathcal{D}\cap Y,\ \|f\|_X\leqq 1,\ \|g\|_Y\leqq 1\}.$$

Combining the above inequalities and the equality

$$\langle K*f, g\rangle=\langle f, \check{K}*g\rangle, \qquad \check{K}=K(-\cdot),$$

we have

$$C\|K\|_{\mathcal{K}(X,Y')}\leqq\|\check{K}\|_{\mathcal{K}(Y,X')}\leqq C'\|K\|_{\mathcal{K}(X,Y')},$$

which is the desired inequality since

$$\|\check{K}\|_{\mathcal{K}(Y,X')}=\|K\|_{\mathcal{K}(Y,X')}.$$

This completes the proof.

THEOREM 3.3. (i) *Let Y be one of the following spaces:*

$$H^1,\ L^p(1<p\leqq\infty),\ BMO,\ \Lambda_s(s\in\boldsymbol{R}),\ L_k^\infty(k\in\boldsymbol{N}).$$

Then $\mathcal{K}(L^1, Y)=Y$ with equality also of the norms.

(ii) *$\mathcal{K}(L^1, L^1)(=\mathcal{K}(L^\infty, L^\infty))$ coincides, with equality also of the norms, with the space of all finite complex Borel measures.*

(iii) *$\mathcal{M}(L^2, L^2)=L^\infty$ with equality also of the norms.*

PROOF. As for (ii) and (iii), see Hörmander [9], Theorems 1.4 and 1.5, pp. 100-101. Let Y be one of the spaces mentioned in (i). Then the following inequality holds:

$$\|K*f\|_Y\leqq\|K\|_Y\|f\|_{L^1}, \qquad f\in\mathcal{D}.$$

Hence $K(L^1, Y)\supset Y$ and

$$\|K\|_{\mathcal{K}(L^1,Y)} \leqq \|K\|_Y .$$

Conversely, suppose that $K \in \mathcal{K}(L^1, Y)$. Then

$$\|\varphi(\cdot\,|t) * K\|_Y \leqq \|K\|_{\mathcal{K}(L^1,Y)}, \qquad t>0, \tag{3.1}$$

where φ is a nonnegative function in $\mathcal{D}$ such that $\hat{\varphi}(0)=1$. Since $\varphi(\cdot\,|t) * K$ tends to K in $\mathcal{D}'$ as $t \to 0$, we have

$$\|K\|_Y \leqq \liminf_{t\downarrow 0} \|\varphi(\cdot\,|t) * K\|_Y \tag{3.2}$$

by Lemma 2.2. From (3.1) and (3.2), we see that $K \in Y$ and

$$\|K\|_Y \leqq \|K\|_{\mathcal{K}(L^1,Y)} .$$

This completes the proof.

The next theorem reflects the characteristic properties of H^p stated in Theorems A and C.

THEOREM 3.4. *Let* $0<p\leqq q<\infty$. *Then*

$$\mathcal{K}(H^p, H^q) = \mathcal{S}' \cap \mathcal{K}(H^p, L^q);$$

the norms of $\mathcal{K}(H^p, H^q)$ *and* $\mathcal{K}(H^p, L^q)$ *are equivalent.*

COROLLARY 3.1. $\mathcal{K}(BMO, BMO) = \mathcal{K}(L^\infty, BMO)$;

$$\mathcal{K}(BMO, \tilde{\Lambda}_s) = \mathcal{K}(L^\infty, \tilde{\Lambda}_s), \qquad s>0;$$

the norms are equivalent.

Corollary 3.1 is a direct consequence of Theorems 3.2 and 3.4.

PROOF OF THEOREM 3.4. The inclusion $\mathcal{K}(H^p, H^q) \subset \mathcal{S}'$ has been mentioned in Remark 2.4. The inclusion $\mathcal{K}(H^p, H^q) \subset \mathcal{K}(H^p, L^q)$ is obvious. We shall prove the converse inclusion $\mathcal{S}' \cap \mathcal{K}(H^p, L^q) \subset \mathcal{K}(H^p, H^q)$. (This is obvious if $q>1$ since $L^q = H^q$, $q>1$. Proof is needed in the case $0<q\leqq 1$.) Suppose that $K \in \mathcal{S}' \cap \mathcal{K}(H^p, L^q)$. The inequality

$$\|K * f\|_{L^q} \leqq M \|f\|_{H^p}, \qquad M = \|K\|_{\mathcal{K}(H^p, L^q)}, \tag{3.3}$$

holds for all $f \in \mathcal{S} \cap H^p$, which can be seen by approximating f by functions of $\mathcal{D} \cap H^p$. Let χ be the function mentioned in the definition of the Littlewood-Paley function $d(f)$. Let $f \in \mathcal{S} \cap H^p$ and $\{\varepsilon_k \,|\, k=0, \pm 1, \pm 2, \cdots\}$ be any sequence consisting of $+1$ and -1. Take any positive integer N and apply the inequality (3.3) to the function

$$\sum_{k=-N}^{N} \varepsilon_k \chi_k * f \in \mathcal{S} \cap H^p,$$

where $\chi_k=\chi(\cdot|2^k)$. Then we have

$$\int\Big|\sum_{k=-N}^{N}\varepsilon_k(\chi_k*K*f)(x)\Big|^q dx\leqq M^q\Big\|\sum_{k=-N}^{N}\varepsilon_k\chi_k*f\Big\|_{H^p}^q .$$

Now we shall use the fact that

$$\Big\|\sum_{k=-N}^{N}\varepsilon_k\chi_k\Big\|_{\mathcal{K}(H^p,H^p)}\leqq C$$

with C independent of N and the (± 1)-sequence $\{\varepsilon_k\}$; this fact can be shown by using Theorem E below. Thus we have

$$\int\Big|\sum_{k=-N}^{N}\varepsilon_k(\chi_k*K*f)(x)\Big|^q dx\leqq(CM)^q\|f\|_{H^p}^q .$$

We average both sides of this inequality over $\{\varepsilon_k\}\in\{\pm 1\}^{Z}$ with respect to the probability measure on $\{\pm 1\}^{Z}$ which is the direct product of the measures μ such that

$$\mu(\{+1\})=\mu(\{-1\})=1/2 .$$

Then Khintchine's inequality (see Zygmund [23], Chapter V, Theorem (8.4)) gives

$$\int\Big(\sum_{k=-N}^{N}|(\chi_k*K*f)(x)|^2\Big)^{q/2}dx\leqq(C'M)^q\|f\|_{H^p}^q$$

with C' independent of N. Thus we obtain

$$\|d(K*f)\|_{L^q}\leqq C'M\|f\|_{H^p}.$$

Hence, by Theorem A, there is a polynomial P such that $K*f-P\in H^q$ and

$$\|K*f-P\|_{H^q}\leqq C''M\|f\|_{H^p}.$$

But P must be equal to zero since $K*f\in L^q$. Thus we have proved that

$$\|K*f\|_{H^q}\leqq C''M\|f\|_{H^p},\qquad f\in\mathcal{S}\cap H^p,$$

which means that $K\in\mathcal{K}(H^p, H^q)$ and

$$\|K\|_{\mathcal{K}(H^p,H^q)}\leqq C''M=C''\|K\|_{\mathcal{K}(H^p,L^q)}.$$

This completes the proof of Theorem 3.4.

THEOREM 3.5. (i) *If* $1\leqq p\leqq\infty$, *then* $\mathcal{K}(L^p, L^p)\subset\mathcal{K}(L^2, L^2)$ *and*

$$\|K\|_{\mathcal{K}(L^2,L^2)}\leqq\|K\|_{\mathcal{K}(L^p,L^p)}.$$

(ii) *If* $0<p\leqq 1$ *and* $K\in\mathcal{K}(H^p, H^p)$, *then there is a polynomial* P *of degree* $\leqq [n/p-n]$ *such that* $K-P\in\mathcal{K}(L^2, L^2)$ *and*

$$\|K-P\|_{\mathcal{K}(L^2,L^2)}\leqq C\|K\|_{\mathcal{K}(H^p,H^p)}.$$

(i) is proved in [9], Corollary 1.3, p. 101. For the proof of (ii), we use the following Lemmas.

LEMMA 3.2. *Let X be any space of functions or distributions and Y be one of the following spaces*:

$$H^p(0<p<\infty),\ L^p(0<p\leqq\infty),\ BMO,\ \Lambda_s(s\in\boldsymbol{R}),\ L_k^\infty(k\in\boldsymbol{N}).$$

If $K_n(n=1, 2, \cdots)$ and K are distributions and $K_n\to K$ in $\mathcal{D}'$, then

$$\|K\|_{\mathcal{K}(X,Y)}\leqq\liminf\|K_n\|_{\mathcal{K}(X,Y)}.$$

LEMMA 3.3. *If $0<p<\infty$ and $t>0$, then*

$$\|m(t\cdot)\|_{\mathcal{M}(H^p,H^p)}=\|m\|_{\mathcal{M}(H^p,H^p)}.$$

Lemma 3.2 can be proved by using Lemma 2.2. Lemma 3.3 is easily deduced from the definitions once we use the equality

$$\|t^{-n/p+n}f(\cdot\,|t)\|_{H^p}=\|f\|_{H^p}. \tag{3.4}$$

PROOF OF THEOREM 3.5. By the equality (3.4) and the inequality

$$|\hat{f}(\xi)|\leqq C\|f\|_{H^p}|\xi|^{n/p-n}$$

(cf. § 6.4), we obtain

$$|\hat{K}(\xi)t^{-n/p+n}\hat{f}(t\xi)|\leqq C\|K\|_{\mathcal{K}(H^p,H^p)}\|f\|_{H^p}|\xi|^{n/p-n} \tag{3.5}$$

with C independent of f and $t>0$. We fix a function $f\in\mathcal{S}\cap H^p$ such that $\hat{f}(\xi)=1$ on $\{|\xi|=1\}$. Then, if we set $t=1/|\xi|$ in (3.5), we obtain

$$|\hat{K}(\xi)|\leqq C\|K\|_{\mathcal{K}(H^p,H^p)},\qquad \xi\neq 0. \tag{3.6}$$

We set

$$K_\varepsilon=\mathcal{F}^{-1}(\hat{K}(\xi)\phi(\xi/\varepsilon)),\qquad \varepsilon>0.$$

Then, from (3.6), we see that $\lim_{\varepsilon\downarrow 0}K_\varepsilon$ exists in $\mathcal{S}'$ and

$$\|\lim_{\varepsilon\downarrow 0}K_\varepsilon\|_{\mathcal{K}(L^2,L^2)}\leqq C\|K\|_{\mathcal{K}(H^p,H^p)}. \tag{3.7}$$

On the other hand, Lemmas 3.2 and 3.3 show that

$$\|\lim_{\varepsilon\downarrow 0}K_\varepsilon\|_{\mathcal{K}(H^p,H^p)}\leqq C\|K\|_{\mathcal{K}(H^p,H^p)}. \tag{3.8}$$

Set $P=K-\lim K_\varepsilon$. P is certainly a polynomial since support $\hat{P}\subset\{0\}$. (3.8) shows that $P\in\mathcal{K}(H^p, H^p)$ and hence degree $P\leqq[n/p-n]$. Thus (3.7) gives the desired inequality. This completes the proof.

We shall refer to the following Fourier multiplier criterions for H^p and Λ_s.

THEOREM E. *Let* $0<p<2$ *and* $k=[n(1/p-1/2)]+1$. *If* m *is a bounded function which is of class* C^k *in* $\boldsymbol{R}^n\setminus\{0\}$ *and if*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha m(\xi)\right|\leqq(A|\xi|^{-1})^{|\alpha|}\qquad for\quad |\alpha|\leqq k$$

with $A\geqq 1$, *then* $m\in\mathcal{M}(H^p, H^p)$ *and*

$$\|m\|_{\mathcal{M}(H^p,H^p)}\leqq CA^{n(1/p-1/2)}.$$

THEOREM F (Fractional integral). *Let* $0<p<\infty$, $0<\mu<n/p$ *and* $1/q=1/p-\mu/n$. *Then the operator*

$$f\longmapsto\mathcal{F}^{-1}(|\xi|^{-\mu}\hat{f}(\xi))$$

is well defined on H^p *and bounded from* H^p *to* H^q.

THEOREM G. (i) *Let* $a>0$, $b>0$, $0<p<2$, $na(1/p-1/2)=b$ *and* $k=[n(1/p-1/2)]+1$. *Suppose that* m *is of class* C^k *on* $\boldsymbol{R}^n$, $m(\xi)=0$ *for* $|\xi|\leqq 1$ *and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha m(\xi)\right|\leqq|\xi|^{-b}(A|\xi|^{a-1})^{|\alpha|}\quad for\quad |\alpha|\leqq k \tag{3.9}$$

with some constant $A\geqq 1$. *Then* $m\in\mathcal{M}(H^p, H^p)$ *and*

$$\|m\|_{\mathcal{M}(H^p,H^p)}\leqq CA^{n(1/p-1/2)}.$$

(ii) *Let* $c>0$, $d>0$, $0<p<2$, $nd(1/p-1/2)=c$ *and* $k=[n(1/p-1/2)]+1$. *Suppose that* m *is a bounded function, of class* C^k *in* $\boldsymbol{R}^n\setminus\{0\}$, $m(\xi)=0$ *for* $|\xi|\geqq 1$ *and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha m(\xi)\right|\leqq|\xi|^{c}(A|\xi|^{-1-d})^{|\alpha|}\quad for\quad |\alpha|\leqq k \tag{3.10}$$

with some constant $A\geqq 1$. *Then* $m\in\mathcal{M}(H^p, H^p)$ *and*

$$\|m\|_{\mathcal{M}(H^p,H^p)}\leqq CA^{n(1/p-1/2)}.$$

THEOREM H. $\mathcal{K}(\Lambda_s, \Lambda_t)=\Lambda(t-s\,;1,\infty)$; *in particular, the class* $K(\Lambda_s, \Lambda_t)$ *depends only on* $t-s$.

As for the space $\Lambda(s\,;1,\infty)$, see Taibleson [20]. We shall also refer to this space in §6.1.

THEOREM I. *Let* $t\in\boldsymbol{R}$ *and* $k=[n/2]+1$. *If* $A\geqq 1$, m *is a function of class* C^k *and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha m(\xi)\right|\leqq(1+|\xi|)^{-t}(A(1+|\xi|)^{-1})^{|\alpha|}\quad for\quad |\alpha|\leqq k,$$

then $m\in\mathcal{M}(\Lambda_s, \Lambda_{s+t})$ *and*

$$\|m\|_{\mathcal{M}(\Lambda_s,\Lambda_{s+t})}\leqq CA^{n/2}.$$

As for the proofs of Theorems E$\sim$I, see the following papers. Theorem E: Calderón-Torchinsky [3], pp. 163-171; Miyachi [13]. Theorem F: Calderón-Torchinsky [3], p. 162. Theorem G: Miyachi [13]; see also § 6.6 of the present paper. Theorem H: Taibleson [20], II. Theorem I: Triebel [21], pp. 30-31, pp. 93-94; cf. also the proof of Theorem 5.1 of the present paper.

§ 4. Results on the singular Fourier multipliers.

We shall consider the Fourier multiplier $m_{a,b}$:

$$m_{a,b}(\xi)=\phi(\xi)|\xi|^{-b}\exp(i|\xi|^{a}), \qquad \xi\in\boldsymbol{R}^{n}, \quad a>0, \quad b\in\boldsymbol{R}.$$

The mapping properties of this Fourier multiplier between the spaces H^p, L^1, L^∞, BMO, Λ_s and L_k^∞ are given in Theorems 4.1$\sim$4.5 and Corollary 4.1. In this section, p and q denote positive numbers, s and t real numbers and k positive integer unless specified otherwise.

THEOREM 4.1. *If* $0<a<1$ *or* $a>1$, *then the following facts hold.*

(I-i) $m_{a,b}\in\mathcal{M}(H^p, H^q)$ *iff*[1] $p\leqq q$, $1/p+1/q\leqq 1$, $(1-a)/p-1/q\leqq b/n-a/2$ *or* $p\leqq q$, $1/p+1/q\geqq 1$, $1/p-(1-a)/q\leqq b/n+a/2$; *in particular,* $m_{a,b}\in\mathcal{M}(H^p, H^p)$ *iff* $|1/p-1/2|\leqq b/(na)$; $m_{a,b}\in\mathcal{M}(H^1, H^1)=\mathcal{M}(BMO, BMO)$ *iff* $b\geqq na/2$.

(I-ii) $m_{a,b}\in\mathcal{M}(L^1, H^1)=\mathcal{M}(BMO, L^\infty)$ *iff* $b>na/2$.

(I-iii) $m_{a,b}\in\mathcal{M}(L^1, L^q)$, $1<q<\infty$, *iff* $1-(1-a)/q<b/n+a/2$.

(I-iv) $m_{a,b}\in\mathcal{M}(L^1, L^1)=\mathcal{M}(L^\infty, L^\infty)$ *iff* $b>na/2$.

(II-i) $m_{a,b}\in\mathcal{M}(H^p, BMO)$ *iff* $0<p\leqq 1$, $1/p\leqq b/n+a/2$ *or* $1<p<\infty$, $(1-a)/p\leqq b/n-a/2$.

(II-ii) $m_{a,b}\in\mathcal{M}(H^p, L^\infty)$ *iff* $0<p\leqq 1$, $1/p\leqq b/n+a/2$ *or* $1<p<\infty$, $(1-a)/p<b/n-a/2$.

(II-iii) $m_{a,b}\in\mathcal{M}(L^1, L^\infty)$ *iff* $b\geqq n-na/2$.

(II-iv) $m_{a,b}\in\mathcal{M}(L^1, BMO)=\mathcal{M}(H^1, L^\infty)$ *iff* $b\geqq n-na/2$.

(III-i) $m_{a,b}\in\mathcal{M}(H^p, \Lambda_s)$ *iff* $0<p\leqq 1$, $1/p\leqq(b-s)/n+a/2$ *or* $1<p<\infty$, $(1-a)/p\leqq(b-s)/n-a/2$.

(III-ii) $m_{a,b}\in\mathcal{M}(H^p, L_k^\infty)$ *iff* $0<p\leqq 1$, $1/p\leqq(b-k)/n+a/2$ *or* $1<p<\infty$, $(1-a)/p<(b-k)/n-a/2$.

(III-iii) $m_{a,b}\in\mathcal{M}(L^1, \Lambda_s)$ *iff* $b-s\geqq n-na/2$.

[1] iff=if and only if

(III-iv) $m_{a,b}\in\mathcal{M}(L^1, L_k^\infty)$ *iff* $b-k\geqq n-na/2$.

THEOREM 4.2. *The case* $a=1$.

(I-i) $m_{1,b}\in\mathcal{M}(H^p, H^q)$ *iff* $p\leqq q$, $1/p+1/q\leqq 1$, $1/p-n/q\leqq b-(n-1)/2$ *or* $p\leqq q$, $1/p+1/q\geqq 1$, $n/p-1/q\leqq b+(n-1)/2$; *in particular*, $m_{1,b}\in\mathcal{M}(H^p, H^p)$ *iff* $(n-1)|1/p-1/2|\leqq b$; $m_{1,b}\in\mathcal{M}(H^1, H^1)=\mathcal{M}(BMO, BMO)$ *iff* $b\geqq(n-1)/2$.

(I-ii) $m_{1,b}\in\mathcal{M}(L^1, H^1)=\mathcal{M}(BMO, L^\infty)$ *iff* $b>(n-1)/2$.

(I-iii) $m_{1,b}\in\mathcal{M}(L^1, L^q)$, $1<q<\infty$, *iff* $b-(n+1)/2>-1/q$.

(I-iv) $m_{1,b}\in\mathcal{M}(L^1, L^1)=\mathcal{M}(L^\infty, L^\infty)$ *iff* $b>(n-1)/2$.

(II-i) $m_{1,b}\in\mathcal{M}(H^p, BMO)$ *iff* $0<p\leqq 1$, $n/p\leqq b+(n-1)/2$ *or* $1<p<\infty$, $1/p\leqq b-(n-1)/2$.

(II-ii) $m_{1,b}\in\mathcal{M}(H^p, L^\infty)$ *iff* $0<p\leqq 1$, $n/p\leqq b+(n-1)/2$ *or* $1<p<\infty$, $1/p<b-(n-1)/2$.

(II-iii) $m_{1,b}\in\mathcal{M}(L^1, L^\infty)$ *iff* $b>(n+1)/2$.

(II-iv) $m_{1,b}\in\mathcal{M}(L^1, BMO)=\mathcal{M}(H^1, L^\infty)$ *iff* $b\geqq(n+1)/2$.

(III-i) $m_{1,b}\in\mathcal{M}(H^p, \Lambda_s)$ *iff* $0<p\leqq 1$, $n/p\leqq b-s+(n-1)/2$ *or* $1<p<\infty$, $1/p\leqq b-s-(n-1)/2$.

(III-ii) $m_{1,b}\in\mathcal{M}(H^p, L_k^\infty)$ *iff* $0<p\leqq 1$, $n/p\leqq b-k+(n-1)/2$ *or* $1<p<\infty$, $1/p<b-k-(n-1)/2$.

(III-iii) $m_{1,b}\in\mathcal{M}(L^1, \Lambda_s)$ *iff* $b-s\geqq(n+1)/2$.

(III-iv) $m_{1,b}\in\mathcal{M}(L^1, L_k^\infty)$ *iff* $b-k>(n+1)/2$.

THEOREM 4.3. (I) *The case* $0<a<1$ *or* $a>1$.

(I-i) $m_{a,b}\in\mathcal{M}(BMO, \Lambda_s)$ *iff* $b-s\geqq na/2$.

(I-ii) $m_{a,b}\in\mathcal{M}(BMO, L_k^\infty)$ *iff* $b-k>na/2$.

(I-iii) $m_{a,b}\in\mathcal{M}(L^\infty, \Lambda_s)$ *iff* $b-s\geqq na/2$.

(I-iv) $m_{a,b}\in\mathcal{M}(L^\infty, L_k^\infty)$ *iff* $b-k>na/2$.

(II) *The case* $a=1$.

(II-i) $m_{1,b}\in\mathcal{M}(BMO, \Lambda_s)$ *iff* $b-s\geqq(n-1)/2$.

(II-ii) $m_{1,b}\in\mathcal{M}(BMO, L_k^\infty)$ *iff* $b-k>(n-1)/2$.

(II-iii) $m_{1,b}\in\mathcal{M}(L^\infty, \Lambda_s)$ *iff* $b-s\geqq(n-1)/2$.

(II-iv) $m_{1,b}\in\mathcal{M}(L^\infty, L_k^\infty)$ *iff* $b-k>(n-1)/2$.

THEOREM 4.4. (i) *The case* $0<a<1$ *or* $a>1$: $m_{a,b}\in\mathcal{M}(\Lambda_s, \Lambda_t)$ *iff* $t-s\leqq b-na/2$.
(ii) *The case* $a=1$: $m_{1,b}\in\mathcal{M}(\Lambda_s, \Lambda_t)$ *iff* $t-s\leqq b-(n-1)/2$.

In the next theorem, we use the following notation:

$$T_{a,b}f=\mathcal{F}^{-1}(m_{a,b}(\xi)\hat{f}(\xi))$$

and

$$\chi_k=\chi(\cdot\,|\,2^{-k}),\qquad k\in\boldsymbol{Z},$$

where χ is a function in $\mathcal{S}$ satisfying (2.1).

THEOREM 4.5. *Let* $a>0$, $a\neq 1$, $b>0$ *and* $1/p-1/2=b/(na)$. *Then*

$$\left(\sum_{k=0}^{\infty}\int_{R^n}|(\chi_k*T_{a,b}f)(x)|^p dx\right)^{1/p}\leqq C\|f\|_{H^p}. \tag{4.1}$$

COROLLARY 4.1. *Suppose that* $a>0$ *and* $a\neq 1$. *Then*: (i) $m_{a,b}\in\mathcal{M}(\Lambda_s, BMO)$ *iff* $-s\leqq b-na/2$; (ii) $m_{a,b}\in\mathcal{M}(\Lambda_s, L_k^\infty)$ *iff* $k-s<b-na/2$.

REMARK 4.1. Theorem 4.5 improves (I-i) of Theorem 4.1 in the case $1/p=1/q=1/2+b/(na)$ since $\|T_{a,b}f\|_{H^p}\approx\|d(T_{a,b}f)\|_{L^p}$ (Theorem A) and $\|d(T_{a,b}f)\|_{L^p}\leqq$ (the left hand side of (4.1)). Hence, by interpolation and duality, we can obtain the corresponding improvements of some of the results in Theorems 4.1, 4.3 and 4.4. One of them is stated in Corollary 4.1. (i) of Corollary 4.1 is indeed an improvement of (i) of Theorem 4.4 since the following inequality holds:

$$\|f\|_{\Lambda_0}\leqq C(t)(\|(1-\Delta)^{t/2}f\|_{L^\infty}+\|f\|_{BMO})\qquad\text{for any } t\in\boldsymbol{R},$$

which can be easily shown by using Proposition 6.1 (§ 6.1).

REMARK 4.2. We can prove similar results if we replace $m_{a,b}$ by

$$\psi(\xi)\{b_1(\xi)\exp(i\phi_1(\xi)^a)+b_2(\xi)\exp(i\phi_2(\xi)^a)\},$$

where $b_1(\xi)$ and $b_2(\xi)$ are smooth homogeneous functions of degree $-b$ and $\phi_1(\xi)$ and $\phi_2(\xi)$ are positive and smooth homogeneous functions of degree 1 such that the Gaussian curvature of the surfaces

$$\{\xi\mid\phi_j(\xi)=1\},\qquad j=1,2,$$

never vanish; cf. Miyachi [14].

We can illustrate the main results of Theorems 4.1, 4.2 and 4.3 in simple graphs. In order to do this, we set

$$X_\rho=\begin{cases}\Lambda_{-n\rho} & \text{if } \rho<0\\ BMO & \text{if } \rho=0\\ H^{1/\rho} & \text{if } \rho>0\end{cases}$$

and graph the set

$$D_{a,b}=\{(\rho,\sigma)\mid \rho\geqq 0,\ \sigma\in\boldsymbol{R},\ m_{a,b}\in\mathcal{M}(X_\rho, X_\sigma)\}$$

in (ρ,σ)-plane. Here we shall give the graphs of $D_{a,b}$ in the cases $0<a\leqq 1$ (Figures 1 and 2) and $a=2$ (Figure 3).

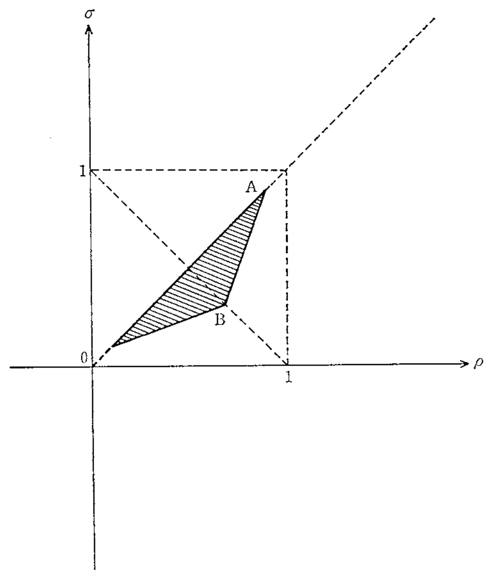

Figure 1 $(0<a\leqq 1)$.

In the case $0<a<1$ and $0<b<n-na/2$,

$$A=\left(\frac{1}{2}+\frac{b}{na},\ \frac{1}{2}+\frac{b}{na}\right)\ \text{and}\ B=\left(\frac{1}{2}+\frac{b}{2n-na},\ \frac{1}{2}-\frac{b}{2n-na}\right).$$

In the case $a=1$ and $0<b<(n+1)/2$,

$$A=\left(\frac{1}{2}+\frac{b}{n-1},\ \frac{1}{2}+\frac{b}{n-1}\right)\ \text{and}\ B=\left(\frac{1}{2}+\frac{b}{n+1},\ \frac{1}{2}-\frac{b}{n+1}\right).$$

We observe two facts from Theorems 4.1, 4.2 and 4.3. Firstly the results have certain discontinuity at $a=1$. More precisely, while the set $D_{a,b}$ depends continuously on a if $0<a<1$ or $a>1$, $D_{1,b}$ does not coincide with the limit of $D_{a,b}$, $a\to 1$:

$$\lim_{a\uparrow 1} D_{a,b}=\lim_{a\downarrow 1} D_{a,b}\not\supset D_{1,b},\ \not\subset D_{1,b}.$$

There may be some connection between this fact and the singularity of the kernel $K_{a,b}=\mathcal{F}^{-1}(m_{a,b})$. The singular support (=the smallest closed set outside of which a distribution is smooth) of $K_{a,b}$ is as follows:

$$SS(K_{a,b})=\{0\}\qquad \text{if } 0<a<1,$$

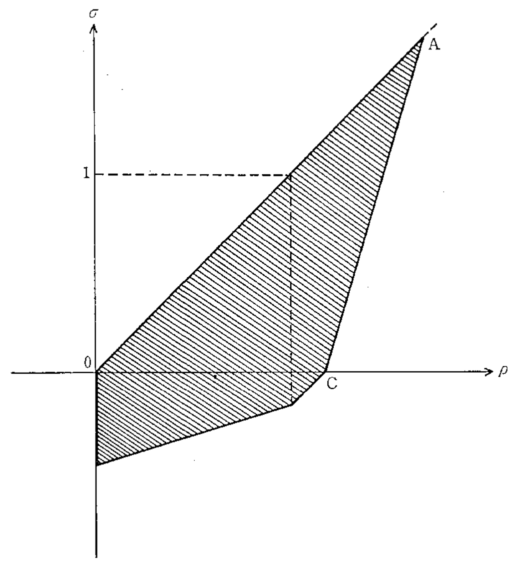

Figure 2 $(0<a\leqq 1)$.

In the case $0<a<1$ and $b>n-na/2$,

$$A=\left(\frac{1}{2}+\frac{b}{na},\ \frac{1}{2}+\frac{b}{na}\right) \quad \text{and} \quad C=\left(\frac{b}{n}+\frac{a}{2},\ 0\right).$$

In the case $a=1$ and $b>(n+1)/2$,

$$A=\left(\frac{1}{2}+\frac{b}{n-1},\ \frac{1}{2}+\frac{b}{n-1}\right) \quad \text{and} \quad C=\left(\frac{b}{n}+\frac{n-1}{2n},\ 0\right).$$

$$SS(K_{a,b})=\{x \mid |x|=1\} \qquad \text{if} \quad a=1,$$

$$SS(K_{a,b})=\varnothing \qquad \text{if} \quad a>1$$

(cf. Proposition 5.1 in § 5).

Secondly we compare the results with that for the multiplier

$$m_{0,b}(\xi)=\psi(\xi)|\xi|^{-b}, \qquad \xi\in \boldsymbol{R}^n,\ b>0.$$

The mapping properties of this Fourier multiplier is well known (cf. Theorems F and I in § 3); the set

$$D_{0,b}=\{(\rho,\ \sigma) \mid \rho\geqq 0,\ \sigma\in\boldsymbol{R},\ m_{0,b}\in\mathcal{M}(X_\rho,\ X_\sigma)\}$$

is given in Figure 4. It is seen that $D_{a,b}\not\subset D_{0,b}$ and $D_{a,b}\not\supset D_{0,b}$; if $1/2<\rho<1/2+b/n$, then

$$\{\sigma\in\boldsymbol{R} \mid m_{a,b}\in\mathcal{M}(X_\rho,\ X_\sigma)\}\supsetneqq\{\sigma\in\boldsymbol{R} \mid m_{0,b}\in\mathcal{M}(X_\rho,\ X_\sigma)\},$$

which means that the oscillating factor $\exp(i|\xi|^a)$, $a>0$, add regularity to functions of H^p if $(1/2+b/n)^{-1}<p<2$. In the case $a=1$, this fact has already been

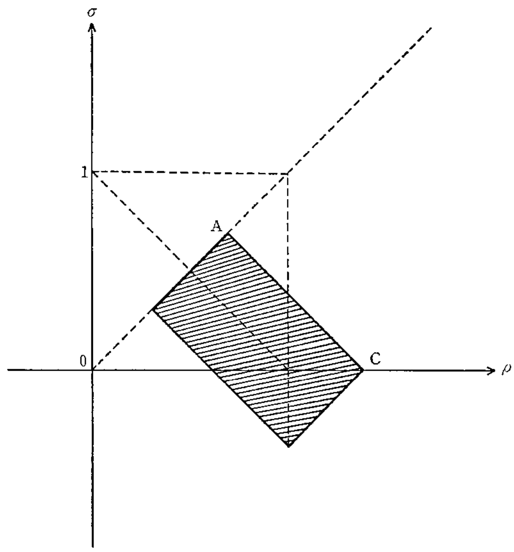

Figure 3 $(a=2)$.

$$A=\left(\frac{1}{2}+\frac{b}{2n}, \frac{1}{2}+\frac{b}{2n}\right) \quad \text{and} \quad C=\left(\frac{b}{n}+1, 0\right).$$

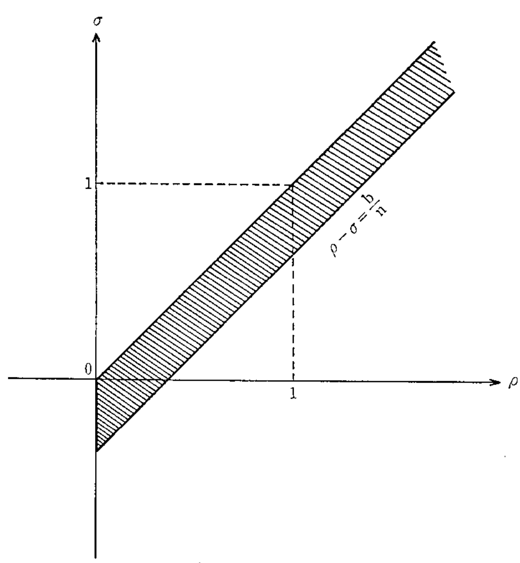

Figure 4 $(a=0)$.

pointed out by Strichartz [19].

Theorem 4.5 and Corollary 4.1 lack the results for the case $a=1$. We leave the following problem unsolved:

PROBLEM. Prove or disprove a fact corresponding to Theorem 4.5 and Corollary 4.1 in the case $a=1$.

§ 5. Proof of Theorems 4.1~4.5 and Corollary 4.1.

In order to prove Theorems 4.1~4.5 and Corollary 4.1, it is important to know the behavior of the kernel $K_{a,b}=\mathcal{F}^{-1}(m_{a,b})$, which is given in the following

PROPOSITION 5.1. (i) *When* $0<a<1$ *and* $b\in\boldsymbol{R}$, $K_{a,b}$ *has the following behavior.* $K_{a,b}$ *is smooth in* $\boldsymbol{R}^n\setminus\{0\}$ *and, for every* β *and* $N>0$,

$$\left(\frac{\partial}{\partial x}\right)^{\beta}K_{a,b}(x)=O(|x|^{-N}) \qquad as\ |x|\to\infty .$$

If k *is a nonnegative integer and* $b-k>n-na/2$, *then* $K_{a,b}$ *is of class* C^k *throughout* $\boldsymbol{R}^n$. *If* $b-|\beta|\leqq n-na/2$, *then*

$$\left(\frac{\partial}{\partial x}\right)^{\beta}K_{a,b}(x)=A\left(\frac{-ix}{|x|}\right)^{\beta}|x|^{\frac{b-|\beta|-n+na/2}{1-a}}\exp\left(iB|x|^{-a/(1-a)}\right)$$
$$+o\left(|x|^{\frac{b-|\beta|-n+na/2}{1-a}}\right)+E(x) \qquad as\ x\to 0, \tag{5.1}$$

where

$$A=\exp\left(\frac{i\pi n}{4}-\frac{i\pi}{2}\right)\cdot(1-a)^{-1/2}a^{\frac{-b+|\beta|+n/2}{1-a}},$$
$$B=a^{a/(1-a)}(1-a)$$

and $E(x)$ *is a smooth function. If the right hand side of* (5.1) *is an integrable function in a neighborhood of the origin, then the distribution* $D^{\beta}K_{a,b}$ *belongs to* L^1.

(ii) *When* $a>1$ *and* $b\in\boldsymbol{R}$, $K_{a,b}$ *has the following behavior.* $K_{a,b}$ *is smooth throughout* $\boldsymbol{R}^n$ *and*

$$\left(\frac{\partial}{\partial x}\right)^{\beta}K_{a,b}(x)=A'\left(\frac{-ix}{|x|}\right)^{\beta}|x|^{\frac{b-|\beta|-n+na/2}{1-a}}\exp\left(iB|x|^{-a/(1-a)}\right)$$
$$+o\left(|x|^{\frac{b-|\beta|-n+na/2}{1-a}}\right) \qquad as\ |x|\to\infty,$$

where

$$A'=\exp\left(\frac{i\pi n}{4}\right)\cdot(a-1)^{-1/2}a^{\frac{-b+|\beta|+n/2}{1-a}}$$

and B *is the constant defined in* (i).

(iii) $K_{1,b}$, $b\in\boldsymbol{R}$, *has the following behavior.* $K_{1,b}$ *is smooth in* $\boldsymbol{R}^n\backslash\{|x|=1\}$ *and, for every* β *and* $N>0$,

$$\left(\frac{\partial}{\partial x}\right)^{\beta}K_{1,b}(x)=O(|x|^{-N}) \qquad as\ |x|\rightarrow\infty .$$

If $b-|\beta|<(n+1)/2$, *then*

$$\left(\frac{\partial}{\partial x}\right)^{\beta}K_{1,b}(x)=A''\left(\frac{x}{|x|}\right)^{\beta}(1-|x|+i0)^{b-|\beta|-(n+1)/2} + o(|1-|x||^{b-|\beta|-(n+1)/2}) \qquad as\ |x|\rightarrow 1, \tag{5.2}$$

where

$$A''=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{i\pi(-b+n)}{2}\right)\cdot\Gamma\left(-b+|\beta|+\frac{n+1}{2}\right).$$

If the right hand side of (5.2) *is an integrable function in a neighborhood of* $\{|x|=1\}$, *then the distribution* $D^{\beta}K_{1,b}$ *belongs to* L^1. *If* $b-|\beta|=(n+1)/2$, *then the distribution* $D^{\beta}K_{1,b}$ *belongs to* L^1 *and*

$$\left(\frac{\partial}{\partial x}\right)^{\beta}K_{1,b}(x)=A'''\left(\frac{-ix}{|x|}\right)^{\beta}\log(1-|x|+i0)+O(1) \qquad as\ |x|\rightarrow 1,$$

where

$$A'''=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{i\pi(n+3)}{4}\right).$$

COROLLARY 5.1. (I) *Let* $0<a<1$ *or* $a>1$. *Then:*

(I-i) $D^{\beta}K_{a,b}\in H^p$, $0<p<\infty$, *iff* $(b-|\beta|-n+na/2)p>-n(1-a)$;

(I-ii) $D^{\beta}K_{a,b}\in L^1$ *iff* $b-|\beta|-n+na/2>-n(1-a)$;

(I-iii) $D^{\beta}K_{a,b}\in BMO$ *iff* $b-|\beta|\geqq n-na/2$;

(I-iv) $D^{\beta}K_{a,b}\in L^{\infty}$ *iff* $b-|\beta|\geqq n-na/2$, *i.e.,* $D^{\beta}K_{a,b}$ *belongs to* L^{∞} *at the same time as it belongs to* BMO;

(I-v) $D^{\beta}K_{a,b}\in\Lambda_s$, $s\in\boldsymbol{R}$, *iff* $b-|\beta|-s\geqq n-na/2$.

(II) *In the case* $a=1$, *we have:*

(II-i) $D^{\beta}K_{1,b}\in H^p$, $0<p<\infty$, *iff* $(b-|\beta|-(n+1)/2)p>-1$;

(II-ii) $D^{\beta}K_{1,b}\in L^1$ *iff* $b-|\beta|-(n+1)/2>-1$;

(II-iii) $D^{\beta}K_{1,b}\in BMO$ *iff* $b-|\beta|\geqq(n+1)/2$;

(II-iv) $D^{\beta}K_{1,b}\in L^{\infty}$ *iff* $b-|\beta|>(n+1)/2$;

(II-v) $\quad D^{\beta}K_{1,b}\in\Lambda_s,\ s\in\boldsymbol{R},\ \ \textit{iff}\ \ b-|\beta|-s\geqq(n+1)/2.$

REMARKS ON THE PROOF OF PROPOSITION 5.1 AND COROLLARY 5.1. In the case $\beta=0$, the estimates in (i) of Proposition 5.1 are proved by Wainger [22], Part II, pp. 41-52. We can derive the estimates for $\beta\neq 0$ from those for $\beta=0$. Expand the polynomial $(i\xi)^{\beta}$ as

$$(i\xi)^{\beta}=\sum_{j=1}^{k}|\xi|^{2j}P_j(\xi),$$

where $k=[|\beta|/2]$ and P_j is a homogeneous harmonic polynomial of degree $|\beta|-2j$ (see Stein-Weiss [18], p. 139), then we have

$$\left(\frac{\partial}{\partial x}\right)^{\beta}K_{a,b}(x)=\sum_{j=1}^{k}\mathcal{F}^{-1}(|\xi|^{2j}P_j(\xi)m_{a,b}(\xi)) \tag{5.3}$$

$$=\sum_{j=1}^{k}i^{|\beta|-2j}\mathcal{F}^{-1}_{n+2|\beta|-4j}(\tilde{m}_{a,b-2j})(|x|)P_j(x),$$

where $\tilde{m}_{a,b-2j}$ is the function $m_{a,b-2j}$ on $\boldsymbol{R}^{n+2|\beta|-4j}$ (we assume that the function $\phi(\xi)$ depends only on $|\xi|$), $\mathcal{F}^{-1}_{n+2|\beta|-4j}$ denotes the inverse Fourier transform on $\boldsymbol{R}^{n+2|\beta|-4j}$ and

$$\mathcal{F}^{-1}_{n+2|\beta|-4j}(\tilde{m}_{a,b-2j})(|x|)$$

shall be interpreted as the function $g(|x|)$, $x\in\boldsymbol{R}^n$, with g such that

$$g(|x|)=\mathcal{F}^{-1}_{n+2|\beta|-4j}(\tilde{m}_{a,b-2j})(x),\qquad x\in\boldsymbol{R}^{n+2|\beta|-4j},$$

(see Stein-Weiss [18], Theorem 3.10, p. 158). We apply the estimates for $\beta=0$ to each term of (5.3) and then obtain the desired estimate for $\beta\neq 0$. The result is that the main term in the asymptotic formula for $(\partial/\partial x)^{\beta}K_{a,b}(x)$ can be obtained by formally differentiating the asymptotic formula for $K_{a,b}(x)$.

In the case $\beta=0$, the asymptotic formula in (ii) of Proposition 5.1 can be found in [13], Lemma 4, p. 174. The case $\beta\neq 0$ can be reduced to the case $\beta=0$ by using the spherical harmonic expansion as above.

We shall show that $K_{a,b}$ is smooth throughout $\boldsymbol{R}^n$ if $a>1$. It will be sufficient to show that

$$\mathcal{F}^{-1}\left(\phi\left(\frac{\xi}{M}\right)f(\xi)\exp(i|\xi|^{a})\right) \tag{5.4}$$

is continuous in $\{|x|\leqq aM^{a-1}/(4\sqrt{n})\}$ for all smooth homogeneous function $f(\xi)$ and all $M\geqq 1$. Let $\{\varphi_j^{\pm}\,|\,j=1,2,\cdots,n\}$ be a partition of unity with the following properties:

$$\begin{cases} \varphi_j^{\pm}\text{'s are smooth homogeneous function of degree } 0\,, \\ \sum_{j=1}^{n}\varphi_j^{+}(\xi)+\sum_{j=1}^{n}\varphi_j^{-}(\xi)=1 \qquad \text{for } \xi\neq 0\,, \\ \text{support } \varphi_j^{\pm}\subset\{\pm\xi_j\geqq|\xi|/(2\sqrt{n})\}\,. \end{cases}$$

We decompose (5.4) by using this partition of unity:

$$\begin{aligned} &\mathcal{F}^{-1}\Big(\psi\Big(\frac{\xi}{M}\Big)f(\xi)\exp(i|\xi|^a)\Big) \\ &=\sum_{j=1}^{n}\mathcal{F}^{-1}\Big(\psi\Big(\frac{\xi}{M}\Big)\varphi_j^{+}(\xi)f(\xi)\exp(i|\xi|^a)\Big) \\ &\quad+\sum_{j=1}^{n}\mathcal{F}^{-1}\Big(\psi\Big(\frac{\xi}{M}\Big)\varphi_j^{-}(\xi)f(\xi)\exp(i|\xi|^a)\Big)\,. \end{aligned}$$

By a limiting argument it is easy to see that the following formal integration by parts is legitimate if $|x|\leqq aM^{a-1}/(4\sqrt{n})$:

$$\begin{aligned} &\mathcal{F}^{-1}\Big(\psi\Big(\frac{\xi}{M}\Big)\varphi_j^{+}(\xi)f(\xi)\exp(i|\xi|^a)\Big)(x) \\ &=(2\pi)^{-n/2}\int_{R^n}\psi\Big(\frac{\xi}{M}\Big)\varphi_j^{+}(\xi)f(\xi)\Big(\frac{1}{i(a|\xi|^{a-2}\xi_j+x_j)}\frac{\partial}{\partial\xi_j}\Big)^N\exp(i(|\xi|^a+\xi\cdot x))d\xi \\ &=(2\pi)^{-n/2}\int_{R^n}\Big[\Big(-\frac{\partial}{\partial\xi_j}\cdot\frac{1}{i(a|\xi|^{a-2}\xi_j+x_j)}\Big)^N\psi\Big(\frac{\xi}{M}\Big)\varphi_j^{+}(\xi)f(\xi)\Big] \\ &\quad\times\exp(i(|\xi|^a+\xi\cdot x))d\xi\,. \end{aligned}$$

The last integral is absolutely convergent and hence continuous in x if N is sufficiently large. Thus we showed that $K_{a,b}$ is smooth throughout $\boldsymbol{R}^n$ if $a>1$.

(iii) of Proposition 5.1 can be found in [14] (cf. Proposition 2 and its proof *loc. cit.*).

The estimates in Proposition 5.1 enables us to see when $D^{\beta}K_{a,b}$ and its Riesz transforms $R_{\alpha}D^{\beta}K_{a,b}(=(-1)^{|\alpha|}D^{\alpha+\beta}K_{a,b+|\alpha|})$ belong to L^p. Thus, by Theorem C, we can prove Corollary 5.1 except (I-iii, -v) and (II-iii, -v).

Proof of (I-iii) *and* (II-iii) *of Corollary* 5.1. It is known that, if $f\in BMO$, then $\exp(\delta|f(x)|)$ is locally integrable for sufficiently small $\delta>0$ (John-Nirenberg [10]) and $|f(x)|(1+|x|)^{-n-\varepsilon}\in L^1$ for every $\varepsilon>0$ (cf. Fefferman-Stein [8], pp. 141-142). But, from the asymptotic behavior of $D^{\beta}K_{a,b}$ given in Proposition 5.1, we see that

$$\int_{|x|\leqq 2}\exp(\delta|D^{\beta}K_{a,b}(x)|)dx=\infty \qquad \text{for all } \delta>0$$

if $0<a<1$, $b-|\beta|<n-na/2$ or $a=1$, $b-|\beta|<(n+1)/2$ and that

$$\int_{R^n}|D^\beta K_{a,b}(x)|(1+|x|)^{-n-\varepsilon}dx=\infty \qquad \text{for sufficiently small } \varepsilon>0$$

if $a>1$ and $b-|\beta|<n-na/2$. Hence $D^\beta K_{a,b}\notin BMO$ for those a, b and β. This proves the "only if" parts of (I-iii) and (II-iii). The "if" part of (I-iii) is evident since $D^\beta K_{a,b}\in L^\infty$ for a, b and β mentioned there. In order to show that $D^\beta K_{1,b}\in BMO$ for $b-|\beta|\geqq(n+1)/2$, it is sufficient to show that $K_{1,(n+1)/2}\in BMO$ since

$$D^\beta K_{1,b}=\mathcal{F}^{-1}\Big(\Big(i\frac{\xi}{|\xi|}\Big)^\beta|\xi|^{-b+|\beta|+(n+1)/2}m_{1,(n+1)/2}(\xi)\Big)$$

and

$$\phi(\xi)\Big(i\frac{\xi}{|\xi|}\Big)^\beta|\xi|^{-b+|\beta|+(n+1)/2}$$

is a Fourier multiplier for BMO if $-b+|\beta|+(n+1)/2\leqq 0$ (Theorem E). From (iii) of Proposition 5.1, we see that

$$K_{1,(n+1)/2}(x)=A\phi(x)\log|1-|x||+\theta(x)\,,$$

where ϕ is the characteristic function of the ball $\{|x|\leqq 2\}$ and $\theta\in L^\infty$. Thus we can claim that $K_{1,(n+1)/2}\in BMO$ once we show that

$$\phi(x)\log|1-|x||\in BMO\,.$$

But this can be shown by slightly modifying the calculations given by John and Nirenberg in [10], pp. 416-417, where it is shown that $\log|x|\in BMO$.

Proof of (I-v) *and* (II-v) *of Corollary* 5.1. The "if" parts are derived from (I-iii) and (II-iii) by using the fact that $\phi(\xi)|\xi|^{-s}\in\mathcal{M}(BMO,\Lambda_s)$. (This fact can be seen from Theorems D and F by using (i) of Theorem 3.2 and Lemma 2.1.) In order to prove the "only if" parts, we utilize the fact that $\phi(\xi)|\xi|^t\in\mathcal{M}(\Lambda_s,\Lambda_{s-t})$ (Theorem I). Using this fact, we see that $D^\beta K_{a,b}\in\Lambda_s$ implies that $D^\beta K_{a,b-t}\in\Lambda_{s-t}$ and hence that $D^\beta K_{a,b-t}\in L^\infty$ for all $t<s$, which is possible only when $a>0$, $a\neq 1$ and $b-s-|\beta|\geqq n-na/2$ or $a=1$ and $b-s-|\beta|\geqq(n+1)/2$. This proves the "only if" parts of (I-v) and (II-v). It is also possible to prove the "only if" parts of (I-v) and (II-v) by direct calculations. For example, let $a>1$ and $0<s<1$. The asymptotic formula in (ii) of Proposition 5.1 gives

$$K_{a,b}\Big(\Big(\frac{2\pi m+\pi}{B}\Big)^{(a-1)/a},0,\cdots,0\Big)-K_{a,b}\Big(\Big(\frac{2\pi m}{B}\Big)^{(a-1)/a},0,\cdots,0\Big)$$

$$=-2A'\Big(\frac{2\pi m}{B}\Big)^{(-b+n-na/2)/a}+o(m^{(-b+n-na/2)/a}) \qquad \text{as } m\to\infty\,.$$

On the other hand

$$\left(\frac{2\pi m+\pi}{B}\right)^{(a-1)/a}-\left(\frac{2\pi m}{B}\right)^{(a-1)/a}\sim\frac{a-1}{a}\left(\frac{2\pi m}{B}\right)^{-1/a}\frac{\pi}{B}\qquad\text{as } m\to\infty\,.$$

Hence, $K_{a,b}\in\Lambda_s$ implies that $m^{(-b+n-na/2)/a}=O(m^{-s/a})$ as $m\to\infty$ or $-b+n-na/2\leqq -s$.

Now we go to the

PROOF OF THEOREMS 4.1, 4.2 AND 4.3. The following results are direct consequences of Corollary 5.1 (cf. Theorem 3.3): (I-ii, -iii, -iv), (II-iii, -iv) and (III-iii, -iv) of Theorems 4.1 and 4.2; (I-iv) and (II-iv) of Theorem 4.3. The other results are, via interpolation and duality, reduced to Propositions 5.2, 5.3 and the counter examples which we shall give below. The duality relations are given in Theorem 3.2. As for interpolation theorems, see Calderón-Torchinsky [3], pp. 135-162, and Bergh-Löfström [1], § 6.4, pp. 152-153. We remark that as for the interpolation between Λ_s-spaces all we have to use is the particular simple result $[L^\infty, \Lambda_s]_\theta\subset\Lambda_{s\theta}$.

Before we state Propositions 5.2, 5.3 and the counter examples, we add two remarks. The first remark concerns with the space Λ_s. By Theorem D, the results for Λ_s, $s\leqq 0$, can be reduced to those for Λ_s, $s>0$. If $s>0$, then, by Lemma 2.1, it holds that $m\in\mathcal{M}(X, \Lambda_s)$ if and only if $m\in\mathcal{M}(X, \tilde{\Lambda}_s)$ and $\hat{\varphi}(\xi)m(\xi)\in\mathcal{M}(X, L^\infty)$, where φ is as mentioned in the lemma. But $m=m_{a,b}$ obviously satisfies the latter condition if $X=H^p$ $(0<p<\infty)$, L^1, L^∞ or BMO since $\mathcal{F}^{-1}(\hat{\varphi}(\xi)m_{a,b}(\xi))\in\mathcal{S}\cap H^1$. Hence, for those X, $m_{a,b}\in\mathcal{M}(X, \Lambda_s)$ if and only if $m_{a,b}\in\mathcal{M}(X, \tilde{\Lambda}_s)$. Thus we can utilize the duality relations of Theorem 3.2, which are given for the $\tilde{\Lambda}_s$-spaces, to obtain the results for the Λ_s-spaces. The second remark concerns with the space L^∞_k. Let $X=H^p$, $0<p<\infty$, or BMO and consider the class $\mathcal{M}(X, L^\infty_k)$. Set

$$\langle\xi\rangle=(1-\psi(\xi))+\psi(\xi)|\xi|\,.$$

If $\langle\xi\rangle^k m(\xi)\in\mathcal{M}(X, L^\infty)$, then $\xi^\beta m(\xi)\in\mathcal{M}(X, L^\infty)$ for all $|\beta|\leqq k$ since $\xi^\beta/\langle\xi\rangle^k\in\mathcal{M}(X, X)$, $|\beta|\leqq k$, by Theorem E. Conversely, if $\xi^\beta m(\xi)\in\mathcal{M}(X, L^\infty)$ for all $|\beta|\leqq k$, then $\langle\xi\rangle^k m(\xi)\in\mathcal{M}(X, L^\infty)$, which can be seen from decomposing $\langle\xi\rangle^k m(\xi)$ as

$$\langle\xi\rangle^k m(\xi)=(1-\psi(\xi))\phi(\xi)m(\xi)+\psi(\xi)^k\sum_{|\beta|=k}f_\beta(\xi)\xi^\beta m(\xi)\,,$$

where f_β's are smooth homogeneous functions of degree 0 such that

$$\sum_{|\beta|=k}f_\beta(\xi)\xi^\beta=|\xi|^k$$

and ϕ is a smooth function ($(1-\psi)\phi$ and $\psi^k f_\beta\in\mathcal{M}(X, X)$ by Theorem E). Thus, for $X=H^p$, $0<p<\infty$, or BMO, it holds that $m\in\mathcal{M}(X, L^\infty_k)$ if and only if $\langle\xi\rangle^k m(\xi)\in\mathcal{M}(X, L^\infty)$. This fact will simplify the proof of the results for L^∞_k. (Note that this remark holds also for $X=\Lambda_s$ but not for $X=L^1$ or L^∞.)

Now we shall continue the proof of the remainder parts of Theorems 4.1~4.3. The "if" parts are obtained by interpolation and duality from the key results of the following three types:

$$m_{a,b}\in\mathcal{M}(H^{p_0}, H^{p_0}), \qquad 0<p_0<2,$$
$$m_{a,b}\in\mathcal{M}(L^{p_1}, L^{p_1'}), \qquad 1<p_1<2,\ 1/p_1+1/p_1'=1,$$
$$m_{a,b}\in\mathcal{M}(H^{p_2}, L^\infty), \qquad 0<p_2\leqq 1,$$

which are given in the following propositions.

PROPOSITION 5.2. (i) *Let* $0<a<1$ *or* $a>1$. *If* $b>0$ *and* $1/p_0-1/2=b/(na)$, *then* $m_{a,b}\in\mathcal{M}(H^{p_0}, H^{p_0})$.

(ii) *The case* $a=1$: *if* $n\geqq 2$, $b>0$ *and* $1/p_0-1/2=b/(n-1)$, *then* $m_{1,b}\in\mathcal{M}(H^{p_0}, H^{p_0})$; *if* $n=1$ *and* $b\geqq 0$, *then* $m_{1,b}\in\mathcal{M}(H^p, H^p)$ *for all* $p>0$.

PROPOSITION 5.3. (I) *Let* $0<a<1$ *or* $1<a<2$ *or* $a>2$. *Then*: (I-i) *if* $0\leqq b\leqq n-na/2$ (*when* $0<a<1$ *or* $1<a<2$) *or* $0\geqq b\geqq n-na/2$ (*when* $a>2$) *and* $1/p_1=1/2+b/(2n-na)$, $1/p_1'=1/2-b/(2n-na)$, *then* $m_{a,b}\in\mathcal{M}(L^{p_1}, L^{p_1'})$; (I-ii) *if* $b>n-na/2$ *and* $1/p_2=b/n+a/2$, *then* $m_{a,b}\in\mathcal{M}(H^{p_2}, L^\infty)$.

(II) *The case* $a=1$: (II-i) *if* $0\leqq b<(n+1)/2$, $1/p_1=1/2+b/(n+1)$ *and* $1/p_1'=1/2-b/(n+1)$, *then* $m_{1,b}\in\mathcal{M}(L^{p_1}, L^{p_1'})$; (II-ii) *if* $b\geqq(n+1)/2$ *and* $1/p_2=b/n+(n-1)/2n$, *then* $m_{1,b}\in\mathcal{M}(H^{p_2}, L^\infty)$.

(III) *The case* $a=2$: (III-i) $m_{2,0}\in\mathcal{M}(L^2, L^2)\cap\mathcal{M}(L^1, L^\infty)$ *and hence* $m_{2,0}\in\mathcal{M}(L^p, L^{p'})$ *for* $1\leqq p\leqq 2$ *and* $1/p'=1-1/p$; (III-ii) *if* $b>0$ *and* $1/p_2=b/n+1$, *then* $m_{2,b}\in\mathcal{M}(H^{p_2}, L^\infty)$.

(i) of Proposition 5.2 is due to Fefferman and Stein [8] (the case $0<a<1$ and $p_0\geqq 1$), Coifman [6] (the case $0<a<1$, $p_0<1$ and $n=1$), Miyachi [13] (the case $0<a<1$, $p_0<1$ and $n\geqq 2$ and the case $a>1$) and Sjölin [17] (the case $0<a<1$, $p_0<1$ and $n\geqq 2$). (ii) of Proposition 5.2 is due to Peral [15] ($p_0\geqq 1$) and Miyachi [14]. As for the proof of Proposition 5.2, see [13] and [14].

PROOF OF PROPOSITION 5.3. We shall give the proof of (II). (I) and (III) can be proved in a similar way. Note that $m_{1,0}\in\mathcal{M}(L^2, L^2)$ and $m_{1,(n+1)/2}\in\mathcal{M}(H^1, L^\infty)$, the latter of which is seen from the fact that $K_{1,(n+1)/2}\in BMO$ ((II-iii) of Corollary 5.1). Consider the following family of Fourier multipliers:

$$m(z;\xi)=\phi(\xi)|\xi|^{-(n+1)z/2}e^{i|\xi|}, \qquad 0\leqq \operatorname{Re} z\leqq 1.$$

From the above results for $m_{1,0}$ and $m_{1,(n+1)/2}$ and the fact that, for $y\in\boldsymbol{R}$,

$$|\xi|^{iy}\in\mathcal{M}(L^2, L^2), \qquad \||\xi|^{iy}\|_{\mathcal{M}(L^2,L^2)}=1$$

and

$$|\xi|^{iy}\in\mathcal{M}(H^1, H^1), \qquad \||\xi|^{iy}\|_{\mathcal{M}(H^1,H^1)}\leqq C(1+|y|)^{n/2},$$

the latter of which is shown by Theorem E, we see that, for $y\in\boldsymbol{R}$,

$$m(iy\,;\,\cdot)\in\mathcal{M}(L^2, L^2), \qquad \|m(iy\,;\,\cdot)\|_{\mathcal{M}(L^2,L^2)}\leqq C$$

and

$$m(1+iy\,;\,\cdot)\in\mathcal{M}(H^1, H^1), \qquad \|m(1+iy\,;\,\cdot)\|_{\mathcal{M}(H^1,H^1)}\leqq C(1+|y|)^{n/2}.$$

Then applying the complex interpolation theorem (Calderón-Torchinsky [3], Theorem 3.4, pp. 151-152), we obtain

$$m(t\,;\,\cdot)\in\mathcal{M}(L^{p(t)}, L^{p(t)'}),$$

where $0<t<1$, $1/p(t)=(1-t)/2+t/1$ and $1/p(t)'=1-1/p(t)$. This proves (II-i).

In order to prove (II-ii), we rewrite $m_{1,b}$ as

$$m_{1,b}(\xi)=m_{1,(n+1)/2}(\xi)|\xi|^{-b+(n+1)/2}.$$

We have $m_{1,(n+1)/2}\in\mathcal{M}(H^1, L^\infty)$. On the other hand, if $b>(n+1)/2$ and p_2 is defined as in (II-ii), then $|\xi|^{-b+(n+1)/2}\in\mathcal{M}(H^{p_2}, H^1)$ (Theorem F). Hence $m_{1,b}\in\mathcal{M}(H^{p_2}, L^\infty)$. This completes the proof of Proposition 5.3.

REMARK 5.1. The result (II-i) of Proposition 5.3 is due to Strichartz [19]. The proof given in [19] uses L^1-L^∞ estimate but does not use L^1-BMO estimate; as a result it needs delicate formulae of Bessel functions and is not easily generalized. Our proof given above which is based on L^1-BMO estimate permits some generalizations as we mentioned in Remark 4.1. Brenner [2] gave another elegant proof of (II-i) of Proposition 5.3 and also obtained some generalizations. The proof of Brenner is based on the Littlewood-Paley theorem (Theorem A).

Now we have established the "if" parts of Theorems 4.1, 4.2 and 4.3. We shall defer giving the counter examples until the end of this section and set about the proof of Theorems 4.4, 4.5 and Corollary 4.1.

PROOF OF THEOREM 4.4. The "if" part of (i) of Theorem 4.4 can be proved by applying the following theorem which is a variant of Theorem I.

THEOREM 5.1. *Let $a>0$ and $t\in\boldsymbol{R}$. Suppose that $A\geqq1$, m is a function of class $C^{[n/2]+1}$ on $\boldsymbol{R}^n$, $m(\xi)=0$ for $|\xi|\leqq2$ and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha m(\xi)\right|\leqq|\xi|^{-t-na/2}(A|\xi|^{a-1})^{|\alpha|} \qquad \textit{for } |\alpha|\leqq[n/2]+1$$

Then $m\in\mathcal{M}(\Lambda_s, \Lambda_{s+t})$ and

$$\|m\|_{\mathcal{M}(\Lambda_s,\Lambda_{s+t})}\leqq CA^{n/2},$$

where the constant C depends only on a, s, t and n.

PROOF. By Theorem H, all we have to show is that

$$\|\mathcal{F}^{-1}m\|_{\Lambda(t;1,\infty)} \leqq CA^{n/2},$$

or, according to the definition of $\Lambda(t;1,\infty)$ (see § 6.1), that

$$\|(\mathcal{F}^{-1}m)*\theta\|_{L^1} \leqq CA^{n/2} \tag{5.5}$$

and

$$\sup_{0<r<1}\{r^{-t}\|\chi(\cdot\,|r)*\mathcal{F}^{-1}m\|_{L^1}\} \leqq CA^{n/2} \tag{5.6}$$

(as for the functions θ and χ, see § 6.1). (5.5) is obvious since $(\mathcal{F}^{-1}m)*\theta=0$. We shall prove (5.6). We have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}(m(\xi)\hat{\chi}(r\xi))\right| \leqq Cr^{t+na/2}(Ar^{-a+1})^{|\alpha|}, \qquad |\alpha| \leqq [n/2]+1,$$

and hence

$$\left\|\left(A^{-1}r^{a-1}\frac{\partial}{\partial\xi}\right)^{\alpha}(m(\xi)\hat{\chi}(r\xi))\right\|_{L^2} \leqq Cr^{t+na/2-n/2}, \qquad |\alpha| \leqq [n/2]+1,$$

from which we obtain

$$\begin{aligned}\|\chi(\cdot\,|r)*\mathcal{F}^{-1}m\|_{L^1} &\leqq C(A^{-1}r^{a-1})^{-n/2}r^{t+na/2-n/2}\\ &= CA^{n/2}r^{t};\end{aligned}$$

this is the desired inequality (5.6). (In the above reasoning we used the inequality

$$\|\mathcal{F}^{-1}f\|_{L^1} \leqq C\inf_{\rho>0}\left\{\rho^{-n/2}\left(\|f\|_{L^2}+\sum_{|\alpha|=[n/2]+1}\left\|\left(\rho\frac{\partial}{\partial\xi}\right)^{\alpha}f(\xi)\right\|_{L^2}\right)\right\},$$

which can be derived from the inequality

$$\|\mathcal{F}^{-1}f\|_{L^1} \leqq C\inf_{\rho>0}\{\rho^{-n/2}\|(1+\rho|x|)^{[n/2]+1}(\mathcal{F}^{-1}f)(x)\|_{L^2}\}$$

by using Plancherel's theorem.) This completes the proof of Theorem 5.1 and at the same time the proof of the "if" part of (i) of Theorem 4.4.

Although Theorem 5.1 is sharp in itself, when we apply the case $a=1$ of this theorem to $m_{1,b}(\xi)$, we cannot obtain the sharp result stated in (ii) of Theorem 4.4. In order to obtain the sharp result, we shall directly deal with the kernel $K_{1,b}=\mathcal{F}^{-1}m_{1,b}$. We shall show that

$$K_{1,(n-1)/2} \in \Lambda(0;1,\infty); \tag{5.7}$$

this implies that $m_{1,(n-1)/2} \in \mathcal{M}(\Lambda_s, \Lambda_s)$ (Theorem H), which together with the fact that $\psi(\xi)|\xi|^{-t}\in\mathcal{M}(\Lambda_s, \Lambda_{s+t})$ (Theorem I) gives the "if" part of (ii) of Theorem 4.4. Take two functions θ and φ such that $\theta\in\mathcal{S}$, $\hat{\theta}(0)\neq0$, $\varphi\in\mathcal{D}$, support $\varphi\subset\{|x|\leqq1\}$, $\hat{\varphi}(0)=0$ and $\hat{\varphi}(\xi)\neq0$ for $\xi\neq0$. In order to prove (5.7), it is sufficient to show that $\theta*K_{1,(n-1)/2}\in L^1$ and

$$\sup_{0<r<1}\{\|\varphi(\cdot|r)*K_{1,(n-1)/2}\|_{L^1}\}<\infty \tag{5.8}$$

(cf. Proposition 6.1 in §6.1). The fact that $\theta*K_{1,(n-1)/2}\in L^1$ is obvious since $\theta*K_{1,(n-1)/2}\in\mathcal{S}$. We shall prove (5.8). We shall abbreviate $K_{1,(n-1)/2}$ to K and $\varphi(\cdot|r)$ to φ_r.

Recall that K is smooth in $\boldsymbol{R}^n\setminus\{|x|=1\}$ and has the following estimates:

$$K(x)=O(|x|^{-N}) \qquad \text{for every } N \text{ as } |x|\to\infty$$

and

$$\operatorname{grad} K(x)=O((1-|x|)^{-2}) \qquad \text{as } |x|\to1$$

(see Proposition 5.1). Hence we have the following estimate uniformly in $0<r<1$: if $|x|\geqq3$, then

$$|(\varphi_r*K)(x)|\leqq C\sup_{|y|\leqq r}\{|K(x-y)|\}\leqq C|x|^{-n-1};$$

if $|x|\leqq3$ and $|1-|x||\geqq2r$, then

$$\begin{aligned}|(\varphi_r*K)(x)|&=\left|\int_{|y|\leqq r}(K(x-y)-K(x))\varphi_r(y)dy\right|\\&\leqq C\sup_{|y|\leqq r}\{|K(x-y)-K(x)|\}\\&\leqq Cr(1-|x|)^{-2}.\end{aligned}$$

Thus we have

$$\int_{|x|\geqq3}|\varphi_r*K(x)|dx\leqq C\int_{|x|\geqq3}|x|^{-n-1}dx=C$$

and

$$\int_{\substack{|x|\leqq3\\|1-|x||\geqq2r}}|\varphi_r*K(x)|dx\leqq Cr\int_{\substack{|x|\leqq3\\|1-|x||\geqq2r}}(1-|x|)^{-2}dx\leqq C$$

with C independent of r, $0<r<1$. Thus the rest of the proof is to show that

$$\sup_{0<r<1}\left\{\int_{|1-|x||\leqq2r}|\varphi_r*K(x)|dx\right\}<\infty.$$

By Plancherel's theorem, we have the following estimates:

$$\begin{aligned}\|\varphi_r*K\|_{L^2}^2&=\int\left|\psi(\xi)|\xi|^{-(n-1)/2}e^{i|\xi|}\hat{\varphi}(r\xi)\right|^2d\xi\\&\leqq C\int_{|\xi|\geqq1}|\xi|^{-n+1}(1+r|\xi|)^{-2}d\xi\\&\leqq Cr^{-1}.\end{aligned}$$

Hence Schwarz' inequality gives

$$\int_{|1-|x||\leq 2r} |\varphi_r * K(x)|\,dx \leq \left(\int_{|1-|x||\leq 2r} dx\right)^{1/2} \|\varphi_r * K\|_{L^2}$$

$$\leq C r^{1/2} r^{-1/2} = C\,.$$

This completes the proof of (5.8).

We have completed the proof of the "if" parts of Theorem 4.4. The "only if" parts of the theorem can be proved by the counter examples given at the end of this section.

Proof of Theorem 4.5 is carried out by applying the following theorem which is an improvement of Theorem G.

THEOREM 5.2. *If m satisfies the assumptions of* (i) *or* (ii) *of Theorem G, then*

$$\left(\sum_{j=-\infty}^{\infty} \int_{R^n} |(\chi(\cdot\,|2^{-j}) * (\mathcal{F}^{-1}m) * f)(x)|^p dx\right)^{1/p} \leq C A^{n(1/p-1/2)} \|f\|_{H^p}\,,$$

where χ is a function in $\mathcal{S}$ satisfying (2.1).

PROOF. It is sufficient to prove the theorem in the case $0<p<1$ since the case $1\leq p<2$ is reduced to the case $0<p<1$ by the complex interpolation method; cf. § 6.6. We shall present here the proof under the assumption that m satisfies the conditions of (i) of Theorem G; the proof can be performed in a similar way when m satisfies the conditions of (ii). We shall abbreviate $\chi(\cdot\,|2^{-j}) * (\mathcal{F}^{-1}m) * f$ to $T_j f$.

By Theorem B, it is sufficient to show that

$$\sum_{j=0}^{\infty} \|T_j f\|_{L^p}^p \leq C A^{np(1/p-1/2)} \qquad \text{for } p\text{-atom } f\,.$$

Since the operator T_j commutes with translations, we may suppose that the p-atoms are centered at the origin. Hence it is sufficient to show that

$$\sum_{j=0}^{\infty} \|T_j f\|_{L^p}^p \leq C A^{np(1/p-1/2)}\,, \qquad f\in \mathcal{A}_r,\ 0<r<\infty\,,$$

where $\mathcal{A}_r$, $0<r<\infty$, is the set of all f's such that

$$\text{support } f \subset \{|x|\leq r\}\,, \qquad \|f\|_{L^\infty} \leq r^{-n/p}$$

and

$$\int f(x) x^\alpha dx = 0 \qquad \text{for } |\alpha| \leq [n/p-n]\,.$$

In [13], it is shown that

$$\sum_{j=0}^{\infty} \|T_j f\|_{L^p(|x|\geq 2r)}^p \leq C A^{np(1/p-1/2)}\,, \qquad f\in \mathcal{A}_r,\ 0<r<\infty\,.$$

Hence all we have to show is the following estimate:

$$\sum_{j=0}^{\infty}\|T_jf\|_{L^p(|x|<2r)}^p \leqq CA^{np(1/p-1/2)}, \qquad f\in \mathcal{A}_r,\ 0<r<\infty. \tag{5.9}$$

Hölder's inequality and Plancherel's theorem gives

$$\begin{aligned}\|T_jf\|_{L^p(|x|<2r)} &\leqq Cr^{n(1/p-1/2)}\|T_jf\|_{L^2}\\ &=Cr^{n(1/p-1/2)}\Big(\int|\hat{\chi}(2^{-j}\xi)|^2|\xi|^{-2b}|\hat{f}(\xi)|^2d\xi\Big)^{1/2}\\ &\leqq Cr^{n(1/p-1/2)}2^{-jb}\Big(\int|\hat{\chi}(2^{-j}\xi)|^2|\hat{f}(\xi)|^2d\xi\Big)^{1/2}.\end{aligned}$$

Hence, by Hölder's inequality, we obtain

$$\begin{aligned}&\sum_{j=0}^{\infty}\|T_jf\|_{L^p(|x|<2r)}^p\\ &\leqq Cr^{np(1/p-1/2)}\Big(\sum_{j=0}^{\infty}2^{-2jbp/(2-p)}\Big)^{(2-p)/2}\Big(\sum_{j=0}^{\infty}\int|\hat{\chi}(2^{-j}\xi)|^2|\hat{f}(\xi)|^2d\xi\Big)^{p/2}\\ &\leqq Cr^{np(1/p-1/2)}\Big(\int_{R^n}|\hat{f}(\xi)|^2d\xi\Big)^{p/2}\\ &=C(r^{n(1/p-1/2)}\|f\|_{L^2})^p\end{aligned}$$

and hence

$$\sum_{j=0}^{\infty}\|T_jf\|_{L^p(|x|<2r)}^p\leqq C \qquad \text{for } f\in\mathcal{A}_r$$

since $\|f\|_{L^2}\leqq Cr^{-n/p+n/2}$ for $f\in\mathcal{A}_r$. Thus we have proved (5.9) and hence completed the proof of Theorem 5.2.

PROOF OF COROLLARY 4.1. The "if" part of (i) is the statement dual to Theorem 4.5 (the case $p=1$). To prove it, observe the following inequality:

$$\Big|\int f(x)g(x)dx\Big|\leqq C\|f\|_{\Lambda(0;\infty,\infty)}\|g\|_{\Lambda(0;1,1)}, \qquad f,\ g\in L^2, \tag{5.10}$$

where

$$\|g\|_{\Lambda(0;1,1)}=\|\theta*g\|_{L^1}+\sum_{k=1}^{\infty}\|\chi(\cdot|2^{-k})*g\|_{L^1}$$

(θ and χ are the functions used in the definition of $\Lambda(s;p,\infty)$; see § 6.1). (5.10) is derived from the following equality:

$$\int f(x)g(x)dx=\langle\theta*f,\ \theta_1*g\rangle+\sum_{k=1}^{\infty}\langle\chi(\cdot|2^{-k})*f,\ \chi_1(\cdot|2^{-k})*g\rangle,$$

where

$$\theta_1=\theta+\chi(\cdot|2^{-1}) \qquad \text{and} \qquad \chi_1=\chi(\cdot|2)+\chi+\chi(\cdot|2^{-1}).$$

Theorem 4.5 (the case $p=1$) shows the estimate

$$\|K_{a,na/2}*g\|_{\Lambda(0;1,1)}\leqq C\|g\|_{H^1}, \tag{5.11}$$

where $K_{a,na/2}=\mathcal{F}^{-1}m_{a,na/2}$. Using (2.5), (5.10) and (5.11), we obtain

$$\begin{aligned}\|K_{a,na/2}*f\|_{BMO}&\approx\sup\{|\langle K_{a,na/2}*f,\ g\rangle|\,\big|\,g\in\mathcal{D}\cap H^1,\ \|g\|_{H^1}\leqq 1\}\\&=\sup\{|\langle f,\ K_{a,na/2}*g\rangle|\,\big|\,g\in\mathcal{D}\cap H^1,\ \|g\|_{H^1}\leqq 1\}\\&\leqq C\|f\|_{\Lambda(0;\infty,\infty)}.\end{aligned}$$

Thus $m_{a,na/2}\in\mathcal{M}(\Lambda_0,\ BMO)$, which, together with Theorems D and I, proves the "if" part of (i).

The "if" part of (ii) is contained in Theorem 4.4 since $\Lambda_{k+\varepsilon}\subset L_k^\infty$, $\varepsilon>0$. Thus we have proved the "if" parts of Corollary 4.1. The "only if" parts are proved by the counter examples given below. (Note that the proof of (ii) is reduced to that for the case $k=0$ as we remarked in the proof of Theorems 4.1～4.3.)

We shall give some counter examples. To state them, we use the following notation:

$$f_\lambda=\mathcal{F}^{-1}(\phi(\xi)|\xi|^{-\lambda}), \qquad \lambda\in\boldsymbol{R}$$

and

$$\tilde{K}_{a,b}=\mathcal{F}^{-1}(\phi(\xi)|\xi|^{-b}\exp(-i|\xi|^a)), \qquad a>0,\ b\in\boldsymbol{R}.$$

COUNTER EXAMPLES TO THEOREMS 4.1, 4.2 AND 4.3. *We assume that* $0<p<\infty$, $0<q\leqq\infty$ *and* $s\in\boldsymbol{R}$.

(I) *The case* $0<a<1$ *or* $a>1$.

(I-i) *Suppose that* $(1-a)/p-1/q>b/n-a/2$; *take* λ *such that* $\lambda-n+na/2>-n(1-a)/p$ *and* $-\lambda-b>n/q-n$. *Then* $\tilde{K}_{a,\lambda}\in H^p$ *but* $K_{a,b}*\tilde{K}_{a,\lambda}\notin H^q$ (*when* $0<q<\infty$) *or* $K_{a,b}*\tilde{K}_{a,\lambda}\notin BMO$ (*when* $q=\infty$).

(I-ii) *Suppose that* $1/p-(1-a)/q>b/n+a/2$; *take* λ *such that* $-\lambda<n/p-n$ *and* $b+\lambda-n+na/2<-n(1-a)/q$. *Then* $f_\lambda\in H^p$ *but* $K_{a,b}*f_\lambda\notin H^q$ (*when* $0<q<\infty$) *or* $K_{a,b}*f_\lambda\notin BMO$ (*when* $q=\infty$).

(I-iii) *Suppose that* $(1-a)/p>(b-s)/n-a/2$; *take* λ *such that* $\lambda-n+na/2>-n(1-a)/p$ *and* $b+\lambda<n+s$. *Then* $\tilde{K}_{a,\lambda}\in H^p$ *but* $K_{a,b}*\tilde{K}_{a,\lambda}\notin\Lambda_s$.

(I-iv) *Suppose that* $1/p>(b-s)/n+a/2$; *take* λ *such that* $-\lambda<n/p-n$ *and* $b+\lambda-s<n-na/2$. *Then* $f_\lambda\in H^p$ *but* $K_{a,b}*f_\lambda\notin\Lambda_s$.

(II) *The case* $a=1$.

(II-i) *Suppose that* $1/p-n/q>b-(n-1)/2$; *take* λ *such that* $\lambda-(n+1)/2>-1/p$ *and* $-\lambda-b>n/q-n$. *Then* $\tilde{K}_{1,\lambda}\in H^p$ *but* $K_{1,b}*\tilde{K}_{1,\lambda}\notin H^q$ (*when* $0<q<\infty$) *or* $K_{1,b}*\tilde{K}_{1,\lambda}\notin BMO$ (*when* $q=\infty$).

(II-ii) *Suppose that* $n/p-1/q>b+(n-1)/2$; *take* λ *such that* $-\lambda<n/p-n$ *and* $b+\lambda-(n+1)/2<-1/q$. *Then* $f_\lambda\in H^p$ *but* $K_{1,b}*f_\lambda\notin H^q$ (*when* $0<q<\infty$) *or* $K_{1,b}*f_\lambda\notin BMO$ (*when* $q=\infty$).

(II-iii) *Suppose that* $1/p>b-s-(n-1)/2$; *take* λ *such that* $\lambda-(n+1)/2>-1/p$ *and* $b+\lambda<n+s$. *Then* $\tilde{K}_{1,\lambda}\in H^p$ *but* $K_{1,b}*\tilde{K}_{1,\lambda}\notin \Lambda_s$.

(II-iv) *Suppose that* $n/p>b-s+(n-1)/2$; *take* λ *such that* $-\lambda<n/p-n$ *and* $b+\lambda-s<(n+1)/2$. *Then* $f_\lambda\in H^p$ *but* $K_{1,b}*f_\lambda\notin \Lambda_s$.

COUNTER EXAMPLES TO THEOREM 4.4 AND COROLLARY 4.1.

(I) *The case* $0<a<1$ *or* $a>1$.

(I-i) *Suppose that* $t-s>b-na/2$; *take* λ *such that* $\lambda-s\geqq n-na/2$ *and* $b+\lambda<n+t$. *Then* $\tilde{K}_{a,\lambda}\in\Lambda_s$ *but* $K_{a,b}*\tilde{K}_{a,\lambda}\notin\Lambda_t$.

(I-ii) *Suppose that* $-s>b-na/2$; *take* λ *such that* $\lambda-s\geqq n-na/2$ *and* $b+\lambda<n$. *Then* $\tilde{K}_{a,\lambda}\in\Lambda_s$ *but* $K_{a,b}*\tilde{K}_{a,\lambda}\notin BMO$.

(I-iii) *Suppose that* $-s=b-na/2$; *take* λ *such that* $\lambda-s=n-na/2$ *and* $b+\lambda=n$. *Then* $\tilde{K}_{a,\lambda}\in\Lambda_s$ *but* $K_{a,b}*\tilde{K}_{a,\lambda}\notin L^\infty$.

(II) *The case* $a=1$.

(II-i) *Suppose that* $t-s>b-(n-1)/2$; *take* λ *such that* $\lambda-s\geqq(n+1)/2$ *and* $b+\lambda<n+t$. *Then* $\tilde{K}_{1,\lambda}\in\Lambda_s$ *but* $K_{1,b}*\tilde{K}_{1,\lambda}\notin\Lambda_t$.

(II-ii) *Suppose that* $-s>b-(n-1)/2$; *take* λ *such that* $\lambda-s\geqq(n+1)/2$ *and* $b+\lambda<n$. *Then* $\tilde{K}_{1,\lambda}\in\Lambda_s$ *but* $K_{1,b}*\tilde{K}_{1,\lambda}\notin BMO$.

(II-iii) *Suppose that* $-s=b-(n-1)/2$; *take* λ *such that* $\lambda-s=(n+1)/2$ *and* $b+\lambda=n$. *Then* $\tilde{K}_{1,\lambda}\in\Lambda_s$ *but* $K_{1,b}*\tilde{K}_{1,\lambda}\notin L^\infty$.

The facts in these counter examples are seen from Corollary 5.1 and the following

LEMMA 5.1. $f_\lambda\in H^p$, $0<p<\infty$, *if and only if* $-\lambda<n/p-n$. $f_\lambda\in L^\infty$ *if and only if* $\lambda>n$. $f_\lambda\in BMO$ *if and only if* $\lambda\geqq n$. $f_\lambda\in\Lambda_s$ *if and only if* $\lambda\geqq n+s$.

Lemma 5.1 can be shown by using the following

LEMMA 5.2. f_λ, $\lambda\in\boldsymbol{R}$, *is a smooth function on* $\boldsymbol{R}^n\setminus\{0\}$. $f_\lambda(x)$ *and all of its derivatives are rapidly decreasing as* $|x|\to\infty$. *If* $0<\lambda<n$, *then*

$$f_\lambda(x)=2^{-\lambda+n/2}\frac{\Gamma((n-\lambda)/2)}{\Gamma(\lambda/2)}|x|^{\lambda-n}+(\textit{smooth function});$$

and

$$f_n(x)=\frac{2^{1-n/2}}{\Gamma(n/2)}\log\frac{1}{|x|}+(\textit{smooth function}).$$

As for Lemma 5.2, see Schwartz [16], § 7 of Chapter 7.

We have now completed the proof of the results of § 4.

§ 6. Appendix.

6.1. *The Lipschitz space or the Besov space.*

We shall define the *Lipschitz space* $\Lambda(s;p,q)$ following Bergh-Löfström [1], Chapter 6, § 6.2, pp. 139–141 (originally it is due to Peetre); we shall use the notation $\Lambda(s;p,q)$ while the notation in [1] is $B^s_{p,q}$ and it is called the *Besov space*. We shall be confined to the case $1\leqq p\leqq\infty$ and $q=\infty$. Let χ be a function in $\mathcal{S}$ satisfying (2.1) and define $\theta\in\mathcal{S}$ by

$$\hat{\theta}(\xi)=\begin{cases}1 & \text{if } \xi=0\,,\\ \sum_{k=0}^{\infty}\hat{\chi}(2^k\xi) & \text{if } \xi\neq 0\,.\end{cases}$$

DEFINITION 6.1. Let $s\in\boldsymbol{R}$ and $1\leqq p\leqq\infty$. For $f\in\mathcal{S}'$, we define $\|f\|_{\Lambda(s;p,\infty)}$ as

$$\|f\|_{\Lambda(s;p,\infty)}=\|\theta*f\|_{L^p}+\sup_{k\in\boldsymbol{N}}\{2^{ks}\|\chi(\cdot\,|2^{-k})*f\|_{L^p}\}.$$

$\Lambda(s;p,\infty)$ is the space of all $f\in\mathcal{S}'$ such that $\|f\|_{\Lambda(s;p,\infty)}<\infty$.

In [20], I, Taibleson defined the Lipchitz space $\Lambda(s;p,q)$ in a different way. But it holds that Taibleson's Lipschitz space coincides with the space defined above. It also holds that the space Λ_s defined in Definition 2.5 (§ 2.1 of the present paper) coincides with $\Lambda(s;\infty,\infty)$ of Definition 6.1.

In the proof of Theorem 4.4, we used a part of the following

PROPOSITION 6.1. *Let* $s\in\boldsymbol{R}$ *and* $1\leqq p\leqq\infty$.

(i) *If* $f\in\Lambda(s;p,\infty)$ *and* $t<s$, *then* $(1-\Delta)^{t/2}f\in L^p$ *and*

$$\|(1-\Delta)^{t/2}f\|_{L^p}\leqq C\|f\|_{\Lambda(s;p,\infty)}\,,$$

where $(1-\Delta)^{t/2}f=\mathcal{F}^{-1}((1+|\xi|^2)^{t/2}\hat{f}(\xi))$ *and* C *is a constant depending only on* n, s, t *and* p.

(ii) *There exist positive integers* N_1, N_2 *and* N_3 *which depend only on* n, s *and* p *such that the following statement is true*: *if* $h\in\mathcal{S}$ *and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\hat{h}(\xi)\right| \leqq \begin{cases} |\xi|^{N_1-|\alpha|} & \textit{for } |\xi|\leqq 1 \textit{ and } |\alpha|\leqq N_3 \\ |\xi|^{-N_2-|\alpha|} & \textit{for } |\xi|\geqq 1 \textit{ and } |\alpha|\leqq N_3, \end{cases}$$

then

$$\sup_{0<r<1} \{r^{-s}\|h(\cdot\,|r)*f\|_{L^p}\} \leqq C\|f\|_{\Lambda(s;\,p,\infty)}$$

with a constant C depending only on n, s and p.

(iii) *Suppose that g and h_j ($j=1, 2, \cdots, m$) be functions in $\mathcal{S}$ with the following properties:* (a) *for every $\xi\neq 0$, there exists an integer k such that*

$$\sum_{j=1}^{m} |\hat{h}_j(2^{-k}\xi)| \neq 0\,;$$

(b) *for every $\xi\in\boldsymbol{R}^n$, there exists positive integer k such that*

$$|\hat{g}(\xi)| + \sum_{j=1}^{m} |\hat{h}_j(2^{-k}\xi)| \neq 0\,.$$

Then we have

$$\|f\|_{\Lambda(s;\,p,\infty)} \leqq C\left[\|g*f\|_{L^p} + \sum_{j=1}^{m} \sup_{k\in N} \{2^{ks}\|h_j(\cdot\,|2^{-k})*f\|_{L^p}\}\right],$$

where the constant C depends only on n, s, p and the functions g and h_j.

PROOF OF PROPOSITION 6.1. *Proof of* (i). We have

$$(1-\Delta)^{t/2}f = \lim_{k\to\infty} \theta(\cdot\,|2^{-k})*(1-\Delta)^{t/2}f \tag{6.1}$$

$$= \theta*(1-\Delta)^{t/2}f + \sum_{k=1}^{\infty} \chi(\cdot\,|2^{-k})*(1-\Delta)^{t/2}f$$

where $\lim_{k\to\infty}$ and $\sum_{k=1}^{\infty}$ converge in $\mathcal{S}'$. Now we use the following Bernstein type inequalities:

$$\|\theta*(1-\Delta)^{t/2}f\|_{L^p} \leqq C\|\theta*f\|_{L^p}$$

and

$$\|\chi(\cdot\,|2^{-k})*(1-\Delta)^{t/2}f\|_{L^p} \leqq C2^{kt}\|\chi(\cdot\,|2^{-k})*f\|_{L^p}, \qquad k\in\boldsymbol{N}.$$

Thus, if $f\in\Lambda(s\,;p,\infty)$ and $t<s$, the series in (6.1) converges in L^p and hence $(1-\Delta)^{t/2}f\in L^p$.

Proof of (iii). By the assumption (a), there exists a finite subset $E\subset \{1, 2, \cdots, m\}\times\boldsymbol{Z}$ and functions $\{\varphi_{j,k}\,|(j, k)\in E\}$ such that

$$\chi = \sum_{(j,k)\in E} h_j(\cdot\,|2^{-k})*\varphi_{j,k}\,.$$

We have

$$\chi(\cdot\,|2^{-i})*f = \sum_{(j,k)\in E} f*(h_j(\cdot\,|2^{-k-i}))*(\varphi_{j,k}(\cdot\,|2^{-i}))$$

and hence

$$\|\chi(\cdot|2^{-i})*f\|_{L^p}\leqq\sum_{(j,k)\in E}\|h_j(\cdot|2^{-k-i})*f\|_{L^p}\|\varphi_{j,k}\|_{L^1}.$$

Thus we obtain

$$\sup_{i>k_0}\{2^{is}\|\chi(\cdot|2^{-i})*f\|_{L^p}\}\leqq C\sum_{j=1}^{m}\sup_{i\in N}\{2^{is}\|h_j(\cdot|2^{-i})*f\|_{L^p}\}$$

where $k_0=\max\{-k|(j, k)\in E$ for some j, $j=1, 2, \cdots, m\}$. In a similar way, we see from the assumption (b) that

$$\sup_{1\leq i\leq k_0}\{2^{is}\|\chi(\cdot|2^{-i})*f\|_{L^p}\}\leqq C\Big[\|g*f\|_{L^p}+\sum_{j=1}^{m}\sup_{i\in N}\{2^{is}\|h_j(\cdot|2^{-i})*f\|_{L^p}\}\Big]$$

and

$$\|\theta*f\|_{L^p}\leqq C\Big[\|g*f\|_{L^p}+\sum_{j=1}^{m}\sup_{i\in N}\{2^{is}\|h_j(\cdot|2^{-i})*f\|_{L^p}\}\Big].$$

This proves (iii).

Proof of (ii). We decompose h as

$$h=\sum_{k=-\infty}^{\infty}h_k \qquad \text{with} \quad h_k=\chi(\cdot|2^k)*h.$$

Then we apply the method in the proof of (iii) to obtain the estimates

$$\sup_{0<r<1}\{r^{-s}\|h_k(\cdot|r)*f\|_{L^p}\}\leqq C(k)\|f\|_{\Lambda(s;p,\infty)}, \qquad k\in \boldsymbol{Z}.$$

If the constants N_1, N_2 and N_3 are sufficiently large, then it is easily verified that $\sum_{k\in Z}C(k)<\infty$ and that the equality

$$h(\cdot|r)*f=\sum_{k\in Z}h_k(\cdot|r)*f \qquad \text{in } \mathcal{S}'$$

is legitimate for all $f\in\Lambda(s;p,\infty)$. Thus we can prove (ii). This completes the proof of Proposition 6.1.

If g and h_j, $j=1, 2, \cdots, m$, are functions in $\mathcal{S}$ which satisfy the conditions (a) and (b) of (iii) of Proposition 6.1 and if further h_j's satisfy the assumption of (ii) of the proposition, then: the norm

$$\|f\|=\|g*f\|_{L^p}+\sum_{j=1}^{m}\sup_{k\in N}\{2^{ks}\|h_j(\cdot|2^{-k})*f\|_{L^p}\}$$

is equivalent to the norm $\|f\|_{\Lambda(s;p,\infty)}$; if $t<s$, the norm

$$\|f\|=\|(1-\Delta)^{t/2}f\|_{L^p}+\sum_{j=1}^{m}\sup_{0<r<1}\{r^{-s}\|h_j(\cdot|r)*f\|_{L^p}\} \tag{6.2}$$

is also equivalent to the norm $\|f\|_{\Lambda(s;p,\infty)}$. In particular, we can take a set $\{h_j\}$ consisting of only one function such that

$$\hat{h}_j(\xi)=|\xi|^{2N}\exp(-|\xi|^2)$$

or

(6.3) $$\hat{h}_j(\xi)=|\xi|^{N}\exp(-|\xi|)$$

if N is sufficiently large. ((6.3) does not belong to $\mathcal{S}$ but is available for (6.2) since $h_j(\cdot|r)*f$ is well defined whenever $(1-\Delta)^{t/2}f\in L^p$ for some t.) Comparing these results with the definition of $\Lambda(s; p, \infty)$ given by Taibleson [20], I (Theorem 3 in p. 421 and Theorem 7 in p. 437), we see that the spaces studied by Taibleson [20] coincide with the spaces of our Definition 6.1. If we take a set $\{h_j\}$ consisting of compactly supported functions, then we can easily show the inclusion $\Lambda_s\subset\Lambda(s; \infty, \infty)$ (Λ_s is defined in Definition 2.5). As for the proof of the converse inclusion $\Lambda(s; \infty, \infty)\subset\Lambda_s$, see the proof of Theorem J′ in the next section (we can modify the proof of "(iii) $\Rightarrow$ (ii)" using the functions $f(x, 2^{-k})=(\theta(\cdot|2^{-k})*f)(x)$, $k=0, 1, 2, \cdots$, in place of the function $f(x, t)$).

6.2. *The Campanato spaces.*

In [5], Campanato introduced the following spaces.

DEFINITION 6.2. Let $s\geqq 0$ and k be a nonnegative integer. If f is a distribution on $\boldsymbol{R}^n$ which is equal to a finite complex measure on every compact set, we define $\|f\|_{\mathcal{L}(k,s)}$ as follows:

$$\|f\|_{\mathcal{L}(k,s)}=\sup_{x_0,r}\left[\inf_{P}\left\{r^{-n-s}\int_{|x-x_0|<r}|f(dx)-P(x)dx|\right\}\right],$$

where the infimum is taken over all polynomials P of degree $\leqq k$ and the supremum is taken over all $x_0\in\boldsymbol{R}^n$ and all $r>0$. $\mathcal{L}(k, s)$ is the set of all f such that $\|f\|_{\mathcal{L}(k,s)}<\infty$. (We have slightly modified the definition given by Campanato [5] and also changed the notation; our $\mathcal{L}(k, s)$ corresponds to $\mathcal{L}_k^{(1,n+s)}$ of [5].)

We shall show that the space $\tilde{\Lambda}_s$ defined in § 2.1 coincides with an $\mathcal{L}(k, s)$-space. We fix a function $\varphi\in\mathcal{D}$ such that

$$\begin{cases}\varphi(x)=\varphi(-x)\\ \text{support }\varphi\subset\{|x|\leqq 1\}\\ \int\varphi(x)dx=1;\end{cases}$$

and, for $f\in\mathcal{D}'$, we define $f(x, t)$ by

$$f(x, t)=(\varphi(\cdot|t)*f)(x), \qquad x\in\boldsymbol{R}^n,\ t>0.$$

Note that, for every α,

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x, t)=(D^{\alpha} f)(x, t) \longrightarrow D^{\alpha} f \quad \text{in } \mathscr{D}' \text{ as } t \downarrow 0 .$$

Now we have the following theorems (the present author learned these theorems from Mr. A. Uchiyama in Tôhoku University):

THEOREM J. *Let k be a nonnegative integer and $s>k$. Then the following three conditions are mutually equivalent*:

(i) $\|f\|_{\mathcal{L}(k,s)}=M_1<\infty$;

(ii) *f is of class C^k and*

$$\sum_{|\alpha|=k} \sup_{x \neq y}\left\{\frac{|D^{\alpha} f(x)-D^{\alpha} f(y)|}{|x-y|^{s-k}}\right\}=M_2<\infty ;$$

(iii) *$f \in \mathscr{D}'$ and*

$$\begin{aligned} &\sum_{|\alpha|=k+1} \sup_{\substack{x \in R^n \\ t>0}}\left\{t^{-s+k+1}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x, t)\right|\right\} \\ &+\sum_{|\alpha|=k} \sup_{\substack{x \in R^n \\ t>0}}\left\{t^{-s+k+1}\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x, t)\right|\right\} \\ &=M_3<\infty . \end{aligned}$$

Moreover the semi-norms $M_i=M_i(f)$, $i=1, 2, 3$, are mutually equivalent.

THEOREM J$'$. *Let k be a nonnegative integer and $s>k$. Then the following three conditions are mutually equivalent*:

(i) $\|f\|_{\mathcal{L}(k+1,s)}=M_1<\infty$;

(ii) *f is of class C^k and*

$$\sum_{|\alpha|=k} \sup_{x \neq y}\left\{\frac{|D^{\alpha} f(x)-2 D^{\alpha} f((x+y)/2)+D^{\alpha} f(y)|}{|x-y|^{s-k}}\right\}=M_2<\infty ;$$

(iii) *$f \in \mathscr{D}'$ and*

$$\begin{aligned} &\sum_{|\alpha|=k+2} \sup_{\substack{x \in R^n \\ t>0}}\left\{t^{-s+k+2}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x, t)\right|\right\} \\ &+\sum_{|\alpha|=k} \sup_{\substack{x \in R^n \\ t>0}}\left\{t^{-s+k+1}\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x, t)\right|\right\} \\ &=M_3<\infty . \end{aligned}$$

Moreover the semi-norms $M_i=M_i(f)$, $i=1, 2, 3$, are mutually equivalent.

COROLLARY 6.1. *For $s>0$, we have*

$$\tilde{\Lambda}_s=\mathcal{L}([s], s);$$

the semi-norms are equivalent.

This corollary is an immediate consequence of Theorems J and J′. For s not equal to an integer, this corollary was shown by Campanato [4] ($0<s<1$), Meyers [12] ($0<s<1$) and Campanato [5] ($s>0$, $s\neq$integer).

Theorems J and J′ are proved by the same method. We shall present here the proof of Theorem J′.

PROOF OF THEOREM J′. (i) $\Rightarrow$ (iii). If P is any polynomial of degree$\leqq k+1$, then

$$\int P(y)\left(\frac{\partial}{\partial x}\right)^{\alpha}\varphi(x-y|t)dy=0 \qquad \text{for } |\alpha|=k+2$$

and

$$\int P(y)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^{\alpha}\varphi(x-y|t)dy=0 \qquad \text{for } |\alpha|=k\,,$$

the latter of which is due to the fact that φ is an even function. Thus we have

$$\begin{aligned}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,t)\right| &= \left|\int\left(\frac{\partial}{\partial x}\right)^{\alpha}\varphi(x-y|t)(f(dy)-P(y)dy)\right| \\ &\leqq \int\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\varphi(x-y|t)\right||f(dy)-P(y)dy| \\ &\leqq Ct^{-n-|\alpha|}\int_{|x-y|<t}|f(dy)-P(y)dy|, \qquad |\alpha|=k+2\,,\end{aligned}$$

and

$$\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,t)\right|\leqq Ct^{-n-|\alpha|-1}\int_{|x-y|<t}|f(dy)-P(y)dy|, \qquad |\alpha|=k\,,$$

where P is any polynomial of degree$\leqq k+1$. Varying P over all polynomials of degree $\leqq k+1$, we obtain $M_3\leqq CM_1$.

(iii) $\Rightarrow$ (i). Suppose that f satisfies the condition (iii). If $|\alpha|=k$, then

$$\int_0^1\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,t)\right|dt\leqq\int_0^1 M_3t^{s-k-1}dt<\infty\,,$$

which implies that $(\partial/\partial x)^{\alpha}f(x,t)$, $|\alpha|=k$, converges, as t tends to 0, uniformly with respect to $x\in\boldsymbol{R}^n$. Hence f is of class C^k. We have

$$\begin{aligned}&\left|\left(f(x)-\sum_{|\alpha|\leqq k}\frac{x^{\alpha}}{\alpha!}D^{\alpha}f(0)\right)-\left(f(x,t)-\sum_{|\alpha|\leqq k}\frac{x^{\alpha}}{\alpha!}D^{\alpha}f(0,t)\right)\right| \\ &\leqq C|x|^k\sum_{|\alpha|=k}\int_0^t\sup_{y\in\boldsymbol{R}^n}\left\{\left|\frac{\partial}{\partial r}D^{\alpha}f(y,r)\right|\right\}dr \\ &\leqq CM_3|x|^kt^{s-k}\,.\end{aligned}$$

On the other hand, we have

$$\left|f(x,t)-\sum_{|\alpha|\leq k+1}\frac{x^{\alpha}}{\alpha!}D^{\alpha}f(0,t)\right|\leqq C|x|^{k+2}\sup_{\substack{y\in R^n\\ |\alpha|=k+2}}\{|D^{\alpha}f(y,t)|\}$$

$$\leqq CM_3|x|^{k+2}t^{s-k-2}.$$

Combining these estimates, we obtain

$$\left|f(x)-\sum_{|\alpha|\leq k}\frac{x^{\alpha}}{\alpha!}D^{\alpha}f(0)-\sum_{|\alpha|=k+1}\frac{x^{\alpha}}{\alpha!}D^{\alpha}f(0,t)\right|$$

$$\leqq CM_3(|x|^k t^{s-k}+|x|^{k+2}t^{s-k-2})$$

and hence

$$r^{-n}\int_{|x|<r}\left|f(x)-\sum_{|\alpha|\leq k}\frac{x^{\alpha}}{\alpha!}D^{\alpha}f(0)-\sum_{|\alpha|=k+1}\frac{x^{\alpha}}{\alpha!}D^{\alpha}f(0,t)\right|dx$$

$$\leqq CM_3(r^k t^{s-k}+r^{k+2}t^{s-k-2}).$$

If we set $t=r$, then

$$\inf_{P;\,\deg P\leq k+1}\left\{r^{-n}\int_{|x|<r}|f(x)-P(x)|\,dx\right\}\leqq CM_3 r^s.$$

The same estimate holds if we replace the ball $\{|x|<r\}$ by any other ball $\{|x-x_0|<r\}$. Thus we have $M_1\leqq CM_3$.

(ii) $\Rightarrow$ (iii). Since φ is an even function, we have

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,t)\right|$$

$$=\left|\frac{1}{2}\int(D^{\alpha}f(x+y)-2D^{\alpha}f(x)+D^{\alpha}f(x-y))\frac{\partial^2}{\partial y_i\partial y_j}\varphi(y|t)dy\right|$$

$$\leqq Ct^{-n-2}\int_{|y|<t}|D^{\alpha}f(x+y)-2D^{\alpha}f(x)+D^{\alpha}f(x-y)|\,dy.$$

Hence

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,t)\right|\leqq CM_2t^{s-k-2}\qquad\text{for } |\alpha|=k.$$

Similarly we have

$$\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,t)\right|\leqq CM_2t^{s-k-1}\qquad\text{for } |\alpha|=k.$$

This proves $M_3\leqq CM_2$.

(iii) $\Rightarrow$ (ii). Suppose that f satisfies the condition (iii). Then f is of class C^k as we have seen in the proof of (iii) $\Rightarrow$ (i). For $|\alpha|=k$, we have

$$|(D^\alpha f(x+y)-2D^\alpha f(x)+D^\alpha f(x-y))-(D^\alpha f(x+y,t)-2D^\alpha f(x,t)+D^\alpha f(x-y,t))|$$

$$=\left|\int_0^t \frac{\partial}{\partial r}(D^\alpha f(x+y,r)-2D^\alpha f(x,r)+D^\alpha f(x-y,r))dr\right|$$

$$\leqq CM_3 t^{s-k}$$

and

$$|D^\alpha f(x+y,t)-2D^\alpha f(x,t)+D^\alpha f(x-y,t)|$$

$$\leqq C|y|^2 \sup_{i,j,z}\left\{\left|\frac{\partial^2}{\partial z_i \partial z_j} D^\alpha f(z,t)\right|\right\}$$

$$\leqq CM_3|y|^2 t^{s-k-2}.$$

Combining these estimates, we obtain

$$\sum_{|\alpha|=k}|D^\alpha f(x+y)-2D^\alpha f(x)+D^\alpha f(x-y)|\leqq CM_3(t^{s-k}+|y|^2 t^{s-k-2}).$$

If we set $t=|y|$, then this estimate shows that $M_2\leqq CM_3$. This completes the proof of Theorem J′.

Now suppose that $0<p\leqq 1$. Then it is easy to see that, for $f\in\mathcal{D}'$,

$$\|f\|_{\mathcal{L}([n/p-n],n/p-n)}\approx\sup\{|\langle f,g\rangle| \mid g\in\mathcal{D},\ g:\ p\text{-atom}\}.$$

Thus Theorem B and Corollary 6.1 shows that

$$\|f\|_{\Lambda_{n/p-n}}\approx\sup\{|\langle f,g\rangle| \mid g\in\mathcal{D}\cap H^p,\ \|g\|_{H^p}\leqq 1\},\qquad 0<p<1,$$

which is (2.7). The inequalities (2.5) can be derived from Theorem B in a similar way; but the proof requires another step, *i.e.*, we must show that $\mathcal{L}(0,0)=BMO$ or that all elements of $\mathcal{L}(0,0)$ is absolutely continuous with respect to the Lebesgue measure. This can be done by introducing $(1,q)$-atom, $q>1$; see Coifman-Weiss [7], pp. 592-594 and pp. 623-634. A proof of (2.6) requires yet another step, *i.e.*, we must show the weak compactness of the unit ball of H^1; see Coifman-Weiss [7], pp. 638-641.

6.3. *Proof of Lemma* 2.1.

Let φ be a function in $\mathcal{D}$ such that

$$\int\varphi(x)dx=1 \quad\text{and}\quad \int\varphi(x)x^\alpha dx=0 \qquad \text{for}\quad 1\leqq|\alpha|\leqq[s].$$

Set $f(x,t)=(\varphi(\cdot|t)*\varphi(\cdot|t)*f)(x)$. Then, in the same way as in the proof of "(i)$\Rightarrow$(iii)" of Theorem J′, we have

$$\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^\alpha f(x,t)\right|\leqq C\|f\|_{\mathcal{L}([s],s)}t^{s-|\alpha|-1}$$

for all α. Hence, if $|\alpha|<s$, we have

$$|D^{\alpha}f(x)|=\left|-\int_{0}^{1}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,t)dt+\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,1)\right|$$

$$\leqq C\|f\|_{\mathcal{L}([s],s)}+\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}f(x,1)\right|$$

$$\leqq C\|f\|_{\tilde{\Lambda}_s}+|(f*\varphi*D^{\alpha}\varphi)(x)|,$$

where the last inequality is due to Corollary 6.1. Thus we obtain

$$\sum_{|\alpha|<s}\|D^{\alpha}f\|_{L^{\infty}}\leqq C\|f\|_{\tilde{\Lambda}_s}+\sum_{|\alpha|<s}\|f*\varphi*D^{\alpha}\varphi\|_{L^{\infty}}$$

and hence

$$\|f\|_{\Lambda_s}\leqq C\|f\|_{\tilde{\Lambda}_s}+C\|f*\varphi\|_{L^{\infty}}.$$

The reverse inequality is obvious. Thus we have proved Lemma 2.1.

6.4. *Proof of Lemma* 2.3.

It is easy to see that there exists a constant C depending only on n and p such that

$$|\hat{f}(\xi)|\leqq C|\xi|^{n/p-n} \qquad \text{for all } p\text{-atoms } f.$$

It is also easy to see that, if f is a p-atom,

$$\hat{f}(\xi)=o(|\xi|^{n/p-n}) \qquad \text{as } \xi\to 0 \text{ or } |\xi|\to\infty. \tag{6.4}$$

Hence, by Theorem B,

$$|\hat{f}(\xi)|\leqq C\|f\|_{H^p}|\xi|^{n/p-n} \qquad \text{for all } f\in H^p$$

and (6.4) holds for all $f\in H^p$. In particular, if $f\in H^p$, then all the derivatives of $\hat{f}$ of order $\leqq[n/p-n]$ vanish at the origin so far as they exist. Thus, if $f\in H^p\cap X_M$ (X_M is as mentioned in Lemma 2.3), then

$$(D^{\alpha}\hat{f})(0)=\int(-ix)^{\alpha}f(x)dx=0 \qquad \text{for } |\alpha|\leqq[n/p-n]. \tag{6.5}$$

Conversely suppose that $f\in X_M$ and (6.5) holds. We shall estimate the maximal function

$$f^{+}(x)=\sup_{0<t<\infty}\{|(\varphi(\cdot|t)*f)(x)|\}$$

with $\varphi\in\mathcal{D}$ such that support $\varphi\subset\{|x|\leqq 1\}$ and $\hat{\varphi}(0)=1$. We assume that $n/p<M<[n/p-n]+n+1$. We abbreviate $[n/p-n]$ to N and $\|(1+|x|)^M f(x)\|_{L^{\infty}}$ to A_f. It is easy to see that

$$|(\varphi(\cdot|t)*f)(x)|\leqq CA_f$$

and

$$\sup_{t \leq |x|/2} \{|(\varphi(\cdot|t)*f)(x)|\} \leqq CA_f|x|^{-M} \qquad \text{for } |x| \geqq 1 .$$

On the other hand, if $|x| \geqq 1$ and $t \geqq |x|/2$, then

$$\begin{aligned}
&|(\varphi(\cdot|t)*f)(x)| \\
&= \left| \int f(y)\Big(\varphi(x-y|t) - \sum_{|\alpha| \leq N} \frac{(-y)^\alpha}{\alpha!}\Big(\frac{\partial}{\partial x}\Big)^\alpha \varphi(x|t)\Big)dy \right| \\
&= \left| \iint_{\substack{y \in R^n \\ 0<r<1}} f(y)(N+1)(1-r)^N \sum_{|\alpha|=N+1} \frac{(-y)^\alpha}{\alpha!}\Big(\frac{\partial}{\partial x}\Big)^\alpha \varphi(x-ry|t)dydr \right| \\
&\leqq CA_f t^{-n-N-1} \iint_{\substack{y \in R^n \\ 0<r<1 \\ |x-ry|<t}} (1+|y|)^{-M}|y|^{N+1}dydr \\
&\leqq CA_f t^{-n-N-1} \iint_{\substack{y \in R^n \\ 0<r<1 \\ r|y|<3t}} (1+|y|)^{-M}|y|^{N+1}dydr \\
&\leqq CA_f t^{-M} \leqq CA_f |x|^{-M} .
\end{aligned}$$

Combining these estimates, we obtain

$$f^+(x) \leqq CA_f(1+|x|)^{-M} .$$

Hence $f \in H^p$ and $\|f\|_{H^p} \approx \|f^+\|_{L^p} \leqq CA_f$. This completes the proof of Lemma 2.3.

6.5. *An approximation of BMO-functions.*

Let f be any element of BMO. We construct a sequence $\{f_n\}$ such that $f_n \in \mathcal{D}$, $f_n \to f$ in $\mathcal{S}'$ and $\|f_n\|_{BMO} \to \|f\|_{BMO}$. Take $\varphi \in \mathcal{D}$ such that $\varphi(x) \geqq 0$, support $\varphi \subset \{|x| \leqq 1\}$ and $\hat{\varphi}(0)=1$. Consider the following function:

$$\chi_{\delta,a}(x) = a+1-\int \pi_{[a,a+1]}(\delta \log|x-y|)\varphi(y)dy ,$$

where $\delta>0$, $a \in \boldsymbol{R}$ and

$$\pi_{[a,a+1]}(t) = \begin{cases} a & \text{if } t<a \\ t & \text{if } a \leqq t \leqq a+1 \\ a+1 & \text{if } t>a+1 . \end{cases}$$

$\chi_{\delta,a}$ is a smooth function with the following properties:

$$0 \leqq \chi_{\delta,a}(x) \leqq 1 ,$$

$$\chi_{\delta,a}(x)=1 \qquad \text{if } \delta \log(|x|+1) \leqq a ,$$

$$\chi_{\delta,a}(x)=0 \qquad \text{if } \delta \log(|x|-1) \geqq a+1 ,$$

and

$$\|\chi_{\delta,a}\|_{BMO} \leqq \delta \|\log|x|\|_{BMO} .$$

Hence, if we take $\delta=\delta(n)$ and $a=a(n)$ appropriately, we obtain $\chi_n=\chi_{\delta(n),a(n)} \in \mathcal{D}$ such that $0 \leqq \chi_n(x) \leqq 1$,

$$\chi_n(x)=1 \quad \text{for } |x| \leqq n \quad \text{and} \quad \|\chi_n\|_{BMO} \leqq n^{-2} .$$

Set

$$f_n(x)=\chi_n(x)(\varphi(\cdot|n^{-1})*g_n)(x) ,$$

where

$$g_n(x)=\begin{cases} f(x) & \text{if } |f(x)| \leqq n \\ nf(x)/|f(x)| & \text{if } |f(x)|>n . \end{cases}$$

Then it is easy to see that $f_n \in \mathcal{D}$ and $f_n \to f$ in $\mathcal{S}'$, and hence

$$\|f\|_{BMO} \leqq \liminf \|f_n\|_{BMO}$$

by Lemma 2.2. On the other hand, we have

$$\begin{aligned} \|f_n\|_{BMO} &\leqq \|\chi_n\|_{L^\infty} \|\varphi(\cdot|n^{-1})*g_n\|_{BMO} + \|\chi_n\|_{BMO} \|\varphi(\cdot|n^{-1})*g_n\|_{L^\infty} \\ &\leqq \|f\|_{BMO} + \frac{1}{n} \end{aligned}$$

and hence

$$\limsup \|f_n\|_{BMO} \leqq \|f\|_{BMO} .$$

Thus $\{f_n\}$ has the desired properties.

6.6. *Note on Theorem G.*

Theorem G is given in [13], where, however, k is defined by

$$k=\max\left\{\left[n\left(\frac{1}{p}-\frac{1}{2}\right)\right]+1, \left[\frac{n}{2}\right]+1\right\} .$$

Here we shall show how we can derive the improved Theorem G from the theorems in [13]; *i.e.*, we assume that Theorem G is established in the case $0<p<1$ and prove the theorem in the case $1 \leqq p<2$. Suppose that m satisfies the assumptions of (i) of Theorem G with $1 \leqq p<2$. Take q such that $q<1$ and

$$k>\frac{[n(1/q-1/2)]+1}{(1/q-1/2)}\left(\frac{1}{p}-\frac{1}{2}\right)$$

and define k_0, θ and b_0 as follows:

$$k_0=[n(1/q-1/2)]+1 ,$$

$$1/p=(1-\theta)/q+\theta/2 ,$$

$$b_0=b/(1-\theta).$$

We construct a family of Fourier multipliers $\{m_z \mid z\in \boldsymbol{C},\ 0\leqq \operatorname{Re} z\leqq 1\}$ as follows:

$$m_z(\xi)=\sum_{j=0}^{\infty} 2^{jb(z-\theta)/(1-\theta)}\eta(2^{-j}\xi)\left(1-(A^{-1}2^{j(1-a)})^2\Delta_\xi\right)^{k_0(z-\theta)/2}[m(\xi)\hat{\chi}(2^{-j}\xi)],$$

where χ is a function in $\mathcal{S}$ satisfying (2.1), η is a function in $\mathcal{D}$ such that $\eta(\xi)=1$ on support $\hat{\chi}$ and support $\eta\subset\{1/3\leqq|\xi|\leqq 3\}$ and

$$\Delta_\xi=\sum_{j=1}^{n}\frac{\partial^2}{\partial\xi_j^2}.$$

Since $k_0(1-\theta)<k$, it is easy to see that, for $y\in\boldsymbol{R}$,

$$\|m_{1+iy}\|_{L^\infty}\leqq C(1+|y|)^{n/2}$$

and

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha m_{iy}(\xi)\right|\leqq C(1+|y|)^{n/2}|\xi|^{-b_0}(A|\xi|^{a-1})^{|\alpha|},\qquad |\alpha|\leqq k_0.$$

Hence

$$\|m_{1+iy}\|_{\mathcal{M}(L^2,L^2)}\leqq C(1+|y|)^{n/2}$$

and

$$\|m_{iy}\|_{\mathcal{M}(H^q,H^q)}\leqq C(1+|y|)^{n/2}A^{n(1/q-1/2)},$$

the latter of which is the consequence of Theorem G for the case $0<p<1$. Thus, applying the complex interpolation theorem (Calderón-Torchinsky [3], Theorem 3.4, pp. 151-152), we obtain

$$m=m_\theta\in\mathcal{M}(H^p,H^p)\quad\text{and}\quad \|m\|_{\mathcal{M}(H^p,H^p)}\leqq CA^{n(1/p-1/2)}.$$

This proves (i) for the case $1\leqq p<2$. (ii) can be proved in a similar way.

We remark that we can weaken the main assumption of Theorem G (inequalities (3.9) or (3.10)) in several ways. The proof given above shows that it is sufficient to require the inequalities (3.9) or (3.10) only for

$$\left(\frac{\partial}{\partial\xi_j}\right)^i m(\xi),\qquad j=1,2,\cdots,n;\ i=0,1,\cdots,[n(1/p-1/2)]+1.$$

Other generalizations of Theorem G are obtained if we use the L^2-scale or the derivatives of fractional order. These remarks are valid also for Theorems E, I and 5.1. Cf. Calderón-Torchinsky [3], §4, pp. 162-171.

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan