The Gel'fand-Levitan theory and certain inverse problems for the parabolic equation

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§1. Introduction and Summary.

Let $(E^1_{p,h,H,a})$ denote the parabolic equation

(1.1) $$\frac{\partial u}{\partial t}+\Big(p(x)-\frac{\partial^2}{\partial x^2}\Big)u=0 \qquad (0<t<\infty,\ 0<x<1)$$

with the boundary condition

(1.2.a) $$\frac{\partial u}{\partial x}-hu\,|_{x=0}=0 \qquad (0<t<\infty)$$

(1.2.b) $$\frac{\partial u}{\partial x}+Hu\,|_{x=1}=0 \qquad (0<t<\infty)$$

and with the initial condition

(1.3) $$u\,|_{t=0}=a(x) \qquad (0<x<1),$$

where $p\in C^1[0, 1]$, $h\in\mathcal{R}$, $H\in\mathcal{R}$ and $a\in L^2(0, 1)$. And let $A^1_{p,h,H}$ be the realization in $L^2(0, 1)$ of the differential operator $(p(x)-\partial^2/\partial x^2)$ with the boundary condition (1.2). We say that $a\in L^2(0, 1)$ is a generating element with respect to $A^1_{p,h,H}$ if and only if a is not orthogonal to any eigenfunction of $A^1_{p,h,H}$.

In a previous work [5], the author showed the following theorem jointly with T. Suzuki:

THEOREM 0. *Suppose that $a\in L^2(0, 1)$ is a generating element with respect to $A^1_{p,0,0}$ and let $u=u(t, x)$ be the solution of $(E^1_{p,0,0,a})$. Then, for any $(q, b)\in C^1[0, 1]\times L^2(0, 1)$, the equality*

(1.4) $$v(t, \xi)=u(t, \xi) \qquad (T_1\leqq t\leqq T_2\,;\ \xi=0, 1)$$

with some T_1, T_2 in $0<T_1<T_2<\infty$ implies

(1.5′) $$(q, b)=(p, a),$$

where $v=v(t, x)$ is the solution of $(E^1_{q,0,0,b})$.

In this paper, we generalize this theorem and show

THEOREM 1. *Let* $(p, h, H, a)\in C^1[0, 1]\times\mathcal{R}\times\mathcal{R}\times L^2(0, 1)$ *be given and let* $u=u(t, x)$ *be the solution of* $(E^1_{p,h,H,a})$. *Then,* (1.4) *for some* $(q, i, I, b)\in C^1[0, 1]\times\mathcal{R}\times\mathcal{R}\times L^2(0, 1)$ *implies*

$$(q, i, I, b)=(p, h, H, a) \tag{1.5}$$

if and only if a *is a generating element with respect to* $A^1_{p,h,H}$, *where* $v=v(t, x)$ *is the solution of* $(E^1_{q,i,I,b})$.

Our proof of Theorem 1 is more heavily based on the Gel'fand-Levitan theory [1] and is more constructive than that of Theorem 0 in [5]. See also Suzuki [6], which gives by the method of [5] a theorem of nonuniqueness in the present problem among other results including a theorem of uniqueness.

Furthermore, for $\alpha\in C^3_+[0,1]\equiv\{\alpha\in C^3[0,1]\,|\,\alpha(x)>0\ (x\in[0,1])\}$ and $a\in L^2(0,1)$, we consider the parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) \qquad (0<t<\infty,\ 0<x<1) \tag{1.6}$$

with the boundary condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,1}=0 \qquad (0<t<\infty) \tag{1.7}$$

and with the initial condition

$$u\,|_{t=0}=a(x) \qquad (0<x<1), \tag{1.8}$$

which is denoted by $(E^2_{\alpha,a})$. Let A^2_α be the realization in $L^2(0, 1)$ of the differential operator $-(\partial/\partial x)(\alpha(x)\partial/\partial x\,\cdot)$ with the Neumann boundary condition (1.7). Noting that the constant function 1 is the eigenfunction of A^2_α corresponding to the eigenvalue 0, we say that $a\in L^2(0, 1)$ is a *weakly* generating element with respect to A^2_α if and only if a is not orthogonal to eigenfunctions of A^2_α other than the constant function. Then, the following theorem is obtained by making use of the Liouville transformation:

THEOREM 2. *Let* $(\alpha, a)\in C^3_+[0, 1]\times L^2(0, 1)$ *be given and let* $u=u(t, x)$ *be the solution of* $(E^2_{\alpha,a})$. *Suppose that* a *is a weakly generating element with respect to* A^2_α. *Then, for any* $(\beta, b)\in C^3_+[0, 1]\times L^2(0, 1)$ *with*

$$\int_0^1\frac{dx}{\sqrt{\alpha(x)}}=\int_0^1\frac{dx}{\sqrt{\beta(x)}}, \tag{1.9}$$

the equality

$$v(t, \xi)=u(t, \xi) \qquad (T_1\leqq t\leqq T_2\,;\ \xi=0, 1) \tag{1.10}$$

implies

$$(\beta, b)=(\alpha, a), \tag{1.11}$$

where $v=v(t, x)$ *is the solution of* $(E^2_{\beta,b})$.

Furthermore, if a *is not a weakly generating element with respect to* A^2_α, *there exists some* $(\beta, b)\in C^3_+[0, 1]\times L^2(0, 1)$ *with* (1.9), *for which* (1.11) *fails to hold in spite of* (1.10).

This paper is composed of four sections. § 2 and § 3 are devoted to the proof of Theorems 1 and 2, respectively. In § 4, we state a few modifications of these theorems as well as a remark on them.

We now want to refer to the circumstances under which we come to publish this paper. Originally, the work was done as a part of master's thesis by the author [4] at the University of Tokyo, which was titled "On certain inverse problems for parabolic equations" and was written in Japanese in 1980. In completing this thesis, the author owed much to Professor H. Fujita, Mr. H. Matano, Mr. A. Miyachi and Mr. T. Suzuki through their valuable advices. For instance, Professor Fujita suggested to the author the study in the direction of the present paper and Mr. Suzuki gave the author crucial ideas for the proof. After his graduation, however, the author was not able to complete the manuscript for publication because of his illness. Again Mr. Suzuki kindly reorganized the result and wrote the English version for the author. The author thanks Mr. Suzuki heartily for his kind help and warm friendship.

§ 2. Proof of Theorem 1.

Let λ_n and $\phi(x, \lambda_n)$ $(n=0, 1, 2, \cdots)$ be the eigenvalues and the eigenfunctions of $A^1_{p,h,H}$, respectively. We normalize $\phi(\cdot, \lambda_n)$ by

$$\phi(0, \lambda_n)=1 \tag{2.1}$$

and put

$$\rho_n=\int_0^1 \phi(x, \lambda_n)^2 dx. \tag{2.2}$$

According to Gel'fand-Levitan [1] and Levitan-Gasymov [2], we call $\{\lambda_n, \rho_n | n=0, 1, 2, \cdots\}$ the spectral characteristics of $A^1_{p,h,H}$.

Suppose that $a\in L^2(0, 1)$ is a generating element with respect to $A^1_{p,h,H}$ and let (1.4) hold for some $(q, i, I, b)\in C^1[0, 1]\times\mathcal{R}\times\mathcal{R}\times L^2(0, 1)$. And let μ_m and $\psi(x, \mu_m)$ $(m=0, 1, 2, \cdots)$ be the eigenvalues and the eigenfunctions of $A^1_{q,i,I}$, respectively. We normalize $\psi(\cdot, \mu_m)$ by

$$\psi(0, \mu_m)=1. \tag{2.1'}$$

By Suzuki-Murayama [5], we have then

$$\lambda_n = \mu_n \qquad (n=0, 1, 2, \cdots) \tag{2.3}$$

and

$$\phi(1, \lambda_n) = \psi(1, \mu_n) \qquad (n=0, 1, 2, \cdots). \tag{2.4}$$

In order to show (1.5), therefore, we have only to derive

$$\rho_n = \sigma_n \qquad (n=0, 1, 2, \cdots), \tag{2.5}$$

where

$$\sigma_m = \int_0^1 \psi(x, \mu_m)^2 dx, \tag{2.2'}$$

since (2.3) and (2.5) yield $q=p$ by means of the Gel'fand-Levitan theory [1], and $b=a$ follows in the same way as in Suzuki-Murayama [5].

Here we prepare the following

LEMMA 1. *Let $\phi=\phi(x, \lambda)$ $(\lambda \in \boldsymbol{C})$ be the solution of*

$$\left(p(x) - \frac{\partial^2}{\partial x^2}\right)\phi = \lambda\phi \tag{2.6}$$

with

$$\phi(0, \lambda) = 1 \tag{2.7}$$

and

$$\phi'(0, \lambda) = h, \tag{2.8}$$

and put

$$\Phi(\lambda) = \phi'(1, \lambda) + H\phi(1, \lambda). \tag{2.9}$$

Then we have

$$\rho_n = -\phi(1, \lambda_n)\Phi'(\lambda_n). \tag{2.10}$$

LEMMA 2. *We put*

$$F(\lambda) = \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right) \tag{2.11}$$

or

$$F(\lambda) = \lambda \prod_{n=0}^{\infty}{}'\left(1 - \frac{\lambda}{\lambda_n}\right), \qquad \textit{according as } 0 \notin \{\lambda_n\} \textit{ or } 0 \in \{\lambda_n\}, \tag{2.11'}$$

where $\prod_{n=0}^{\infty}{}'(1-\lambda/\lambda_n)$ means the infinite product of $(1-\lambda/\lambda_n)$ except the term for

$\lambda_n=0$. *Then* $F(\lambda)$ *is an entire function of order one half and we have*

$$\Phi(\lambda)=c_1 F(\lambda) \tag{2.12}$$

for some constant $c_1\neq 0$.

Lemma 2 was proved in Levitan-Gasymov [2]. Lemma 1 is shown as follows. The equality

$$\phi''(x,\lambda_n)\phi(x,\lambda)-\phi''(x,\lambda)\phi(x,\lambda_n)=(\lambda-\lambda_n)\phi(x,\lambda)\phi(x,\lambda_n)$$

is obtained by (2.6). Therefore, we have

$$\begin{aligned}(2.13)\qquad &(\lambda-\lambda_n)\int_0^1\phi(x,\lambda)\phi(x,\lambda_n)dx\\ &=\phi'(1,\lambda_n)\phi(1,\lambda)-\phi'(1,\lambda)\phi(1,\lambda_n) \qquad (\because (2.7),\ (2.8))\\ &=-H\phi(1,\lambda_n)\phi(1,\lambda)-\phi'(1,\lambda)\phi(1,\lambda_n)\\ &=-\phi(1,\lambda_n)\Phi(\lambda),\end{aligned}$$

which gives (2.10).

By means of the relations

$$\rho_n=\frac{1}{2}+O\left(\frac{1}{n^2}\right) \qquad (n\to\infty) \tag{2.14.a}$$

and

$$\phi(1,\lambda_n)=(-1)^n+O\left(\frac{1}{n}\right) \qquad (n\to\infty), \tag{2.14.b}$$

due to Levitan-Sargsjan [3], we have

$$\begin{aligned}(2.15)\qquad c_1&=\lim_{n\to\infty}\frac{\Phi'(\lambda_n)}{F'(\lambda_n)} \qquad (\because (2.12))\\ &=-\frac{1}{2}\lim_{n\to\infty}\frac{(-1)^n}{F'(\lambda_n)} \qquad (\because (2.10)).\end{aligned}$$

Therefore, $\{\lambda_n\}$ determines uniquely $F(\lambda)$, c_1 and $\Phi(\lambda)$ in turn, hence (2.5) follows from (2.4) and (2.10).

Suppose conversely that $a\in L^2(0,1)$ is not a generating element with respect to $A^1_{p,h,H}$. Then, there exists λ_N such that

$$(a,\phi(\cdot,\lambda_N))=0. \tag{2.16}$$

Set

$$\alpha_n=\begin{cases}\phi(1,\lambda_n) & (n\neq N)\\ \frac{1}{2}\phi(1,\lambda_n) & (n=N)\end{cases} \tag{2.17}$$

and put

$$\sigma_n = -\alpha_n \Phi'(\lambda_n), \tag{2.18}$$

where $\Phi(\lambda)$ is defined by (2.12), (2.11) (or (2.11')) and (2.15). By the Gel'fand-Levitan theory [1], there exists $(q, i, I) \in C^1[0, 1] \times \mathcal{R} \times \mathcal{R}$ such that the spectral characteristics of $A^1_{q,i,I}$ coincides with $\{\lambda_n, \sigma_n | n=0, 1, 2, \cdots\}$. Obviously $(q, i, I) \neq (p, h, H)$. Let $\{\mu_m (=\lambda_m)\}$ and $\{\psi(\cdot, \mu_m)\}$ be the eigenvalues and the eigenfunctions of $A^1_{q,i,I}$ normalized by (2.1') respectively. Then, the equalities

$$\lambda_n = \mu_n \qquad (n=0, 1, 2, \cdots) \tag{2.19}$$

and

$$\phi(1, \lambda_n) = \psi(1, \mu_n) \qquad (n \neq N) \tag{2.20}$$

hold by the way of definition of $A^1_{q,i,I}$. We now take $b \in L^2(0, 1)$ such that

$$(b, \psi(\cdot, \mu_n))/\sigma_n = (a, \phi(\cdot, \lambda_n))/\rho_n \qquad (n=0, 1, 2, \cdots), \tag{2.21}$$

and denote the solution of $(E^1_{q,i,I,b})$ by $v=v(t, x)$. Then, expanding u and v by eigenfunctions, we obtain (1.4) by virtue of (2.19), (2.20), (2.21) and (2.16).

§ 3. Proof of Theorem 2.

By the Liouville transformation

$$z = z(x) = \int_0^x \frac{dy}{\sqrt{\alpha(y)}} \tag{3.1.a}$$

and

$$\tilde{u}(t, z) = u(t, x)\alpha(x)^{1/4}, \tag{3.1.b}$$

$(E^2_{\alpha,a})$ is transformed into

$$\frac{\partial \tilde{u}}{\partial t} + \left(p(z) - \frac{\partial^2}{\partial z^2}\right)\tilde{u} = 0 \qquad (0<t<\infty,\ 0<z<l), \tag{3.2}$$

$$\frac{\partial \tilde{u}}{\partial z} - h\tilde{u}\,\Big|_{z=0} = 0, \tag{3.3.a}$$

$$\frac{\partial \tilde{u}}{\partial z} + H\tilde{u}\,\Big|_{z=l} = 0 \tag{3.3.b}$$

and

$$\tilde{u}\,|_{t=0} = \tilde{a}(z), \tag{3.4}$$

where

$$p(z) = f''(z)/f(z), \tag{3.5}$$

(3.6.a) $$h=f'(0)/f(0),$$

(3.6.b) $$H=-f'(l)/f(l),$$

(3.7) $$\tilde{a}(z)=a(x)f(z)$$

and

(3.8) $$l=\int_0^1 \frac{ds}{\sqrt{\alpha(s)}},$$

with

(3.9) $$f(z)=\alpha(x)^{1/4}.$$

Similarly, $(E^2_{\beta,b})$ is transformed by

(3.10.a) $$w=w(y)=\int_0^y \frac{ds}{\sqrt{\beta(s)}}$$

and

(3.10.b) $$\tilde{v}(t,\ w)=v(t,\ y)\beta(y)^{1/4},$$

and we get

(3.11) $$\frac{\partial\tilde{v}}{\partial t}+\left(q(w)-\frac{\partial^2}{\partial w^2}\right)\tilde{v}=0 \qquad (0<t<\infty,\ 0<w<l),$$

(3.12.a) $$\frac{\partial\tilde{v}}{\partial w}-i\tilde{v}\,|_{w=0}=0,$$

(3.12.b) $$\frac{\partial\tilde{v}}{\partial w}+I\tilde{v}\,|_{w=l}=0$$

and

(3.13) $$\tilde{v}\,|_{t=0}=\tilde{b}(w),$$

noting (1.9), where

(3.14) $$q(w)=g''(w)/g(w),$$

(3.15.a) $$i=g'(0)/g(0),$$

(3.15.b) $$I=-g'(l)/g(l)$$

and

(3.16) $$\tilde{b}(w)=b(y)g(w)$$

with

(3.17) $$g(w)=\beta(y)^{1/4}.$$

Then, (1.10) is equivalent to

(3.18.a) $$\alpha(0)^{-1/4}\tilde{u}(t,\ 0)=\beta(0)^{-1/4}\tilde{v}(t,\ 0) \qquad (T_1\leqq t\leqq T_2)$$

and

$$\alpha(1)^{-1/4}\tilde{u}(t, l)=\beta(1)^{-1/4}\tilde{v}(t, l) \qquad (T_1 \leqq t \leqq T_2). \tag{3.18.b}$$

Let λ_n and $\phi(\cdot, \lambda_n)$ $(n=0, 1, 2, \cdots)$ be the eigenvalues and the eigenfunctions of A_α^2, respectively, and let μ_m and $\psi(\cdot, \mu_m)$ $(m=0, 1, 2, \cdots)$ be those of A_β^2, respectively. We normalize the eigenfunctions as usual by

$$\phi(0, \lambda_n)=\psi(0, \mu_m)=1. \tag{3.19}$$

Then, we see easily that the eigenvalues and the eigenfunctions of $A_{p,h,H}^1$ are given by λ_n and $\tilde{\phi}(z, \lambda_n)=\phi(x, \lambda_n)\alpha(x)^{1/4}/\alpha(0)^{1/4}$ $(n=0, 1, 2, \cdots)$, respectively, and those of $A_{q,i,I}^1$ are given by μ_m and $\tilde{\psi}(w, \mu_m)=\psi(y, \mu_m)\beta(y)^{1/4}/\beta(0)^{1/4}$ $(m=0, 1, 2, \cdots)$, respectively. Furthermore, we have

$$(\tilde{a}, \tilde{\phi}(\cdot, \lambda_n))_{L^2(0,l)}=\alpha(0)^{-1/4}(a, \phi(\cdot, \lambda_n))_{L^2(0,1)}$$

and

$$(\tilde{b}, \tilde{\psi}(\cdot, \mu_m))_{L^2(0,l)}=\beta(0)^{-1/4}(b, \psi(\cdot, \mu_m))_{L^2(0,1)}.$$

Suppose, in the first place, that $a \in L^2(0, 1)$ is a weakly generating element with respect to A_α^2 and that (1.10) holds. Then we have (3.18) and

$$(\tilde{a}, \tilde{\phi}(\cdot, \lambda_n))_{L^2(0,l)}=\alpha(0)^{-1/4}(a, \phi(\cdot, \lambda_n))_{L^2(0,1)} \neq 0 \qquad (n=1, 2, \cdots). \tag{3.20}$$

In the same way as in Suzuki-Murayama [5], we expand $\tilde{u}$ and $\tilde{v}$ by eigenfunctions, compare each side of (3.18) and get

$$\mu_n=\lambda_n \qquad (n=1, 2, \cdots) \tag{3.21}$$

and

$$\left(\frac{\beta(0)}{\beta(1)}\right)^{1/4}\tilde{\psi}(l, \mu_n)=\left(\frac{\alpha(0)}{\alpha(1)}\right)^{1/4}\tilde{\phi}(l, \lambda_n) \qquad (n=1, 2, \cdots). \tag{3.22}$$

Here we recall the relations

$$\tilde{\phi}(l, \lambda_n)=(-1)^n+O\left(\frac{1}{n}\right) \qquad (n\to\infty) \tag{3.23.a}$$

and

$$\tilde{\psi}(l, \mu_n)=(-1)^n+O\left(\frac{1}{n}\right) \qquad (n\to\infty), \tag{3.23.b}$$

and we obtain

$$\left(\frac{\beta(0)}{\beta(1)}\right)^{1/4}=\left(\frac{\alpha(0)}{\alpha(1)}\right)^{1/4} \tag{3.24}$$

and

$$\tilde{\psi}(l, \mu_n)=\tilde{\phi}(l, \lambda_n) \qquad (n=1, 2, \cdots). \tag{3.25}$$

On the other hand, we note

(3.26) $$\mu_0=\lambda_0=0$$

and

(3.27) $$\psi(1,\ \mu_0)=\phi(1,\ \lambda_0)=1\,.$$

Summing up, we get

(3.28) $$\mu_n=\lambda_n \qquad (n=0,\ 1,\ 2,\ \cdots)$$

and

(3.29) $$\tilde{\psi}(l,\ \mu_n)=\tilde{\phi}(l,\ \lambda_n) \qquad (n=0,\ 1,\ 2,\ \cdots)\,.$$

Therefore, in the same way as in the proof of Theorem 1, we have

(3.30) $$q(z)=p(z) \qquad (0\leqq z\leqq l)\,,$$

(3.31.a) $$i=h$$

and

(3.31.b) $$I=H\,.$$

On the other hand, we have

(3.32.a) $$1=\int_0^l \frac{dx}{dz}dz=\int_0^l \sqrt{\alpha(x)}dz=\int_0^l f(z)^2 dz\,,$$

and similarly

(3.32.b) $$1=\int_0^l g(z)^2 dz\,.$$

Since the positive solution $e=e(z)$ of the equation

$$e''(z)=p(z)e(z) \qquad (0\leqq z\leqq l)$$

with

$$e'(0)/e(0)=h$$

and with

$$\int_0^l e(z)^2 dz=1$$

is unique, we have

(3.33) $$g(z)=f(z) \qquad (0\leqq z\leqq l)$$

by (3.30), (3.31.a), (3.32), (3.5) and (3.14). Therefore, since

(3.34.a) $$\frac{dz}{dx}=\frac{1}{f(z)^2} \qquad (0\leqq x\leqq 1)\,,$$

(3.34.b) $$z(0)=0\,,$$

(3.35.a) $$\frac{dw}{dx}=\frac{1}{g(w)^2} \qquad (0\leqq x\leqq 1)$$

and

(3.35.b) $$w(0)=0\,,$$

we have

(3.36) $$w(x)=z(x) \qquad (0\leqq x\leqq 1)\,,$$

hence

(3.37) $$\begin{aligned}\beta(x)&=\frac{1}{w'(x)^2}\\ &=\frac{1}{z'(x)^2}=\alpha(x) \qquad (0\leqq x\leqq 1)\,.\end{aligned}$$

From this stage, the proof of

(3.38) $$b(x)=a(x) \qquad (\text{a.e. } x\in(0,\,1))$$

is similar to that of Suzuki-Murayama [5].

Suppose, conversely, that $a\in L^2(0, 1)$ is not a weakly generating element with respect to A_α^2. Then there exists some $N\geqq 1$ such that

(3.39) $$(a,\,\phi(\cdot,\,\lambda_N))_{L^2(0,1)}=\alpha(0)^{-1/4}(\tilde{a},\,\tilde{\phi}(\cdot,\,\lambda_N))_{L^2(0,l)}=0\,.$$

Set

(3.40) $$\alpha_n=\begin{cases}\tilde{\phi}(l,\,\lambda_n) & (n\neq N)\\[2ex] \dfrac{1}{2}\tilde{\phi}(l,\,\lambda_n) & (n=N)\,.\end{cases}$$

According to the latter part of §2, we obtain $\{\sigma_n\,|\,n=0, 1, 2, \cdots\}$ and construct $(q, i, I)\in C^1[0, l]\times\mathcal{R}\times\mathcal{R}$ whose spectral characteristics are $\{\lambda_n, \sigma_n\,|\,n=0, 1, 2, \cdots\}$. Let $\tilde{\varphi}(\cdot, \lambda_n)$ $(n=0, 1, 2, \cdots)$ be the eigenfunctions of $A^1_{q,i,I}$ normalized by $\tilde{\varphi}(0, \lambda_n)=1$ and put

(3.41) $$g(w)=\tilde{\varphi}(w,\,\lambda_0)\Big/\Big(\int_0^l\tilde{\varphi}(w,\,\lambda_0)^2dw\Big)^{1/2}\,.$$

Then $g=g(w)$

(*) is positive definite on $[0, l]$

and

(**) satisfies (3.14), (3.15) and (3.32.b).

Take $\tilde{b}\in L^2(0, l)$ such that

$$\frac{1}{g(0)}(\tilde{b}, \tilde{\psi}(\cdot, \lambda_n))_{L^2(0,l)}/\sigma_n = \frac{1}{f(0)}(\tilde{a}, \tilde{\phi}(\cdot, \lambda_n))_{L^2(0,l)}/\rho_n \quad (n=0, 1, 2, \cdots), \tag{3.42}$$

and $\tilde{v}=\tilde{v}(t, w)$ be the solution of (3.11) with (3.12) and (3.13). Then, we have

$$\frac{1}{g(0)}\tilde{v}(t, 0) = \frac{1}{f(0)}\tilde{u}(t, 0) \qquad (0<t<\infty). \tag{3.43}$$

On the other hand, since $N \neq 0$, we have

$$\tilde{\psi}(l, \lambda_0) = \tilde{\phi}(l, \lambda_0) \tag{3.44}$$

by (3.40). We note here

$$\tilde{\phi}(l, \lambda_0) = \phi(1, \lambda_0)\alpha(1)^{1/4}/\alpha(0)^{1/4} = f(l)/f(0) \qquad (\because \phi(\cdot, \lambda_0) \equiv 1).$$

On the other hand, we have

$$\tilde{\psi}(l, \lambda_0) = g(l)/g(0)$$

by (3.41) and $\tilde{\psi}(0, \lambda_0)=1$, hence

$$g(l)/g(0) = f(l)/f(0). \tag{3.45}$$

Therefore, since

$$\tilde{\phi}(l, \lambda_n) = \tilde{\psi}(l, \lambda_n) \qquad (n \neq N)$$

and

$$(\tilde{a}, \tilde{\phi}(\cdot, \lambda_N)) = (\tilde{b}, \psi(\cdot, \lambda_N)) = 0,$$

$$\frac{1}{g(l)}\tilde{v}(t, l) = \frac{1}{f(l)}\tilde{u}(t, l) \qquad (0<t<\infty). \tag{3.46}$$

We now transform (3.11), (3.12) and (3.13) by

$$y = y(w) = \int_0^w g^2(s)ds \tag{3.47.a}$$

and

$$v(t, y) = \tilde{v}(t, w)g(w)^{-1/4}. \tag{3.47.b}$$

Since $g=g(w)$ satisfies $(*)$ and $(**)$, v satisfies $(E^{2}_{\beta,b})$ for

$$\beta(y) = g(w)^4$$

and

$$b(y) = \tilde{b}(w)\beta(y)^{-1/4},$$

and (1.10) holds by (3.43) and (3.46). On the other hand, $(\beta, b) \neq (\alpha, a)$ and (1.9) are obvious.

§ 4. Concluding remarks.

1. For $(p, a) \in C^1[0, 1] \times L^2(0, 1)$, let $(E^3_{p,a})$ denote the parabolic equation

$$\frac{\partial u}{\partial t}+\left(p(x)-\frac{\partial^2}{\partial x^2}\right)u=0 \qquad (0<t<\infty,\ 0<x<1) \tag{4.1}$$

with the boundary condition

$$u|_{x=0,1}=0 \qquad (0<t<\infty) \tag{4.2}$$

and with the initial condition

$$u|_{t=0}=a(x) \qquad (0<x<1). \tag{4.3}$$

And let A^3_p be the realization in $L^2(0, 1)$ of the differential operator $(p(x)-\partial^2/\partial x^2)$ with the Dirichlet boundary condition (3.2). On the other hand, for $(\alpha, a) \in C^3_+[0, 1] \times L^2(0, 1)$, let $(E^4_{\alpha,a})$ denote the parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) \tag{4.4}$$

with the boundary condition

$$u|_{x=0,1}=0 \qquad (0<t<\infty) \tag{4.5}$$

and with the initial condition

$$u|_{t=0}=a(x) \qquad (0<x<1). \tag{4.6}$$

And let A^4_α be the realization in $L^2(0,1)$ of the differential operator $-(\partial/\partial x)(\alpha(x)\partial/\partial x\,\cdot)$ with the Dirichlet boundary condition (4.5). Then, in the same way as in Theorems 1 and 2, we can show the following Theorems 3 and 4, whose proofs are omitted.

THEOREM 3. *Let $(p, a) \in C^1[0, 1] \times L^2(0, 1)$ be given and let $u=u(t, x)$ be the solution of $(E^3_{p,a})$. Then,*

$$\left.\frac{\partial v}{\partial x}\right|_{x=0,1}=\left.\frac{\partial u}{\partial x}\right|_{x=0,1} \qquad (T_1 \leqq t \leqq T_2) \tag{4.7}$$

for some $(q, b) \in C^1[0, 1] \times L^2(0, 1)$ implies

$$(q, b)=(p, a), \tag{4.8}$$

if and only if a is a generating element with respect to A^3_p, where $v=v(t, x)$ is the solution of $(E^3_{q,b})$.

THEOREM 4. *Let $(\alpha, a) \in C^3_+[0, 1] \times L^2(0, 1)$ be given, and let $u=u(t, x)$ be the* *solution of* $(E^4_{\alpha,a})$. *Suppose that* a *is a generating element with respect to* A^4_α. *Then, for any* $(\beta, b)\in C^3_+[0, 1]\times L^2(0, 1)$ *with*

$$\int_0^1 \frac{dx}{\sqrt{\beta(x)}}=\int_0^1 \frac{dx}{\sqrt{\alpha(x)}}, \tag{4.9}$$

the equality

$$\beta(\xi)\frac{\partial v}{\partial x}(t, \xi)=\alpha(\xi)\frac{\partial u}{\partial x}(t, \xi) \qquad (T_1\leqq t\leqq T_2\,;\ \xi=0, 1) \tag{4.10}$$

implies

$$(\beta, b)=(\alpha, a), \tag{4.11}$$

where $v=v(t, x)$ *is the solution of* $(E^4_{\beta,b})$.

Furthermore, if a *is not a generating element with respect to* A^4_α, *there exists some* $(\beta, b)\in C^3_+[0, 1]\times L^2(0, 1)$ *with* (4.9), *for which* $(\beta, b)\neq(\alpha, a)$ *in spite of* (4.10).

2. In Theorems 2 and 4, the conditions (1.9) and (4.9) are essential. Without them, (1.11) or (4.11) doesn't follow, even if a is a (weakly) generating element with respect to A^2_α or A^4_α. For example, let $u=u(t, x)$ be the solution of

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}u \qquad (0<t<\infty,\ 0<x<1), \tag{4.12}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=\left.\frac{\partial u}{\partial x}\right|_{x=1}=0 \qquad (0<t<\infty) \tag{4.13}$$

and

$$\begin{aligned} u|_{t=0}&=a(x) \\ &\equiv 1+\sum_{n=1}^{\infty}\frac{1}{(n\pi)^3}\cos n\pi x \qquad (0<x<1), \end{aligned} \tag{4.14}$$

and let $v=v(t, x)$ be the solution of

$$\frac{\partial v}{\partial t}=\frac{1}{9}\frac{\partial^2}{\partial x^2}v \qquad (0<t<\infty,\ 0<x<1), \tag{4.15}$$

$$\left.\frac{\partial v}{\partial x}\right|_{x=0}=\left.\frac{\partial v}{\partial x}\right|_{x=1}=0 \qquad (0<t<\infty) \tag{4.16}$$

and

$$\begin{aligned} v|_{t=0}&=b(x) \\ &\equiv 1+\sum_{n=1}^{\infty}\frac{1}{(n\pi)^3}\cos 3n\pi x \qquad (0<x<1). \end{aligned} \tag{4.17}$$

Then, we have

$$u(t,\ x)=1+\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\frac{1}{(n\pi)^3}\cos n\pi x \tag{4.18}$$

and

$$v(t,\ x)=1+\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\frac{1}{(n\pi)^3}\cos 3\,n\pi x\ , \tag{4.19}$$

so that

$$v(t,\ \xi)=u(t,\ \xi) \qquad (0<t<\infty\ ;\ \xi=0,\ 1)\ . \tag{4.20}$$

Furthermore, a is a generating element with respect to A_α^2 $(\alpha\equiv 1)$, whereas (1.11) doesn't hold.

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