Micro-local theory of boundary value problems II
—Theorems on regularity up to the boundary for reflective and diffractive operators—

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Abstract

We prove the solvability of boundary value problems for pseudo-differential operators which are semi-hyperbolic in one side of the boundary. We also prove a kind of regularity up to the boundary for operators anti-semi-hyperbolic in one side of the boundary and for diffractive operators. That is, the micro-analyticity of micro-local solutions of these operators propagates up to the boundary.

Introduction

In [17], we have introduced the notion of "mildness" for hyperfunctions on a real analytic boundary (say, $x_1=0$); that is, a subclass of hyperfunctions defined in one side of the boundary which have boundary values for any normal derivative of finite or infinite order. Furthermore we have developed several operations on mild hyperfunctions, and have simultaneously formulated them micro-locally. In this paper, employing these tools, we formulate boundary value problems in a micro-local situation and study solvability or a kind of regularity for them.

Set $M=\boldsymbol{R}^n \ni (x_1, x')$, $M_+=\{x\in M;\ x_1\geqq 0\}$ and $N=\{x\in M;\ x_1=0\}$. Let $P(x, D)=D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D')$ (where $D_j=\partial/\partial x_j$) be a differential operator of order m with real analytic coefficients defined on $\{x\in M;\ |x|<r\}$. Then, as seen in [17], every hyperfunction solution $f(x)$ of $Pf(x)=0$ on $\{x\in M;\ x_1>0, |x|<r\}$ is mild from the positive side of N on $\{x'\in N;\ |x'|<r\}$. So the boundary values $(D_1^j f)(+0, x')$ $j=0, 1, \cdots$ are always well-defined. But, once we attempt to formulate them using only the theory of microfunctions, we will confront two essential difficulties. Actually we can not define boundary values of microfunction solutions (say, defined on $\{(x;\ i\eta)\in iS^*M;\ x_1>0, |x|<r\}$) in general.

Even if so, these boundary values are not unique as microfunctions on N (consider the case $P=D_1^2+\cdots+D_n^2$). To avoid these difficulties we use the micro-localized notion of mildness, that is, the sheaf $\hat{\mathcal{C}}_{N|M_+}$ on iS^*N introduced in [17].

A germ $u(x)$ of $\hat{C}_{N|M_+}$ at $p_0'=(x_0'; i\eta_0')\in iS^*N$ defines a section of microfunctions on $\{(x; i\eta)\in iS^*M; r>x_1>0, |x'-x_0'|<r, |\eta'-\eta_0'|<r\}$ for some small $r>0$. However the converse is not true. Further, sections of pseudo-differential operators defined on $\{(z; \zeta)\in T^*X; z_1=0, (z'; \zeta')=p_0', \zeta_1\in \boldsymbol{C}\}$ operate on the stalk of $\hat{C}_{N|M_+}$ at p_0'. As seen in [17], if $P(x, D)$ is a pseudo-differential operator of order m of the form $D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D')$, the correspondence $\{u\in\hat{C}_{N|M_+}; Pu=0\}\to {}^t(u(+0, x'), \cdots, (D_1^{m-1}u)(+0, x'))\in(\mathcal{C}_N)^m$ is an injective sheaf homomorphism on iS^*N. Thus by way of $\hat{C}_{N|M_+}$ boundary value problems are micro-localized in a natural manner.

In the first section, we explicitly seek relationship among boundary values of micro-local solutions corresponding to elliptic factors. It has already been obtained implicitly in [15] and [26]. Then we construct the solution $u(x)$ in $(\hat{C}_{N|M_+})^k$ to the following system of pseudo-differential equations at $p_0'\in iS^*N$,

$$\begin{cases} (D_1I-A(x, D'))u(x)=f(x) \\ u(+0, x')=u_0(x'). \end{cases}$$

Here $A(x, D')$ is a (k, k)-matrix of first-order pseudo-differential operators defined at $(0, x_0'; i\eta_0')\in \boldsymbol{R}\times iS^*N$, $f(x)$ is a germ of $(\hat{C}_{N|M_+})^k$ and $u_0(x')$ is a germ of $(\mathcal{C}_N)^k$ at p_0' such that $D_1I-A(x, D')$ is semi-hyperbolic in $\{x_1>0\}$ at p_0'; that is, $\det(\zeta_1I-\sigma_1(A)(x, i\eta'))=0$ has no root with positive real part with respect to ζ_1 when $0\leqq x_1\leqq\varepsilon$, $|x'-x_0'|\leqq\varepsilon$, $|\eta'-\eta_0'|\leqq\varepsilon$ for some $\varepsilon>0$. Theorems of this type have been obtained by many authors ([3], [12], [29], [8], [23], [14]), though they assumed that P was a differential operator or P was micro-hyperbolic in both sides or $f=0$. In the proof we employ the argument of analytic continuation of defining functions due to Bony-Schapira and Kashiwara-Kawai ([3], [12]). Furthermore, employing the micro-local Green formula in [17], we obtain the dual version of this theorem. That is, for any $\hat{C}_{N|M_+}$-solution $u(x)$ at p_0' of $P(x, D)u(x)=0$ such that $P(x, D_1, -D')$ is semi-hyperbolic in $\{x_1>0\}$ at p_0', the micro-analyticity of $u(x)$ as a section of $\mathcal{C}_M$ in $\{x_1>0\}$ leads to the micro-analyticity of all the boundary values at p_0'. This is a generalization of the theorem by Kaneko [8], where P is a differential operator and $u(x)$ is a hyperfunction solution. For operators micro-hyperbolic in both sides ([26]) and for non-micro-characteristic operators ([27]), theorems of the same type have been obtained by Schapira.

In the second section, we prove the N_+-regularity of diffractive pseudo-differential operators. That is, assume that $P(x, D)=D_1^2+P_1(x, D')D_1+P_2(x, D')$ is a second-order pseudo-differential operator with real principal symbol defined at $p_0=(0, x_0'; i\eta_0)\in iS^*M\underset{M}{\times}N$ satisfying: $\sigma(P)(p_0)=\{\sigma(P), x_1\}(p_0)=0$, $\{\{\sigma(P), x_1\}, \sigma(P)\}(0, x_0', \eta_0)<0$ and $d\sigma(P)\wedge dx_1\wedge\omega(p_0)\neq 0$. Here $\{,\}$ is the Poisson bracket and ω is the fundamental 1-form. Then for every $\hat{C}_{N|M_+}$-solution $u(x)$ of $Pu(x)=0$

at $p_0'=(x_0'; i\eta_0')$, all the boundary values $(D_1^j u)(+0, x')$ $(j=0, 1)$ are micro-analytic at p_0' if and only if $u(x)$ is micro-analytic as a section of microfunctions on $\gamma_{p_0}-\{p_0\}(\subset\{x_1>0\})$. Here γ_{p_0} is the bicharacteristic strip through p_0 (cf. [21], [6], [28]). In the last step of the proof of this theorem we employ Bony's and Schapira's results ([1], [27]) on non-micro-characteristic pseudo-differential operators.

A remark added in proof: In a recent paper of J. Sjöstrand ([32]) a similar theorem for diffractive operators is obtained. In it, he considers solutions satisfying the Dirichlet condition. On the other hand our result covers any solutions. The methods are completely different from each other.

§ 1. Micro-local boundary value problems

Let $P(x, D)$ be a differential operator of order m defined in $M=\{x\in \boldsymbol{R}^n; |x|<r\}$. Suppose that $N=\{x\in M; x_1=0\}$ is non-characteristic with respect to P. Then, according to Komatsu-Kawai and Schapira's theory of boundary value problems (Komatsu-Kawai [19], Schapira [25]), any hyperfunction solution of $P(x, D)u=0$ in $\{x\in M; x_1>0\}$ has the unique extension $\tilde{u}(x)\in\Gamma_{M_+}(M, \mathcal{B}_M)$ and the "boundary values" ${}^t(f_0, \cdots, f_{m-1})\in\Gamma(N, \mathcal{B}_N)^m$ such that $\tilde{u}$ coincides with u in $\{x\in M; x_1>0\}$ and $P\tilde{u}=\sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1)$. On the other hand this is directly explained by the theory of mild hyperfunctions. In fact, since u is mild on N from the positive side of N (see [17]), $\operatorname{ext}(u)=u(x)Y(x_1)\in\Gamma_{M_+}(M, \mathcal{B}_M)$ is well defined and satisfies

$$P(x, D)\operatorname{ext}(u)=\sum_{j,k=0}^{m-1}\delta^{(j)}(x_1)Q_{jk}(x', D')(D_1^k u)(+0, x').$$

Here $\{Q_{jk}(x', D')\}$ are differential operators of order less than $m-j-k$ induced by $P(x, D)$ and N. So we know that $\tilde{u}=\operatorname{ext}(u)$ and $f_j(x')=\sum_{k=0}^{m-1} Q_{jk}(x', D')(D_1^k u)(+0, x')$. Furthermore, by the theory of the sheaf $\hat{\mathcal{C}}_{N|M_+}$ and the exact sequence

$$0\longrightarrow \mathcal{A}_M|_N\longrightarrow \hat{\mathcal{B}}_{N|M_+}\longrightarrow \pi_{N*}\hat{\mathcal{C}}_{N|M_+}\longrightarrow 0,$$

we can treat the solution $u(x)$ or the boundary values $(f_0, \cdots, f_{m-1})$ micro-locally on iS^*N (recall that ext and Trace are defined also for the sections of $\hat{\mathcal{C}}_{N|M_+}$). For example, on account of the above exact sequence and the Cauchy-Kowalewsky theorem, the local hyperfunction solution $u(x)$ of the problem

$$\begin{cases} P(x, D)u(x)=0 & x_1>0 \\ \partial^j u/\partial x_1^j(+0, x')=f_j(x') & j=0, \cdots, m-1 \end{cases} \tag{1.1}$$

for given hyperfunctions $(f_0(x'), \cdots, f_{m-1}(x'))$ exists if and only if the problem (1.1) has a $\hat{C}_{N|M_+}$-solution at every point of iS^*N (such a solution is unique at every point of iS^*N because $P(x, D)\operatorname{ext}(u)$ is uniquely determined by $(f_0, \cdots, f_{m-1})$ as is shown below). The operator $P(x, D)$ is also micro-localizable, too. Indeed $\hat{C}_{N|M_+}$ is a $\iota_*\mathcal{P}_X$-module which contains $\pi_N{}^{-1}\mathcal{D}_X$ as a subsheaf. Thus the micro-local boundary value problems for pseudo-differential operators are formulated on iS^*N.

DEFINITION 1.1. *$P(x, D)\in\iota_*\mathcal{P}^f_X(\iota: S^*_NX\backslash S^*_YX\to iS^*N)$ is called to have N as a non-characteristic hypersurface* if and only if the map $\iota: (S^*_NX\backslash S^*_YX)\cap\{\sigma(P)=0\}\to iS^*N$ is proper.

Easily to see, each fiber of $\iota: \{(0, x'; \zeta_1, i\eta')\in S^*_NX\backslash S^*_YX;\ \sigma(P)=0\}\to(x'; i\eta')\in iS^*N$ is finite and its number counting multiplicities is locally constant. Let m be this number. Then by Weierstrass' division theorem for pseudo-differential operators $P(x, D)$ is decomposed into the product $Q\cdot R$. Here Q and R are sections of $\iota_*\mathcal{P}^f_X$, Q is invertible and R has the following form:

$$R(x, D)=D_1^m+R_1(x, D')D_1^{m-1}+\cdots+R_m(x, D'),$$

with order $R_j(x, D')\leqq j$. Therefore we will study essentially the pseudo-differential operators of this form.

PROPOSITION 1.2. *Let $P(x, D)=D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D')$ (where order $P_j\leqq j$) be a section of $\iota_*\mathcal{P}^f_X$. Then the $\hat{C}_{N|M_+}$-solution of the problem:*

$$P(x, D)u(x)=0 \text{ and } \partial^ju/\partial x_1^j(+0, x')=f_j(x')\quad j=0, \cdots, m-1$$

for given microfunctions ${}^t(f_0, \cdots, f_{m-1})\in C_N^m$ is unique at every point of iS^*N if it exists.

PROOF. Let $u(x)$ be a germ of $\hat{C}_{N|M_+}$ such that $Pu=0$ and $(D_1^ju)(+0, x')=0$ for $j=0, 1, \cdots, m-1$. Then, easily to see, $P(x, D)\operatorname{ext}(u)=0$ holds as a section of $C_{M_+|X}$. Therefore by Proposition 1.2.1 in [17] we have $\operatorname{ext}(u)=0$.

Considering $\operatorname{ext}(u)$ instead of u, the analysis of micro-local boundary value problems is brought to the analysis on $S^*_{M_+}X$. One of the advantages of the micro-localization of the boundary value problems is that we can use not only pseudo-differential operators, but also quantized contact transformations keeping $S^*_{M_+}X$ fixed (see Theorem 4.2.17 in [16]). We give direct applications of these tools.

PROPOSITION 1.3 (Relationship among boundary values, cf. [15], [26]). *Let $P(x, D)=D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D')$* (with order $P_j\leqq j$) *be a section of $\iota_*\mathcal{P}^f_X$ and let s be the number of points of zeros $\{\zeta_1\in\boldsymbol{C};\ \sigma(P)(0, x_0'; \zeta_1, i\eta_0')=0,$*

$\operatorname{Re}\zeta_1>0\}$ *counting multiplicities for a point* $(x_0'; i\eta_0')\in iS^*N$. *Then there exist sections of* $\mathscr{P}_Y^f$, $Q_{jk}(x', D')$ $(m-s\leqq j\leqq m-1,\ 0\leqq k\leqq m-s-1)$, *defined in a neighborhood of* $(x_0'; i\eta_0')$ *such that every* $\hat{C}_{N|M_+}$*-solution* $u(x)$ *of* $Pu(x)=0$ *at* $(x_0'; i\eta_0')$ *satisfies the following equations.*

$$(1.2)\qquad (D_1^j u)(+0, x')=\sum_{k=0}^{m-s-1} Q_{jk}(x', D')(D_1^k u)(+0, x') \quad \text{for } m-s\leqq j\leqq m-1.$$

If $P(x, D)$ *is elliptic, that is,* $\{\zeta_1\in \boldsymbol{C};\ \sigma(P)(0, x_0'; \zeta_1, i\eta_0')=0\}\cap i\boldsymbol{R}=\varnothing$, *then* (1.2) *is a necessary and sufficient condition for the solvability of the boundary value problem* $P(x, D)u(x)=0$.

We call the equations (1.2) *"the relationship among boundary values".*

PROOF. Consider the canonical extension of u,

$$P(x, D)\operatorname{ext}(u)=\sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1).$$

By the division theorem for pseudo-differential operators $P(x, D)$ is decomposed into the product $P'P''$ of two pseudo-differential operators $P', P''\in \iota_*\mathscr{P}_X^f$ which have the following properties: P'' is invertible on $\{(0, x'; \zeta_1, i\eta')\in S_N^*X;\ x'=x_0', \eta'=\eta_0', \operatorname{Re}\zeta_1>0\}$ and $P'(x, D)=D_1^s+A_1(x, D')D_1^{s-1}+\cdots+A_s(x, D')$ (where order $A_j\leqq j$). Therefore we have $\sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1)\in P'(x, D)\Gamma(\{x'=x_0', \eta'=\eta_0', \operatorname{Re}\zeta_1>0\}, \mathcal{C}_{N|X})$. We may assume $\eta_{0,n}>0$ and apply the quantized contact transform β_n^+ (use the formula in Prop. 1.1.4 of [17]). Hence it follows that

$$\sum_{j=0}^{m-1}(i\zeta_1)^j(-D_n)^{j+1}f_j(x')\in \tilde{P}'(\zeta_1, x', D_{\zeta_1}, D_{x'})\Gamma(\{x'=x_0', \eta'=\eta_0', \operatorname{Re}\zeta_1>0\}, \mathcal{C}\mathcal{O}),$$

where $\tilde{P}'=\beta_n^+\cdot P'(\beta_n^+)^{-1}$ (see §1 in [17]). Now, using the formula (1.5) in the proof of Theorem 1.2.3 in [17] expressing the residue modulo $\tilde{P}'$, we obtain the following equation

$$\int_\gamma \sum_{r=0}^{s-1}(\zeta_1-w)^r\Big(\sum_{l=-\infty}^{s}\sum_{j=0}^{\infty}\frac{(-1)^j}{(r+j+1)!}\frac{\partial^{2j+r+1}\tilde{P}_l'}{\partial w^{j+r+1}\partial\tau^j}(w, x'; 0, D_{x'})\Big)$$
$$\times\Big\{(\tilde{P}')^{-1}\cdot\sum_{q=0}^{m-1}(iw)^q(-D_n)^{q+1}f_q(x')\Big\}dw=0.$$

Here $\tilde{P}_l'(w, x'; \tau, \zeta')$ is the homogeneous part of $\tilde{P}'(w, x', D_w, D_{x'})$ of order l and γ is a real analytic closed curve in $\boldsymbol{C}$ enclosing all the zeros of $\sigma(P')(0, x_0'; \zeta_1, i\eta_0')$ $((\eta_0')_n=+1)$. Since the left-hand side of this equation is a polynomial in ζ_1 of degree less than $s-1$, it reduces to the following pseudo-differential equations for

$(f_0, \cdots, f_{m-1})$.

$$\sum_{q=0}^{m-1} B_{tq}(x', D')f_q(x')=0 \quad \text{for} \quad t=0, \cdots, s-1, \tag{1.3}$$

where $\{B_{tq}(x', D')\}$ are pseudo-differential operators defined at $(x'_0; i\eta'_0)\in iS^*N$ given as follows:

$$B_{tq}(x', D')f(x')=\frac{\partial^t}{\partial\zeta_1^t}\left\{\int_\gamma \sum_{r=0}^{s-1}(\zeta_1-w)^r\left(\sum_{l=-\infty}^{s}\sum_{j=0}^{\infty}\frac{(-1)^j}{(r+j+1)!}\right.\right.$$
$$\left.\left.\times\frac{\partial^{2j+r+1}\tilde{P}'_l}{\partial w^{j+r+1}\partial\tau^j}(w, x'; 0, D_{x'})\right)((\tilde{P}')^{-1}(iw)^q(-D_n)^{q+1}f(x'))dw\right\}_{\zeta_1=0}.$$

Easily to see, the order of B_{tq} is less than $q+1$ and

$$\sigma_{q+1}(B_{tq})(x'; \zeta')=\frac{\partial^t}{\partial\zeta_1^t}\left\{\int_\gamma \sum_{r=0}^{s-1}(\zeta_1-w)^r\frac{1}{(r+1)!}\frac{\partial^{r+1}\tilde{P}'_s}{\partial w^{r+1}}(w, x'; 0, \zeta')\right.$$
$$\left.\times(\tilde{P}'_s(w, x'; 0, \zeta'))^{-1}(iw)^q(-\zeta_n)^{q+1}dw\right\}_{\zeta_1=0}.$$

Let $a_1(x', \zeta'), \cdots, a_s(x', \zeta')$ be the zeros of $\tilde{P}'_s(w, x'; 0, \zeta')=\sigma(\tilde{P}')(w, x'; 0, \zeta')=\sigma(P')(0, x'; -i\zeta_n w, \zeta')$ with respect to w. Then, noting that $\tilde{P}'_s(w, x'; 0, \zeta')=(-i\zeta_n)^s(w-a_1(x', \zeta'))\cdots(w-a_s(x', \zeta'))$, we have

$$\sigma_{q+1}(B_{tq})(x', \zeta')=(-\zeta_n)^{q+1}\frac{\partial^t}{\partial\zeta_1^t}\left\{\int_\gamma (iw)^q\prod_{j=1}^{s}(w-a_j(x', \zeta'))^{-1}\right.$$
$$\left.\times\sum_{r=0}^{s-1}\frac{(\zeta_1-w)^r}{(r+1)!}\frac{\partial^{r+1}}{\partial w^{r+1}}\left(\prod_{j=1}^{s}(w-a_j(x', \zeta'))\right)dw\right\}_{\zeta_1=0}$$
$$=(-\zeta_n)^{q+1}\frac{\partial^t}{\partial\zeta_1^t}\left\{\int_{|w|=R}(iw)^q\prod_{j=1}^{s}(w-a_j(x', \zeta'))^{-1}(\zeta_1-w)^{-1}\right.$$
$$\left.\times\left(\prod_{j=1}^{s}(\zeta_1-a_j)-\prod_{j=1}^{s}(w-a_j)\right)dw\right\}_{\zeta_1=0}.$$

Here R is a sufficiently large number such that $|a_j|<R$ for every j. Therefore,

$$\sigma_{q+1}(B_{tq})(x', \zeta')=2\pi(-i\zeta_n)^{q+1}q!\,\delta_{tq} \quad \text{for} \quad 0\leqq\forall q\leqq s-1. \tag{1.4}$$

Hence the pseudo-differential equations in (1.3) are solvable with respect to $(f_0, \cdots, f_{s-1})$. That is, there exist pseudo-differential operators $C_{jk}(x', D')\in\mathscr{P}^f_Y|_{(x'_0; i\eta'_0)}$ for $j=0, \cdots, s-1$, $k=s, \cdots, m-1$ such that (1.3) is equivalent to the following equations:

$$f_j(x')=\sum_{k=s}^{m-1} C_{jk}(x', D')f_k(x') \quad \text{for} \quad j=0, \cdots, s-1. \tag{1.5}$$

To obtain the relationship among $u_j(x')=(D_1^j u)(+0, x')$, we write $f_j(x')$ as linear combinations of $u_0, \cdots, u_{m-1}$:

$$f_j(x')=u_{m-j-1}(x')+\sum_{k+l\leq m-j-2}(-1)^l\binom{j+l}{j}\frac{\partial^l P_{m-j-k-l-1}}{\partial x_1^l}(0, x', D')u_k(x'). \tag{1.6}$$

(Use the formula: $D_1^k(Y(x_1)f(x))=Y(x_1)D_1^k f(x)+\sum_{j=1}^{k} D_1^{j-1}(\delta(x_1)D_1^{k-j}f(x))$.) Remark that this equation is solvable with respect to $u_0, \cdots, u_{m-1}$ in the following way.

$$\begin{aligned} u_j(x')=f_{m-j-1}(x')+E_{j,m-j}(x', D')f_{m-j}(x')+\cdots \\ +E_{j,m-1}(x', D')f_{m-1}(x') \quad \text{for every} \quad j. \end{aligned} \tag{1.7}$$

Combination of (1.5)~(1.7) yields the desired relationship (1.2). Conversely, assume the relationship (1.2) among the boundary values $u_0(x'), \cdots, u_{m-1}(x')$. Then $v(x)=P(x, D)^{-1}\left(\sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1)\right)$ is well-defined as a section of $\mathcal{C}_{N|X}$ on $\{(0, x_0'; \zeta_1, i\eta_0')\in S_N^*X;\ \sigma(P)(0, x_0', \zeta_1, i\eta_0')\neq 0 \text{ or } \mathrm{Re}\,\zeta_1>0\}$ (where $\{f_j\}_j$ are defined by (1.6)). Therefore, if $P(x, D)$ is elliptic, $v(x)$ is extended to $\{(0, x_0'; \zeta_1, i\eta_0');\ \sigma(P)(0, x_0', \zeta_1, i\eta_0')\neq 0 \text{ or } \mathrm{Re}\,\zeta_1\geqq 0\}$ as a section of $\mathcal{C}_{N|X}$. In particular, $v(x)$ defines a germ $u(x)$ of $\hat{\mathcal{C}}_{N|M_+}$ at $(x_0'; i\eta_0')$ (recall that $\hat{\mathcal{C}}_{N|M_+}=\iota_*^\dagger\mathcal{C}_{M_+|X}\cap\mathcal{C}_{N|X}^\infty|_{iS^*N\times\{\infty\}}/\iota_*\mathcal{C}_{N|X}$ in Def. 2.1.5 of [17]). It is easy to see that $Pu(x)=0$ and $(D_1^j u)(+0, x')=u_j(x')$ for $j=0, \cdots, m-1$. Thus the proof is completed.

LEMMA 1.4. *Let $(0, y_0'; i\tau_0)$ and $(0, x_0'; i\eta_0)$ be two points in $S_N^*X\cap iS^*M$ and Φ be a real quantized contact transformation from a neighborhood of $(0, y_0'; i\tau_0)$ into a neighborhood of $(0, x_0'; i\eta_0)$. Put $S^j(x, D_x)=\Phi y_j\Phi^{-1}$ and $R^j(x, D_x)=\Phi D_{y_j}\Phi^{-1}$. Assume that $S^1(x, D_x)\in\mathcal{P}_X\cdot x_1$. Then Φ defines sheaf isomorphisms: $\mathcal{C}_{N|X}\xrightarrow{\sim}\mathcal{C}_{N|X}$, $\mathcal{C}_{M_+|X}\xrightarrow{\sim}\mathcal{C}_{M_+|X}$. Let $k(x', y')$ be the kernel function of the real quantized contact transformation Φ' in N induced by Φ in the following way:*

$$\Phi' y_j\Phi'^{-1}=S^j(*, D_{x'}, 0, x'),\ \Phi' D_{y_j}\Phi'^{-1}=R^j(*, D_{x'}, 0, x') \tag{1.8}$$

for $j=2, \cdots, n$. Here $S^j(D_x, x)$ or $R^j(D_x, x)$ are the transposed normal expressions of S^j or R^j respectively; that is, all x-operators in each term are disposed in the place latter than D_x-operators. Then the following formula holds:

$$\Phi(\delta(y_1)f(y'))=\delta(x_1)\cdot\int k(x', y')f(y')dy' \quad \textit{for} \quad \forall f(y')\in\mathcal{C}_N. \tag{1.9}$$

In fact, the contact transformation induced by Φ keeps S_N^*X fixed and so Φ

defines sheaf isomorphisms $\mathcal{C}_{N|X} \xrightarrow{\sim} \mathcal{C}_{N|X}$ or $\mathcal{C}_{M_+|X} \xrightarrow{\sim} \mathcal{C}_{M_+|X}$ (see Lemma 4.2.13 and Theorem 4.2.17 in [16]). Furthermore the operators defined by (1.8) commute with x_1 and satisfy the relationship: $[R^j(*, D_{x'}, 0, x'), S^k(*, D_{x'}, 0, x')] = \delta_{jk}, \cdots$ etc. Therefore these operators define a real quantized contact transformation Φ' in N.

PROOF. Let $K(x, y)$ be the kernel function of Φ. Then $\Phi(\delta(y_1) \times \delta(y' - \tilde{y}'))$ is given by $K(x, 0, \tilde{y}')$. Use the theory on holonomic systems. We omit the details.

REMARK. An arbitrary real quantized contact transformation keeping S_N^*X fixed is written as the composite of an inner automorphism and a quantized contact transformation as above.

The following example is not covered by the proposition above. Nevertheless, the micro-local Green formula is available to calculate the relationship between the boundary values.

EXAMPLE 1.5. The following boundary value problem (where $k=0, 1, \cdots$) is solvable micro-locally at $(x_0'; i\eta_0') \in iS^*N$ (with $\eta_{0,2} > 0$) from the positive side of N

$$\text{(I)} \qquad \begin{cases} Pu = (D_1^2 + x_1^k D_2^2) u(x) = 0 & x_1 > 0, \\ (D_1^j u)(+0, x') = u_j(x') & j = 0, 1, \end{cases}$$

if and only if the relationship

$$u_1(x') + \frac{\Gamma(1 - 1/(k+2))}{\Gamma(1 + 1/(k+2))} \left(\frac{1}{i(k+2)} D_2 \right)^{2/(k+2)} u_0(x') = 0 \tag{1.10}$$

holds at $(x_0'; i\eta_0')$. Thus a fractional order derivative appears contrary to the above proposition. Indeed the adjoint equation

$$\text{(II)} \qquad \begin{cases} {}^tPv = (D_1^2 + x_1^k D_2^2) v(x, y') = 0 & x_1 > 0 \\ v(+0, x', y') = \delta(x' - y') \end{cases}$$

has a hyperfunction solution

$$v(x, y') = C_k \cdot \delta(x_3 - y_3) \cdots \delta(x_n - y_n) \times \int_{-\infty}^{+\infty} |\eta|^{1/(k+2)} \sqrt{x_1} H_{1/k+2}^{(1)} \left(\frac{2i|\eta|}{k+2} x_1^{(k+2)/2} \right) e^{i\eta(x_2 - y_2)} d\eta,$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind and C_k is a constant depending only on k. Since the singular support of

$$\partial v / \partial x_1(+0, x', y') = C_k' \cdot \delta(x'' - y'') \times \int_{-\infty}^{+\infty} |\eta|^{2/(k+2)} e^{i\eta(x_2 - y_2)} d\eta$$

is also contained in $\{(x', y'; i\eta', i\tau'); x'=y', \eta'+\tau'=0\}$, we can apply the microlocal Green formula (see [17]). Therefore $\int u_0(x')(D_1 v)(+0, x', y')dx' - \int u_1(x')v(+0, x', y')dx'=0$ holds at $(x'_0; i\eta'_0)$. This is just the relationship (1.10). The sufficiency is also proved by using $v(x, y')$ as the fundamental solution of (I).

Next, we treat pseudo-differential operators which are semi-hyperbolic in one side of the boundary (cf. [3], [12], [14], [8], [23]).

DEFINITION 1.6. Let $P(x, D)=D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D')$ (order $P_j \leqq j$) be a section of $\iota_* \mathcal{P}_X^f$. $P(x, D)$ is said to be semi-hyperbolic in the positive side of N at $(x'_0; i\eta'_0) \in iS^*N$ if the equation $\sigma(P)(x; \zeta_1, i\eta')=0$ with respect to ζ_1 has no root with positive real part when $\varepsilon \geqq x_1 \geqq 0$ and $|x'-x'_0| \leqq \varepsilon$, $|\eta'-\eta'_0| \leqq \varepsilon$ for some positive constant ε. We employ the same terminology for an equation or for a symbol corresponding to such an operator.

As typical examples, we have $P=D_1^2-x_1^k D_2^2$ at $(0; \pm i dx_2)$, $P=D_1-i(x_1+x_2^2)D_2$ at $(0; i dx_2)$ etc.

REMARK. The terminology of semi-hyperbolicity is introduced by A. Kaneko in [9]. This notion is deeply connected with "partial micro-hyperbolicity" defined by Kashiwara-Kawai in [12]. That is, an operator semi-hyperbolic in the positive side of N at $(x'_0; i\eta'_0) \in iS^*N$ is partially micro-hyperbolic with respect to the dx_1-direction on $\{(x; i\eta) \in iS^*M; 0<x_1<\varepsilon, \eta' \neq 0, |x'-x'_0|<\varepsilon, |\eta'-\eta'_0|<\varepsilon\}$ for some $\varepsilon>0$.

Now we give a generalization of the results on the solvability of boundary value problems obtained by many authors (see Introduction). We first recall the following lemma.

LEMMA 1.7. *Let U be an open subset of iS^*N with proper convex fibers. Then the following $\mathcal{D}_X$-sheaf isomorphism holds* (cf. Proposition 2.1.21 in [17]).

$$(\pi_N|_U)_! R^1 \iota_! \mathcal{C}_{N|X} \xleftarrow{\sim} (\tau_N|_{U^\circ})_* (\tilde{\mathcal{A}}_{M+}|_{F_0}) / \mathcal{A}_M|_N ,$$

where $F_0=F_+ \cap F_- = \{(0, x'; w_1, iv') \in S_N X; w_1=0\} \subset iSM$ and particularly $\tilde{\mathcal{A}}_{M+}|_{F_0} = \tilde{\mathcal{A}}_{M-}|_{F_0}$ holds (see Definition 2.1.15 in [17]).

PROOF. Calculate $\boldsymbol{R}(\pi_N|_U)_! \boldsymbol{R}\iota_! \mathcal{C}_{N|X}$ in the same way as in Proposition 2.1.21 in [17]. We omit the details.

THEOREM 1.8. *Let $A(x, D')$ be a $k \times k$-matrix of pseudo-differential operators of order less than* 1 *defined in a neighborhood of* $(0, x'_0; i\eta'_0) \in \boldsymbol{R} \times iS^*N$. *Suppose that* $\det(\zeta_1 I_k - \sigma_1(A(x, D')))$ *is semi-hyperbolic in the positive side of N at $(x'_0; i\eta'_0)$. Then for every germ $f(x)={}^t(f_1(x), \cdots, f_k(x)) \in \mathcal{C}^k_{N|M_+}$ and every data $v(x')={}^t(v_1(x'), \cdots, v_k(x')) \in \mathcal{C}^k_N$ at $(x'_0; i\eta'_0)$, there exists a unique solution $u(x)={}^t(u_1(x), \cdots, u_k(x))$*

$\in \hat{C}^k_{N|M_+}$ *such that:*

$$\begin{cases} (D_1 I_k - A(x, D'))u(x) = f(x) \\ u(+0, x') = v(x') \end{cases} \qquad at\ (x'_0;\ i\eta'_0).$$

REMARK. Easily to see, inhomogeneous initial value problems for single pseudo-differential operators which are semi-hyperbolic in the positive side of N are reduced to this theorem.

PROOF. Without loss of generality we may assume that $v(x')=0$, $(x'_0;\ i\eta'_0) =(0;\ idx_2)$ and that $A(x, D')$ is expanded into the following power series

$$A(x, D') = \sum_{L=(l_2,\cdots,l_n)} a_L(x) D^L_{x'}.$$

Here L moves over all multi-indices such that $l_2 \in \boldsymbol{Z}$, $l_3 \geqq 0, \cdots, l_n \geqq 0$ and $|L| = l_2 + \cdots + l_n \leqq 1$. $\{a_L(z)\}$ are holomorphic functions defined in

$$\Omega = \{z \in \boldsymbol{C}^n;\ |z_1| < R,\ |z'| < R\}$$

satisfying

$$\sup_{\Omega} |a_L(z)| \leqq B(1-|L|)!\ b^{|L|} c^{1_2 - |L|},$$

where $|a_L(z)| = \max_{p,q} |a^{p,q}_L(z)|$ and B, b, c are positive constants (cf. [4] §2 and [12] §2). In particular $A(z, D')$ is a section on $\{(z;\ \zeta) \in P^*X;\ |z_1| < R,\ |z'| < R,\ |\zeta_j| < c|\zeta_2|$ for $j=3, \cdots, n\}$. Furthermore by the semi-hyperbolicity in the positive side of N we may assume that:

(1.11) $\det(\zeta_1 I_k - \sigma(A)(z, \zeta'))$ never vanishes on $\{(z;\ \zeta) \in S^*X;\ y_1 = 0,\ 0 \leqq x_1 < R,\ |z'| < R,\ |\zeta_j| < c|\zeta_2|$ for $j=3, \cdots, n$, $-\mathrm{Im}(\zeta_1/\zeta_2) > B(|y'| + \sum_{j=3}^{n} |\mathrm{Im}(\zeta_j/\zeta_2)|)\}$.

(Apply the ordinary hyperbolic inequality to $\det(\zeta_1 I_k - \sigma(A)(z_1^2, z_2, \cdots, z_n, \zeta'))$; [3], [12]). By the softness of $\hat{C}_{N|M_+}$, it may be assumed that $f(x)$ is a section of $\hat{C}^k_{N|M_+}$ on $\{|x'| < R\}$ with support in

$$U = \{(x';\ i\eta');\ |x'| < r,\ |\eta_j| < r\eta_2 \quad \text{for } j=3, \cdots, n\},$$

where $r < \min(c/2, b/6, R/3)$ is taken small enough later (depending only on R, b, c, n). Since $D_1 I_k - A(x, D')$ is invertible on $\{(0, x';\ \zeta_1, i\eta') \in S^*_N X;\ |x'| < R,\ |\eta_j| < \frac{1}{2} c\eta_2$ for $j=3, \cdots, n$ and $|\zeta_1| > T\eta_2\}$ for a sufficiently large number T, $u'(x) = (D_1 I_k - A(x, D'))^{-1} \operatorname{ext}(f(x))$ is defined as a section of $C^k_{N|X}$ on $\{(0, x';\ \zeta_1, i\eta') \in S^*_N X;\ |x'| < R,\ |\zeta_1| > T|\eta'|\}$ with support in $\{(x';\ i\eta') \in U,\ |\zeta_1| > T|\eta'|\}$. By way of the $\iota_* \mathcal{P}_X$-sheaf homomorphisms

$$\mathcal{C}^{\infty}_{N|X}|_{iS^*N\times\infty} \longrightarrow \mathcal{C}^{\infty}_{N|X}|_{iSN^*\times\infty}/\iota_*\mathcal{C}_{N|X} \hookrightarrow R^1\iota_!\mathcal{C}_{N|X},$$

$u'(x)$ (modulo $(\iota_*\mathcal{C}_{N|X})^k$) is identified with a section of $(R^1\iota_!\mathcal{C}_{N|X})^k$ on $\{(x'; i\eta'); |x'|<R\}$ with support in U. By the preceding lemma and the cohomological triviality of the sheaf $\mathcal{A}_M|_N$, we have a section $G(z)$ of $(\tilde{\mathcal{A}}_{M+}|_{F_0})^k$ on $\{(x'; iv')\in iSN; |x'|<R, (x'; iv')\in U^\circ\}=\{(x'; iv'); r\leqq|x'|<R\}\cup\{(x'; iv'); |x'|<R, v_2\geqq r(|v_3|+\cdots+|v_n|)\}$ such that $\left[-\frac{1}{2\pi i}G(z)\log z_1\right]$ coincides with u' as a section of $(\mathcal{C}^{\infty}_{N|X}|_{iS^*N\times\infty})^k$ modulo $(\iota_*\mathcal{C}_{N|X})^k$. On the other hand by Prop. 2.1.21 [17], $f(x)$ is identified with a section $F(z)$ of $(\tilde{A}_{M+})^k$ on $\{(x'; i\eta'); |x'|<R, v_2\geqq r(|v_3|+\cdots+|v_n|)\}$ modulo $(\mathcal{A}_M|_N)^k$. Especially for sufficiently small numbers R', R'' $G(z)$ is holomorphic on $\left\{z\in\boldsymbol{C}^n; |x'|<\frac{2}{3}R, R'>y_2>r\left(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''}\right)\right\}\cup\left\{z\in\boldsymbol{C}^n; r<|x'|<\frac{2}{3}R, z_1=0, y'=0\right\}$ and $F(z)$ is holomorphic on $\left\{z\in\boldsymbol{C}^n; |x'|<\frac{2}{3}R, |x_1|<R', |y|<R', R'>y_2>r\left(|y_3|+\cdots+|y_n|+\frac{|y_1|}{R''}+\frac{|x_1|}{R''}Y(-x_1)\right)\right\}\cup\left\{z\in\boldsymbol{C}^n; r<|x'|<\frac{2}{3}R, z_1=0, y'=0\right\}$. From a technical reason we divide $G(z)$ into a sum $G'(z)+G''(z)$ of vectors of holomorphic functions such that $G'(z)$ is holomorphic on $D'=\left\{z\in\boldsymbol{C}^n; |x'|<\frac{2}{3}R, R'>y_2>r\left(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''}\right)\right\}\cup\left\{z\in\boldsymbol{C}^n; |x'|<\frac{2}{3}R, y_2>2r\left(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''}\right)-R', y_2\geqq R'\right\}$ and G'' is holomorphic on $D''=\left\{z\in\boldsymbol{C}^n; |x'|<\frac{2}{3}R, R'>y_2>2r\left(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''}\right)-R'\right\}$. In fact because $D'\cup D''=\left\{z\in\boldsymbol{C}^n; |x'|<\frac{2}{3}R, y_2>2r\left(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''}\right)-R'\right\}$ is a Stein domain, this is possible. From the assumption on $G(z)$ it follows that $G'(z)$ is holomorphic on $D=\left\{z\in\boldsymbol{C}^n; |x'|<\frac{2}{3}R, y_2>2r\left(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''}\right)\right\}\cup\left\{r<|x'|<\frac{2}{3}R, z_1=0, y'=0\right\}$. Furthermore the boundary value of $G'(z)$ coincides with $u'(x)$ as a section of $(R^1\iota_!\mathcal{C}_{N|X})^k$ on $\left\{(x'; i\eta'); |x'|<\frac{2}{3}R\right\}$. So, from now on, we use $G'(z)$ instead of $G(z)$.

Now we recall the operations of pseudo-differential operators on holomorphic functions ([4], [12]). Let $H(z)$ be a vector-valued holomorphic function defined in an open set W. Then for a positive number ε,

$$A^\varepsilon(z, D')H(z)=\sum_{l_2\geq 0}a_L(z)D_{z'}^L H(z)+\sum_{l_2\leq -1}\frac{a_L(z)}{(|l_2|-1)!}\int_{i\varepsilon}^{z_2}(z_2-s)^{|l_2|-1}$$

$$\times D_{z_3}^{l_3}\cdots D_{z_n}^{l_n}H(z_1, s, z_3, \cdots, z_n)ds$$

is well defined if $z\in W\cap\Omega$, $|z_2-i\varepsilon|<b$ and $\gamma\left(\{z\}\cup\left\{w\in\boldsymbol{C}^n; w_1=z_1, w_2=i\varepsilon,\right.\right.$

$|w_j-z_j|\leqq\frac{2}{c}|z_2-i\varepsilon|\ \ j=3,\cdots,n\}\Big)\subset W$, where γ denotes the convex hull (see [4]). Set $\varepsilon=\min(b/2, cR/6n)$. Then $A^\varepsilon(z, D')G'(z)$ is holomorphic on $\Big\{z\in\boldsymbol{C}^n;\ |z_1|<\delta$ $|x'|<2r,\ |y'|<r,\ y_2>2r\Big(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''}\Big)\Big\}\cup\{z\in\boldsymbol{C}^n;\ z_1=0,\ y'=0,\ r<|x'|$ $<2r\}$ if $0<\delta<R$, $|\varepsilon+3r|<b$, $2r+\frac{2}{c}\sqrt{n}(3r+\varepsilon)<\frac{1}{2}R$ and $\frac{\delta}{R''}+\sqrt{n}\Big(r+\frac{2}{c}\sqrt{n}$ $\cdot(3r+\varepsilon)\Big)<\frac{\varepsilon}{16r}$. Easily to see, we can take r and δ satisfying these conditions. Note that the boundary value of $A^\varepsilon(z, D')G'(z)$ is equal to $A(x, D')u'(x)$ as a section of $(R^1\iota_!\mathcal{C}^k_{N|X})$ on $\{(x';i\eta');\ |x'|<2r\}$. So $(D_1I_k-A^\varepsilon(z, D'))G'(z)-F(z)$ is holomorphic on $\{z\in\boldsymbol{C}^n;\ z_1=0,\ y'=0,\ |x'|<2r\}$. Consequently for a sufficiently small positive number r_0 $\Big(r_0<\frac{10nr}{c}(3r+\varepsilon)$, and $r_0<r\Big)$, $G'(z)$ is holomorphic on

$$V=\Big\{z\in\boldsymbol{C}^n;\ |x'|<\frac{2}{3}R,\ y_1=0,\ y_2>3r\Big(\sqrt{n}\,|y''|+\frac{|x_1|}{R''}\Big)\Big\}$$

$$\cup\Big\{\frac{5}{4}r<|x'|<\frac{R}{2},\ y_1=0,\ y_2>3r\Big(\sqrt{n}\,|y''|+\frac{|x_1|}{R''}\Big)-r_0\Big\},$$

and $(D_1I_k-A^\varepsilon(z, D'))G'(z)$ is holomorphic on

$$\Big\{z\in\boldsymbol{C}^n;\ |x'|<\frac{3}{2}r,\ y_1=0,\ 0\leqq x_1<r_0,\ r_0>y_2>3r\sqrt{n}\,|y''|\Big\},$$

where $y''=(y_3,\cdots,y_n)$. Now using the assumption (1.11) on the semi-hyperbolicity of P, we will show that $G'(z)$ is extended analytically to $\Big\{z\in\boldsymbol{C}^n;$ $0\leqq x_1<r_1,\ |x'|<\frac{3}{2}r,\ y_1=y_3=\cdots=y_n=0,\ 0<y_2<r_1\Big\}$ for a sufficiently small number $r_1>0$. By Lemma 2.2.7 in [17] this implies that the boundary value $u(x)=G'(x_1, x_2+i0, x_3,\cdots,x_n)$ of $G'(z)$ defines a mild hyperfunction from the positive side of N. Hence $u(x)$ is the solution in $\hat{\mathcal{C}}^k_{N|M+}$ of $(D_1I_k-A(x, D'))u(x)=f(x)$ at $(0;idx_2)$. To do so, we employ Lemma 4.3 in [12]. Consider a family of real analytic functions which are convex with respect to y'':

$$\varphi_\lambda(x_1, y_3,\cdots,y_n)=\{4r\sqrt{n}+(e^{Bx_1}-1)(4r\sqrt{n}+1)\}\sqrt{|y''|^2+\lambda^2}+\lambda\Big(\frac{1}{\beta-x_1}-\frac{1}{\beta}\Big)$$

for $1\geqq\lambda>0$, $\beta>0$. We claim that $G'(z)$ is holomorphic on $V\cup\bigcup_{0<\lambda\leqq1}\{z\in\boldsymbol{C}^n;\ y_1=0,$ $y_2=\varphi_\lambda,\ 0\leqq x_1<\beta,\ |x'|<R/2\}\supset\{z\in\boldsymbol{C}^n;\ y_1=0,\ y''=0,\ y_2>0,\ 0\leqq x_1<\beta,\ |x'|<R/2\}$ if the following conditions 1)~5) are all satisfied: Set $S_\lambda=\{z\in\boldsymbol{C}^n;\ y_1=0,\ y_2=\varphi_\lambda,$ $0\leqq x_1<\beta,\ |x'|<R/2\}$.

1) $S_1\subset V$; this is satisfied if $\beta^2<R''/3r$.

2) $S_\lambda \cap \{|x'|>5r/4 \text{ or } |y''|>\mu\} \subset V$, where μ is a constant to be specified below; this is satisfied if $\beta<r_0R''/3r$ and $3\beta/\sqrt{n}\, R''<\mu$.

3) $S_\lambda \setminus V \subset \{y_1=0,\ 0\leqq x_1<r_0,\ |x'|\leqq 5r/4,\ 3r\sqrt{n}\,|y''|<y_2<r_0\}$ for $\forall\lambda\in(0, 1]$; this is satisfied if $\beta<r_0$ and $3r\left(\sqrt{n}\,\mu+\dfrac{\beta}{R''}\right)<r_0$.

4) For every $z^0\in S_\lambda\setminus V$, $\gamma(\{z^0\}\cup\{w\in \boldsymbol{C}^n;\ w_1=z_1^0,\ w_2=i\varepsilon,\ |w_j-z_j^0|\leqq \dfrac{2}{c}|z_2^0-i\varepsilon|$ for $j=3, \cdots, n\})\subset V\cup \bigcup_{\lambda\leqq\lambda'\leqq 1} S_{\lambda'}$; taking account of the convexity of $\{y_2=\varphi_\lambda\}\cap\{z_1=z_1^0\}$ and the inequality $r_0+\dfrac{16rn}{c}(\varepsilon+2r+r_0)<\varepsilon$, this is satisfied if $(e^{B\beta}-1)\cdot(4r\sqrt{n}+1)<4r\sqrt{n}$.

5) At every point of $S_\lambda\setminus V$, the inequality $\partial\varphi_\lambda/\partial x_1>B(|y''|+\varphi_\lambda)$ holds (see Lemma 4.3 [12]), where B is the constant appearing in (1.11); this is satisfied if $\beta<1/B$.

Clearly we can take β and μ so that they satisfy all the conditions 1)~5). So by using Lemma 4.3 in [12] and Holmgren's argument, our claim is justified. Since $\operatorname{ext}(u)-(D_1I_k-A(x, D'))^{-1}\operatorname{ext}(f)\in\iota_*\mathcal{C}_{N|X}^k$, we have $(D_1I_k-A(x, D'))\operatorname{ext}(u)-\operatorname{ext}(f)\in(D_1I_k-A(x, D'))\cdot\iota_*\mathcal{C}_{N|X}^k$. On the other hand $(D_1I_k-A(x, D'))\operatorname{ext}(u)-\operatorname{ext}(f)=\operatorname{Trace}(u)\times\delta(x_1)$ always holds. Hence from the division theorem for $\mathcal{C}_{N|X}$ (matrix case) we obtain $\operatorname{Trace}(u)=0$. Thus the proof is completed.

COROLLARY 1.9 (Half solvability, cf. [8], [23]). *Let* $P(x, D)=D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D')$ *be a semi-hyperbolic pseudo-differential operator of order* m *in the positive side of* N *defined on* $\iota^{-1}((x_0';\ i\eta_0'))$. *Then the sheaf homomorphism*

$$P(x, D):\ \mathcal{C}_{M_+|X}\ni u \longrightarrow Pu\in\mathcal{C}_{M_+|X}$$

is isomorphic on $(\iota^+)^{-1}((x_0';\ i\eta_0'))$. *In particular when* $P(x, D)$ *is a hyperbolic differential operator in the positive side of* N *defined at* $(0, x_0')\in N$, *the sheaf homomorphism*

$$P(x, D):\ \mathcal{H}_{M_+}^0(\mathcal{B}_M)\ni u \longrightarrow Pu\in\mathcal{H}_{M_+}^0(\mathcal{B}_M)$$

is isomorphic at $(0, x_0')$.

PROOF. It suffices to show the solvability of $Pu=f$ in $\mathcal{C}_{M_+|X}$ at every point $p_0\in(\iota^+)^{-1}((x_0';\ i\eta_0'))\cap iS^*M$ for every germ $f\in\mathcal{C}_{M_+|X}$. Considering the surjectivity $\mathcal{H}_{M_+}^0(\mathcal{B}_M)\to\mathcal{C}_{M_+|X}/\mathcal{C}_{N|X}$ at p_0 (Proposition 4.2.10 in [16]), f is written as $f=[f']+g$ at p_0, where $f'\in\mathcal{H}_{M_+}^0(\mathcal{B}_M)$ at $(0, x_0')$ and $g\in\mathcal{C}_{N|X}|_{p_0}$. Now we consider $f'(x)$ as a section of $\mathcal{C}_{N|M_+}$. Then according to Proposition 2.1.10 in [17], the quantized Legendre transform $\beta_k^\varepsilon(f')(\zeta_1, x')$ $((x_0';\ i\eta_0')\in V_k^\varepsilon)$ is represented by a section $A(\zeta_1, x')$ of $\mathcal{BO}$ defined on $\{(\zeta_1, x')\in\boldsymbol{C}\times\boldsymbol{R}^{n-1};\ \operatorname{Re}\zeta_1>0,\ |x'-x_0'|<\delta\}$. Divide $A(\zeta_1, x')$ into a sum $A_1(\zeta_1, x')+A_2(\zeta_1, x')$ of sections of $\mathcal{BO}$, where A_1 is defined on $\{\zeta_1\in C;\ \mathrm{Re}\,\zeta_1>0$ or $|\zeta_1|>2|t_0|\}\times\{x';\ |x'-x'_0|<\delta\}$ and A_2 is defined on $\{\zeta_1\in C;\ \mathrm{Re}\,\zeta_1>0$ or $|\zeta_1|<2|t_0|\}\times\{x';\ |x'-x'_0|<\delta\}$ (where $p_0=(0,\ x'_0;\ it_0,\ i\eta'_0)$ and $\eta_k=\pm 1=\varepsilon$). Note that $(\beta_k^\varepsilon)^{-1}(A_2)\in\mathcal{C}_{N|X}|_{p_0}$ and $(\beta_k^\varepsilon)^{-1}(A_1)\in(\iota_*^{\ddagger}\mathcal{C}_{M+|X}\cap \mathcal{C}^\infty_{N|X}|_{iS^*N\times\infty})_{(x'_0;\ i\eta'_0)}$. Consequently f' is written as a sum $\mathrm{ext}(f'')+g'$ at p_0, where $f''\in\hat{\mathcal{C}}_{N|M_+}|_{(x'_0;\ i\eta'_0)}$ and $g'\in\mathcal{C}_{N|X}|_{p_0}$. So we have $f=\mathrm{ext}(f'')+(g+g')$ with $f''\in\hat{\mathcal{C}}_{N|M_+}|_{(x'_0;\ i\eta'_0)}$ and $g+g'\in\mathcal{C}_{N|X}|_{p_0}$. Furthermore by using the division theorem for $\mathcal{C}_{N|X}$, $g+g'$ is written as $Ph+\sum_{j=0}^{m-1}v_j(x')\delta^{(j)}(x_1)$ with $h\in\mathcal{C}_{N|X}|_{p_0}$ and $(v_j)_j\in \mathcal{C}_N^m|_{(x'_0;\ i\eta'_0)}$. Thus the equation is reduced to $Pu=\mathrm{ext}(f'')+\sum_{j=0}^{m-1}v_j(x')\delta^{(j)}(x_1)$. This is solved by the preceding theorem.

In the rest of this section we shall treat the problems of propagation of micro-analyticity of solutions up to the boundary. First of all we formulate these problems in a micro-local view point.

DEFINITION 1.10. Let $P(x,D)=D_1^m+P_1(x,D')D_1^{m-1}+\cdots+P_m(x,D')$ be a section of $\iota_*\mathcal{P}_X^f$ of order m. *Then* $P(x,D)$ *is said to be* N_+*-regular* (N_-*-regular*) *at* $p_0\in iS^*M\underset{M}{\times}N-S_Y^*X$ if the following condition is fulfilled: If a germ $u(x)$ of $\mathcal{C}_{M+|X}\cap\mathcal{H}^0_{iS^*M\underset{M}{\times}N}(\mathcal{C}_M)$ (resp. $\mathcal{C}_{M-|X}\cap\mathcal{H}^0_{iS^*M\underset{M}{\times}N}(\mathcal{C}_M)$) at p_0 satisfies $P(x,D)u\in\mathcal{C}_{N|X}$, then u belongs to $\mathcal{C}_{N|X}|_{p_0}$. We remark that this concept is invariant under quantized contact transformations keeping $S_{M_+}^*X$ fixed.

REMARK. P. Schapira defined the N-regularity in [26]. This is obtained by replacing $\mathcal{C}_{M+|X}$ by $\mathcal{C}_M$ in the above definition.

COROLLARY 1.11. *The operator* $P(x,D)$ *is* N*-regular* (see Schapira [26]) *at* p_0 *if and only if* P *is* N_+- and N_-*-regular at* p_0.

PROOF. Recall the definition of N-regularity. Then the assertion follows directly from the exact sequence,

$$0\longrightarrow\mathcal{C}_{N|X}\longrightarrow\mathcal{C}_{M+|X}\oplus\mathcal{C}_{M-|X}\longrightarrow\mathcal{C}_M\longrightarrow 0$$

at p_0 (see Proposition 4.2.10 in [16]).

The meaning of N_+-regularity is explained as follows (cf. Schapira [26]). We assume that $\sigma(P)$ has a zero of order s at $p_0=(0,\ x'_0;\ i\eta_{0,1},\ i\eta'_0)$ with respect to ζ_1. Let $v(x)$ be a $\hat{\mathcal{C}}_{N|M_+}$-solution of $P(x,D)$ at $(x'_0;\ i\eta'_0)$. Suppose that $v(x)$ is micro-analytic near p_0 in the positive side of N. By this terminology we will mean that $\mathrm{ext}(v)$ is zero as a microfunction on $\{(x;\ i\eta)\in iS^*M;\ \varepsilon>x_1>0,\ |x'-x'_0|<\varepsilon,\ |\eta-\eta_0|<\varepsilon\}$ for some $\varepsilon>0$. Considering that $\mathrm{ext}(v)\in\mathcal{C}_{M+|X}\cap \mathcal{H}^0_{iS^*M\underset{M}{\times}N}(\mathcal{C}_M)$ at p_0, we then obtain that $\mathrm{ext}(v)\in\mathcal{C}_{N|X}$ at p_0 from the N_+-regularity of P at p_0. On the other hand by the same argument as in Proposition 1.3, we can show that this is equivalent to the s-relations among boundary values $v(+0, x'), \cdots, D_1^{m-1}v(+0, x')$. In other words the s-boundary values corresponding to the zero $\zeta_1 = i\eta_{0,1}$ of multiplicity s vanish at $(x_0'; i\eta_0')$. Therefore this means that the micro-analyticity of solutions propagates from the positive side of the boundary up to the boundary.

THEOREM 1.12. *Let $P(x, D) = D_1^m + P_1(x, D')D_1^{m-1} + \cdots + P_m(x, D')$ be a pseudo-differential operator of order m defined on $\iota^{-1}((x_0'; i\eta_0'))$ with $(x_0'; i\eta_0') \in iS^*N$. Suppose that ${}^tP(x, D)$ is semi-hyperbolic in the positive side of N at $(x_0'; -i\eta_0')$. In other words, the equation $\sigma(P)(x; \zeta_1, i\eta') = 0$ with respect to ζ_1 has no root with negative real part when $\varepsilon \geqq x_1 \geqq 0$, $|x' - x_0'| \leqq \varepsilon$, $|\eta' - \eta_0'| \leqq \varepsilon$ for some $\varepsilon > 0$. Then $P(x, D)$ is N_+-regular at every point of $\iota^{-1}((x_0'; i\eta_0')) \cap iS^*\dot{M}$.* (Cf. Kaneko [8] and Schapira [26]).

PROOF. Fix a point $p_0 = (0, x_0'; i\eta_{0,1}, i\eta_0') \in iS^*M \underset{M}{\times} N$. To show the N_+-regularity of P at p_0, we may assume that $\{\zeta_1 \in \boldsymbol{C};\ \sigma(P)(0, x_0'; \zeta_1, i\eta_0') = 0\} = \{i\eta_{0,1}\}$. Set $M' = M \times \boldsymbol{R}^{n-1} \ni (x, y') = (x_1, \cdots, x_n, y_2, \cdots, y_n)$ and $N' = N \times \boldsymbol{R}^{n-1}$. We remark that, as an operator on functions in (x, y'), ${}^tP(x, D)$ is semi-hyperbolic in the positive side of N' at $(x_0', x_0'; -i\eta_0', i\eta_0') \in iS^*N'$. So the following boundary value problem has a $\hat{\mathcal{C}}_{N'|M'_+}$-solution $u_k(x, y')$ for every $k = 0, \cdots, m-1$:

$$\begin{cases} {}^tP(x, D)u_k(x, y') = 0\,, \\ D_{x_1}^j u_k(+0, x', y') = \delta_{jk} \cdot \delta(x' - y') \qquad j = 0, \cdots, m-1 \end{cases}$$

at $(x_0', x_0'; -i\eta_0', i\eta_0')$. Let $v(x)$ be any germ of $\mathcal{C}_{M+|X} \cap \mathcal{H}^0_{iS^*M \underset{M}{\times} N}(\mathcal{C}_M)$ at p_0 with $Pv(x) \in \mathcal{C}_{N|X}$. By the division theorem for $\mathcal{C}_{N|X}$ we have a germ $v'(x) \in \mathcal{C}_{N|X}$ at p_0 and germs $f_0(x'), \cdots, f_{m-1}(x')$ of $\mathcal{C}_N$ at $(x_0'; i\eta_0')$ such that $P(x, D)(v(x) - v'(x)) = \sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1)$. Recalling that P is invertible on $\iota^{-1}((x_0'; i\eta_0')) - \{p_0\}$, $w(x) = v(x) - v'(x)$ is extended to a germ of $\iota_*^{\ddagger}\mathcal{C}_{M+|X} \cap \mathcal{C}^\infty_{N|X}|_{iS^*N \times \infty}$ at $(x_0'; i\eta_0')$. That is, $[w(x)]$ is a $\hat{\mathcal{C}}_{N|M_+}$-solution of P at $(x_0'; i\eta_0')$. In order to prove our theorem it suffices to show that $f_1(x') = \cdots = f_{m-1}(x') = 0$ at $(x_0'; i\eta_0')$. We apply the micro-local Green formula to this case (§2.2 [17]). Indeed since $\mathrm{SS}\,(D_{x_1}^j u_k(+0, x', y')) \subset \{(x', y'; i\eta', i\tau');\ x' = y',\ \eta' + \tau' = 0\}$ for $j, k = 0, 1, \cdots, m-1$ and $\mathrm{SS}\,(\mathrm{ext}\,[w(x)]) = \mathrm{SS}\,(w(x)) \subset \{x_1 = 0\}$, the conditions are all fulfilled. Therefore we have $f_j(y') = 0$ at $(x_0'; i\eta_0')$ for $j = 0, 1, \cdots, m-1$. Thus the proof is completed.

To conclude this section we give an opposite example.

EXAMPLE 1.13. $P = D_1^2 + x_1^{2k}D_2^2$ $(k = 1, 2, \cdots)$ is neither N_+- nor N_--regular at $(0; +idx_2)$ (see Example 1.5), because it has a solution with singularity in $x_1 = 0$.

§2. An application to diffractive boundary value problems

We apply the results in §1 and [17] to prove the N_+-regularity of diffractive operators, for example $P=D_1^2-(x_1-x_2)D_3^2$ (cf. [6], [28]). They are neither operators treated in §1 nor operators studied by Schapira in [27].

Let $P(x, D)$ be a pseudo-differential operator of finite order with real principal symbol defined at $p_0=(0, x_0'; i\eta_0)\in iS^*M\underset{M}{\times}N-iS_N^*M$. We consider *the most generic case of diffraction*, that is,

$$\begin{cases} \sigma(P)(p_0)=0\,, \quad \{\sigma(P),\, x_1\}(p_0)=0\,, \quad \{\{\sigma(P),\, x_1\},\, \sigma(P)\}(p_0)\neq 0\,, \\ \{\{\sigma(P),\, x_1\},\, x_1\}(p_0)\neq 0\,, \quad d\sigma(P)\wedge dx_1(p_0)\neq 0\,. \end{cases} \tag{2.1}$$

In fact, let $(x(t); i\eta(t))$ be the bicharacteristic strip for P passing through $p_0=(x(0); i\eta(0))$. Then we have $dx_1/dt(0)=\frac{1}{i}\{\sigma(P), x_1\}(p_0)=0$ and $d^2x_1/dt^2(0)=-\{\sigma(P), \{\sigma(P), x_1\}\}(p_0)\neq 0$. So the bicharacteristic strip is strictly tangent to $\{(x; i\eta); x_1=0\}$ at p_0.

By the condition $\{\{\sigma(P), x_1\}, x_1\}=\partial^2\sigma(P)/\partial\zeta_1^2\neq 0$ we may assume that P is a pseudo-differential operator of second order written in the form: $\sigma(P)=\zeta_1^2+a_1(x, \zeta')\zeta_1+a_2(x, \zeta')$. Here a_1, a_2 are real valued when x, ζ' are real and $\sigma(P)=0$ has a double root $\zeta_1=i\eta_{0,1}$ for $(x; \zeta')=(x_0; i\eta_0')$ by the condition $\{\sigma(P), x_1\}(p_0)=0$. Therefore by a suitable real contact transformation keeping $\{x_1=0\}$ invariant, $\sigma(P)$ and p_0 are transformed into $\sigma(P)=\zeta_1^2+r(x, \zeta')$ and $p_0=(0; 0, i\eta_0')$. Noting that $r(0, i\eta_0')=0$ and $\partial r/\partial x_1(0, i\eta_0')=(1/2)\{\{\sigma(P), x_1\}, \sigma(P)\}\neq 0$, we can write $r(x, \zeta')$ as $-(x_1-\varphi(x', \zeta'))a(x, \zeta')$. Here $\varphi(x', \xi')$ and $a(x, \xi')$ are real valued analytic functions homogeneous of degree 0 and 2 with respect to ξ' respectively. Further $\varphi(0, \eta_0')=0$, $a(0, \eta_0')\neq 0$. Since $d\varphi\neq 0$ follows from $d\sigma(P)\wedge dx_1\neq 0$, we can take $\varphi(x', \zeta')\equiv x_2$. Thus $\sigma(P)$ is transformed into the following form:

$$\sigma(P)=\zeta_1^2-(x_1-x_2)a(x, \zeta') \qquad \text{at} \quad p_0=(0; 0, i\eta_0')\,. \tag{2.2}$$

Here $a(x, \xi')$ is a positive valued real analytic function homogeneous of degree 2 with respect to ξ' defined on a neighborhood of $(x; \xi')=(0; \eta_0')$ (we have chosen $a>0$ because in this case the bicharacteristic strip passing through p_0 is contained in $\{x_1\geq 0\}$).

PROPOSITION 2.1. *We inherit the notation from above. Let $u(x)$ be a section of $\hat{C}_{N|M_+}$ defined on a neighborhood U of $p_0'=\iota(p_0)$. Assume that $P(x, D)u(x)=0$ on U (particularly $P(x, D)$ is defined on $\iota^{-1}(U)$). Then there exists a section $v(x)$ of $\hat{C}_{N|M_+}$ on U whose support is compactly contained in U such that $Pv(x)=0$ on U and $v(x)=u(x)$ at p_0'.*

PROOF. We may assume $P(x, D)=D_1^2-(x_1-x_2)a(x, D')$. We denote by

$\gamma_p(t)=(x(t, p); i\eta(t, p))$ $(-\delta\leqq t\leqq\delta)$ the bicharacteristic strip passing through $p=\gamma_p(0)\in\{\sigma(P)=0\}$, that is, $(x(t, p); \eta(t, p))$ is the integral curve for $H_{\sigma(P)}$. Without loss of generality we may assume that $\partial/\partial x_1((x_1-x_2)a(x, \eta'))>0$ on $\{|x_1|<\alpha, (x'; i\eta')\in U\}$, which implies $d^2x_1(t, p)/dt^2=2d\eta_1(t, p)/dt>0$ there. Choose positive numbers δ, ε and a neighborhood $V\Subset U$ of p_0' such that the integral curve $(x(t, p); \eta(t, p))$ is defined as an analytic mapping $(t, p)\mapsto(x(t, p), \eta(t, p))$ from $[-\delta, \delta]\times\{(x; \eta); \sigma(P)(x, \eta)=0, |x_1|\leqq\varepsilon, (x'; i\eta')\in\bar{V}\}$ to $\{(x; \eta); |x_1|<\alpha, (x'; i\eta')\in U\}$ and satisfies $x_1(\pm\delta, p)\geqq 2\varepsilon$ for every $p\in\{\sigma(P)(x, \eta)=0, |x_1|\leqq\varepsilon, (x'; i\eta')\in\bar{V}\}$. By the softness of $\hat{\mathcal{C}}_{N|M_+}$, there exists a section $w(x)$ of $\hat{\mathcal{C}}_{N|M_+}$ on U with support in V such that $w(x)=u(x)$ on a neighborhood of p_0'. So the support of $P(x, D)w(x)$ is contained in $V\backslash\{p_0'\}$. Consider $f(x)=\operatorname{ext}(P(x, D)w(x))$ which is a section of $\mathcal{C}_{M_+|X}$ on $\{(x_1, x'; \zeta_1, i\eta')\in S^*_{M_+}X; x_1<\lambda, \eta'\neq 0\}$ with support in $\{(x_1, x'; \zeta_1, i\eta')\in S^*_{M_+}X; x_1=0, \eta'\neq 0, (x'; i\eta')\in V\backslash\overline{W}\}\cup\{(x_1, x'; \zeta_1, i\eta')\in S^*_{M_+}X; 0\leqq x_1<\lambda, \lambda|\eta_1|\leqq|\eta'|, (x'; i\eta')\in V\backslash\overline{W}\}$ for a small number $0<\lambda<\varepsilon$, and a small neighborhood $W\Subset V$ of p_0' in iS^*N. By the flabbiness of $\mathcal{C}_M$ we can cut the support of $f(x)$ in $\{x_1>0\}$ such that (the support of $f(x))\cap\{x_1>0\}$ is contained in $\{(x_1, x'; i\eta_1, i\eta')\in iS^*M; 0<x_1\leqq\mu, \lambda|\eta_1|\leqq|\eta'|, (x'; i\eta')\in V\backslash\overline{W}\}$. Here $\mu<\lambda$ is a positive number such that the intersection of the bicharacteristic $\gamma_{p_0}(t)$ passing through p_0 with $\{0\leqq x_1\leqq\mu\}$ is contained in $\{(x; i\eta)\in iS^*M; (x'; i\eta')\in W\}$. $P(x, D)$ is of real principal type and the support of $f(x)$ has a compact intersection with every bicharacteristic strip $\{\gamma_p(t)\}$. Therefore we can find a section $g(x)$ of $\mathcal{C}_M$ satisfying $P(x, D)g(x)=f(x)$ defined on $\{(x; i\eta)\in iS^*M; |x_1|<\lambda, \eta'\neq 0, (x'; i\eta')\in U\}$ with support in $K=[(\text{support } f)\cap iS^*M]\cup[\{|x_1|<\lambda\}\cap\{\gamma_p(t); |t|\leqq\delta, p\in(\text{support } f)\cap\{\sigma(P)(x, \eta)=0\}\}]$. By the flabbiness of $\mathcal{C}_M$ and $\mathcal{B}_M$, there exist sections $G_+(x)$, $G_-(x)$ of $\mathcal{B}_M$ on $\{|x_1|<\lambda, x'\in\pi_N(U)\}$ with support in $\{x_1\geqq 0\}$, $\{x_1\leqq 0\}$ respectively such that $g(x)=[G_+(x)]+[G_-(x)]$ holds as a microfunction on $\{|x_1|<\lambda, (x'; i\eta')\in U\}$. Consider the difference $r(x)=\operatorname{ext}(w(x))-[G_+(x)]$. This is a section of $\mathcal{C}_{M_+|X}$ on $\{(x_1, x'; \zeta_1, i\eta')\in S^*_{M_+}X; 0\leqq x_1<\lambda, \eta'\neq 0, (x'; i\eta')\in U\}$ and it satisfies $P(x, D)r(x)=w(+0, x')\delta'(x_1)+(D_1w)(+0, x')\delta(x_1)+f(x)-P(x, D)[G_+(x)]=w(+0, x')\delta'(x_1)+(D_1w)(+0, x')\delta(x_1)+P(x, D)[G_-(x)]\in\Gamma(U, (\iota^-)_*\mathcal{C}_{M_-|X})$. Noting that $(\iota^+)_*\mathcal{C}_{M_+|X}\cap(\iota^-)_*\mathcal{C}_{M_-|X}=\iota_*\mathcal{C}_{N|X}$, we have $P(x, D)r(x)\in\Gamma(U, \iota_*\mathcal{C}_{N|X})$ and $r(x)\in\Gamma(U, (\iota^+)_*\mathcal{C}_{M_+|X}\cap\mathcal{C}^\infty_{N|X}|_{iS^*N\times\infty})$. Since $[G_+]=-[G_-]\in\mathcal{C}_{N|X}$ on $\{(x; i\eta)\in iS^*M; x_1=0, \eta'\neq 0, (x'; i\eta')\in U\backslash\iota(K\cap\{x_1=0\})\}$, $[G_+(x)]$ represents a section of $\iota_*\mathcal{C}_{N|X}$ on $U\backslash\iota(K\cap\{x_1=0\})$. Therefore $r(x)$ defines a section $v(x)$ of $\hat{\mathcal{C}}_{N|M_+}$ on U with support compactly contained in U. Then $Pv(x)=0$ on U and $v(x)=u(x)$ at p_0', because $\iota(K\cap\{x_1=0\})$ is compactly contained in $U\backslash\{p_0'\}$. This completes the proof.

Assume that $P(x, D)=D_1^2-(x_1-x_2)A(x, D')$ is a second-order differential operator defined in a neighborhood of the origin and that $\sigma(A)(x, \xi')\geqq 0$ for every x and every $\xi'\in\boldsymbol{R}^{n-1}$. On account of hyperbolicity of P in $\{x_1-x_2>0\}$, any hyperfunction solution of $Pu=0$ defined on $\{x_1>0, |x|<R\}$ can be continued to $\{|x|<r, x_1-x_2>0\}\cup\{|x|<R, x_1>0\}$ as a solution for a small $r>0$. Then this solution is identified with a solution defined on $\{(t, x)\in\boldsymbol{R}\times\boldsymbol{R}^n ; 0<t<1, |x|<r, x_1-tx_2>0\}$ of the following system of differential equations:

$$\begin{cases} (D_1^2-(x_1-x_2)A(x, D'))u(t, x)=0, \\ D_t u(t, x)=0. \end{cases}$$

We can consider the boundary value of $u(t, x)$ to $x_1-tx_2=0$, which in its turn satisfies a certain equation. Thus by means of the artificial variable t we can paraphrase the problem of propagation of regularity by this new equation.

We want to apply this argument to the general case; that is, $P(x, D)$ is a pseudo-differential operator in (2.2) and $u(x)$ is a micro-local solution (that is, $\hat{\mathcal{C}}_{N|M_+}$-solution) of $P(x, D)u=0$. To do so we must employ the method used in Theorem 1.8.

LEMMA 2.2. *Set*

$$U=\{(0, x' ; \zeta_1, i\eta')\in S_N^*X ; \eta_n=+1, |\zeta_1|^2+\eta_2^2+\cdots+\eta_{n-1}^2<\varepsilon^2\},$$

and

$$K_+=\{(0, x' ; w_1, iv')\in S_{M_+}X|_N ; v_n\geqq\varepsilon\sqrt{(-u_1)_+^2+v_1^2+\cdots+v_{n-1}^2}\}$$

(where $(t)_+=t$ if $t\geqq 0$, $=0$ if $t<0$). Let $f(x)$ be a section of $\hat{\mathcal{B}}_{N|M_+}$ such that $\mathrm{ext}(f(x))$ *represents a section of $R^1(\pi_{N/X}|_U)_!\mathcal{C}_{N|X}$; in other words, for a suitable closed set $A\subset U$ which is compact in every fiber of $\pi_{N/X}$,* $\mathrm{ext}(f(x))$ *can be continued to S_N^*X-A as a section of $\mathcal{C}_{N|X}$. Then $f(x)$ is written as the boundary value of a section of $(\tau|_{K_+})_*\tilde{\mathcal{A}}_{M_+}$.* (As for the definitions of $S_{M_+}X$, $\tilde{\mathcal{A}}_{M_+}$, see §2.1 in [17].)

PROOF. We denote $R^1\Gamma_{\widetilde{S_N X}}\tau_{N/X}^{-1}\mathcal{O}_X[1]$ by $q_{N|X}$, where $\tau_{N/X}$ is the projection from the monoidal transform ${}^N\widetilde{X}$ of X with center N to X (see CH I in [24]). Then, Proposition 1.2.2 in CH I in [24] shows that $\mathcal{C}_{N|X}=\boldsymbol{R}\tau'_*\pi'^{-1}_* q_{N|X}\otimes\omega_{N/X}[n-1]$, where $\tau' : D_N X=(1/2)S_N X\underset{N}{\times}S_N^*X\to S_N^*X$, $\pi' : D_N X\to S_N X$ are canonical projections. Using this expression, after a direct calculation of derived functors (cf. Proposition 2.1.21 [17]), we obtain $\boldsymbol{R}(\pi_{N/X}|_U)_!\mathcal{C}_{N|X}\simeq\boldsymbol{R}(\tau|_{K_0})_* q_{N|X}[-1]$ with $K_0=\{(0, x' ; w_1, iv')\in S_N X ; v_n\geqq\varepsilon\sqrt{|w_1|^2+v_2^2+\cdots+v_{n-1}^2}\}$ (the dual cone of U). Set $K_-=\{(0, x' ; w_1, iv')\in S_{M_-}X ; v_n\geqq\varepsilon\sqrt{(u_1)_+^2+v_1^2+\cdots+v_{n-1}^2}\}$. Note that $K_+\cap K_-=K_0$, $q_{N|X}=q_{M_+}=q_{M_-}=q_{M_+\cup M_-}$ on K_0 (as for the definitions of q_{M_+}, q_{M_-}, $q_{M_+\cup M_-}$, see Definition 2.1.15 and the proof of Proposition 2.1.21 in [17]) and that the fibre of $K=K_+\cup K_-$ is cohomologically trivial (that is, a Stein manifold) for the sheaf $q_{M_+\cup M_-}$. Thus we have the exact sequence

$$0 \longrightarrow (\tau|_K)_* q_{M_+\cup M_-} \longrightarrow (\tau|_{K_+})_* q_{M_+} \oplus (\tau|_{K_-})_* q_{M_-} \longrightarrow (\tau|_{K_0})_* q_{N|X} \longrightarrow 0\,.$$

This implies the lemma. We omit the details.

From now on, we assume $P(x, D)=D_1^2-(x_1-x_2)a(x, D')$, where $a(x, \zeta')$ is the one defined in (2.2). In fact by inner automorphisms lower order terms are negligible because $d\sigma(P)\wedge\omega\neq 0$, where $\omega=\zeta_1 dx_1+\cdots+\zeta_n dx_n$ is the fundamental 1-form.

PROPOSITION 2.3. *Let $P(x, D)=D_1^2-(x_1-x_2)a(x, D')$ be the pseudo-differential operator as above, and $f(x)$ be a $\hat{C}_{N|M_+}$-solution of $Pf(x)=0$ at $p_0'=(0\,;\,i\eta_0')$. Assume that $(\eta_{0,3}', \cdots, \eta_{0,n}')\neq 0$, in other words, $d\sigma(P)\wedge dx_1\wedge\omega\neq 0$ at p_0. Set $M'=\boldsymbol{R}\times M\ni(t, x_1, \cdots, x_n)$, $N'=\{(t, x)\in M'\,;\, x_1-tx_2=0\}\ni(t, x')$ and $M_+'=\{(t, x)\in M'\,;\, x_1-tx_2\geqq 0\}$. Then there exists a hyperfunction $g(t, x)$ defined on $\Omega=\{(t, x)\in M'\,;\, 0<t<1,\ x_1-tx_2>0,\ |x|<r\}$ with small $r>0$ satisfying the following:*

i) *$D_t g(t, x)=0$ on Ω. The canonical flabby extension $G(t, x)=g(t, x)Y(t)Y(1-t)$ is mild from the positive side of N' at every point of $\{(t, x')\in N'\,;\, 0\leqq t\leqq 1,\ |x'|<r\}$.*

ii) *$G(t, x)$ satisfies the pseudo-differential equation $P(x, D)\operatorname{ext}(G(t, x))=0$ as a section of $C_{M_+'|X'}/C_{N'|X'}$ in a neighborhood of $\{(t, x\,;\, i\tau dt+i\eta' dx'+\zeta_1 d(x_1-tx_2))\in S_{M_+'}^*X'\,;\, x=0,\ 0\leqq t\leqq 1,\ \zeta_1=\tau=0,\ \eta'=\eta_0'\}$.*

iii) *$g(+0, x)$ and $g(1-0, x)$ are mild on $\{x\in N\,;\, |x|<r\}$ and $\{x\in M\,;\, |x|<r,\ x_1-x_2=0\}$ from the positive side of $x_1=0$ and $x_1-x_2=0$ respectively. $g(+0, x)$ coincides with $f(x)$ as a germ of $\hat{C}_{N|M_+}$ at $p_0'=(0\,;\,i\eta_0')\in iS^*N$ and $g(1-0, x)$ coincides with $\operatorname{ext}(f(x))$ as a section of C_M on $\{(x\,;\,i\eta)\in iS^*M\,;\, x_1>0,\ x_1-x_2>0,\ |x|<r,\ |\eta-\eta_0|<r\}$.*

PROOF. After a suitable change of coordinates, we can take $p_0=(0\,;\,idx_n)$ $(n\geqq 3)$. In the coordinate system $u_1=x_1-x_2$, $u_2=x_2, \cdots, u_n=x_n$, $P(x, D)$ is written as $D_{u_1}^2-u_1 a(u_1+u_2, u', D_{u_2}-D_{u_1}, D_{u_3}, \cdots, D_{u_n})$. Hence the Weierstrass division theorem for pseudo-differential operators admits the following decomposition

$$P(x, D)=E(u, D_u)(D_{u_1}^2-u_1 B(u, D_{u'})D_{u_1}-u_1 C(u, D_{u'}))$$

in $W=\{(w\,;\,\lambda)=(u+iv\,;\,\mu+i\nu)\in S^*X\,;\, |w_1|<2R,\ |w'|<R,\ |\lambda_j|<R|\lambda_n|$ for every $j\neq n\}$ for some $R>0$. Here $E(u, D_u)$ is elliptic on W; $B(u, D_{u'})$ and $C(u, D_{u'})$ are pseudo-differential operators of order 1 and 2 respectively defined on $\overline{W}$ such that $\sigma_1(B)(w, \lambda)$, $\sigma_2(C)(w, \lambda)$ are real for real w, λ and that $\sigma_2(C)(u, \mu)>0$ for every $(u\,;\,\mu)\in S^*M\cap\overline{W}$. Further, by taking R small enough, we may assume that:

(2.3) $\lambda_1^2-u_1\sigma_1(B)(u_1, w', \lambda')\lambda_1-u_1\sigma_2(C)(u_1, w', \lambda')$ never vanishes on

$$\{(w\,;\,\lambda)\in\overline{W}\,;\ \operatorname{Im} w_1=0, \text{ and } \operatorname{Im}(\lambda_1/\lambda_n)>I\sqrt{u_1}\,(|\operatorname{Im} w'|+|\operatorname{Im}(\lambda'/\lambda_n)|)$$

if $u_1 \geqq 0$, $\operatorname{Im}(\lambda_1/\lambda_n) > I\sqrt{-u_1}$ if $u_1 < 0\}$.

Here I is a suitable positive constant. Then, as in Theorem 1.8, there exists a constant $d>0$ such that, with respect to $\{w_n = i\varepsilon\}$, $B^\varepsilon(w, D_{w'})H(w)$ and $C^\varepsilon(w, D_{w'})H(w)$ are well-defined if $|w_n - i\varepsilon| < d$, $|w_1| < 2R$, $|w'| < R$ and $H(\tilde{w})$ is holomorphic on $\gamma(\{w\} \cup \{\tilde{w}\,;\, \tilde{w}_1 = w_1,\ \tilde{w}_n = i\varepsilon,\ |\tilde{w}_j - w_j| \leqq (1/R)|\tilde{w}_n - w_n|$ for every $j = 2, \cdots, n-1\})$.

Now return to the solution $f(x)$. By Proposition 2.1, we may assume that $f(x)$ is a section of $\hat{\mathscr{B}}_{N|M_+}$ on N whose support as a section of $\hat{\mathcal{C}}_{N|M_+}$ is contained in a sufficiently small neighborhood U of $p_0' = (0\,;\, idx_n)$; and that $P(x, D)f(x) = 0$ holds as a section of $\hat{\mathcal{C}}_{N|M_+}$ everywhere on iS^*N as well as P is defined. Since $P(x, D)\operatorname{ext}(f(x)) \in \Gamma_c(U, \iota_*\mathcal{C}_{N|X})$ and P is invertible on $\{(0\,;\, \zeta_1 dz_1 + idx_n) \in S_N^*X\,;\, \zeta_1 \neq 0\}$, for every $k>0$ we can take U small enough such that $\operatorname{ext}(f(x))$ is extensible as a section of $\mathcal{C}_{N|X}$ to

$$S_N^*X - \left\{(0,\ x'\,;\, \zeta_1,\ i\eta') \in S_N^*X\,;\ |x'| < k,\ \eta_n > \frac{1}{k}\sqrt{|\zeta_1|^2 + \eta_2^2 + \cdots + \eta_{n-1}^2}\right\}.$$

Hereafter we fix k and $f(x)$. The constant $k(<(1/2)R)$ will be chosen small enough depending only on n, R, d as specified later. Then by Lemma 2.2 $f(x)$ is identified with a holomorphic function $F(z)$ defined on $\{z \in \boldsymbol{C}^n\,;\, \delta > y_n > k((-x_1)_+ + (x_1 - \delta)_+ + |y_1| + \cdots + |y_{n-1}|),\ |x'| < R\} \cup \{z \in \boldsymbol{C}^n\,;\ y = 0,\ x_1 = 0,\ k \leqq |x'| < R\}$ for some $\delta > 0$. Replacing $F(z)$ modulo an element in $\mathcal{A}_M|_N$ by the same argument as in Theorem 1.8, we may assume that $F(z)$ is holomorphic on

$$\begin{aligned} D = \{&z \in \boldsymbol{C}^n\,;\ y_n > 2k\{(-x_1)_+ + (x_1 - \delta)_+ + |y_1| + \cdots + |y_{n-1}|\},\ |x'| < R\} \\ &\cup \{y_n + \delta > 2k\{(-x_1)_+ + (x_1 - \delta)_+ + |y_1| + \cdots + |y_{n-1}|\},\ k < |x'| < 2R/3\} \end{aligned}$$

for some smaller $\delta > 0$. In the coordinates w, D is written as $\{w \in \boldsymbol{C}^n\,;\ v_n > 2k\{(-u_1 - u_2)_+ + (u_1 + u_2 - \delta)_+ + |v_1 + v_2| + |v_2| + \cdots + |v_{n-1}|\},\ |u'| < R\} \cup \{v_n + \delta > 2k\{\cdots\},\ k < |u'| < 2R/3\}$. Setting $\varepsilon = \min\{d/2,\ R^2/2\sqrt{n}\}$,

$$Q^\varepsilon(w, D_w)F(w) = (D_{w_1}^2 - w_1 B^\varepsilon(w, D_{w'})D_{w_1} - w_1 C^\varepsilon(w, D_{w'}))F(w)$$

is holomorphic on $\{|u'| \leqq 2k,\ |w_1| < 2R,\ |w'| < R,\ k > v_n > 2k\{(-u_1 - u_2)_+ + (u_1 + u_2 - \delta)_+ + |v_1 + v_2| + |v_2| + \cdots + |v_{n-1}|\}\}$ if $2k + (\sqrt{n}/R)(3k + \varepsilon) < R$, $k + 2k((n+1)/R)\cdot(3k + \varepsilon) < \varepsilon$, and $3k + \varepsilon < d$. These are fulfilled if k is taken small enough. Remark that the boundary value of $Q^\varepsilon(w, D_w)F(w)$ is equal to

$$Q(u, D_u)f = (D_{u_1}^2 - u_1 B(u, D_{u'})D_{u_1} - u_1 C(u, D_{u'}))f$$

as a section of $R^1(\pi_{N/X}|_{W \cap S_N^* X})_! \mathcal{C}_{N|X}$ on $\{|u'| \leqq 2k\}$ (see Lemma 2.2). Since $E(u, D_u)$ is elliptic on W and $E(u, D_u)Q(u, D_u)f = P(x, D_x)f = 0$ holds as a section of $R^1(\pi_{N/X}|_{W\cap S^*_N X})_! C_{N|X}$ (this is true when $k<R/2\sqrt{2n}$), $Q^\varepsilon(w, D_w)F(w)$ is holomorphic on $Z=\{|u_1+u_2|\leqq\delta', |u'|\leqq 2k, |v_n|\leqq\delta', v_1^2+\cdots+v_{n-1}^2\leqq\delta'^2\}$ for some $\delta'>0$. To prolong $F(u_1, w')$ analytically (cf. §4 in [12]), we introduce a family of piecewise real analytic hypersurfaces $\{S_\lambda\}$ $(0<\lambda\leqq 1)$:

$$S_\lambda=\{(u_1, w')\in \boldsymbol{R}\times \boldsymbol{C}^{n-1};\ v_n=\varphi_\lambda(u_1, u_2, v_2, \cdots, v_{n-1})$$
$$+2k(u_1+u_2-\delta)_+ +2k(-u_1-u_2-2\alpha)_+,\ |u'|<2R/3,\ u_1<\alpha\}\ .$$

Here φ_λ is given by

$$\{(8k\sqrt{n}+1)(e^{I(\alpha-u_1)}-1)+8k\sqrt{n}\}\sqrt{\lambda^2+v_2^2+\cdots+v_{n-1}^2}+h_\lambda(u_1)$$
$$+h_\lambda\Big(u_1-2\sqrt{u_2^2+\frac{1}{8}\alpha^2}+\frac{3}{4}\alpha\Big),$$

with $h_\lambda(t)=8k(\sqrt{t^2+\lambda^2}-t)$, and α is a small positive constant specified later. Choose $\alpha<\min\{\delta/2, \delta/4k, \delta'/2, \delta'/12k\}$. Then we have $S_\lambda\backslash D\subset\{|u'|\leqq k, \alpha>u_1 \geqq 2\sqrt{u_2^2+(1/8)\alpha^2}-\alpha, v_2^2+\cdots+v_{n-1}^2\leqq 4\alpha^2/n\}$ and $S_\lambda\backslash D\subset Z$ for every $\lambda\in(0, 1]$. Therefore F is continued analytically to

$$D\cup\bigcup_{1\geqq\lambda>0}S_\lambda=D\cup\Big\{|u'|<2R/3,\ u_1\leqq\alpha,\ v_n>$$
$$((8k\sqrt{n}+1)(e^{I(\alpha-u_1)}-1)+8k\sqrt{n})\sqrt{v_2^2+\cdots+v_{n-1}^2}$$
$$+16k\Big((-u_1)_+ +(-u_1+2\sqrt{u_2^2+\frac{1}{8}\alpha^2}-\frac{3}{4}\alpha)_+\Big)\Big\}$$

if the following conditions are satisfied:

1) $S_1\subset D$; this is satisfied if $8k\sqrt{n}>2k(2\alpha+2\sqrt{n}(2\alpha/\sqrt{n}))$, that is, $\alpha<(2/3)\sqrt{n}$.

2) For every $(u_1^0, w^{0\prime})\in S_\lambda\backslash D$, $\gamma(\{(u_1^0, w^{0\prime})\}\cup\{(u_1^0, w');\ w_n=i\varepsilon,\ |w_j-w_j^0|\leqq (1/R)|i\varepsilon-w_n^0|$ for every $j=2, \cdots, n-1\})$ is contained in $\bigcup_{\lambda'\geqq\lambda}S_{\lambda'}\Big(\subset\bigcup_{1\geqq\lambda'\geqq\lambda}S_{\lambda'}\cup D\Big)$; considering the convexity of $S_\lambda\cap\{u_1=u_1^0\}$, this is satisfied if $k+(\sqrt{n}/R)(\varepsilon+k+12k\alpha)<2R/3$ and $\varepsilon>12k\alpha+[((8k\sqrt{n}+1)(e^{2\alpha I}-1)+8k\sqrt{n})\sqrt{n}+32k](1/R)(\varepsilon+k+12k\alpha)$ (use the formula $h_\lambda(t_1+t_2)\leqq h_\lambda(t_1)+16k|t_2|$).

3) At every $(u_1^0, w^{0\prime})\in S_\lambda\backslash D$ the surface S_λ is real analytic and non-characteristic for $Q(w, D_w)$; that is, S_λ is written locally as $\{v_n=\varphi_\lambda\}$ and satisfies the following inequalities at this point:

$$\begin{cases} -\dfrac{\partial\varphi_\lambda}{\partial u_1}>I\sqrt{u_1}\Big(|\varphi_\lambda|+\sqrt{v_2^2+\cdots+v_{n-1}^2}+\Big|\dfrac{\partial\varphi_\lambda}{\partial u_2}\Big|\Big) & \text{if}\quad u_1\geqq 0\,, \\ -\dfrac{\partial\varphi_\lambda}{\partial u_1}>I\sqrt{-u_1} & \text{if}\quad u_1<0\,, \end{cases}$$

(see Lemma 4.3 in [12] and the assumption (2.3)).
Easily to see, these are fulfilled if $\alpha<1$, $I\sqrt{\alpha}<\min\{1/6, 8k\}$. Surely we can take k and α small enough such that they satisfy all the conditions listed till now and that k depends only on n, R and d. Consequently it follows that $F(z)$ is holomorphic on $\{z\in \boldsymbol{C}^n;\ |x'|<R,\ y_n>2k((-x_1)_+ +(x_1-\delta)_+ +|y_1|+\cdots+|y_{n-1}|)\}$ $\cup\{|x'|<2R/3,\ x_1-x_2\leqq\alpha,\ y_1=y_2,\ y_n>((8k\sqrt{n}+1)(e^{I(\alpha-x_1+x_2)}-1)+8k\sqrt{n})$ $\cdot\sqrt{y_2^2+\cdots+y_{n-1}^2}+16k((x_2-x_1-(3/4)\alpha+2\sqrt{x_2^2+(1/8)\alpha^2})_+ +(x_2-x_1)_+)\}$ hence on

$$\{z\in \boldsymbol{C}^n;\ |x|<\beta,\ y_n>16k\cdot\min\{(-x_1)_+,\ (x_2-x_1)_+\},\ y_1=\cdots=y_{n-1}=0\}$$

for some small $\beta>0$. So $G'(t, x)=F(x_1, \cdots, x_{n-1}, x_n+i0)Y(x_1-tx_2)Y(t)Y(1-t)$ is a well-defined hyperfunction on $\{(t, x)\in \boldsymbol{R}\times\boldsymbol{R}^n;\ |x|<\beta\}$ with support in $\{x_1-tx_2\geqq 0,\ 0\leqq t\leqq 1\}$. Set $g(t, x)=G'(t, x)|_{\{0<t<1, x_1-tx_2>0\}}$. Then $g(t, x)$ is a hyperfunction defined in $\{0<t<1,\ x_1-tx_2>0,\ |x|<\beta\}$ and satisfies $D_t g(t, x)=0$ there. Hence $G'(t, x)|_{\{x_1-tx_2>0\}}$ is equal to the canonical flabby extension $g(t, x)Y(t)Y(1-t)$. Further, though we omit the proof, we can show the mildness of $G'(t, x)$ on $\{x_1-tx_2=0\}$ by prolonging $F(w)$ analytically (cf. § 2 in [12]). Note that $G'(t, x)$ defines a section $[G'(t, x)]$ of $\mathcal{C}_{M'_+|X'}/\mathcal{C}_{N'|X'}$ on a neighborhood of $L=\{(t, x;\ i\tau dt+idx_n+\zeta_1 d(x_1-tx_2)\in S^*_{M'_+}X';\ x=0,\ 0\leqq t\leqq 1,\ \zeta_1=\tau=0\}$; and that $Q(u, D_u)[G'(t, x)]$ coincides on L with

$$[\{Q^{\varepsilon}(w, D_w)F(w)\}_{\mathrm{Im}\,w_1=\cdots=\mathrm{Im}\,w_{n-1}=0}\times Y(x_1-x_2t)Y(t)Y(1-t)]_{\mathrm{Im}\,w_n=+0}$$
$$\in\mathcal{O}_{z=0}\otimes Y(x_1-x_2t)Y(t)Y(1-t).$$

(This is not a formal triviality. It requires a proof which is long, but easy. So we omit it.) Therefore $Q(u, D_u)[G'(t, x)]=0$ holds on L, which is equivalent to $P(x, D_x)[G'(t, x)]=0$ on L. It is easy to verify the claim iii) in the statement of this proposition. Thus the proof is completed.

THEOREM 2.4. *Let $P(x, D)$ be a pseudo-differential operator with real principal symbol defined at $p_0=(0, x_0';\ i\eta_0)\in iS^*M\underset{M}{\times}N-iS^*_NM$. Assume that:*

$$\begin{cases}\sigma(P)(p_0)=\{\sigma(P), x_1\}(p_0)=0,\ \{\{\sigma(P), x_1\}, x_1\}(p_0)\neq 0,\\ \{\{\sigma(P), x_1\}, \sigma(P)\}(0, x_0', \eta_0)<0,\ d\sigma(P)\wedge dx_1\wedge\omega\neq 0 \text{ at } p_0.\end{cases}$$

Then $P(x, D)$ is N_+-regular at p_0. In other words, this is equivalent to the following statement. Without loss of generality we may assume that $P(x, D)$ is a second-order pseudo-differential operator of the form $D_1^2+P_1(x, D')D_1+P_2(x, D')$. Then any boundary value of a $\hat{\mathcal{C}}_{N|M_+}$-solution $f(x)$ of $P(x, D)f(x)=0$ at $p_0'=\iota(p_0)$ is micro-analytic at p_0' if $\mathrm{SS}(\mathrm{ext}(f))\cap\gamma_{p_0}\cap\{x_1>0\}=\varnothing$. Here γ_{p_0} is the bicharacteristic strip passing through p_0 defined in a small neighborhood of p_0.

PROOF. We may assume $P(x, D)=D_1^2-(x_1-x_2)a(x, D')$ and $p_0=(0; idx_n)$ as in Proposition 2.3. So by this proposition, for the given solution $f(x)$ there exists a hyperfunction $g(t, x)$ as described there. Hereafter we use the coordinates $(s, u_1, \cdots, u_n)=(t, x_1-x_2t, x_2, \cdots, x_n)$. Therefore $P(x, D)$ and D_t are written in the form:

$$\begin{cases} P=D_{u_1}^2-(u_1-(1-s)u_2)a(u_1+su_2, u', D_{u_2}-sD_{u_1}, D_{u_3}, \cdots, D_{u_n}), \\ D_t=D_s-u_2D_{u_1}. \end{cases}$$

By Weierstrass' preparation theorem for pseudo-differential operators P can be decomposed into the product $R'R$ of pseudo-differential operators on $L=\{(s, u; idu_n)\in S_{M'_+}^*X'; 0\leqq s\leqq 1, u=0\}$. Here R' is elliptic on L and R is a second-order pseudo-differential operator of the form

$$R=D_{u_1}^2-(u_1-(1-s)u_2)B(s, u, D_{u'})D_{u_1}-(u_1-(1-s)u_2)C(s, u, D_{u'}).$$

Hence from ii) in Proposition 2.3 we obtain a pseudo-differential equation for sections of $C_{M'_+|X'}/C_{N'|X'}$ on L,

$$\{D_{u_1}^2-(u_1-(1-s)u_2)B(s, u, D_{u'})D_{u_1}-(u_1-(1-s)u_2)C(s, u, D_{u'})\}[\mathrm{ext}(G)]=0.$$

Since $R(s, u, D_u)$ is invertible on $\{(s, u; \lambda_1du_1+idu_n)\in S_{N'}^*X'; u=0, 0\leqq s\leqq 1, \lambda_1\in \boldsymbol{C}\setminus\{0\}\}$, this implies

$$R(s, u, D_u)G=0$$

as a section of $\hat{C}_{N'|M'_+}$ on $\{(s, u'; idu_n)\in iS^*N'; 0\leqq s\leqq 1, u'=0\}$. On the other hand from i) it follows that $(D_s-u_2D_{u_1})G=D_t(g(t, x)Y(t)Y(1-t))=g(+0, u)\delta(s)-g(1-0, u_1+u_2, u')\delta(s-1)$. Here $g(+0, u)=f(u)$ as a germ of $\hat{C}_{N|M_+}$ at $(0; idu_n)\in iS^*N$, and, because $P(D_s-u_2D_{u_1})G=(D_s-u_2D_{u_1})PG=0$, $g(1-0, u_1+u_2, u')$ is a $\hat{C}_{N''|M''_+}$-solution of $R(1, u, D_u)\psi=0$ at $(0; idu_n)\in iS^*N''$, where $M''_+=\{(s, u)\in M'; s=1, u_1\geqq 0\}$, $N''=\{u\in M''_+; u_1=0\}$. Furthermore, recalling that $g(1-0, u_1+u_2, u')$ coincides with $f(u_1+u_2, u')$ as a section of $C_{M''}$ on $\{(u; i\nu du)\in iS^*M''; r>u_1>0, r>u_1+u_2>0, |u'|<r, |\nu-(0, \cdots, 0, 1)|<r\}$, we can apply the N''_+-regularity of $R(1, u, D_u)$ to this case. In fact $R(1, u, D_u)$ is just the operator $D_{u_1}^2-u_1B(u, D_{u'})D_{u_1}-u_1C(u, D_{u'})$ introduced in Proposition 2.3 which is hyperbolic to the codirection du_1 in $\{r>u_1>0\}$ for small $r>0$. So this is N''_+-regular by virtue of Theorem 1.12. Remark that the punched bicharacteristic strip $\gamma_{p_0}-\{p_0\}$ is contained not only in $\{x_1>0\}$, but also in $\{x_1-x_2>0\}$. Hence the assumption $\mathrm{SS}(\mathrm{ext}(f))\cap\gamma_{p_0}\cap\{x_1>0\}=\emptyset$ leads to $\mathrm{SS}(\mathrm{ext}(g(1-0, u_1+u_2, u')))\subset\{u_1=0\}$ near $(0; idu_n)\in iS^*M''$. Therefore by the N''_+-regularity we have $g(1-0, u_1+u_2, u')=0$ at $(0; idu_n)\in iS^*N''$ as a germ of $\hat{C}_{N''|M'_+}$. Consequently it follows that

$$(D_s - u_2 D_{u_1}) G(s, u) = f(u)\delta(s)$$

holds on $\{(s, u'; idu_n) \in iS^*N'; 0 \leqq s \leqq 1, u'=0\}$ as a section of $\mathcal{C}_{N'|M'_+}$. Hence, set

$$h_0(s, u') = G(s, +0, u') \quad \text{and} \quad h_1(s, u') = D_{u_n}^{-1}(D_{u_1}G)(s, +0, u'),$$

and take the boundary values on $u_1 = +0$ in these equations:

$$\begin{cases} R(s, u, D_u) G(s, u) = 0, \\ (D_s - u_2 D_{u_1}) G(s, u) = f(u)\delta(s). \end{cases}$$

Then we have the following system of pseudo-differential equations of first-order for sections of $\mathcal{C}_{N'}$:

$$\{D_s I_2 - A(s, u', D_{u'})\} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} f(+0, u')\delta(s) \\ D_{u_n}^{-1}(D_{u_1} f)(+0, u')\delta(s) \end{pmatrix} \tag{2.4}$$

on a neighborhood of $\{(s, u'; idu_n) \in iS^*N'; 0 \leqq s \leqq 1, u'=0\}$, where

$$A = \begin{pmatrix} 0, & u_2 D_{u_n} \\ -(1-s)u_2^2 D_{u_n}^{-1} C(s, 0, u', D_{u'}), & -(1-s)u_2^2 D_{u_n}^{-1} B(s, 0, u', D_{u'}) D_{u_n} \end{pmatrix}.$$

Recall that the supports of $G(s, +0, u')$ and $(D_{u_1}G)(s, +0, u')$ are contained in $\{(s, u') \in N'; 0 \leqq s \leqq 1\}$ by the definition of G. Hence $h_0(s, u')$ and $h_1(s, u')$ belong to $\mathcal{C}_{N'_+|Y'}$ at $\{(s, u'; idu_n); s=1, u'=0\}$, where $N'_+ = \{(s, u') \in N'; 1-s \geqq 0\}$ and Y' is a complex neighborhood of N'. Considering the injectivity of $D_s I_2 - A$: $\mathcal{C}_{N'_+|Y'} \to \mathcal{C}_{N'_+|Y'}$ at $(1, 0; idu_n)$, we have, by (2.4), $h_0 = h_1 = 0$ at $(1, 0; idu_n)$. So, $\{\mathrm{SS}(h_0) \cup \mathrm{SS}(h_1)\} \cap U \subset \{0 \leqq s < 1-\delta\}$ holds for some small $\delta > 0$ and some neighborhood U of $\{(s, u'; idu_n) \in iS^*N'; 0 \leqq s \leqq 1, u'=0\}$. Note that the determinant of the principal symbol of $D_s I_2 - A(s, u', D_{u'})$ is given by

$$\lambda_s^2 + (1-s)u_2^2 \sigma_1(B)(s, 0, u', \lambda')\lambda_s + (1-s)u_2^3 \sigma_2(C)(s, 0, u', \lambda'). \tag{2.5}$$

Here B, C have real principal symbols with $\sigma_2(C) > 0$ on $\{0 \leqq s \leqq 1, u'=0, \lambda' = (0, \cdots, 0, 1)\}$. In particular $D_s I_2 - A$ is elliptic on $\Omega_+ = \{(s, u'; i\nu_s, i\nu') \in iS^*N'; 0 < s < 1, 0 < u_2 < \varepsilon, |u'| < \varepsilon, |\nu_2| + \cdots + |\nu_{n-1}| < \varepsilon\nu_n\}$ and hyperbolic on $\Omega_- = \{0 < s < 1, -\varepsilon < u_2 < 0, |u'| < \varepsilon, |\nu_2| + \cdots + |\nu_{n-1}| < \varepsilon\nu_n\}$ for small $\varepsilon > 0$. Therefore $\{\mathrm{SS}(h_0) \cup \mathrm{SS}(h_1)\} \cap U' \subset \{0 \leqq s < 1-\delta, u_2 = 0, \nu_s = 0\} \cup \{s=0\}$ for some smaller neighborhood U'. Indeed, in Ω_-, $D_s I_2 - A$ is a hyperbolic operator with small velocity of order $|u_2|^{3/2}$ ($\ll |u_2|$). Hence, for sufficiently small $\varepsilon' > 0$, every point of $\Omega_- \cap U \cap \{\det(\nu_s I_2 - \sigma_1(A)(s, u', \nu')) = 0, |u'| < \varepsilon', 0 < s \leqq 1-\delta, |\nu_2| + \cdots + |\nu_{n-1}| < \varepsilon'\nu_n\}$ is combined with some point of $U \cap \{s = 1-\delta\}$ by a bicharacteristic strip (for the symbol (2.5)) contained in $\Omega_- \cap U$. Consequently we can take U' as above. On the other hand in the regular involutory submanifold:

$$V=\{(s, u'; i\nu_s, i\nu')\in iS^*N'; \nu_s=0, u_2=0\},$$

the micro-Holmgren theorem of Bony (Théorème 3.10 in [1]) is available. In fact, since the micro-principal symbol of the symbol (2.5) along V is λ_s^2, the micro-analyticity of (h_0, h_1) propagates along integral curves of $\partial/\partial s$ in $U' \cap \{0<s<1, u_2=0, \nu_s=0\}$. Thus we have $U' \cap \{SS(h_0) \cup SS(h_1)\} \subset \{s=0\}$. At the last step of the proof we use Schapira's theory in [27]. Since $N=\{x\in M; x_1=0\} = \{(s, u')\in N'; s=0\}$ is non-micro-characteristic for $D_sI_2-A(s, u', D_{u'})$, D_sI_2-A is a N-regular operator according to his theory (Théorème 2.2 in [27]). Thus $(D_sI_2-A)^t(h_0, h_1)={}^t(f(+0, u'), D_{u_n}^{-1}(D_{u_1}f)(+0, u'))\otimes\delta(s)\in(\mathcal{C}_{N|Y'})^2$ (where Y' is a complex neighborhood of N') leads to ${}^t(h_0, h_1)\in(\mathcal{C}_{N|Y'})^2$ at $(0, 0; idu_n)\in iS^*N' \underset{N'}{\times} N$. Finally by the division theorem for $\mathcal{C}_{N|Y'}$ (see §1 in [17]) we have $f(+0, u') = D_{u_n}^{-1}(D_{u_1}f)(+0, u')=0$ at $(0; idu_n)\in iS^*N$. This completes the proof.

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