On the conditional stability of non-minimal solutions of $w'' + e^w = 0$

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1. Introduction and Theorem.

In general it seems to be difficult to know whether or not a given initial value is on a stable manifold. The purpose of the present paper is to give a comprehensive example of initial values on the stable manifold through the well-known equation (1.1) below (I. M. Gel'fand [3], H. Fujita [2]).

In the preceding paper [4], the author showed that non-minimal[1)] solutions w of $\frac{d^2w}{dx^2}+e^w=0\ (-l<x<l)$ with $w(-l)=w(l)=0$ are conditionally stable as a stationary solution of

$$(1.1)\qquad \begin{cases} \dfrac{\partial v}{\partial t}=\dfrac{\partial^2 v}{\partial x^2}+e^v & (-l<x<l,\ t>0), \\ v(x,0)=a(x) & (-l<x<l), \\ v(-l,t)=v(l,t)=0 & (t>0), \end{cases}$$

where l is a positive constant. That is to say, there exists a manifold S_w in $\mathring{W}_2^1(-l,l)$, called a stable manifold, having the following property: if $a\in S_w$, then the solution v of (1.1) converges to w as $t\to+\infty$, where $\mathring{W}_2^1(-l,l)$ is the subspace of the usual Sobolev space $W_2^1(-l,l)$ with norm $\|\ \ \|_1$ which consists of elements satisfying the homogeneous Dirichlet boundary condition. On the other hand, we know that w is unstable in the sense: if $a\leqq w$ $(a\not\equiv w)$ then v of (1.1) converges to the minimal stationary solution $w_{\min}$ as $t\to+\infty$ uniformly in x, and if $a\geqq w$ $(a\not\equiv w)$ then v grows up or blows up, where we write simply $a\underset{(\leqq)}{\geqq}w$ for $a(x)\underset{(\leqq)}{\geqq}w(x)$ $(-l<x<l)$ (Fujita [2]). Thus we immediately see that initial values on S_w must cross w. We shall describe the set S_w more precisely later.

In order to state our theorem, we recall a famous result by Gel'fand [3]:

1) $w_{\min}$ is said to be minimal if $w(x)\geqq w_{\min}(x)$ $(-l<x<l)$ for any solution w of the equation.

There exists l_0 $(=0.93\cdots)$ such that, when $l<l_0$ (1.1) has two stationary solutions, when $l=l_0$ it has only one stationary solution, and when $l>l_0$ it has no stationary solution. These solutions are given by

$$w_\alpha(x)=\alpha-2\log\cosh(2e^\alpha)^{1/2}x \qquad (-l<x<l), \tag{1.2}$$

where α is a positive root of the equation

$$e^{\alpha/2}=\cosh(2e^\alpha)^{1/2}l. \tag{1.3}$$

[Note: When $l<l_0$ (1.3) has two solutions α_1, α_2 $(0<\alpha_1<\alpha_2)$, when $l=l_0$ it has only one solution $\alpha_0>0$, and when $l>l_0$ it has no solution.] Since we are interested in the non-minimal solution, we confine our consideration to w_{α_2} in the case $l<l_0$. In what follows, we simply write w, α for w_{α_2}, α_2 respectively. Note that $\dfrac{dw}{dx}<0$ for $0<x<l$.

Our theorem now follows.

THEOREM. *Assume that* $l<l_0$. *Let* w *be as above. Suppose that given* a_0, $a_1\in\mathring{W}_2^1(-l, l)$ *satisfy*

$$a_0\leqq w\leqq a_1. \tag{1.4}$$

Then, there exists a γ in $0\leqq\gamma\leqq1$ such that the solution of (1.1) with the

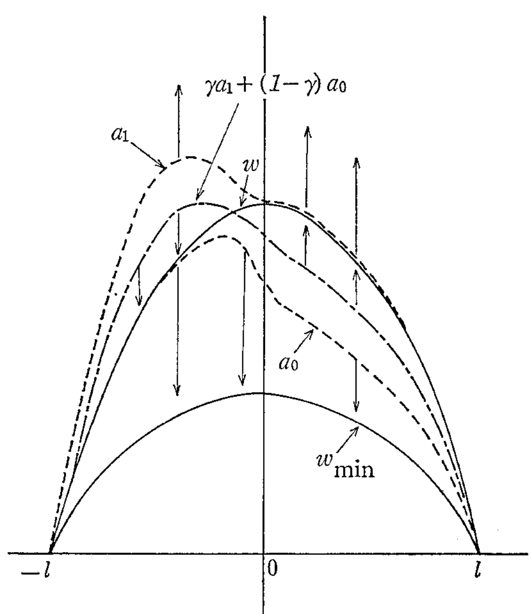

Fig. 1

initial value $\gamma a_1+(1-\gamma)a_0$ converges exponentially to w as $t\to+\infty$ in the norm $\|\ \|_1$, if $\|a_0-w\|_1$, $\|a_1-w\|_1$ are sufficiently small. (See Fig. 1.)
Note: If $a_i\not\equiv w$ $(i=0, 1)$, then solutions v_i of (1.1) with $v_i(x, 0)=a_i(x)$ never converge to w as $t\to+\infty$ (Fujita's result).

In the next section, we prove this theorem by showing that $\gamma a_1+(1-\gamma)a_0$ lies on the stable manifold S_w constructed in [4].

2. Proof of Theorem.

In the work [4], S_w was defined as

$$S_w=\{a\in \overset{\circ}{W}{}_2^1(-l, l): a-w\in S\}, \tag{2.1}$$

where S is a stable manifold for the trivial solution $u\equiv 0$ of

$$\begin{cases} \dfrac{\partial u}{\partial t}=\Big(\dfrac{\partial^2}{\partial x^2}+e^w\Big)u+e^w(e^u-1-u) & (-l<x<l,\ t>0), \\ u(x, 0)=a(x)-w(x) & (-l<x<l), \\ u(-l, t)=u(l, t)=0 & (t>0). \end{cases} \tag{2.2}$$

This S has the following property: If $a-w\in S$, then the solution u of (2.2) decays exponentially as $t\to+\infty$ in the norm $\|\ \|_1$. (See Theorem 2 in [4].) For the proof of Theorem, therefore, it is sufficient to show that $\gamma a_1+(1-\gamma)a_0-w\in S$. To show this, we prepare some notations and lemmas.

Let X be $\overset{\circ}{W}{}_2^1(-l, l)$ with the norm $\|\cdot\|=\|(A_0+\beta_0)^{1/2}\cdot\|_{L^2(-l,l)}$, where A_0 is the operator $-\Big(\dfrac{d^2}{dx^2}+e^w\Big)$ with Dirichlet boundary condition in $L^2(-l, l)$, and β_0 is a positive constant such that $A_0+\beta_0$ is positive. We note that $\|\ \|$ is equivalent to $\|\ \|_1$, and that X can be endowed with the inner product $(\cdot,\cdot)=((A_0+\beta_0)^{1/2}\cdot, (A_0+\beta_0)^{1/2}\cdot)_{L^2}$. Let Y be the subspace generated by the eigenspace corresponding to non-positive eigenvalues of $-\Big(\dfrac{d^2}{dx^2}+e^w\Big)$ in X, and Z be the complementary subspace of Y. Let P, Q be projections onto Y, Z respectively.

LEMMA 2.1. *The above S may be represented by*

$$S=\{\theta(b): b\in B\}, \tag{2.3}$$

where $B=\{b\in Z: \|b\|<r\}$ (r is some positive constant), and θ is a continuous map: $B\to X$, with $Q\theta(b)=b$ for $b\in B$.

(For the proof, see the proof of Theorem 2 in [4].)

LEMMA 2.2. *Let λ_1, λ_2 be the smallest eigenvalue and the second smallest one, respectively, of the eigenvalue problem*

$$
\left\{\begin{array}{ll}\left(\frac{d^2}{dx^2}+e^w\right)\phi+\lambda\phi=0 & (-l<x<l), \\ \phi(-l)=\phi(l)=0. & \end{array}\right. \tag{2.4}
$$

Then

$$\lambda_1\leqq 0<\lambda_2. \tag{2.5}$$

Also, let ϕ_1, ϕ_2 be eigenfunctions corresponding to λ_1, λ_2, respectively. Then, ϕ_1 does not change its sign in $(-l, l)$, and ϕ_2 has only one zero in it.

(We shall give the proof of Lemma 2.2 later.)

We show that $\gamma a_1+(1-\gamma)a_0-w\in S$ for some $0<\gamma<1$. We may assume $\phi_1>0$. By Lemma 2.1 and Lemma 2.2, S may be expressed as

$$S=\{f(b)\phi_1+b: b\in B\},$$

where $f(b)=(\theta(b), \phi_1)$. This f is a continuous functional on B. Set $C^+=\{u\in X: u\geqq 0, u\not\equiv 0\}$, $C^-=\{u\in X: u\leqq 0, u\not\equiv 0\}$. Then C^+ and C^- are convex cones in X with the origin as their vertices, and they satisfy

$$\{\mu\phi_1: \mu>0\}\subset C^+\subset\{\mu\phi_1+z: \mu>0, z\in Z\}, \tag{2.6}$$

$$\{\mu\phi_1: \mu<0\}\subset C^-\subset\{\mu\phi_1+z: \mu<0, z\in Z\}. \tag{2.7}$$

The fist inclusion of (2.6) (or (2.7)) is trivial. For any $u\in X$,

$$
\begin{aligned}
(u, \phi_1)&=((A_0+\beta_0)^{1/2}u, (A_0+\beta_0)^{1/2}\phi_1)_{L^2} \\
&=(u, (A_0+\beta_0)\phi_1)_{L^2}=(\lambda_1+\beta_0)(u, \phi_1)_{L^2}.
\end{aligned}
$$

Since $A_0+\beta_0$ is positive definite, $\lambda_1+\beta_0>0$. Hence, $(u, \phi_1)>0$ if $u\in C^+$, and $(u, \phi_1)<0$ if $u\in C^-$. This shows the second inclusion of (2.6) (or (2.7)).

By Fujita's instability results about w, $C^+\cap S=\emptyset$ and $C^-\cap S=\emptyset$. Furthermore we have

$$C^+\cap\{\mu\phi_1+b: \mu\in \boldsymbol{R}\}\subset\{\mu\phi_1+b: \mu>f(b)\} \qquad (b\in B), \tag{2.8}$$

$$C^-\cap\{\mu\phi_1+b: \mu\in \boldsymbol{R}\}\subset\{\mu\phi_1+b: \mu<f(b)\} \qquad (b\in B). \tag{2.9}$$

Indeed, if there exists $\mu\leqq f(b)$ with $\mu\phi_1+b\in C^+$, then $0<\mu\leqq f(b)$. Thus, by (2.6) and the fact that C^+ is a convex cone,

$$f(b)\phi_1+b=(f(b)-\mu)\phi_1+(\mu\phi_1+b)\in C^+.$$

This contradict the fact $C^+\cap S=\emptyset$. Thus (2.8) has been proved. Similarly (2.9) can be proved.

Let $\|a_i-w\|<r$ $(i=0, 1)$. We may represent a_i-w as

$$a_i-w=\mu_i\phi_1+b_i \qquad (i=0, 1), \tag{2.10}$$

where $\mu_i=(a_i-w, \phi_1)$, $b_i=a_i-w-\mu_i\phi_1\in B$. By (1.4), $a_1-w\in C^+$, $a_0-w\in C^-$. Hence, by (2.8) and (2.9), we have $\mu_0<f(b_0)$ and $\mu_1>f(b_1)$. Setting

$$g(\xi)=f(\xi b_1+(1-\xi)b_0)-(\xi\mu_1+(1-\xi)\mu_0) \qquad (0\leqq\xi\leqq1)$$

and noting that f is continuous, we see that g is a continuous function on $[0, 1]$ such that

$$g(0)=f(b_0)-\mu_0>0,$$

$$g(1)=f(b_1)-\mu_1<0.$$

Hence, there exists $0<\gamma<1$ such that $g(\gamma)=0$. By (2.10),

$$\begin{aligned}\gamma a_1+(1-\gamma)a_0-w&=(\gamma\mu_1+(1-\gamma)\mu_0)\phi_1+\gamma b_1+(1-\gamma)b_0\\&=f(\gamma b_1+(1-\gamma)b_0)\phi_1+\gamma b_1+(1-\gamma)b_0\\&\in S.\end{aligned}$$

Thus Theorem has been shown except for the proof of Lemma 2.2.

3. Proof of Lemma 2.2.

Since w is unstable in Fujita's sense, so is w in the usual Lyapunov sense (in the norm $\|\ \ \|$). By Theorem 1 in [4], therefore, we see that $\lambda_1\leqq0$. The claim about ϕ_1, ϕ_2 is well-known (R. Courant & D. Hilbelt [1]).

We show that $0<\lambda_2$. We first note that the symmetry of w yields that $x=0$ is the only zero of ϕ_2. We may assume that $\phi_2(x)>0$ for $0<x<l$. By (2.4) and the fact that $\phi_2(0)=0$, λ_2 is an eigenvalue of the eigenvalue problem

$$\begin{cases}\left(\dfrac{d}{dx}+e^w\right)\phi+\lambda\phi=0 \qquad (0<x<l),\\ \phi(0)=\phi(l)=0,\end{cases} \tag{3.1}$$

and the restriction of ϕ_2 on $[0, l]$ is the eigenfunction corresponding to λ_2.

LEMMA 3.1. *Let $\varphi\in C^2[0, l]$, and $\varphi(x)>0$ for all $0\leqq x\leqq l$. Then*

$$\inf_{0\leq x\leq l}\left[-\frac{\varphi''(x)+e^{w(x)}\varphi(x)}{\varphi(x)}\right]\leqq\lambda_2. \tag{3.2}$$

where $''=\dfrac{d^2}{dx^2}$.

PROOF OF LEMMA 3.1. Since λ_2 and ϕ_2 satisfy (3.1), we have, noting $\phi_2(0)=\phi_2(l)=0$, $\phi_2'(0)\geqq 0$ and $\phi_2'(l)\leqq 0$,

$$\begin{aligned}0&=\int_0^l(\phi_2''+e^w\phi_2+\lambda_2\phi_2)\varphi\,dx\\&=[\phi_2'\varphi-\phi_2\varphi']_0^l+\int_0^l\phi_2(\varphi''+e^w\varphi+\lambda_2\varphi)\,dx\\&\leqq\int_0^l\phi_2(\varphi''+e^w\varphi+\lambda_2\varphi)\,dx\,.\end{aligned}$$

Hence

$$\begin{aligned}\lambda_2\int_0^l\phi_2\varphi\,dx&\geqq-\int_0^l\phi_2(\varphi''+e^w\varphi)\,dx\\&\geqq\inf_{0\leq x\leq l}\left[-\frac{\varphi''+e^w\varphi}{\varphi}\right]\int_0^l\phi_2\varphi\,dx\,.\end{aligned}$$

Since $\int_0^l\phi_2\varphi\,dx>0$, we obtain (3.2).

Let us construct a function φ satisfying assumptions in Lemma 3.1, and $\inf_{0\leq x\leq l}[-(\varphi''+e^w\varphi)/\varphi]>0$.
Set

$$\varphi(x)=-w'(x)+(\varepsilon-w'(x))\cos\frac{\pi(x+\rho)}{2(l+\rho)}\,,$$

where ρ is an arbitrary positive constant, and ε is a positive constant to be determined later. By (1.2), $\varphi\in C^2[0, l]$ and $\varphi>0$ (since $w'(x)<0$ for $0<x<l$). Moreover, $0\leqq w\leqq\alpha$, and $\left(\dfrac{d^2}{dx^2}+e^w\right)w'=0$. Hence if we take ε so that it satisfies

$$\left[e^\alpha-\left(\frac{\pi}{2(l+\rho)}\right)^2\right]\varepsilon<\frac{\pi}{l+\rho}\sin\frac{\pi\rho}{2(l+\rho)}\,,$$

then we have

$$\begin{aligned}\varphi''+e^w\varphi&=\left(\frac{d^2}{dx^2}+e^w\right)\left[(\varepsilon-w')\cos\frac{\pi(x+\rho)}{2(l+\rho)}\right]\\&=\left[\varepsilon\left[e^w-\left(\frac{\pi}{2(l+\rho)}\right)^2\right]+\left(\frac{\pi}{2(l+\rho)}\right)^2w'\right]\cos\frac{\pi(x+\rho)}{2(l+\rho)}\end{aligned}$$

$$-\frac{\pi}{l+\rho}e^{w}\sin\frac{\pi(x+\rho)}{2(l+\rho)}$$

$$\leqq\varepsilon\left[e^{\alpha}-\left(\frac{\pi}{2(l+\rho)}\right)^{2}\right]-\frac{\pi}{l+\rho}\sin\frac{\pi\rho}{2(l+\rho)}$$

$$<0.$$

Therefore, we have

$$0<\inf_{0\leq x\leq l}\left[-\frac{\varphi''+e^{w}}{\varphi}\right]\leqq\lambda_{2}.$$

This completes the proof of Lemma 2.2.

REMARK. The stable manifold S_w belongs to the "lower half space" $\{a\in X: (a-w, \phi_1)\leqq 0\}$.

PROOF. Since $A_0+\beta_0$ is positive, it is sufficient to show that $(a-w, \phi_1)_{L^2}\leqq 0$ for any $a\in S_w$. Let $a\in S_w$ and v be the solution of (1.1) with the initial value a. Then v converges to w in $\| \;\;\|$ as $t\to+\infty$. On the other hand,

$$\frac{d}{dt}(v-w, \phi_1)_{L^2}=\left(\frac{\partial v}{\partial t}, \phi_1\right)_{L^2}$$

$$=\left(\frac{\partial^2}{\partial x^2}(v-w)+(e^{v}-e^{w}), \phi_1\right)_{L^2},$$

(by the mean value theorem and integration by parts)

$$=\left(v-w, \left(\frac{d^2}{dx^2}+e^{w}\right)\phi_1\right)_{L^2}$$

$$+(e^{w+\sigma(v-w)}(v-w)^2, \phi_1)_{L^2} \qquad (0\leqq\sigma\leqq 1)$$

$$\geqq-\lambda_1(v-w, \phi_1)_{L^2}.$$

Therefore, $(v-w, \phi_1)_{L^2}\geqq e^{-\lambda_1 t}(a-w, \phi_1)_{L^2}$ if $(a-w, \phi_1)_{L^2}>0$. Since $-\lambda_1\geqq 0$, this contradicts the fact that $\|v-w\|\to 0$ as $t\to+\infty$. This completes the proof.

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