On compact Kähler manifolds with semipositive bisectional curvature

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§1. Introduction

Let M be a Kähler manifold of complex dimension m and $\boldsymbol{T}M$ be its holomorphic tangent bundle. As in [5], we define (holomorphic) bisectional curvature $H(\xi, \eta)$, $\xi, \eta \in \boldsymbol{T}_zM$, $z \in M$, by

$$H(\xi, \eta) = -R(\xi, \bar{\xi}, \eta, \bar{\eta}),$$

where R is the Riemannian curvature tensor. We say that M has semipositive bisectional curvature of rank r if for all $z \in M$ and for all $\xi, \eta \in \boldsymbol{T}_zM$,

$$H(\xi, \eta) \geqq 0$$

and if r is the smallest integer such that for all $z \in M$, for all nonzero $\xi \in \boldsymbol{T}_zM$ and for all linearly independent $(r+1)$ vectors $\eta_1, \cdots, \eta_{r+1} \in \boldsymbol{T}_zM$,

$$\sum_{i=1}^{r+1} H(\xi, \eta_i) > 0.$$

When M has semipositive bisectional curvature of rank r, $\wedge^{r+1}\boldsymbol{T}M$ is quasi-positive in the sense of Wu [18]. As a special case of Theorem E in [18], we have

Fact: Let M be a compact Kähler manifold whose bisectional curvature is semipositive of rank $r \leqq (m-1)/2$. Then M is simply connected and $H^2(M; \boldsymbol{Z}) \cong \boldsymbol{Z}$.

In this paper we shall prove

THEOREM 1. *Let M be a Kähler manifold and $f: \boldsymbol{P}^1(\boldsymbol{C}) \to M$ be a locally energy minimizing harmonic map. If M has semipositive bisectional curvature of rank $r \leqq (m-1)/2$ then f is holomorphic or antiholomorphic.*

Theorem 1 has been proved by Siu and Yau [16] when M has positive bisectional curvature. Making use of it they succeeded in giving an alternate proof of Frankel conjecture which had been proved by Mori [13] with the method of algebraic geometry of characteristic $p>0$. In this paper we shall use Theorem 1 to prove the following theorem.

THEOREM 2. *Let M be a compact Kähler manifold of dimension $m\geqq 3$ with semipositive bisectional curvature of rank 1. If M satisfies either A or B below, then M is biholomorphic to complex projective space $\boldsymbol{P}^m(\boldsymbol{C})$ or complex quadric $\boldsymbol{Q}^m(\boldsymbol{C})$.*

A. *$c_1(M)$ is not a generator of $H^2(M;\boldsymbol{Z})\cong\boldsymbol{Z}$.*

B. *$A(M)=\{[\xi]\in\boldsymbol{P}(TM)\mid$ for some nonzero $\eta\in TM$, $H(\xi,\eta)=0\}$ is an irreducible subvariety of codimension 1 in $\boldsymbol{P}(TM)$.*

Let M be an m-dimensional compact irreducible Hermitian symmetric space. Then M is known to be a Kähler manifold with nonnegative curvature. In the last section of this paper we shall compute the rank r of semipositivity of bisectional curvature of M. The remarkable result obtained from the computations there is

$$c_1(M)=(m-r+1)\alpha$$

where α is the positive generator of $H^2(M;\boldsymbol{Z})\cong\boldsymbol{Z}$.

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§ 2. Harmonic maps

First we review the definition and some properties of harmonic maps. Let (M, g) and (N, h) be compact Riemannian manifolds. A smooth map $f: N\rightarrow M$ is called harmonic if f is a critical map of the energy functional

$$E(f)=\frac{1}{2}\int_N e(f)$$

where

$$e(f)=h^{ij}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}g_{\alpha\beta}.$$

We confine ourselves to consider the case when N is 1-dimensional complex projective space $\boldsymbol{P}^1$ with the Fubini-Study metric and M is a compact Kähler manifold with Kähler form

$$\omega=\sqrt{-1}\,g_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta.$$

We regard $\boldsymbol{P}^1$ as $\boldsymbol{C}\cup\{\infty\}$ and take w as the coordinate on $\boldsymbol{C}$. Then

$$\int_{P^1}f^*\omega=E'(f)-E''(f)$$

$$E(f)=E'(f)+E''(f)$$

where

$$E'(f)=\int_{P^1} g_{\alpha\bar\beta}\frac{\partial f^\alpha}{\partial w}\frac{\overline{\partial f^\beta}}{\partial w}\sqrt{-1}\,dw\wedge d\bar w$$

$$E''(f)=\int_{P^1} g_{\alpha\bar\beta}\frac{\partial f^\alpha}{\partial \bar w}\frac{\overline{\partial f^\beta}}{\partial \bar w}\sqrt{-1}\,dw\wedge d\bar w\,.$$

Hence

$$2E'(f)=E(f)+\int_{P^1} f^*\omega$$

and

$$2E''(f)=E(f)-\int_{P^1} f^*\omega\,.$$

By the homotopy invariance of $\int_{P^1} f^*\omega$, we can see that a smooth map $f: \boldsymbol{P}^1\to M$ is harmonic if and only if

$$\nabla_{\bar w}\frac{\partial f^\alpha}{\partial w}=\frac{\partial^2 f^\alpha}{\partial w\partial\bar w}+\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial w}\frac{\partial f^\gamma}{\partial \bar w}=0\,.$$

Hence if f is holomorphic or antiholomorphic then f is harmonic. Let $f_t: \boldsymbol{P}^1\to M$, $0\leqq t\leqq 1$, be a smooth family of smooth maps such that f_0 is holomorphic. Then

$$\begin{aligned}E(f_0)&=E'(f_0)=E'(f_0)-E''(f_0)\\&=E'(f_1)-E''(f_1)\leqq E(f_1)\,.\end{aligned}$$

This implies that a holomorphic map attains the minimum of E in its homotopy class.

PROOF OF THEOREM 1. f^*TM can be considered as the holomorphic vector bundle over $\boldsymbol{P}^1$ associated to the analytic locally free sheaf generated by local sections $s_1, \cdots, s_m$ of f^*TM such that $\nabla_{\bar w}s_i=0$ ([16]). Then by the theorem of Grothendieck ([7]), f^*TM splits into holomorphic line bundles $L_1, \cdots, L_m$.

Step 1. Suppose there are $(r+1)$ nonnegative line bundles $L_1, \cdots, L_{r+1}$. Then there exists a holomorphic section s_i of L_i for $1\leqq i\leqq r+1$. We may regard s_i as a holomorphic section of f^*TM. Let $f_{i,t}: \boldsymbol{P}^1\to M$ be a variation of f parametrized by $\{t\in\boldsymbol{C}\mid |t|<\varepsilon\}$ such that

$$f_{i,0}=f\,,$$

$$\frac{\partial f_{i,t}}{\partial t}=s_i \text{ at } t=0 \text{ and}$$

$$\frac{\partial f_{i,t}}{\partial \bar{t}}=0 \text{ at } t=0.$$

Then the second variation of E'' at $t=0$ is

$$0\leqq \frac{\partial^2}{\partial t\partial \bar{t}}\bigg|_{t=0} E''(f_{i,t})$$

$$=\int_{P^1} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \frac{\partial f^\alpha}{\partial \bar{w}} \overline{\frac{\partial f^\beta}{\partial \bar{w}}} s_i^\gamma \overline{s_i^\delta} \sqrt{-1}\, dw \wedge d\bar{w}$$

since $\nabla_{\bar{w}} s_i=0$. Since s_i's are holomorphic sections, their zeros are isolated. From this and the curvature assumption it follows that f must be holomorphic.

Step 2. Suppose now that the number of the nonnegative line bundles in L_i's are at most r. Since $r\leqq(m-1)/2$ there exist at least $(r+1)$ negative line bundles, say $L_{m-r}, \cdots, L_m$. We shall construct $(r+1)$ smooth sections $s_{m-r}, \cdots, s_m$ of f^*TM such that $\nabla_w s_i=0$ for $m-r\leqq i\leqq m$ and that they are linearly independent except at their zeros. Since each L_i is negative, the dual bundle L_i^* of L_i is positive and there exists a holomorphic section s_i^* of L_i^* for $m-r\leqq i\leqq m$. Next we extend s_i^* to a smooth section of $f^*\boldsymbol{T}^*M$ so that $s_i^*(t_j)=0$ for all local holomorphic sections t_j of L_j, $j\neq i$. Then $\nabla_{\bar{w}} s_i^*=0$ since

$$\langle \nabla_{\bar{w}} s_i^*,\ t\rangle=\frac{\partial}{\partial \bar{w}}\langle s_i^*,\ t\rangle=0$$

for all local holomorphic sections t of $f^*\boldsymbol{T}M$. Set

$$s_i^*=s_{i,\alpha}^* dz^\alpha$$

and define

$$s_i'=g^{\alpha\bar{\beta}} s_{i,\alpha}^* \frac{\partial}{\partial \bar{z}^\beta}.$$

Then $\nabla_{\bar{w}} s_i'=0$. Finally if we set $s_i=\overline{s_i'}$ then we obtain $\nabla_w s_i=0$. We can easily see that $s_{m-r}, \cdots, s_m$ are linearly independent except at their zeros. Let $f_{i,t}: \boldsymbol{P}^1 \to M$ be a variation of f parametrized by $\{t\in\boldsymbol{C} \mid |t|<\varepsilon\}$ such that

$$f_{i,0}=f,$$

$$\frac{\partial f_{i,t}}{\partial t}=s_i \text{ at } t=0 \quad\text{and}$$

$$\frac{\partial f_{i,t}}{\partial \bar{t}}=0 \text{ at } t=0.$$

Then we obtain

$$0 \leqq \frac{\partial^2}{\partial t \partial \bar{t}}\Big|_{t=0} E'(f_{i,t})$$

$$= \int_{P^1} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \frac{\partial f^\alpha}{\partial w} \overline{\frac{\partial f^\beta}{\partial w}} s_i^\gamma \bar{s}_i^\delta \sqrt{-1}\, dw \wedge d\bar{w}$$

for $m-r \leqq i \leqq m$. From this and the curvature assumption it follows that f is antiholomorphic, completing the proof of Theorem 1.

§ 3. Types of rational curves

Partly using the arguments in the proof of Theorem 1, we can prove the following theorem.

THEOREM 3. *Let M be a Kähler manifold whose bisectional curvature is semipositive of rank r. Given a nonconstant holomorphic map $f: \boldsymbol{P}^1 \to M$, $f^*\boldsymbol{T}M$ splits into nonnegative line bundles $L_1, \cdots, L_m$ in which there are at least $(m-r)$ positive line bundles. Furthermore there exists at least one positive line bundle of degree greater than or equal to* 2.

PROOF. By the theorem of Grothendieck, $f^*\boldsymbol{T}M$ splits into holomorphic line bundles $L_1, \cdots, L_m$. Each L_i is a nonnegative line bundle since in general the curvature of a quotient bundle E/F of a holomorphic vector bundle E by a subbundle F is greater than that of E (p. 79, [6]).

Assume there were at least $(r+1)$ trivial line bundles in L_i's. We repeat the same argument as in Step 2 in the proof of Theorem 1. We can construct smooth sections $s_1, \cdots, s_{r+1}$ of $f^*\boldsymbol{T}M$ such that $\nabla_w s_i = 0$ and that they are linearly independent except at their zeros. Computing the second variation of E' at f with the variation vectors s_i, $1 \leqq i \leqq r+1$, we see that $\partial f/\partial w = 0$. This implies that f is a constant map and contradicts our assumption. Thus there are at least $(m-r)$ positive line bundles.

On $\boldsymbol{P}^1 = \boldsymbol{C} \cup \{\infty\}$ there exists the holomorphic vector field $\partial/\partial w$ with the zero of order 2 at ∞. If we project $f_*(\partial/\partial w)$ onto each L_i, then at least one of them is nontrivial. Thus we have at least one line bundle of degree greater than o equal to 2. This completes the proof of Theorem 3.

We will say that a holomorphic map $f: \boldsymbol{P}^1 \to M$ is of type I (resp. type II) if $f^*\boldsymbol{T}M$ splits into positive line bundles (resp. otherwise). A holomorphic map of type II is characterized as follows;

LEMMA 4. *Let M be a Kähler manifold whose bisectional curvature is semipositive of rank $r \geqq 1$. If a holomorphic map f is of type* II, *then there exists a nonzero $\xi \in \boldsymbol{T}_{f(w)}M$ such that $H(\xi, (\partial f/\partial w)(w)) = 0$ for each $w \in \boldsymbol{P}^1$.*

PROOF. If f is of type II, then f^*TM admits a trivial line subbundle and the argument in the proof of Theorem 1 shows that there exists a nowhere zero smooth section s of f^*TM such that $\nabla_w s=0$. From the second variation formula with the variation vector s, we see that

$$\int_{P^1} R\left(\frac{\partial f}{\partial w}, \overline{\frac{\partial f}{\partial w}}, s, \bar{s}\right)\sqrt{-1}\, dw \wedge d\bar{w}=0 .$$

Thus $H(s(w), \partial f/\partial w)=0$. This completes the proof.

§4. Convergence and splitting property of a sequence of rational curves

Let M be a compact simply connected Riemannian manifold. Then $\pi_2(M)$ is in one-to-one correspondence with the free homotopy classes $\boldsymbol{P}^1 \to M$ (p. 384, [17]). So we can define the sum in the free homotopy classes $\boldsymbol{P}^1 \to M$.

Given a smooth map $f: \boldsymbol{P}^1 \to M$ we define

$$E([f])=\inf\left\{\sum_{i=1}^{k} E(f_i)\,\middle|\, k \in \boldsymbol{N},\, f_i: \boldsymbol{P}^1 \longrightarrow M,\, f=\sum_{i=1}^{k} f_i\right\},$$

where the sum means that the sum of free homotopy classes represented by f_i's is the free homotopy class represented by f.

In this section we shall prove

LEMMA 5. *Let M be a compact Kähler manifold with semipositive bisectional curvature of rank* 1 *and with $m \geqq 3$. And let $f_j: \boldsymbol{P}^1 \to M$, $j \in \boldsymbol{N}$, be a sequence of holomorphic maps such that*

1) *f_j's are homotopic to one another and*
2) *$|(\partial f_j/\partial w)(0)|=1$ for all $j \in \boldsymbol{N}$ and $(\partial f_j/\partial w)(0)$ converges to some $\xi \in \boldsymbol{T}_z M$ as $j \to \infty$, where* 0 *is the origin of $\boldsymbol{C} \subset \boldsymbol{P}^1$.*

Then one of the following holds:

a) *There exists a holomorphic map $f_\infty: \boldsymbol{P}^1 \to M$ such that $(\partial f_\infty/\partial w)(0)=\xi$ and f_∞ is homotopic to f_1.*

b) *There exists an integer $\nu \geqq 2$ and nonconstant holomorphic or antiholomorphic maps $g_1, \cdots, g_\nu: \boldsymbol{P}^1 \to M$ such that $\sum_{i=1}^{\nu} g_i=f_1$ and that $E([f_1])=\sum_{i=1}^{\nu} E(g_i)$.*

c) *There exists a holomorphic map $g: \boldsymbol{P}^1 \to M$ of type* II *such that g is homotopic to f_1 and that $g(0)=z$ and $(\partial g/\partial w)(0)=0$.*

PROOF. First we claim that $E([f_1])=E(f_j)$ for all $j \in \boldsymbol{N}$. Since f_j's are holomorphic and homotopic to one another, we have $E(f_i)=E(f_j)$ for all i, j. If there were $h_1, \cdots, h_k: \boldsymbol{P}^1 \to M$ such that $\sum_{i=1}^{k} h_i=f_1$ and $\sum_{i=1}^{k} E(h_i)+\varepsilon<E(f_1)$, then we can construct a piecewise smooth map $h: \boldsymbol{P}^1 \to M$ parametrizing $h_1: \boldsymbol{P}^1 \to M$ and a curve

between $h_1(\infty)$ and $h_2(0)$, $h_2: \boldsymbol{P}^1 \to M$ and a curve between $h_2(\infty)$ and $h_3(0)$, continuing this way at last $h_k: \boldsymbol{P}^1 \to M$. Then

$$\text{Area}(h) = \sum_{i=1}^{k} \text{Area}(h_i) \leqq \sum_{i=1}^{k} E(h_i) < E(f_1) - \varepsilon .$$

Since $\dim_{\boldsymbol{R}} M \geqq 6$ we may approximate h by a smooth immersion $\tilde{h}: \boldsymbol{P}^1 \to M$ such that $\text{Area}(\tilde{h}) < E(f_1) - \varepsilon/2$. There is an orientation preserving diffeomorphism from $\boldsymbol{P}^1$ to $\boldsymbol{P}^1$ pulling back the conformal structure defined by $\tilde{h}^* ds_M^2$ to the standard conformal structure. We define $\tilde{\tilde{h}}$ to be $\tilde{h}$ composed with this diffeomorphism so that $\tilde{\tilde{h}}$ is conformal. Then

$$E(\tilde{\tilde{h}}) = \text{Area}(\tilde{\tilde{h}}) = \text{Area}(\tilde{h}) < E(f_1) - \frac{\varepsilon}{2} .$$

This contradicts the fact that f_1 is energy minimizing in its homotopy class. Thus the claim is proved.

Since MAIN ESTIMATE 3.15 in [14] is valid for our f_j and $E(f_j) = E([f_1])$ for $j \in \boldsymbol{N}$, the arguments of § 4 in [14] apply to our f_j. If a subsequence of f_j converges in $C^1(\boldsymbol{P}^1, M)$, this reduces to case a). Otherwise there exists a finite subset $\{x_1, \cdots, x_n\}$ of $\boldsymbol{P}^1$ with the following property. Taking a subsequence if necessary, f_j converges to a harmonic map f_∞ in $C^1(\boldsymbol{P}^1 - \{x_1, \cdots, x_n\}, M)$ but not in $C^1(\boldsymbol{P}^1 - \{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}, M)$ for $1 \leqq i \leqq n$. At each x_i a nontrivial harmonic map g_i from $\boldsymbol{P}^1$ to M is made up and we have

$$E(f_\infty) + \sum_{i=1}^{n} E(g_i) \leqq \overline{\lim_{j\to\infty}} E(f_j) = E([f_1]) .$$

As in [16] we can see that each g_i is energy minimizing in its homotopy class. If $n \geqq 2$, then $E([f_1]) - E(g_1) > 0$. It follows that there exist nontrivial harmonic maps $g_2', \cdots, g_k'$ from $\boldsymbol{P}^1$ to M such that $f_1 = g_1 + g_2' + \cdots + g_k'$ and that $E(f_1) = E(g_1) + E(g_2') + \cdots + E(g_k')$ as is seen in [16]. It is clear that each g_j' is energy minimizing. From Theorem 1 each g_j' is holomorphic or antiholomorphic. This reduces to case b). The case when $n=1$ and $E(f_\infty) > 0$ is also reduced to case b) by the same argument.

Now we consider the case when $n=1$ and $E(f_\infty) = 0$. By assumption 2) we must have $x_1 = 0$ and there must exists a sequence y_j, $j \in \boldsymbol{N}$, such that $y_j \to x_1 = 0$ and that

$$b_j = |df_j(y_j)| = \max_{y \in P^1} |df_j(y)| \longrightarrow \infty \quad \text{as} \quad j \longrightarrow \infty$$

for otherwise $|df_j|$ is bounded on a neighborhood of 0 and f_j converges on this neighborhood. We define $\tilde{f}_j: \boldsymbol{P}^1 \to M$ by $\tilde{f}_j(w) = f_j(w/b_j + y_j)$ where $\boldsymbol{P}^1$ is considered as $\{w \in \boldsymbol{C}\} \cup \{\infty\}$. Let $\varphi_j: \boldsymbol{P}^1 \to \boldsymbol{P}^1$ be $\varphi_j(w) = w/b_j + y_j$. Then $\tilde{f}_j = f_j \circ \varphi_j$ and

$|d\tilde{f}_j(w)|=|df_j(w/b_j+y_j)|\cdot|d\varphi_j(w)|$. We have

$$|d\varphi_j(w)|^2=(1+|w|^2)^2\frac{1}{b_j^2}\left(1+\left|\frac{w}{b_j}+y_j\right|^2\right)^{-2}$$

and so

$$|d\tilde{f}_j(w)|^2\leqq(1+|w|^2)^2\left(1+\left|\frac{w}{b_j}+y_j\right|^2\right)^{-2}.$$

This inequality implies that, taking a subsequence if necessary, $\tilde{f}_j$ converges to some $\tilde{f}_\infty: \boldsymbol{P}^1\to M$ in $C^1(\boldsymbol{P}^1-\{\infty\}, M)$ since Lemma 4.2 in [14] holds for our $\tilde{f}_j$. If $|d\tilde{f}_j|$ is unbounded on every neighborhood of ∞, then by the same argument as in the case $n=1$ and $E(f_\infty)>0$ this reduces to b) since $|d\tilde{f}_j(0)|\to 1$ as $j\to\infty$ and $E(\tilde{f}_\infty)>0$. If $|d\tilde{f}_j|$ is bounded on some neighborhood of ∞, then $\tilde{f}_j$ converges to $\tilde{f}_\infty$ in $C^1(\boldsymbol{P}^1, M)$ and $-b_jy_j$ converges to some $y_\infty\in\boldsymbol{P}^1$. We may assume that $y_\infty=0$ by composing an automorphism of $\boldsymbol{P}^1$. If $\tilde{f}_\infty$ is an immersion at 0, then it is easy to see that $\tilde{f}_\infty(0)=z$ and $(\partial\tilde{f}_\infty/\partial w)(0)=c\xi$, $c\in\boldsymbol{C}^*$. This reduces to a). If $\tilde{f}_\infty$ is not an immersion at 0 and if $\tilde{f}_\infty$ is of type II, then this reduces to case c).

Now we suppose $\tilde{f}_\infty$ is of type I and that $\tilde{f}_\infty$ is not an immersion at 0. Let $\partial/\partial w$ be the holomorphic vector field on $\boldsymbol{P}^1$ with zero of order 2 at ∞. Then $\tilde{f}_{\infty *}(\partial/\partial w)$ is a holomorphic section of $\tilde{f}_\infty^*TM$ and has 3 zeros counted with multiplicity. Since $\tilde{f}_\infty$ is of type I we see that

$$\tilde{f}_\infty^*c_1(M)[\boldsymbol{P}^1]\geqq 3+(m-1)=m+2.$$

By Fact in §1 M is algebraic and $\pi_2(M)$ is isomorphic to $H_2(M;\boldsymbol{Z})$. Noting these facts we conclude that this reduces to case b) by Theorem 3 in [13]. Now the proof is complete.

§5. Proof of Theorem 2

By Fact in §1 there exists a positive line bundle F such that $c_1(F)$ is a generator of $H^2(M;\boldsymbol{Z})\cong\boldsymbol{Z}$, and $\pi_2(M)$ is isomorphic to $H_2(M;\boldsymbol{Z})$. Let $f_0: \boldsymbol{P}^1\to M$ be a smooth map which represents a generator of the free part of $\pi_2(M)$ such that $f_0^*c_1(F)[\boldsymbol{P}^1]=1$. Then there exist energy minimizing harmonic maps $f_1, \cdots, f_k: \boldsymbol{P}^1\to M$ such that $f_1+\cdots+f_k=f_0$ and $E(f_1)+\cdots+E(f_k)=E([f_0])$ by Proposition 2 in [16]. By Theorem 1 they are holomorphic or antiholomorphic. If f_j is antiholomorphic, we put $\bar{f}_j(w)=f_j(\bar{w})$ which is holomorphic. By Theorem 3.16 in [14], there exists $\varepsilon>0$ such that if s is a harmonic map and $E(s)<\varepsilon$ then s is a constant map. Thus there exists $k_0\in\boldsymbol{N}$ such that

$$k_0=\max\left\{n\in\boldsymbol{N}\,\middle|\,\sum_{i=1}^n f_i=f_0,\ \sum_{i=1}^n E(f_i)=E([f_0])\right\}.$$

So we may assume that k is chosen as large as possible. In this situation we see from Theorem 3 in this paper and Theorem 3 in [13], each f_j^*TM (or $\bar{f}_j^*TM$ when f_j is antiholomorphic) is isomorphic to

$$\%\qquad \left\{\begin{array}{lll} \text{i)} & o(2)\oplus o(1)\oplus \cdots\oplus o(1)\oplus o(1) & \text{or} \\ \text{ii)} & o(3)\oplus o(1)\oplus \cdots\oplus o(1)\oplus o & \text{or} \\ \text{iii)} & o(2)\oplus o(2)\oplus o(1)\oplus\cdots\oplus o(1)\oplus o & \text{or} \\ \text{iv)} & o(2)\oplus o(1)\oplus \cdots\oplus o(1)\oplus o\,. & \end{array}\right.$$

Let $c_1(TM)=\lambda c_1(F)$. First we assume $\lambda\neq 1$ which is assumption A. Then one sees easily that $f_i^*c_1(TM)[P^1]=\pm(m+1)$ for all i, or $f_i^*c_1(TM)[P^1]=\pm m$ for all i. Since we know that λ is positive by curvature assumption, we obtain

$$f_0^*c_1(TM)[P^1]\geqq m\,.$$

We conclude from the theorem of Kobayashi and Ochiai [11] that M is biholomorphic to $P^m(C)$ or $Q^m(C)$.

Next we consider under the assumption B. We will say that a holomorphic map $f: P^1\to M$ of type II is of type II_a if f can be (holomorphically) deformed to a map of type I. A holomorphic map of type II_b will be a map of type II which cannot be deformed to a map of type I. In the list of %, if f^*TM is isomorphic to iv) then f is of type II_b. If f^*TM is isomorphic to ii) or iii), then we cannot discriminate whether f is of type II_a or of type II_b.

The following claim is proved in [16], § 5.

Claim 1. Let $f: P^1\to M$ be a holomorphic map of type I or of type II_a. Then there exists a proper subvariety Z_1 of $P(TM)$ such that for all $[\xi]\in P(TM)-Z_1$ there exists a holomorphic map $\tilde{f}: P^1\to M$ homotopic to f with $(\partial\tilde{f}/\partial w)(0)=\xi$.

By the similar argument with the aid of Lemma 4, we can prove the following claim.

Claim 2. Let $f: P^1\to M$ be a holomorphic map of type II_b. Then there exists a proper subvariety Z_2 of $A(M)$ such that for all $[\xi]\in A(M)-Z_2$ there exists a holomorphic map $\tilde{f}: P^1\to M$ homotopic to f with $(\partial\tilde{f}/\partial w)(0)=\xi$.

The proof of Theorem 2 will be finished if we prove the following claim.

Claim 3. There exists a holomorphic map $\tilde{f}: P^1\to M$ such that $f_0=\tilde{f}$ modulo torsion in $H_2(M; Z)$. To be more precise, arranging $f_1, \cdots, f_k$ we have holomorphic maps $f_1, \cdots, f_\nu, f$, antiholomorphic maps $g_1, \cdots, g_\nu$, $2\nu+1=k$ and

$$\{f_i^*c_1(TM)+g_i^*c_1(TM)\}[P^1]=0 \quad \text{for} \quad 1\leqq i\leqq \nu\,.$$

Proof of Claim 3. If $k\geqq 2$, one sees that there exist a holomorphic map

f_1 and an antiholomorphic map g_1 in f_i's. We have only to consider the three cases below.

Case 1. Both f_1 and $\bar{g}_1$ are of type I or of type II_a.

Case 2. Both f_1 and $\bar{g}_1$ are of type II_b.

Case 3. f_1 is of type I or of type II_a and $\bar{g}_1$ is of type II_b, or the converse. In the cases 1 and 2, we can lead a contradiction as in §6 of [16] by making use of Claim 1 and Claim 2.

Now we consider Case 3. We may assume that $\bar{g}_1$ is an immersion at $0\in \boldsymbol{P}^1$ and that $(\partial \bar{g}_1/\partial w)(0)=\xi$. By Claim 1 there exists a sequence of holomorphic maps f_{1j}: $\boldsymbol{P}^1\to M$ homotopic to f_1 such that $(\partial f_{1j}/\partial w)(0)\to\xi$ as $j\to\infty$. We apply Lemma 5 to f_{1j}. If a) occurred, then we can lead a contradiction as in §6 of [16]. By the choice of k, b) also cannot occur. Thus only c) can occur and we may replace f_1 so that f_1 is of type II, $(\partial f_1/\partial w)(0)=0$ and

$$f_1^*TM\cong o(3)\oplus o(1)\oplus\cdots\oplus o(1)\oplus o\,.$$

Choose $w\in \boldsymbol{P}^1$ at which f_1 is an immersion and put $(\partial f_1/\partial w)(w)=\eta$. By Lemma 4 we have $\eta\in A(M)$. By Claim 2 there exists a sequence of holomorphic maps g_{1j}: $\boldsymbol{P}^1\to M$ homotopic to $\bar{g}_1$ such that $(\partial g_{1j}/\partial w)(0)\to\eta$ as $j\to\infty$. We apply Lemma 5 again. We conclude as before that we may replace $\bar{g}_1$ so that $\bar{g}_1$ is of type II, $(\partial \bar{g}_1/\partial w)(0)=0$ and

$$\bar{g}_1^*TM\cong o(3)\oplus o(1)\oplus\cdots\oplus o(1)\oplus o\,.$$

This implies that $f_1+g_1=0$ modulo torsion in $H_2(M;\boldsymbol{Z})$.

If $k-2\geqq 2$ we repeat the same argument.

Finally we obtain

$$f_0=f_1+g_1+\cdots+f_\nu+g_\nu \text{ modulo torsion in } H_2(M;\boldsymbol{Z})$$

or

$$f_0=\tilde{f}+f_1+g_1+\cdots+f_\nu+g_\nu \text{ modulo torsion in } H_2(M;\boldsymbol{Z})\,.$$

If the former occured, we have $f_0^*c_1(F)[\boldsymbol{P}^1]=0$ and this is a contradiction. If the latter occured, $\tilde{f}$ must be holomorphic. Otherwise $\tilde{f}$ is antiholomorphic and $f_0^*c_1(F)[\boldsymbol{P}^1]<0$ and this is a contradiction. Thus Claim 3 is proved.

The proof of Theorem 2 is complete from Theorem 3 and the theorem of Kobayashi and Ochiai [11].

§6. Irreducible Hermitian symmetric spaces of compact type

In this section we calculate the rank of semipositivity of bisectional curvature for irreducible Hermitian symmetric spaces of compact type. We shall use the same notation as in [8].

Let G/K be an irreducible Hermitian symmetric space of compact type with

$\mathfrak{g}$ and $\mathfrak{k}$ as the Lie algebras of G and K, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ the Cartan decomposition, J the complex structure and R the Riemannian curvature tensor with respect to some G-invariant Kähler metric g. Then it is known that

$$R(X, Y)Z=-[[X, Y], Z] \quad \text{for} \quad X, Y, Z\in\mathfrak{p}$$

and that

$$g(R(X, Y)Y, X)=g([X, Y], [X, Y]).$$

Thus $g(R(X, Y)Y, X)=0$ if and only if $[X, Y]=0$. On the other hand we have

$$H(\xi, \eta)=-R(X-iJX, X+iJX, Y-iJY, Y+iJY)$$
$$=4R(X, Y, X, Y)+4R(X, JY, X, JY),$$

where $\xi=X-iJX$, $\eta=Y-iJY$. Hence $H(\xi, \eta)=0$ if and only if $[X, Y]=0$ and $[X, JY]=0$.

Type BD I: complex quadric $SO(m+1)/SO(2)\times SO(m-1)$, $r=1$.
The notations being as above, we have

$$\mathfrak{g}=o(m+1),\ \mathfrak{k}=o(2)\oplus o(m-1) \text{ and}$$

$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & 0 & -{}^t u \\ 0 & 0 & -{}^t v \\ u & v & 0 \end{pmatrix} \middle|\ u, v\ (m-1)\text{-column vectors}\right\}.$$

We identify $\mathfrak{p}=\boldsymbol{R}^{m-1}\oplus\boldsymbol{R}^{m-1}$ and an element of $\mathfrak{p}$ will be denoted by (u, v). The complex structure is then expressed by $J(u, v)=(-v, u)$. By a direct computation, for $X=(u, v)$ and $Y=(s, t)$, $R(X, Y, X, Y)=0$ holds if and only if $\langle u, t\rangle=\langle v, s\rangle$ and $u\wedge s+v\wedge t=0$, where $\langle\ ,\ \rangle$ is the usual inner product of $\boldsymbol{R}^{m-1}$. Hence for $\xi=(u, v)-iJ(u, v)$ and $\eta=(s, t)-iJ(s, t)$, $H(\xi, \eta)=0$ holds if and only if $|u|=|v|$, $\langle u, v\rangle=0$ and $s-it=c(u+iv)$ for some $c\in\boldsymbol{C}$. It follows from this that bisectional curvature of $SO(m+1)/SO(2)\times SO(m-1)$ is semipositive of rank 1. It is interesting to note that the fiber of $A(M)$ in Theorem 2 is complex quadric again.

Type A III: $U(p+q)/U(p)\times U(q)$, $r=(p-1)\times(q-1)$.
The Cartan decomposition is

$$u(p+q)=u(p)\oplus u(q)\oplus\mathfrak{p},$$

where

$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & -{}^t\bar{X} \\ X & 0 \end{pmatrix} \middle|\ X\quad p\times q \text{ complex matrix}\right\}.$$

As complex vector spaces, $\mathfrak{p}$ with the complex structure J is isomorphic to $\boldsymbol{C}^{p\times q}$ with the usual complex structure. From now on we will identify them. For

$X, Y\in\mathfrak{p}$, $H(X, Y)=0$ if and only if $X{}^t\bar{Y}=0$ and ${}^t\bar{X}Y=0$. Let $A_X(Y)=(Y{}^t\bar{X}, {}^t\bar{X}Y)$. To compute the rank of semipositivity of bisectional curvature, we have only to seek a nonzero vector $X\in\mathfrak{p}$ such that the kernel of A_X has the greatest dimension. Let $X\in\mathfrak{p}$ be any nonzero vector. We assume the ij-element of ${}^t\bar{X}$ is nonzero. We denote $\overset{\downarrow}{x}_j$ the j-th column vector of ${}^t\bar{X}$ and $\vec{x}_i$ the i-th row vector of ${}^t\bar{X}$. If Y belongs to the kernel $\mathrm{Ker}(A_X)$ of A_X then $Y\overset{\downarrow}{x}_j=0$ and $\vec{x}_iY=0$. Thus

$$\dim_C \mathrm{Ker}\,(A_X)\leqq\dim_C\{Y\in\mathfrak{p}\mid Y\overset{\downarrow}{x}_j=0,\ \vec{x}_iY=0\}=(p-1)(q-1)\,.$$

On the other hand if we take

$$X=\begin{pmatrix} 1 & 0 & \\ 0 & 0 & 0 \\ & 0 & 0 \end{pmatrix}$$

then $\dim_C\mathrm{Ker}\,(A_X)=(p-1)(q-1)$.

Type D III: $SO(2n)/U(n)$, $r=(n-2)(n-3)/2$ for $n\geqq2$.
The notations being as before we have

$$\mathfrak{g}=\left\{\begin{pmatrix} A & -{}^tB \\ B & D \end{pmatrix}\middle|\ A, D\in\mathfrak{so}(n),\ B\ n\times n \text{ matrix}\right\},$$

$$\mathfrak{k}=\left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\middle|\ A\in\mathfrak{so}(n),\ B \text{ symmetric matrix}\right\},$$

$$\mathfrak{p}=\left\{\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\middle|\ A, B\in\mathfrak{so}(n)\right\} \text{ and}$$

$$J=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

We denote (A, B) for $\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$ for short. Then $J(A, B)=(-B, A)$. Thus for $\xi=(A, B)-iJ(A, B)$ and $\eta=(C, D)-iJ(C, D)$, $H(\xi, \eta)=0$ holds if and only if $AD=BC$ and $AC=-BD$. We only need to seek a nonzero vector (A, B) such that the kernel of the linear map $(C, D)\mapsto(AD-BC, AC+BD)$ has the greatest dimension. It is easy to check that $(A, 0)$ where

$$A=\begin{pmatrix} 0 & 1 & \\ -1 & 0 & 0 \\ & 0 & 0 \end{pmatrix}$$

is one of the desired vectors and that the dimension over C of the kernel is

$(n-2)(n-3)/2$.

Type C I: $Sp(n)/U(n)$, $r=n(n-1)/2$.
The Cartan decomposition is

$$\mathfrak{sp}(n)=u(n)\oplus i\mathfrak{p}\,,$$

where

$$\mathfrak{p}=\left\{\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \middle|\ A,\ B \text{ symmetric } n\times n \text{ matrices}\right\}.$$

We denote (A, B) for $\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$ for short. Then the complex structure J is expressed by $J(A, B)=(-B, A)$. Thus for $\xi=(A, B)-iJ(A, B)$ and $\eta=(C, D)-iJ(C, D)$, $H(\xi, \eta)=0$ holds if and only if $AC+BD=0$ and $AD-BC=0$. We want to seek a nonzero vector (A, B) such that the kernel of the linear map $(C, D)\mapsto(AC+BD, AD-BC)$ has the greatest dimension. One easily sees that $(A, 0)$ where

$$A=\begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}$$

is one of the desired vectors and that the dimension over $\boldsymbol{C}$ of the kernel is $n(n-1)/2$.

To study in two exceptional cases, we recall the root system. We keep the notations as in the beginning of this section. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$ and let $\mathfrak{g}_C$, $\mathfrak{k}_C$ and $\mathfrak{h}_C$ be the complexification of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$. Then $\mathfrak{h}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$. Let Δ_c be the system of roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$ and $\alpha_1, \cdots, \alpha_t$ be the simple roots such that the center of $\mathfrak{h}_C$ is given by $\alpha_2=\cdots=\alpha_t=0$. We put

$$Q_+=\{\alpha\in\Delta_c\,|\,\alpha>\alpha_1\}\,,$$

$$\mathfrak{p}^+=\sum_{\alpha\in Q_+}\mathfrak{g}^\alpha\,,\qquad \mathfrak{p}^-=\sum_{-\alpha\in Q_+}\mathfrak{g}^\alpha\,,$$

where $\mathfrak{g}^\alpha$ is the root subspace corresponding to $\alpha\in\Delta_c$. Then the complexification of the tangent space $(G/K)_0\otimes\boldsymbol{C}$ is identified with $\mathfrak{p}^+\oplus\mathfrak{p}^-$ and $E_\alpha\mapsto E_{-\alpha}$ is the complex conjugation where E_α is a basis of $\mathfrak{g}^\alpha$. Thus for $\xi=\sum_{\alpha\in Q_+}\xi^\alpha E_\alpha$ and $\eta=\sum_{\alpha\in Q_+}\eta^\alpha E_\alpha$ in $\mathfrak{p}^+$, $H(\xi, \eta)=0$ holds if and only if $\sum_{\alpha,\beta\in Q_+}\xi^\alpha\overline{\eta^\beta}[E_\alpha, E_{-\beta}]=0$ since $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras. To compute the rank of semipositivity of bisectional curvature we have only to seek a nonzero vector $X\in\mathfrak{p}^+$ such that the linear map

$$ad(X)\cdot Y=[X, Y],\ Y\in\mathfrak{p}^-$$

has the smallest rank.

Type E III: $E_6/\mathrm{Spin}(10)\times SO(2)$, $r=5$.
The positive roots of E_6 are

$$x_i-x_j\ (i<j),\ x_i+x_j+x_k\ (i<j<k),\ \sum_{i=1}^{6} x_i,$$

where $1\leqq i, j, k\leqq 6$, and the simple roots are

$$\alpha_i=x_i-x_{i+1}\ (1\leqq i\leqq 5),\ \alpha_6=x_4+x_5+x_6,$$

see [1]. Hence Q_+ consists of

$$x_1-x_i\ (2\leqq i\leqq 6),\ x_1+x_i+x_j\ (2\leqq i<j\leqq 6),\ \text{ and }\ \sum_{i=1}^{6} x_i.$$

By a tedious but easy calculation of the matrix representation of $ad(X)$, we can see that any vector corresponding to a root in Q_+ is one of the desired vectors and that the dimension of the kernel is 5.

Type E VII: $E_7/E_6\times SO(2)$, $r=10$.
The positive roots of E_7 are

$$x_i-x_j\ (1\leqq i<j\leqq 7),\ x_i+x_j+x_k\ (1\leqq i<j<k\leqq 7)$$
$$\text{and }\ \sum_{j=1}^{7} x_j-x_i\ (2\leqq i\leqq 7).$$

and the simple roots are

$$\alpha_i=x_i-x_{i+1}\ (1\leqq i\leqq 6),\ \alpha_7=x_5+x_6+x_7,$$

see [1]. Hence Q_+ consists of

$$x_1-x_i\ (2\leqq i\leqq 7),\ x_1+x_i+x_j\ (2\leqq i<j\leqq 7)$$
$$\text{and }\ \sum_{j=1}^{7} x_j-x_i\ (2\leqq i\leqq 7).$$

One sees that any vector corresponding to a root in Q_+ is one of the desired vectors and that the dimension of the kernel is 10.

Let D be an irreducible bounded symmetric domain. Calabi and Vesentini [2] and Borel [1] computed a number $\gamma(D)$ defined by $\gamma(D)=S/m\lambda_1$ where S is the scalar curvature and λ_1 is the smallest eigenvalue of the curvature operator. In comparison with Table 1 in [2] we have $\gamma(D)=m-r+1$. Let M be the compact Hermitian symmetric space corresponding to D. Then $c_1(M)=\gamma(D)\cdot\alpha$ where α is the positive generator of $H^2(M;\boldsymbol{Z})\cong\boldsymbol{Z}$. Hence we obtain $c_1(M)=(m-r+1)\alpha$.

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