# *On smooth $S^1$-actions on cohomology complex projective spaces. The case where the fixed point set consists of four connected components.*

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(Communicated by A. Hattori)

## 0. Introduction

In [7], T. Petrie made the following conjecture:

Let $X$ be a closed, smooth $2n$-dimensional homotopy complex projective space which admits a smooth non-trivial $S^1$-action, and let $h: X \to \boldsymbol{C}P^n$ be a homotopy equivalence. Then

$$h^*\hat{\mathcal{A}}(\boldsymbol{C}P^n)=\hat{\mathcal{A}}(X),$$

where $\hat{\mathcal{A}}(X)$ is the total $\hat{\mathcal{A}}$-class of $X$ defined by

$$\hat{\mathcal{A}}(X)=\Pi(x_i/2)(\sinh x_i/2)^{-1}\in H^*(X; \boldsymbol{Q}),$$

and where the elementary symmetric functions of the $x_i^2$ give the Pontrjagin classes of $X$.

This conjecture is equivalent to the following statement.

Let $X$ be as above and $x$ a generator of $H^2(X; \boldsymbol{Z})$. Then the total Pontrjagin class $p(X)$ of $X$ is of the form

$$p(X)=(1+x^2)^{n+1}. \tag{0.1}$$

DEFINITION. A closed, smooth $2n$-dimensional manifold $X$ is a cohomology complex projective space ($X$ is a cohomology $\boldsymbol{C}P^n$) if its cohomology ring has the form

$$H^*(X; \boldsymbol{Z})=Z[x]/(x^{n+1}), \quad x\in H^2(X; \boldsymbol{Z}).$$

In this paper we shall consider the above conjecture for cohomology complex projective spaces. Concerning this, A. Hattori proved the following proposition in [4].

PROPOSITION 0.1 (Hattori). *Let $X$ be a cohomology $\boldsymbol{C}P^n$. If $X$ admits a non-trivial smooth $S^1$-action of the linear type or of the Petrie type, then the total Pontrjagin class of $X$ must be of the form* (0.1).

Here the definitions of the linear type and the Petrie type will be given in Section 1. Then our main theorem is the following.

THEOREM 0.2. *Let $X$ be a cohomology $CP^n$. If $X$ admits a smooth $S^1$-action whose fixed point set consists of four connected components, then the $S^1$-action on $X$ is of the linear type or of the Petrie type.*

Combining this with Proposition 0.1, we have

COROLLARY 0.3. *Let $X$ be the same as Theorem 0.2. Then the total Pontrjagin class of $X$ must be of the form* (0.1).

Previously to our result, the following facts are known.

Let $X$ be a cohomology $CP^n$. If $X$ admits a smooth $S^1$-action whose fixed point set consists of two connected components, that is, if the action is semi-free, then T. Yoshida and K. Wang proved independently the above conjecture in [15], [16]. In this case, however, the $S^1$-action on $X$ is of the linear type. If $X$ admits a smooth $S^1$-action whose fixed point set consists of three connected components, it is known that the $S^1$-action on $X$ is also of the linear type ([14]).

With regard to our result T. Petrie constructed the manifolds, which are homeomorphic to $CP^n$, with the Petrie type $S^1$-actions whose fixed point sets consist of four connected components ([8], [10]).

This paper is organized as follows. In Section 1 we shall state some results due to G. E. Bredon, J. C. Su, W. Y. Hsiang and T. Petrie on cohomology complex projective spaces with non-trivial $S^1$-actions, and define the linear type and the Petrie type. Finally the equivariant Gysin homomorphism, which is our main tool, will be introduced and its some properties will be stated. In Section 2 an alternative proof of Proposition 0.1 will be given. Then it will also be proved that the total Pontrjagin class of each $S^1$-fixed point set component is of the form (0.1). In Section 3 we study a $Z_m$-fixed point set component $Y$ in $X$ such that the $S^1$-fixed point set in $Y$ consists of two connected components, where $Z_m$ denotes a finite cyclic subgroups of $S^1$ with order $m$. Section 4 is devoted to prove one proposition. In Section 5 Theorem 0.2 will be proved.

*Notation.* 1) Let $X$ be a $G$-space. Then $F(H, X)$ denotes the $H$-fixed point set in $X$ where $H$ is a subgroup of $G$ ($H$ may equal $G$). If we put $F(G, X)=\cup F_i$ where $\{F_i\}$ are its connected components, then $F(H, X)_i$ denotes the connected component of $F(H, X)$ containing $F_i$.

2) Let $X$ be a smooth manifold and let $Y$ be a smooth submanifold of $X$. Then $N(Y, X)$ denotes the normal bundle of $Y$ in $X$.

3) For a compact Lie group $G$, let $X$ be a left $G$-space and $EG\to BG$ a universal $G$-bundle which may be regarded as a right $G$-space. Then $X_G$ denotes

the orbit space obtained from $EG\times X$ by identifying $(ug, g^{-1}x)$ with $(u, x)$ for $u\in EG$, $x\in X$ and $g\in G$.

In concluding this introduction, I would like to express my hearty thanks to Prof. A. Hattori who gave me many useful suggestions, and some methods used in this paper are due to his lecture.

## 1. Preliminaries and definitions

In this section we shall recall some results due to G. E. Bredon, J. C. Su, W. Y. Hsiang and T. Petrie on cohomology complex projective spaces with non-trivial $S^1$-actions, and give the definitions of the linear type and the Petrie type stated in our introduction. Finally we shall introduce the equivariant Gysin homomorphism and state its some properties.

From now on let $X$ denote a cohomology $CP^n$ with a smooth non-trivial $S^1$-action. Let $G$ denote the circle group $S^1$ or $\boldsymbol{Z}_{p^r}$ which is the cyclic subgroup of $S^1$ with prime power order $p^r$. We put the $G$-fixed point set $F(G, X)=\cup F_i$ where $\{F_i\}$ are its connected components. Then, as is well known, each $F_i$ is a closed orientable smooth submanifold of $X$. Let $x$ be a generator of $H^2(X; \boldsymbol{Z})$ and $x_i$ its restriction to $F_i$. Then the cohomological property of $F(G, X)$ is given by the following.

Proposition 1.1 (Bredon [1], Su [12]). *Let the situation and notations be as above. Let $L$ denote* $\boldsymbol{Z}$ (resp. $\boldsymbol{Z}_p$) *if* $G=S^1$ (resp. $\boldsymbol{Z}_{p^r}$), *where $p$ is a prime number. Then the cohomology ring of each $F_i$ with $L$ coefficient is of the form*

$$H^*(F_i; L)=L[x_i]/(x_i)^{n_i+1}$$

*for some integer $n_i<n$, and moreover the relation $\sum_i(n_i+1)=n+1$ holds.*

We shall recall the concept of the equivariant cohomology.

Definition. Let $G$ be a compact Lie group and $M$ a $G$-space. Then we define the equivariant cohomology $H_G^*(M)$ of $M$ by

$$H_G^*(M)=H^*(M_G).$$

For a space $X$, which is a cohomology $CP^n$ with a non-trivial smooth $S^1$-action, W. Y. Hsiang determined the ring structure of $H_{S^1}^*(X; \boldsymbol{Z})$ which plays an important role in this paper. Before stating the structure, we shall give some remarks on $H_{S^1}^*(X; \boldsymbol{Z})$. Until this section ends, all cohomology groups will be assumed to be with $\boldsymbol{Z}$ coefficient unless otherwise indicated.

Let us consider the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & X_{S^1} & \xrightarrow{\rho} & BS^1 \\ & & \uparrow{\scriptstyle\tau} & & \\ & & F_{S^1} & & \end{array}$$

where $\tau$ and $\varphi$ are the inclusions, $\rho$ is the projection and $F=F(S^1, X)$. We always regard $H^*_{S^1}(X)$ as the $H^*(BS^1)$-algebra through $\rho^*: H^*(BS^1)\to H^*_{S^1}(X)$. Since the ordinary cohomologies of $X$ and $BS^1$ vanish at every odd degree, the Serre spectral sequence of $\rho$ collapses. This means that $\varphi^*: H^*_{S^1}(X)\to H^*(X)$ is surjective. Therefore there exists a lifting $y$ of a generator $x$ of $H^2(X)$, and we shall fix it. Then from the definition of $y$ and Proposition 1.1 we see that

$$H^2_{S^1}(X) \xrightarrow{\tau^*} H^2_{S^1}(F) \cong \sum_i (H^2(F_i)\oplus H^2(BS^1))$$

$$y \longmapsto \sum_i (x_i+a_i\alpha)$$

where $\{F_i\}$ are connected components of $F=F(S^1, X)$, $\alpha$ is a generator of $H^2(BS^1)$ and $\{a_i\}$ are integers.

Now we can state the ring structure of $H^*_{S^1}(X)$.

PROPOSITION 1.2 (Hsiang [2], [5]). *Let the situation and notations be as above. By Proposition* 1.1 *each $F_i$ is a cohomology $\boldsymbol{CP}^{n_i}$ for some $n_i$. Then the ring structure of $H^*_{S^1}(X)$ is given by*

$$H^*_{S^1}(X)=H^*(BS^1)[y]/\prod_i (y-a_i\alpha)^{n_i+1}.$$

*Moreover integers $\{a_i\}$ are mutually distinct.*

We remark that the integers $\{a_i\}$ depend on a choice of a lifting $y$, but that the differences $\{a_i-a_j\}$ are independent of its choice. In fact, if we let $\{a'_i\}$ be the integers induced by another lifting of $x$, then $a_i=a'_i+a$ for some integer $a$ independent of $i$. Conversely, for any integer $a$, there exists another lifting of $x$ satisfying the relation $a_i=a'_i+a$ for any $i$.

We shall put $GW_i=\{a_i-a_j \mid j\neq i\}$ for each $i$ and call them the global weights at $F_i$. T. Petrie indicated that the global weights $GW_i$ at $F_i$ have the following geometrical meaning.

PROPOSITION 1.3 (Petrie [7]). *Let $\boldsymbol{Z}_m$ be a cyclic subgroup of $S^1$ with order $m$. If $S^1$-fixed point set components $F_i$ and $F_j$ are contained in a same connected component of $F(\boldsymbol{Z}_m, X)$, then the difference $a_i-a_j\equiv 0 \pmod m$. Moreover if $m$ is a prime power, then its converse holds. Therefore our $S^1$-action is effective if and only if g.c.d. of $GW_i$ equals* 1 *for each $i$.*

Let $M$ be an arbitrary smooth manifold with a non-trivial smooth $S^1$-action and let $\{F_i\}$ be connected components of $F(S^1, M)$. We shall give some remarks on the normal bundle $N(F_i, M)$.

As is well known, $N(F_i, M)$ has a unique decomposition into Whitney sum of complex vector bundles

$$N(F_i, M)=\sum_{m\in Z,\, m>0} N_i^\dagger(m)$$

such that, for each $m$, $v\in N_i^\dagger(m)$ and $g\in S^1\subset \boldsymbol{C}$, we have

$$g_*v=g^m v$$

where $g_*$ denotes the differential induced by $g: M\to M$.

If, for each $m$,

$$N_i^\dagger(m)=V\oplus V'$$

is a decomposition into complex vector bundles, then

$$N_i^\dagger(m)=V\oplus \bar{V}'$$

is also a decomposition of real vector bundle $N_i^\dagger(m)$ into complex vector bundles where $\bar{V}'$ denotes the complex conjugate bundle of $V'$. Here we note that for $v\in \bar{V}'$ and $g\in S'\subset \boldsymbol{C}$, we have

$$g_*v=g^{-m}v\,.$$

Therefore setting

$$V=N_i(m)\,,\ \bar{V}'=N_i(-m)\,,$$

we obtain a decomposition of $N(F_i, M)$

$$N(F_i, M)=\sum_{m\in Z,\, m\neq 0} N_i(m)$$

which is called an admissible decomposition.

Again we shall consider about a cohomology complex projective space $X$ and let $\{F_i\}$ be connected components of $F(S^1, X)$. Fix a point $p_i\in F_i$. Then an admissible decomposition of $N(F_i, X)$ makes $N(F_i, X)|p_i$ a complex $S^1$-module. Let $t$ denote the standard 1-dimensional $S^1$-module.

DEFINITION. A smooth $S^1$-action on $X$ is called of the linear type at $F_i$ if there exists an admissible decomposition of $N(F_i, X)$ such that

$$N(F_i, X)=\sum_{j\neq i} N_i(a_i-a_j)$$

where $\dim_C N_i(a_i-a_j)=n_j+1$. This means $N(F_i, X)|p_i=\sum_{j\neq i}(n_j+1)t^{a_i-a_j}$. If there exists the above admissible decomposition for every $F_i$, then the $S^1$-action is simply called of the linear type.

DEFINITION. A smooth $S^1$-action on $X$ is called of the Petrie type if there exists an admissible decomposition of $N(F_i, X)$ for each $i$ such that

$$N(F_i, X)|p_i+t+t^{pq}=\sum_{j\neq i}(n_j+1)t^{a_i-a_j}+t^p+t^q \tag{1.1}$$

where $p$ and $q$ are mutually coprime positive integers independent of $i$.

For an admissible decomposition $\sum_j N_i(w_{ij})$ of $N(F_i, X)$, we put $LW_i=\{w_{ij}\}$ and call them the local weights at $F_i$ associated with an admissible decomposition $\sum_j N_i(w_{ij})$. Recall the global weights $GW_i=\{a_i-a_j|j\neq i\}$ at $F_i$. The definition of the linear type at $F_i$ is equivalent that there exists an admissible decomposition of $N(F_i, X)$ such that the global weights $GW_i$ coincide with the local weights $LW_i$ associated with it. For an arbitrary smooth $S^1$-action on $X$, $GW_i$ and $LW_i$ do not coincide. However there is the following relation between them.

PROPOSITION 1.4 (Petrie [7]). *Let $X$ be as above and $\sum_j N_i(w_{ij})$ an arbitrary admissible decomposition of $N(F_i, X)$. Then an identity*

$$\prod_{j\neq i}|a_i-a_j|=\prod_j|w_{ij}|$$

*holds for each $i$.*

We need one more result on the dimension of a connected component of $F(\boldsymbol{Z}_m, X)$ where $\boldsymbol{Z}_m$ is a cyclic subgroup of $S^1$ with order $m$.

PROPOSITION 1.5 (Su [13]). *Let $X$ and $\boldsymbol{Z}_m$ be as above and $Y$ a connected component of $F(\boldsymbol{Z}_m, X)$. If $Y\cap F(S^1, X)\neq\emptyset$, then*

$$\dim Y\leqq 2(\chi(Y)-1)$$

*where $\chi(Y)$ denotes the euler number of $Y$. When the $S^1$-action on $X$ is of the linear type, the equality holds for any subgroup $\boldsymbol{Z}_m$ of $S^1$.*

REMARK. When $m$ is a prime power, the equality holds by Proposition 1.1 and the fact $\chi(Y)=\chi(F(S^1, Y))$.

REMARK. Since $X$ is a cohomology $CP^n$ with a non-trivial smooth $S^1$-action and $\boldsymbol{Z}_m$ is a subgroup of $S^1$, it is not difficult to see that $Y$ is an even dimensional orientable closed smooth manifold.

Now we shall introduce the equivariant Gysin homomorphism, which is our main tool, and state its four properties, see for example [11].

Let $M$ and $N$ be $m$ and $n$ dimensional closed oriented $S^1$-manifolds respectively and let $f: M\to N$ be an $S^1$-map. Then the equivariant Gysin homomorphism

$$f_!: H^q_{S^1}(M)\longrightarrow H^{q+n-m}_{S^1}(N)$$

is defined for any $q$ and has similar properties to the ordinary Gysin homomorphism. Throughout this paper the notation $f_!$ will represent the equivariant Gysin homomorphism of an $S^1$-map $f$ unless otherwise indicated.

*Property* 1. Let $M$, $N$ and $V$ be closed oriented $S^1$-manifolds and let $f: M \to N$, $g: N \to V$ be $S^1$-maps. Then we have

$$(g \circ f)_! = g_! \circ f_! .$$

*Property* 2. Let $M$ and $N$ be closed oriented $S^1$-manifolds and $f: M \to N$ an $S^1$-map. Then for $u \in H^*_{S^1}(M)$ and $v \in H^*_{S^1}(N)$, we have

$$f_!(u \cup f^*(v)) = f_!(u) \cup v .$$

*Property* 3. Let $M$ be a closed oriented $S^1$-manifold and let $M_1$ and $M_2$ be closed oriented $S^1$-submanifolds of $M$. Let $j_i: M_i \to M$ be the inclusion maps $(i=1, 2)$. If $M_1 \cap M_2 = \emptyset$, then both $j_1^* \circ j_{2!}$ and $j_2^* \circ j_{1!}$ are zero maps.

Let $N$ be a closed oriented smooth $S^1$-manifold and $M$ a closed oriented smooth $S^1$-submanifold of $N$. Then the normal bundle $N(M, N)$ becomes an orientable $S^1$-bundle.

DEFINITION. The natural orientation of the normal bundle $N(M, N)$ means such the orientation that the given base orientation plus the fiber orientation of $N(M, N)$ coincides with the given one of $N$.

*Property* 4. Let $N$, $M$ be as above and $j: M \to N$ the inclusion. Then we have

$$e(N(M, N)_{S^1}) = j^* j_!(1)$$

where 1 represents the standard generator of $H^0_{S^1}(M)$ and $e(N(M, N)_{S^1})$ denotes the Euler class of the vector bundle $N(M, N)_{S^1} \to M_{S^1}$ with the orientation induced from the natural orientation of $N(M, N)$.

DEFINITION. For an oriented $S^1$-vector bundle $E \to M$, we define the equivariant Euler class $e^{S^1}(E)$ by

$$e^{S^1}(E) = e(E_{S^1}) .$$

Similarly for an complex (resp. real) $S^1$-vector bundle $E \to M$, we define the equivariant $i$-th Chern (resp. Pontrjagin) class $c_i^{S^1}(E)$ (resp. $p_i^{S^1}(E)$) by $c_i(E_{S^1})$ (resp. $p_i(E_{S^1})$).

The following lemma shows one way to compute $e^{S^1}(N(M, N))$ when $M$ is a connected component of $F(S^1, N)$.

LEMMA 1.6. *Let $E \to M$ be a complex $S^1$-vector bundle in which $M$ is fixed under the $S^1$-action, and assume that this bundle has no trivial factor, that is, if* *an element v of E is fixed under the $S^1$-action, then v must be zero. Then this bundle has a decomposition*

$$E=\sum_{m\in \boldsymbol{Z},\, m\neq 0} E(m) \tag{1.2}$$

*by the previous remark, where* $g\cdot v=g^m v$ *for* $v\in E(m)$ *and* $g\in S^1\subset \boldsymbol{C}$.

*If we put formally*

$$c(E(m))=\prod_i (1+\xi_i(m))$$

*and orient the bundle* $E\to M$ *by the right side of* (1.2), *then*

$$e^{S^1}(E)=\prod_{m,\, i} (m\alpha+\xi_i(m))$$

*where* $\alpha\in H^2(BS^1)$ *is the generator given by the first Chern class of the associated vector bundle with the universal* $S^1$*-bundle* $ES^1\to BS^1$.

PROOF. From the definition of the equivariant Euler class we see

$$e^{S^1}(E)=\prod_m e^{S^1}(E(m)).$$

Therefore it suffices to prove

$$e^{S^1}(E(m))=\prod_i (m\alpha+\xi_i(m)). \tag{1.3}$$

Let $H\to BS^1$ be the associated vector bundle with the universal $S^1$-bundle $ES^1\to BS^1$. Then we can construct a bundle isomorphism $h$ between the bundles $E(m)_{S^1}\to M_{S^1}$ and $H^m\hat{\otimes}E(m)\to M_{S^1}$ as follows, where $H^m$ denotes the $m$-fold tensor bundle of $H$, and $M_{S^1}=BS^1\times M$ by our assumption.

For $u\in H$ and $v\in E(m)$ we define

$$h([u,\ v])=u^m\hat{\otimes} v.$$

It is easy to check that this definition is well-defined and that $h$ is a bundle isomorphism.

On the other hand, from the definition of $\alpha$ we have

$$c(H)=1+\alpha,$$

hence it follows that

$$c(H^m\hat{\otimes} E(m))=\prod_i (1+(m\alpha+\xi_i(m))),$$

in particular

$$e(H^m\hat{\otimes} E(m))=\prod_i (m\alpha+\xi_i(m)).$$

Thus the identity (1.3) follows. q. e. d.

DEFINITION. Let $N$ be a closed oriented smooth $S^1$-manifold and $M$ a connected component of $F(S^1, N)$. We know that $N(M, N)$ decomposes uniquely into

$$N(M, N)=\sum_{m\in Z,\, m>0} N(m) \tag{1.4}$$

by the previous remark. Then the normal orientation of the normal bundle $N(M, N)$ means the orientation of $N(M, N)$ induced by the ordinary orientation of the right side of (1.4).

## 2. $p^{S^1}(X)$ of the linear type and the Petrie type

Let $X$ be a cohomology $CP^n$ with a non-trivial smooth $S^1$-action as before. Then the tangent bundle $TX\to X$ is naturally regarded as an $S^1$-vector bundle via the differential $g_*: TX\to TX$ induced by $g\in S^1$. Hence we can consider the vector bundle $(TX)_{S^1}\to X_{S^1}$. In this section we shall compute the equivariant Pontrjagin class $p^{S^1}(X)=p^{S^1}(TX)$ for the case where the $S^1$-action on $X$ is of the linear type or of the Petrie type, and give an alternative proof of Proposition 0.1.

Our main result in this section is the following.

THEOREM 2.1. *Let $X$ be as above and notations the same as Proposition* 1.2. *If the given $S^1$-action on $X$ is of the linear type, then*

$$p^{S^1}(X)=\prod_i (1+(y-a_i\alpha)^2)^{n_i+1}.$$

*If the given $S^1$-action on $X$ is of the Petrie type defined by* (1.1), *then*

$$p^{S^1}(X)=\prod_i (1+(y-a_i\alpha)^2)^{n_i+1}\frac{(1+p^2\alpha^2)(1+q^2\alpha^2)}{(1+\alpha^2)(1+p^2q^2\alpha^2)}.$$

We remark that there is a natural bundle map

$$\begin{array}{ccc} TX & \xrightarrow{\tilde{\varphi}} & (TX)_{S^1} \\ \downarrow & \varphi & \downarrow \\ X & \longrightarrow & X_{S^1} \end{array}$$

where $\tilde{\varphi}$ and $\varphi$ are inclusion maps defined by $\tilde{\varphi}(v)=[e, v]$ and $\varphi(x)=[e, x]$ for $v\in TX$, $x\in X$ and an arbitrary fixed $e\in ES^1$. Therefore we have

$$\varphi^* p^{S^1}(X)=p(X) \tag{2.1}$$

by the naturality of characteristic classes. Moreover we have

$$\varphi^*(y)=x,\quad \varphi^*(\alpha)=0 \tag{2.2}$$

by the definition of $y$ and $\alpha$, where $x$ is a generator of $H^2(X; \boldsymbol{Z})$. Therefore, if we use (2.1) and (2.2) and recall the relation $\sum(n_i+1)=n+1$, then we can deduce Proposition 0.1 from Theorem 2.1.

Furthermore we shall prove the following in the proof of Theorem 2.1.

THEOREM 2.2. *Let $X$ be a cohomology $\boldsymbol{C}P^n$ with a non-trivial smooth $S^1$-action. Then each connected component $F_i$ of $F(S^1, X)$ is also a cohomology $\boldsymbol{C}P^{n_i}$ for some integer $n_i$ by Proposition* 1.1. *If the given $S^1$-action on $X$ is of the linear type or of the Petrie type, then we have*

$$p(F_i)=(1+x_i^2)^{n_i+1}$$

*for each $F_i$, where $x_i$ is a generator of $H^2(F_i; \boldsymbol{Z})$.*

In order to prove Theorem 2.1 we need several lemmas. In the following, to simplify our discussion, let $Y$ denote a connected component of $F(\boldsymbol{Z}_m, X)$ satisfying $F(S^1, Y)\neq\emptyset$, where $\boldsymbol{Z}_m$ is a subgroup of $S^1$.

Let $F_k$ be a connected component of $F(S^1, Y)$ and $j_k: F_k\to Y$ the inclusion. We shall study the normal bundle $N(F_k, Y)$, in particular, calculate its total Pontrjagin class. For that purpose we shall calculate its total Chern class under a certain complex structure induced by the $S^1$-action on $Y$. Our idea is to compute $e^{S^1}(N(F_k, Y))$ in two different ways and compare them. Namely, in one way we make use of the equivariant Gysin homomorphism $j_{k!}$ and Property 4. In the other way we consider a decomposition of $N(F_k, Y)$ into complex vector bundles induced by the $S^1$-action and use Lemma 1.6. If we compare these coefficients of each power of $\alpha$, we will obtain various informations about the Chern classes of $N(F_k, Y)$.

We shall compute $e^{S^1}(N(F_k, Y))$ in the first way. To this end, it is desirable to know the structure of $H_{S^1}^*(Y; \boldsymbol{Q})$ as an $H^*(BS^1; \boldsymbol{Q})$-algebra. But it seems difficult to determine it. Fortunately the structure of the localized ring $S^{-1}H_{S^1}^*(Y; \boldsymbol{Q})$ as an $S^{-1}H^*(BS^1; \boldsymbol{Q})$-algebra is determined, where $S=H^*(BS^1; \boldsymbol{Q})-\{0\}$, and it suffices for our purpose.

LEMMA 2.3. *Let $X$, $Y$ and $S$ be as above and let $j: Y\to X$ be the inclusion. Then the structure of $S^{-1}H_{S^1}^*(Y; \boldsymbol{Q})$ as an $S^{-1}H^*(BS^1; \boldsymbol{Q})$-algebra is given by*

$$S^{-1}H_{S^1}^*(Y; \boldsymbol{Q})\cong S^{-1}H^*(BS^1; \boldsymbol{Q})[\bar{y}]/\prod_{F_k\subset Y}(\bar{y}-a_k\alpha)^{n_k+1}$$

*where $\bar{y}=j^*(y)$.*

PROOF. Let $\phi: F(S^1; Y)\to Y$ be the inclusion. Then we have the natural commutative diagram

$$\begin{array}{ccc} S^{-1}H^*_{S^1}(Y;\boldsymbol{Q}) & \xrightarrow{\phi^*} & \sum_{F_k\subset Y} S^{-1}H^*_{S^1}(F_k;\boldsymbol{Q}) \\ \uparrow j^* & & \uparrow \\ S^{-1}H^*_{S^1}(X;\boldsymbol{Q}) & \longrightarrow & \sum_{F_k\subset X} S^{-1}H^*_{S^1}(F_k;\boldsymbol{Q}) \end{array}$$

where all the maps are induced by the natural inclusions. We remark that the horizontal maps in this diagram are isomorphisms by the localization theorem, see for example [5, Chapter III, Theorem III. 1]. Moreover since the right vertical map is trivially surjective, $j^*$ is a surjection. This fact and Proposition 1.2 mean that $S^{-1}H^*_{S^1}(Y;\boldsymbol{Q})$ is generated by the set $\{1, \bar{y}, \bar{y}^2, \cdots, \bar{y}^n\}$ as an $S^{-1}H^*(BS^1;\boldsymbol{Q})$-module.

However we have

$$\phi^*\Big(\prod_{F_k\subset Y}(\bar{y}-a_k\alpha)^{n_k+1}\Big)=\prod_{F_k\subset Y}x_k^{n_k+1}=0\,,$$

so it follows that

$$\prod_{F_k\subset Y}(\bar{y}-a_k\alpha)^{n_k+1}=0 \tag{2.3}$$

from the fact that $\phi^*$ is an isomorphism. Hence $S^{-1}H^*_{S^1}(Y;\boldsymbol{Q})$ is already generated by the set $\{1, \bar{y}, \bar{y}^2, \cdots, \bar{y}^{h-1}\}$ as an $S^{-1}H^*(BS^1;\boldsymbol{Q})$-module, where $h=\sum_{F_k\subset Y}(n_k+1)$. On the other hand the rank of $\sum_{F_k\subset Y} S^{-1}H^*_{S^1}(F_k;\boldsymbol{Q})$ as an $S^{-1}H^*(BS^1;\boldsymbol{Q})$-module equals $\sum_{F_k\subset Y}(n_k+1)=h$, because $S^{-1}H^*_{S^1}(F_k;\boldsymbol{Q})$ is isomorphic to $S^{-1}H^*(BS^1;\boldsymbol{Q})\otimes H^*(F_k;\boldsymbol{Q})$ for each $F_k$ by Künneth formula. Therefore the set $\{1, \bar{y}, \bar{y}^2, \cdots, \bar{y}^{h-1}\}$ forms an additive base of $S^{-1}H^*_{S^1}(Y;\boldsymbol{Q})$ as an $S^{-1}H^*(BS^1;\boldsymbol{Q})$-module. Our lemma follows from this fact and the relation (2.3). q.e.d.

Now we shall give some remarks about orientations. We choose a generator $x$ of $H^2(X;\boldsymbol{Z})$ and fix it. We shall orient $X$ by $x^n[X]=1$. Recall that each connected component $F_k$ of $F(S^1, X)$ is also a cohomology $\boldsymbol{C}P^{n_k}$ and that $x_k$ is a generator of $H^2(F_k, \boldsymbol{Z})$ where $x_k$ is the restriction of $x$ to $F_k$. If $F_k$ is not a point, we orient $F_k$ by $x_k^{n_k}[F_k]=1$. If $F_k$ is a point, we give it the standard orientation. Moreover, for a connected component $Y$ of $F(\boldsymbol{Z}_m, X)$, we shall choose an arbitrary orientation on $Y$ and fix it for a while.

The following lemma is useful to compute $e^{S^1}(N(F_k, Y))=j_k^*j_{k!}(1)$.

LEMMA 2.4. *Let $F_k$, $Y$ and $X$ be as above and let $j_k\colon F_k\to Y$, $j\colon Y\to X$ be the inclusion maps. Then we have*

$$f(\bar{y}, \alpha)j_{k!}(1)=\prod_{F_i\subset Y,\, i\neq k}(\bar{y}-a_i\alpha)^{n_i+1} \quad \textit{in } S^{-1}H^*_{S^1}(Y;\boldsymbol{Q}), \tag{2.4}$$

*where $f(\bar{y}, \alpha)$ is a polynomial of $\bar{y}$ and $\alpha$ determined by*

$$j_!(1)=f(y, \alpha)\prod_{F_s\cap Y=\varnothing}(y-a_s\alpha)^{n_s+1}.$$

*In particular* $f(\bar{y}, \alpha)$ *has the following two properties*:

(2.5) $$\frac{1}{2}\deg f(\bar{y}, \alpha)=\chi(Y)-1-\frac{1}{2}\dim Y$$

*where* $\deg f(\bar{y}, \alpha)$ *denotes the degree of* $f(\bar{y}, \alpha)$ *in cohomology.*

(2.6) $$f(\bar{y}, \alpha)\in H^*_{S^1}(Y; \boldsymbol{Z}).$$

PROOF. Although $j_{k!}(1)$ belongs to $H^*_{S^1}(Y; \boldsymbol{Z})$, we regard it as an element of $S^{-1}H^*_{S^1}(Y; \boldsymbol{Q})$ by the natural homomorphism: $H^*_{S^1}(Y; \boldsymbol{Z})\to H^*_{S^1}(Y; \boldsymbol{Q})\to S^{-1}H^*_{S^1}(Y; \boldsymbol{Q})$. Since $F_i\cap F_k=\varnothing$ for $i\neq k$, we have, by Property 3,

$$j_i^* j_{k!}(1)=0 \qquad \text{for } i\neq k \text{ such that } F_i\subset Y.$$

Hence it follows from Lemma 2.3 that $j_{k!}(1)$ must be divided by $(\bar{y}-a_i\alpha)^{n_i+1}$ for such $i$. Thus it is written as

(2.7) $$j_{k!}(1)=g_k(\bar{y}, \alpha)\prod_{F_i\subset Y,\, i\neq k}(\bar{y}-a_i\alpha)^{n_i+1}$$
$$=j^*\Big(g_k(y, \alpha)\prod_{F_i\subset Y,\, i\neq k}(y-a_i\alpha)^{n_i+1}\Big),$$

where $g_k(y, \alpha)$ is an element of $S^{-1}H^*_{S^1}(X, \boldsymbol{Q})$.

For $j_!(1)\in H^*_{S^1}(X; \boldsymbol{Z})$, we can do the above argument within $H^*_{S^1}(X; \boldsymbol{Z})$ by Proposition 1.2. Thus we have

$$j_!(1)=f(y, \alpha)\prod_{F_s\cap Y=\varnothing}(y-a_s\alpha)^{n_s+1}$$

where we remark that $f(y, \alpha)$ belongs to $H^*_{S^1}(X; \boldsymbol{Z})$ by Proposition 1.2.

Therefore, using Property 2 and (2.7), we obtain

(2.8) $$j_!(j_{k!}(1))=j_!(1)g_k(y, \alpha)\prod_{F_i\subset Y,\, i\neq k}(y-a_i\alpha)^{n_i+1}$$
$$=f(y, \alpha)g_k(y, \alpha)\prod_{i\neq k}(y-a_i\alpha)^{n_i+1}.$$

On the other hand, for the composition $j\circ j_k: F_k\to X$, we have

$$(j\circ j_k)_!(1)=c\prod_{i\neq k}(y-a_i\alpha)^{n_i+1}$$

by the same discussion as the case of $j_!(1)$, where $c$ is an integer. If we restrict this identity to the ordinary cohomology and use the fact that the restriction of $(j\circ j_k)_!$ is then the ordinary Gysin homomorphism, we see that the integer $c$ must be 1 (here $X$ and $F_k$ are oriented by the previous remark). Hence we have

(2.9) $$j_!(j_{k!}(1))=(j\circ j_k)_!(1)=\prod_{i\neq k}(y-a_i\alpha)^{n_i+1}$$

by Property 1.

Equating (2.8) with (2.9) and using Proposition 1.2, we deduce the relation

$$f(y, \alpha)g_k(y, \alpha)\equiv 1 \mod (y-a_k\alpha)^{n_k+1} \quad \text{in } S^{-1}H^*_{S^1}(X; \boldsymbol{Q}).$$

Therefore if we multiply $f(\bar{y}, \alpha)$ on both sides in (2.7) and use this relation and Lemma 2.3, then we obtain the desired identity (2.4).

Finally if we compare the degrees of both sides in (2.4), then the property (2.5) follows. q. e. d.

We remark that $f(y, \alpha)$ depends on $Y$ but it is independent of each connected component of $F(S^1, Y)$. We shall call it the defect of $Y$. In the following we shall compute the defect $f(y, \alpha)$ of $Y$ in the cases of the linear type and the Petrie type.

LEMMA 2.5. *Let $\boldsymbol{Z}_m$ be an arbitrary cyclic subgroup of $S^1$. If the given $S^1$-action on $X$ is of the linear type, then the defect $f(y, \alpha)$ of $Y$ must be $\pm 1$ for any connected component $Y$ of $F(\boldsymbol{Z}_m, X)$.*

PROOF. Recall that in this case we have

$$\dim Y = 2(\chi(Y)-1)$$

by Proposition 1.5. Hence it follows from the property (2.5) that deg $f(y, \alpha)$ is zero. This and the property (2.6) show that $f(y, \alpha)$ is a constant integer.

We note that since our $S^1$-action is of the linear type, the restriction of $N(F_k, Y)$ to a point $p_k \in F_k$ is equal to $\sum_{F_i\subset Y, i\neq k}(n_i+1)t^{a_k-a_i}$ where $t$ denotes the standard 1-dimensional $S^1$-module as before. Therefore we have

$$\varphi_k^*(j_{k!}(1)) = \varepsilon \prod_{F_i\subset Y, i\neq k}((a_k-a_i)\alpha)^{n_i+1}$$

where $\varphi_k : p_k \to Y$ is the inclusion, $\varepsilon = \pm 1$. Here $\varepsilon$ depends on a choice of an orientation of $Y$.

On the other hand since $\varphi_k^*(\bar{y}) = a_k\alpha$, we get

$$\varphi_k^*\Big(\prod_{F_i\subset Y, i\neq k}(\bar{y}-a_i\alpha)^{n_i+1}\Big) = \prod_{F_i\subset Y, i\neq k}((a_k-a_i)\alpha)^{n_i+1}.$$

Therefore if we apply $\varphi_k^*$ to the identity (2.4), we see that $f(y, \alpha)$ must be $\varepsilon = \pm 1$. q. e. d.

Next we shall compute $f(y, \alpha)$ in the Petrie type. Recall that

$$N(F_k, X)|p_k + t + t^{pq} = \sum_{j\neq k}(n_j+1)t^{a_k-a_j} + t^p + t^q \tag{2.10}$$

where $p$ and $q$ are mutually coprime positive integers and they are independent of each connected component of $F(S^1, X)$. We notice that the above identity

(2.10) implies that there exists $a_i$ such that $|a_k-a_i|=pq$.

Since $Y$ is a connected component of $F(\boldsymbol{Z}_m, X)$, it is not difficult to see that

$$\dim Y=2(\chi(Y)-1) \text{ or } 2(\chi(Y)-2). \tag{2.11}$$

In fact the identity $\dim Y=2(\chi(Y)-2)$ holds if and only if $m$ divides $pq$ but divides neither $p$ nor $q$. The fact (2.11) means

$$\deg f(y, \alpha)=0 \text{ or } 2,$$

that is, $f(y, \alpha)$ is a constant integer or a linear combination of $y$ and $\alpha$ over $\boldsymbol{Z}$.

Let us assume $\dim Y=2(\chi(Y)-2)$ and show $f(y, \alpha)=\pm pq\alpha$. First we remark that we have

$$N(F_k, Y)|p_k=\sum_{\substack{m|a_k-a_j\\ j\neq k,i}}(n_j+1)t^{a_k-a_j}+n_i t^{a_k-a_i} \tag{2.12}$$

$$N(F_i, Y)|p_i=\sum_{\substack{m|a_i-a_j\\ j\neq k,i}}(n_j+1)t^{a_i-a_j}+n_k t^{a_i-a_k} \tag{2.13}$$

if we choose suitable complex structures in $N(F_k, Y)$ and $N(F_i, Y)$. Further notice that we have

$$N(Y, X)|\tilde{p}+t_m=\sum_{m\nmid a_k-a_s}(n_s+1)t_m^{a_k-a_s}+t_m^p+t_m^q \tag{2.14}$$

where $\tilde{p}\in Y$ and $t_m$ is the restriction of $t$ to the character ring $R(\boldsymbol{Z}_m)$. Since we assume $\dim Y=2(\chi(Y)-2)$, $m$ is not equal to 2 by Proposition 1.1. Therefore the right hand side of (2.14) and $t_m$ have the ordinary orientations. We may assume, if necessary by giving the reverse orientation to $Y$, that the orientations of both sides in (2.14) coincide when the natural orientation is given to $N(Y, X)$.

On the other hand, if we give the natural orientation to $N(F_k, X)$, then the orientations of both sides in (2.10) coincide because we have

$$\tilde{\varphi}_k^*\tilde{j}_{k!}(1)=\prod_{j\neq k}((a_k-a_j)\alpha)^{n_j+1}$$

by (2.9), where $\tilde{\varphi}_k: p_k\to X$ and $\tilde{j}_k: F_k\to X$ are the inclusions.

Let us give the natural orientation to $N(F_k, Y)$. Then, by the above argument, it is not difficult to see that the difference of the orientations of both sides in (2.12) is given by $\operatorname{sign}(a_k-a_i)$. Similarly, in (2.13), their difference is given by $\operatorname{sign}(a_i-a_k)$.

LEMMA 2.6. *Let $X$ support the Petrie type $S^1$-action given by* (2.10) *and let $Y$ be a connected component of $F(\boldsymbol{Z}_m, X)$. If* $\dim Y=2(\chi(Y)-2)$, *then the defect $f(y, \alpha)$ of $Y$ is $\pm pq\alpha$.*

PROOF. The assumption $\dim Y=2(\chi(Y)-2)$ implies that $f(y, \alpha)$ is written as

$Ay+B\alpha$, where $A$ and $B$ are integers. By Lemma 2.4 we have

$$(A\bar{y}+B\alpha)j_{k!}(1)=\prod_{j\neq k}(\bar{y}-a_j\alpha)^{n_j+1}$$

$$(A\bar{y}+B\alpha)j_{i!}(1)=\prod_{j\neq i}(\bar{y}-a_j\alpha)^{n_j+1}.$$

Restrict these equations to points $p_k\in F_k$ and $p_i\in F_i$ respectively, so we obtain

$$Aa_k+B=|a_k-a_i|$$

$$Aa_i+B=|a_i-a_k|$$

by (2.12), (2.13) and the remark followed by (2.14). Hence it follows that

$$A=0, \quad B=|a_k-a_i|=pq.$$

Finally we note that if we change an orientation of $Y$, then $f(y, \alpha)=pq\alpha$ turns into $-pq\alpha$. q. e. d.

Now we shall compute the equivariant Pontrjagin class of the linear type. In the following we fix $F_k$.

DEFINITION. Let $\Lambda_k$ denote the indices of the $S^1$-fixed point components $\{F_i\}$ except $F_k$. Then we define a partial ordering "$\leqq$" among them as follows:

$$i\leqq j \quad \text{means that } (a_k-a_i)|(a_k-a_j).$$

If $i\leqq j$ and $|a_k-a_i|\neq|a_k-a_j|$, then we write $i\prec j$.

We shall compute the equivariant Chern class of $N_k(a_k-a_j)$ inductively on this ordering.

Let $Y$ be $F(\boldsymbol{Z}_m, X)_k$, that is, a connected component containing $F_k$ in $F(\boldsymbol{Z}_m, X)$. Since our $S^1$-action is of the linear type, $N(F_k, Y)$ decomposes into

$$N(F_k, Y)=\sum_{m|a_k-a_i}N_k(a_k-a_i)$$

where $\dim_C N_k(a_k-a_i)=n_i+1$.

LEMMA 2.7. *If we put* $c(N_k(a_k-a_i))=\prod_q(1+\xi_q(i))$ *formally, then*

$$\prod_{\substack{m|a_k-a_i\\ i\neq k}}\prod_q((a_k-a_i)\alpha+\xi_q(i))=\prod_{\substack{m|a_k-a_i\\ i\neq k}}((a_k-a_i)\alpha+x_k)^{n_i+1}.$$

PROOF. This lemma follows immediately from Lemma 1.6, 2.4 and 2.5. q. e. d.

LEMMA 2.8. *Let* $N(F_k, X)$ *decompose into* $\sum_{i\neq k}N_k(a_k-a_i)$ *where* $\dim_C N_k(a_k-a_i)$ $=n_i+1$. *Then, for each* $i$, *we have*

$$c^{S^1}(N_k(a_k-a_i))=(1+(a_k-a_i)\alpha+x_k)^{n_i+1},$$

$$p^{S^1}(N_k(a_k-a_i))=(1+((a_k-a_i)\alpha+x_k)^2)^{n_i+1},$$

*hence*

$$p^{S^1}(N(F_k, X))=\prod_{i\neq k}(1+((a_k-a_i)\alpha+x_k)^2)^{n_i+1}.$$

PROOF. We have only to compute $c^{S^1}(N_k(a_k-a_i))$ for each $i$ because the rest of the above statement is easily deduced from this.

*Step* 1. First we shall compute it for a maximal element $i$ with respect to the partial ordering $\leqq$.

When $|a_k-a_i|\neq|a_k-a_j|$ for all $j\neq i$, we consider $F(\boldsymbol{Z}_{|a_k-a_i|}, X)_k$. Then, by Lemma 2.7, we have

$$\prod_q((a_k-a_i)\alpha+\xi_q(i))=((a_k-a_i)\alpha+x_k)^{n_i+1}$$

where $c(N_k(a_k-a_i))=\prod_q(1+\xi_q(i))$. We regard both sides of this identity as polynomials of $\alpha$, and replace $\alpha$ by $(1+(a_k-a_i)\alpha)/(a_k-a_i)$. Then we obtain an identity

$$\prod_q(1+(a_k-a_i)\alpha+\xi_q(i))=(1+((a_k-a_i)\alpha+x_k)^{n_i+1}$$

where the left hand side of this identity exactly represents the equivariant Chern class of $N_k(a_k-a_i)$.

When $|a_k-a_i|=|a_k-a_h|$ for some $h\neq i$ and $|a_k-a_i|\neq|a_k-a_j|$ for all $j\neq i$, $h$, we obtain the desired result by the similar argument to the above and we shall omit the details.

*Step* 2. Suppose that this lemma holds for all $u$ such that $i\prec u$.

When $|a_k-a_i|\neq|a_k-a_j|$ for all $j\neq i$, we have

$$\prod_q((a_k-a_i)\alpha+\xi_q(i))\prod_{i<u}((a_k-a_u)\alpha+x_k)^{n_u+1}=\prod_{i\leq j}((a_k-a_j)\alpha+x_k)^{n_j+1}$$

by the assumption of induction and Lemma 2.7, where $c(N_k(a_k-a_i))=\prod_q(1+\xi_q(i))$. Comparing the coefficients of various powers of $\alpha$ both sides of this identity and repeat the same argument as Step 1, we obtain the desired result.

When $|a_k-a_i|=|a_k-a_h|$ for some $h\neq i$ and $|a_k-a_i|\neq|a_k-a_j|$ for all $j\neq i$, $h$, the similar discussion holds and the desired result is obtained. We shall omit the details. q. e. d.

By Proposition 1.2, $p^{S^1}(X)$ can be written uniquely as

$$p^{S^1}(X)=\prod(1+(y-a_i\alpha)^2)^{n_i+1}h(y, \alpha)$$

where $h(y, \alpha)$ is a formal power series belonging to $Z[[y, \alpha]]/\Pi(y-a_i\alpha)^{n_i+1}$. Let $\tilde{j}_k: F_k \to X$ be the inclusion. Then we have

$$\tilde{j}_k^* p^{S^1}(X)=(1+x_k^2)^{n_k+1}\prod_{i\neq k}(1+((a_k-a_i)\alpha+x_k)^2)^{n_i+1}\tilde{j}_k^* h(y, \alpha). \tag{2.15}$$

On the other hand, since there is a bundle isomorphism

$$TX|F_k = TF_k \oplus N(F_k, X),$$

we have

$$\begin{aligned}\tilde{j}_k^* p^{S^1}(X) &= p^{S^1}(F_k)p^{S^1}(N(F_k, X)) \\ &= p^{S^1}(F_k)\prod_{i\neq k}(1+((a_k-a_i)\alpha+x_k)^2)^{n_i+1}\end{aligned} \tag{2.16}$$

by Lemma 2.8.

Comparing (2.15) with (2.16), we obtain

$$p^{S^1}(F_k)=(1+x_k)^{n_k+1}\tilde{j}_k^* h(y, \alpha).$$

We note that the total Pontrjagin class of $F_k$ belongs to the subgroup $H^*(F_k; \boldsymbol{Z}) \otimes H^0(BS^1; \boldsymbol{Z})$ of $H^*_{S^1}(F_k; \boldsymbol{Z})=H^*(F_k; \boldsymbol{Z})\otimes H^*(BS^1; \boldsymbol{Z})$. Hence $\tilde{j}_k^* h(y, \alpha)$ must also belong to $H^*(F_k; \boldsymbol{Z})\otimes H^0(BS^1; \boldsymbol{Z})$. Since the above argument holds for every connected component of $F(S^1, X)=F$, the restriction of $h(y, \alpha)$ to $F_{S^1}$ must belong to $H^*(F, \boldsymbol{Z})\otimes H^0(BS^1; \boldsymbol{Z})$. Here we have the following fact, see [6], Proposition 3.7.

LEMMA 2.9. *Let $\tau: F_{S^1} \to X_{S^1}$ be the inclusion and let $u$ be an element of $H^{2q}_{S^1}(X; \boldsymbol{Z})$, where $q$ is a non-zero integer. If $\tau^* u$ belongs to the subgroup $H^{2q}(F; \boldsymbol{Z})\otimes H^0(BS^1; \boldsymbol{Z})$, then $u$ must be zero.*

Let us put $h(y, \alpha)=1+\sum_{q>0} h^q(y, \alpha)$ where $h^q(y, \alpha)$ is the element of degree $2q$ in $h(y, \alpha)$. We apply this lemma to each $h^q(y, \alpha)$, so we see $h(y, \alpha)=1$. Thus Theorem 2.1 has been proved in the case of the linear type $S^1$-action.

In the case of the Petrie type $S^1$-action, the similar argument holds using Lemma 2.6. We shall omit the proof.

Finally we note that Theorem 2.2 follows from Theorem 2.1 and Lemma 2.8.

## 3. Some properties about $Y$ such that $F(S^1, Y)$ consists of two connected components

Let $Y$ be a connected component of $F(\boldsymbol{Z}_m, X)$ such that $F(S^1, Y)$ consists of two connected components. Then we shall deduce some properties about $Y$ using several lemmas in Section 2.

Recall that each normal bundle $N(F_i, Y)$ has a unique decomposition

$$N(F_i, Y)=\sum_{m_s\in Z,\, m_s>0} N_i(m_s)$$

where $\{F_i\,;\ i=1, 2\}$ are connected components of $F(S^1, Y)$. Let $m_j$ be the maximum element of $\{m_s\}$. Then we shall consider $F(\boldsymbol{Z}_{m_j}, Y)$, and again write it $Y$ and $m_j$ as $m$. Therefore, from now on until this section ends, $Y$ will be assumed to denote a connected component of $F(\boldsymbol{Z}_m, X)$ such that $F(S^1, Y)$ consists of two connected components and that $g_*v=g^m v$ for any $v\in N(F_i, Y)$ and $g\in S^1\subset \boldsymbol{C}$.

Since each $F_i$ is a certain connected component of $F(S^1, X)$, $F_i$ is a cohomology $\boldsymbol{C}P^{n_i}$ and an integer $a_i$ is assigned to $F_i$.

PROPOSITION 3.1. *Let $Y$ satisfy the following two conditions*:

(1) $\dim Y=2(\chi(Y)-1)$,

(2) *either $n_1$ or $n_2$ is non-zero.*

*We put*

$$h=\max\{b\in \boldsymbol{Z}\,;\ 2^b\,|\,a_1-a_2\}.$$

*If $2^h$ divides $m$, then $m$ must equal $|a_1-a_2|$.*

PROOF. First we remark that $N(F_i, Y)=N_i(m)$ for each $i$. By the condition (2), we may assume $n_1\geqq n_2$ and $n_1\geqq 1$. It follows from the condition (1) and Lemma 2.4 that

$$c j_1^* j_{1!}(1)=j_1^*(\bar{y}-a_2\alpha)^{n_2+1}=(x_1+(a_1-a_2)\alpha)^{n_2+1}$$

where $c$ is an integer.

On the other hand, if we put $c(N_1(m))=\prod_q(1+\xi_q)$ formally, then we deduce

$$\varepsilon j_1^* j_{1!}(1)=e^{S^1}(N_1(m))=\prod_q(\xi_q+m\alpha)$$

from Property 4, Lemma 1.6 and the condition (1), where $\varepsilon=\pm 1$ denotes the difference between the natural orientation and the normal orientation of $N(F_1, Y)$.

Compare the above two identities, so we obtain

$$\varepsilon c\prod_q(\xi_q+m\alpha)=(x_1+(a_1-a_2)\alpha)^{n_2+1}, \tag{3.1}$$

in particular

$$\varepsilon c m^{n_2+1}=(a_1-a_2)^{n_2+1}. \tag{3.2}$$

We regard both sides of the identity (3.1) as polynomials of $\alpha$ and replace $\alpha$ by $(1+m\alpha)/m$. Then it turns into

$$(3.3)\qquad \varepsilon c\prod_q(1+m\alpha+\xi_q)=(x_1+(a_1-a_2)(1+m\alpha)/m)^{n_2+1}.$$

Here the class $\prod_q(1+m\alpha+\xi_q)$ represents exactly the equivariant Chern class of $N_1(m)$, hence it belongs to the integral cohomology $H^*_{S^1}(F_1; \boldsymbol{Z})$. If $n_2=0$, then $n_1\geqq 1=n_2+1$ by the condition (2). Hence, if we compare the coefficients of $x_1^{n_1}$ of both sides in (3.3), then we see $|c|=1$. This and (3.2) show $m=|a_1-a_2|$. Thus we may assume $n_2\geqq 1$.

By Proposition 1.1 and the assumption $n_1\geqq n_2$, the cohomology element $x_1^{n_2}$ is non-zero, and the coefficient of $x_1^{n_2}$ in the right hand side of (3.3) is $(n_2+1)\times(a_1-a_2)/m$. Therefore, if we divide both sides of (3.3) by $\varepsilon c$, then we see that

$$(3.4)\qquad (n_2+1)(a_1-a_2)/\varepsilon\, cm=\frac{n_2+1}{\left(\dfrac{a_1-a_2}{m}\right)^{n_2}}$$

must be an integer, where (3.2) is used.

Suppose that $m$ and $|a_1-a_2|$ are distinct. Then we have $|a_1-a_2/m|\geqq 3$, because $m$ is a divisor of $a_1-a_2$ by Proposition 1.3 and we assume that $2^h$ divides $m$. Hence we have

$$|a_1-a_2/m|^{n_2}\geqq 3^{n_2}>n_2+1 \qquad \text{for} \quad n_2\geqq 1,$$

which contradicts the integrality of (3.4). q. e. d.

REMARK. The condition (2) is essentially necessary. In fact, T. Petrie constructs the counterexamples ([8], [10]).

In the following we shall study the case where $\dim Y<2(\chi(Y)-1)$.

PROPOSITION 3.2. *If* $\dim Y<2(\chi(Y)-1)$, *then the dimensions of* $F_1$ *and* $F_2$ *must coinside, that is,* $n_1=n_2$.

This proposition follows immediatelly from the following proposition.

PROPOSITION 3.3. *If* $n_1\neq n_2$, *then* $\dim Y=2(\chi(Y)-1)$ *and* $N(F_i, Y)=N_i(|a_1-a_2|)$ *for each* $i$.

Before the proof of Proposition 3.3, we shall prepare the following lemma.

LEMMA 3.4. *Let* $R$ *denote the subring of* $H^*_{S^1}(Y; \boldsymbol{Q})$ *generated by* $\bar{y}$ *and* $\alpha$, *and let* $j_1\colon F_1\to Y$ *be the inclusion. Then*

$$j_1^*;\ R\longrightarrow H^*_{S^1}(F_1; \boldsymbol{Q})$$

*is injective in degrees not more than* $2n_1$.

PROOF. The subring $R$ is also generated by $(\bar{y}-a_1\alpha)$ and $\alpha$. Recall that $j_1^*(\alpha^s(\bar{y}-a_1\alpha)^t)=\alpha^s x_1^t$ for all integers $s$, $t$ and that the class $\{\alpha^s x_1^t;\ s+t\leqq n_1\}$

forms an additive base of $H^*_{S^1}(F_1; \boldsymbol{Q})$. Therefore this lemma clearly follows. q. e. d.

PROOF OF PROPOSITION 3.3. Since $n_1$ and $n_2$ are distinct integers, we may assume $n_1 \geqq n_2+1$. Using Lemma 2.4 for the inclusion $j_1: F_1 \to Y$, we obtain

$$j_1^* f(\bar{y}, \alpha) j_1^* j_{1!}(1) = (x_1 + (a_1 - a_2)\alpha)^{n_2+1}.$$

Note that both $j_1^* f(\bar{y}, \alpha)$ and $j_1^* j_{1!}(1)$ are homogeneous elements of $H^*_{S^1}(F_1; \boldsymbol{Z})$ and that $x_1^s$ does not vanish for $s \leqq n_2+1$ by the assumption $n_1 \geqq n_2+1$. Therefore the above identity implies that $j_1^* f(\bar{y}, \alpha)$ must be of the form $\pm(x_1 + (a_1 - a_2)\alpha)^q$ for some $q \leqq n_2+1$. Hence, it follows from Lemma 3.4 and the assumption $n_1 \geqq n_2+1$ that $f(\bar{y}, \alpha)$ must be of the form $\pm(\bar{y} - a_2\alpha)^q$.

On the other hand, using Lemma 2.4 for the inclusion $j_2: F_2 \to Y$, we obtain

$$(3.5) \qquad j_2^* f(\bar{y}, \alpha)\, j_2^* j_{2!}(1) = (x_2 + (a_2 - a_1)\alpha)^{n_1+1}.$$

Since $f(\bar{y}, \alpha) = \pm(\bar{y} - a_2\alpha)^q$ by the above discussion, we have $j_2^* f(\bar{y}, \alpha) = \pm x_2^q$.

Suppose that $q$ is positive. If we restrict the identity (3.5) to a point of $F_2$, then its left side vanishes, and its right side equals $((a_2 - a_1)\alpha)^{n_1+1}$ which is nonzero because $a_1$ and $a_2$ are distinct by Proposition 1.2. Thus $q$ must be zero, that is, $f(\bar{y}, \alpha) = \pm 1$. This means our proposition. q. e. d.

If $\dim Y < 2(\chi(Y) - 1)$, then $n_1 = n_2$ by Proposition 3.2. From now on we shall put $n_1 = n_2 = r$ and assume

$$(A) \qquad r \geqq 1.$$

Our aim is to show that the defect of $Y$ is equal to $\pm(a_1 - a_2)\alpha$ under certain circumstances.

By Lemma 2.4 we have

$$(3.6) \qquad \begin{aligned} j_1^* f(\bar{y}, \alpha) j_1^* j_{1!}(1) &= (x_1 + (a_1 - a_2)\alpha)^{r+1} \\ j_2^* f(\bar{y}, \alpha) j_2^* j_{2!}(1) &= (x_2 + (a_2 - a_1)\alpha)^{r+1}. \end{aligned}$$

We shall represent $j_1^* f(\bar{y}, \alpha)$ and $j_1^* j_{1!}(1)$ as polynomials of $x_1$ and $\alpha$ in $H^*_{S^1}(F_1; \boldsymbol{Z})$, and regard them as elements of the polynomial ring $\boldsymbol{Z}[x_1, \alpha]$. Then, since $x_1^{r+1} = 0$ and $x_1^q \neq 0$ for $q \leqq r$ in $H^*_{S^1}(F_1; \boldsymbol{Z})$, we have an identity as elements of $\boldsymbol{Z}[x_1, \alpha]$

$$(3.7) \qquad j_1^* f(\bar{y}, \alpha) j_1^* j_{1!}(1) = (x_1 + (a_1 - a_2)\alpha)^{r+1} - h x_1^{r+1}$$

where $h$ is an integer.

*Assertion* 3.5. $|h| = 1$.

PROOF. Suppose $h=0$, then (3.7) shows that $j_1^*f(\bar{y}, \alpha)$ must be of the form $\pm(x_1+(a_1-a_2)\alpha)^q$ for some $q$. Then, the same argument as the proof of Proposition 3.3 implies that $q$ must be zero, that is, $\deg f(y, \alpha)=0$. This contradicts our assumption $\dim Y<2(\chi(Y)-1)$.

Next suppose that $h$ is a positive integer. Let us consider the identity (3.7) in $C[x_1, \alpha]$. The right hand side of (3.7) decomposes into $\prod_{\zeta}(x_1+(a_1-a_2)\alpha-\zeta b x_1)$ where $b=\sqrt[r+1]{h}$ and $\zeta$ moves over all $(r+1)$-roots of 1. Hence $j_1^*f(\bar{y}, \alpha)$ is of the form

$$j_1^*f(\bar{y}, \alpha)=c\prod_{\zeta'}(x_1+(a_1-a_2)\alpha-\zeta' b\, x_1)$$

where $c\in Q$ and $\zeta'$ moves over a certain subset of $(r+1)$-roots of 1. Since $\dim Y\geqq\dim F_1+2=2r+2$, we have

$$\deg f(\bar{y}, \alpha)=2(\chi(Y)-1)-\dim Y$$
$$=4r+2-\dim Y\leqq 2r\,.$$

Therefore, by Lemma 3.4, $f(\bar{y}, \alpha)$ must be of the form

$$f(\bar{y}, \alpha)=c\prod_{\zeta'}((\bar{y}-a_2\alpha)-\zeta' b(\bar{y}-a_1\alpha))\,. \tag{3.8}$$

Recall that we assume $N(F_i, Y)=N_i(m)$. Therefore, if we restrict each identity of (3.6) to a point of $F_i$ using (3.8), then we obtain

$$|c||a_1-a_2|^{2r+1-d}m^d=|a_1-a_2|^{r+1}\,,$$
$$|c||b(a_2-a_1)|^{2r+1-d}m^d=|a_2-a_1|^{r+1}\,,$$

where $2d=\dim Y$. Since we assume that $h$ is a positive integer, we deduce the fact $h=1$ from these equations.

For the case where $h$ is a negative integer, we deduce $h=-1$ from the same discussion as the above case. q.e.d.

Thus we have the following two equations from (3.7):

$$j_1^*f(\bar{y}, \alpha)=c\prod_{\zeta'}(x_1+(a_1-a_2)\alpha+\zeta' x_1) \tag{3.9}$$

$$c\, j_1^*j_{1!}(1)=\prod_{\zeta''}(x_1+(a_1-a_2)\alpha+\zeta'' x_1) \tag{3.10}$$

where $\zeta'$ and $\zeta''$ move over certain subsets of $2(r+1)$-roots of 1.

For the sake of simplicity, we moreover assume the following:

(B) $2^h|m$, where $h=\max\{b\in Z;\ 2^b|a_1-a_2\}$ as before,

(C) there exists a connected component $\tilde{Y}$ of $F(Z_p, X)$ containing $Y$ with $F(S^1, \tilde{Y})=F(S^1, Y)$, where $p$ is 2 if $a_1-a_2$ is even and an arbitrary prime number dividing $m$ otherwise.

We remark that $\tilde{Y}$ is fixed pointwise under the restricted $\boldsymbol{Z}_{2h}$-action on $X$.

LEMMA 3.6. *Let $\tilde{Y}$ be as above and $\rho: Y \to \tilde{Y}$ the inclusion. We shall put $\rho_!(1)=\tilde{f}(\tilde{y}, \alpha)$ where $\tilde{y}$ denotes the restriction of $y$ to $\tilde{Y}_{S^1}$. Then we have*

$$s\tilde{f}(y, \alpha)=f(y, \alpha)$$

*where $s$ is a non-zero integer.*

PROOF. Consider the diagram

$$Y \xrightarrow{\rho} \tilde{Y} \xrightarrow{\tilde{j}} X, \qquad j: Y \to X$$

where all the maps are the inclusions. On one hand we have

$$j_!(1)=f(y, \alpha)\prod_{F_j \cap Y=\varnothing}(y-a_j\alpha)^{n_j+1}.$$

On the other hand we have

$$\begin{aligned} j_!(1)&=\tilde{j}_!(\rho_!(1)) \\ &=\tilde{j}_!(\tilde{j}^*\tilde{f}(y, \alpha)) \\ &=\tilde{f}(y, \alpha)\tilde{j}_!(1) \\ &=\tilde{f}(y, \alpha)\, s\prod_{F_j \cap Y=\varnothing}(y-a_j\alpha)^{n_j+1} \end{aligned}$$

where $\tilde{j}_!(1)=s\prod_{F_j \cap \tilde{Y}=\varnothing}(y-a_j\alpha)^{n_j+1}$ for some non-zero integer $s$ because $\dim \tilde{Y}=2(\chi(\tilde{Y})-1)$. Equating the above two identities, we see

$$f(y, \alpha)\equiv s\tilde{f}(y, \alpha) \qquad \bmod (y-a_1\alpha)^{r+1}(y-a_2\alpha)^{r+1}.$$

Here we note that

$$\deg \tilde{f}(y, \alpha)=\deg f(y, \alpha)=2(\chi(Y)-1)-\dim Y=4r+2-\dim Y \leqq 2r.$$

Thus $f(y, \alpha)$ and $s\tilde{f}(y, \alpha)$ coincide. q. e. d.

Now we shall determine $f(y, \alpha)$ under the above circumstances. First we shall prove the following.

*Assertion* 3.7. The coefficient of $x_1^{2r+1-d}$ in $j_1^* f(\tilde{y}, \alpha)$ vanishes, if the conditions (A), (B) and (C) are satisfied.

PROOF. Let $\tilde{j}_1: F_1 \to \tilde{Y}$ be the inclusion. By (3.9) and Lemma 3.6 we have

$$s\tilde{j}_1^*\tilde{f}(\tilde{y}, \alpha)=c\prod_{\zeta'}(x_1+(a_1-a_2)\alpha+\zeta' x_1). \tag{3.11}$$

Let $A$ denote the coefficient of $x_1^{2r+1-d}$ in $\prod_{\zeta'}(x_1+(a_1-a_2)\alpha+\zeta' x_1)$, then

(3.12) $$|A|=|\prod_{\zeta'}(1+\zeta')|\leqq 2^{2r+1-d}.$$

The term $\tilde{j}_1^*\tilde{f}(\tilde{y}, \alpha)=\tilde{j}_1^*\rho_!(1)$ represents the equivariant Euler class of the restricted $S^1$-bundle $j_1^*N(Y, \tilde{Y})$ over $F_1$. Let $j_1^*N(Y, \tilde{Y})$ decompose into

$$j_1^*N(Y, \tilde{Y})=\sum_{m_j\in Z,\, m_j>0} N(m_j).$$

We shall put $\dim_C N(m_j)=d_j$ and $c(N(m_j))=\prod_q(1+\xi_q(j))$. Then

(3.13) $$\tilde{j}_1^*\tilde{f}(\tilde{y}, \alpha)=\tilde{j}_1^*\rho_!(1)=\varepsilon\prod_{j,q}(m_j\alpha+\xi_q(j))$$

by Property 4 and Lemma 1.6, where $\varepsilon=\pm 1$.

Thus, by (3.11) and (3.13), we have

(3.14) $$\varepsilon s\prod_{j,q}(m_j\alpha+\xi_q(j))=c\prod_{\zeta'}(x_1+(a_1-a_2)\alpha+\zeta' x_1).$$

in particular

(3.15) $$|s\prod m_j^{d_j}|=|c(a_1-a_2)^{2r+1-d}|.$$

Since the term $\prod_{j,q}(m_j\alpha+\xi_q(j))$ represents the total equivariant Chern class of $j_1^*N(Y, \tilde{Y})$, it belongs to $H_{S^1}^*(F_1; \boldsymbol{Z})$. Hence

(3.16) $$\left|\frac{cA}{s}\right|=\frac{|A|}{\left|\dfrac{(a_1-a_2)^{2r+1-d}}{\prod m_j^{d_j}}\right|}$$

must be an integer by (3.14), where (3.15) is used.

We assert that $|a_1-a_2|/m_j\geqq 3$ for any $j$. In fact, Proposition 1.3 and the assumption $\dim Y<2(\chi(Y)-1)$ imply that any $m_j$ is a divisor of $a_1-a_2$ but it is not equal to $|a_1-a_2|$. Moreover the assumption about the prime $p$ means that each $m_j$ is divided by $2^h$. Thus the above inequality holds.

Therefore, if we note $\sum d_j=2r+1-d$, then we have

(3.17) $$\left|\frac{(a_1-a_2)^{2r+1-d}}{\prod m_j^{d_j}}\right|\geqq 3^{2r+1-d}.$$

Since $2r+1-d=\chi(Y)-1-\dfrac{1}{2}\dim Y>0$ by our assumption, the fact $A=0$ follows from (3.12), (3.17) and the integrality of (3.16). q. e. d.

*Assertion* 3.8. In the same situation as Assertion 3.7, $m=|a_1-a_2|$.

PROOF. Let $B$ denote the coefficient of $x_1^{d-r}$ in the term $\prod_{\zeta'}(x_1+(a_1-a_2)\alpha$

$+\zeta'' x_1)$ in (3.10), hence $|B|=|\prod_{\zeta'}(1+\zeta'')|\leqq 2^{d-r}$. The identity (3.7) means that the coefficient of $\alpha x_1^r$ in $j_1^* f(\bar{y}, \alpha) j_1^* j_{1!}(1)$ is non-zero. Hence $B$ must be non-zero by Assertion 3.7.

Suppose $m\neq|a_1-a_2|$. We know that $m$ is a divisor of $a_1-a_2$ and assume that $m$ is divided by $2^h$, so it follows that $|a_1-a_2|/m\geqq 3$.

Restrict the identity (3.10) to a point of $F_1$, so we obtain

$$|cm^{d-r}|=|a_1-a_2|^{d-r}, \tag{3.18}$$

hence

$$|c|=|a_1-a_2/m|^{d-r}\geqq 3^{d-r}.$$

Note that $j_1^* j_{1!}(1)$ is an element of $H_{S^1}^*(Y, \boldsymbol{Z})$, so the quotient $B/c$ must be an integer. However we know that $0\neq|B|\leqq 2^{d-r}$ and $d-r>0$, hence we have an inequality

$$0\neq|B/c|\leqq(2/3)^{d-r}<1$$

which contradicts the integrality of $B/c$. q. e. d.

Thus it follows from Assertion 3.8 and (3.18) that

$$|c|=1. \tag{3.19}$$

PROPOSITION 3.9. *If the conditions* (A), (B) *and* (C) *are satisfied, then the defect* $f(y, \alpha)$ *of* $Y$ *is equal to* $\pm(a_1-a_2)\alpha$. *Therefore we have* $N(F_i, Y)=N(|a_1-a_2|)$ *for each* $i$ *and* $\dim Y=2(\chi(Y)-2)$.

PROOF. Recall that we have

$$j_1^* f(\bar{y}, \alpha)=\varepsilon s\prod_{j,q}(m_j\alpha+\xi_q(j))=c\prod_{\zeta'}(x_1+(a_1-a_2)\alpha+\zeta' x_1)$$

by (3.14). Assertion 3.7 means that a certain $\zeta'$ must be $-1$ and a certain $\xi_q(j)$ must vanish. We may assume $\xi_1(u)$ vanishes. Therefore the above identity reduces to

$$\varepsilon s m_u\alpha\prod_{q=2}^{d_u}(m_u\alpha+\xi_q(u))\prod_{j\neq u}\prod_{q=1}^{d_j}(m_j\alpha+\xi_q(j))$$
$$=c(a_1-a_2)\alpha\prod_{\zeta'\neq -1}(x_1+(a_1-a_2)\alpha+\zeta' x_1).$$

Then, if we repeat the same argument as Assertion 3.7, then we deduce the fact $2r-d=0$. This fact, Assertion 3.7 and 3.8 imply this proposition. q. e. d.

Summing up the results obtained in this section, we have

COROLLARY 3.10. *For every prime number* $p$ *dividing* $a_1-a_2$, *we assume* $F(S^1, Y_p)=F_1\cup F_2$ *where* $Y_p=F(\boldsymbol{Z}_p, X)_1=F(\boldsymbol{Z}_p, X)_0$. *We orient* $Y_p$ *suitably and give the natural orientation to* $N(F_i, Y_p)$ *for* $i=1, 2$. *Then*

$$N(F_1, Y_p)|p_1 = n_2 t^{a_1-a_2} + t^{w_p},$$

$$N(F_2, Y_p)|p_2 = n_1 t^{a_2-a_1} + t^{-w_p},$$

*where $p_i \in F_i$ and $w_p$ is an integer satisfying $p|w_p|a_1-a_2$. Here, if $n_1 \neq n_2$, then $|w_p|$ must equal $|a_1-a_2|$.*

*Moreover, we have*

$$e^{S^1}(N(F_1, Y_p)) = \frac{(x_1+(a_1-a_2)\alpha)^{n_2+1}}{(a_1-a_2)\alpha}(w_p\alpha),$$

$$p^{S^1}(N(F_1, Y_p)) = \frac{(1+(x_1+(a_1-a_2)\alpha)^2)^{n_2+1}}{1+(a_1-a_2)^2\alpha^2}(1+w_p^2\alpha^2),$$

*where $N_1(w_p)$ is a trivial complex line bundle if $|w_p| \neq |a_1-a_2|$, and the similar statement holds for $N(F_2, Y_p)$.*

PROOF. If $n_1 \neq n_2$, then this corollary easily follows from Lemma 2.4 and Proposition 3.3. If $n_1 = n_2 = 0$, then the statement is trivial. Therefore we assume $n_1 = n_2 \geqq 1$. Let $N(F_i, Y_p)$ decompose into $\sum_{m_i>0} N_i(m_i)$ and consider $F(\boldsymbol{Z}_{m_{i0}}, X)_i$ where $m_{i0}$ is the maximum element in $\{m_i\}$. Then, if $a_1-a_2$ is odd, our statement easily follows from Lemma 2.4, Lemma 3.6 and Proposition 3.9. When $a_1-a_2$ is even, we consider the case where $p=2$ first. In this case, our corollary follows by the same reason as the case where $a_1-a_2$ is odd. This means that the first half of this corollary holds for any $p$. If $|w_p| = |a_1-a_2|$, the rest of the statements follows from Lemma 1.6, 2.4 and Property 4. If $|w_p| \neq |a_1-a_2|$, then the consideration of the case where $p=2$ implies

$$e^{S^1}(N_1(a_1-a_2)) = (x_1+(a_1-a_2)\alpha)^{n_2+1}/(a_1-a_2)\alpha.$$

This means $c(N_1(w_p))=1$ for any $p$. Therefore the rest of the statements follows. q. e. d.

## 4. One proposition

Henceforth we consider the case where $X$ admits an effective smooth $S^1$-action such that $F(S^1, X)$ consists of four connected components. Put $F(S^1, X) = \bigcup_{i=0}^{3} F_i$, then each $F_i$ is a cohomology $CP^{n_i}$ and an integer $a_i$ is assigned to it, where $\{a_i\}$ are mutually distinct. This section is devoted to prove the following proposition.

PROPOSITION 4.1. *Let $X$ and $\{a_i\}$ be as above. If there exists some $i$ such that $|a_j-a_i|>1$ for all $j \neq i$, then the $S^1$-action is of the linear type.*

Our proof is based on the method used in [14]. In the case of four components, however, the proof is more complicated than that of three components.

First we note that we can translate $\{a_i\}$ into $\{a_i+a\}$ for any integer $a$ by changing $y$ to $y+a\alpha$. Therefore we may assume that $a_0=0$ and $|a_i|>1$ for $i=1, 2, 3$. Since our $S^1$-action is assumed to be effective, g.c.d. $(a_1, a_2, a_3)=1$ by Proposition 1.3. From now on until this section ends, if we write $a_i$, $a_\nu$, $a_\mu$ etc., then the indices $i$, $\nu$, $\mu$ etc. are not zero, that is, $a_0$ is excluded.

We shall consider the following three cases:

Case I. $a_\nu \nmid a_\mu$ for all distinct $\nu$ and $\mu$.

Let $(i, j, k)$ denote a permutation of $(1, 2, 3)$ until Lemma 4.5 has been proved.

Case II. For a suitable choice of $(i, j, k)$ we have $a_i | a_j$ but $a_k \nmid a_j$.

Case III. For a suitable choice of $(i, j, k)$ we have $a_i | a_j$ and $a_k | a_j$.

Here we notice that the case where $a_j | a_i$ and $a_j | a_k$ or $a_i | a_j$ and $a_j | a_k$ does not occur because g.c.d.$(a_1, a_2, a_3)=1$. Therefore the above three cases cover all the cases which actually occur.

Before the investigation into these three cases, we shall prepare two lemmas.

Let $s$ be an arbitrary prime power dividing $a_i$ and let $t$ be that dividing $a_j$. We assume $s \nmid a_j$ and $t \nmid a_i$. Then the following three cases on $a_k$ occur:

Case A. $s \nmid a_k$; $t \nmid a_k$,

Case B. $s | a_k$; $t \nmid a_k$ or $s \nmid a_k$; $t | a_k$,

Case C. $s | a_k$; $t | a_k$.

Let $Y_s$ and $Y_t$ denote $F(Z_s, X)_0$ and $F(Z_t, X)_0$ respectively. Then the intersection of the normal bundles $N(F_0, Y_s)$ and $N(F_0, Y_t)$ becomes also an $S^1$-vector bundle over $F_0$. In fact, if we put

$$N(F_0, Y_s)=\sum_{s|s_a,\, s_a>0} N_0(s_a)\,, \quad N(F_0, Y_t)=\sum_{t|t_b,\, t_b>0} N_0(t_b),$$

then their intersection is given by $\sum N_0(u)$ where $\{u\}$ is the intersection of the sets $\{s_a\}$ and $\{t_b\}$.

LEMMA 4.2. *If Case* A *or Case* B *occurs, then the intersection of* $N(F_0, Y_s)$ *and* $N(F_0, Y_t)$ *is trivial, that is,* $N(F_0, Y_s)\cap N(F_0, Y_t)=F_0$. *If Case* C *occurs, then* $\dim_C N(F_0, Y_s)\cap N(F_0, Y_t)=n_k+1$.

PROOF. First consider Case A and Case B. Suppose that the intersection of $N(F_0, Y_s)$ and $N(F_0, Y_t)$ is non-trivial. Then there exists a local weight $u$ at $F_0$ divided by the multiplication $st$. Consider $F(Z_u, X)_0$ and write it $Y$, then obviously $\dim Y>\dim F_0$. Hence, by Proposition 1.5, $Y$ must contain a certain connected component of $F(S^1, X)$ other than $F_0$. However, since we assume $s \nmid a_j$ and $t \nmid a_i$, $u$ does not divide any $a_\nu$. This contradicts Proposition 1.3. Thus the first statement of this lemma is proved.

Let us consider Case C. First we shall show

$$\dim_C N(F_0, Y_s) \cap N(F_0, Y_t) \geqq n_k + 1 . \tag{4.1}$$

From our assumption we easily deduce

$$\begin{aligned} \dim_C N(F_0, Y_s) &= n_i + 1 + n_k + 1 , \\ \dim_C N(F_0, Y_t) &= n_j + 1 + n_k + 1 . \end{aligned} \tag{4.2}$$

On the other hand, we have clearly

$$\dim_C N(F_0, X) = n_i + 1 + n_j + 1 + n_k + 1 = n - n_0 .$$

Therefore the inequality (4.1) follows.

Next we shall show its converse. Let us consider $F(\boldsymbol{Z}_{st}, X)_0$ and write it $Y$. The multiple $st$ divides neither $a_i$ nor $a_j$, and $\dim Y > \dim F_0$ by (4.1). Hence, by Proposition 1.3 and 1.5, $Y$ contains $F_k$ but it contains neither $F_i$ nor $F_j$. Thus, if we use Proposition 1.5 again, then we have

$$\dim Y \leqq 2(n_0 + n_k + 1) .$$

This shows that the converse of the inequality (4.1) holds. q. e. d.

REMARK 4.3. In this proof, we use the assumption that $s$ and $t$ are prime powers, only to show (4.2). Therefore, even if $s$ or $t$ is not a prime power, this lemma holds if (4.2) is satisfied.

LEMMA 4.4. *Suppose that $N(F_0, X)$ decomposes into*

$$\sum_{m_1 | a_1} N_0(m_1) \oplus \sum_{m_2 | a_2} N_0(m_2) \oplus \sum_{m_3 | a_3} N_0(m_3)$$

*where* $\dim_C \sum N_0(m_1) = n_1 + 1$, $\dim_C \sum N_0(m_2) = n_2 + 1$ *and* $\dim_C \sum N_0(m_3) = n_3 + 1$. *Then every $|m_\nu|$ must equal $|a_\nu|$ for each $\nu$.*

PROOF. Since we assume $m_\nu | a_\nu$ for each $\nu$, we have

$$\Pi |m_1| \Pi |m_2| \Pi |m_3| \leqq |a_1|^{n_1+1} |a_2|^{n_2+1} |a_3|^{n_3+1} .$$

However, by Proposition 1.4, the equality must hold. Thus this lemma follows. q. e. d.

REMARK. For any other normal bundle $N(F_\nu, X)$, this lemma clearly holds if the similar assumption is satisfied.

Now we shall deal with Case I. First we shall prove the following.

LEMMA 4.5. *If an $S^1$-action on $X$ satisfies the condition of Case* I, *then it must be of the linear type at $F_0$.*

PROOF. Since we assume that each $a_\nu$ does not divide any other $a_\mu$ and that $|a_\nu| > 1$ for every $\nu$, the following four cases can occur with respect to prime powers dividing each $a_\nu$.

Let the condition $(*\nu)$ mean that a prime power $p_\nu$ divides $a_\nu$ but it does not divide any other $a_\mu$.

Case I.1. There exists a prime power $p_\nu$ satisfying the condition $(*\nu)$ for every $\nu$.

Case I.2. There exist prime powers $p_i$ and $p_j$ satisfying the condition $(*i)$ and $(*j)$ respectively, but no prime power satisfies the condition $(*k)$.

Case I.3. There exists a prime power $p_i$ satisfying the condition $(*i)$, but no prime power satisfies the condition $(*j)$ or $(*k)$.

Case I.4. The other case.

First let us deal with Case I.1. We shall consider $F(\boldsymbol{Z}_{p_\nu};\ X)_0$ for each $\nu$ and write it $Y_\nu$. Then each $N(F_0, Y_\nu)$ has the trivial intersection with the others by Lemma 4.2. Since $p_\nu$ is a prime power dividing $a_\nu$, $\dim_C N(F_0, Y_\nu)=n_\nu+1$. Therefore, if we note $\sum_{\nu=1}^{3}(n_\nu+1)=n-n_0$, we see that $N(F_0, X)$ decomposes into $\sum N(F_0, Y_\nu)$.

Let each $N(F_0, Y_\nu)$ decompose into

$$N(F_0, Y_\nu)=\sum_{m_\nu>0} N_0(m_\nu).$$

We remark that any $m_\nu$ divides $a_\nu$. In fact, if we consider $F(\boldsymbol{Z}_{m_\nu};\ X)_0$ it must contain $F_\nu$ by Proposition 1.5. Hence $m_\nu$ must divide $a_\nu$ by Proposition 1.3.

Thus, in Case I.1, our lemma follows from Lemma 4.4.

Next we shall deal with Case I.2. If we consider prime powers dividing $a_k$, we see that the condition of Case I.2 implies that there exists prime powers $q_i$ and $q_j$ satisfying

$$q_i | a_j,\ a_k\ ;\ q_i \nmid a_i\,,$$

$$q_j | a_k,\ a_i\ ;\ q_j \nmid a_j\,.$$

Let us consider $F(\boldsymbol{Z}_{p_i}, X)_0$, $F(\boldsymbol{Z}_{p_j}, X)_0$, $F(\boldsymbol{Z}_{q_i}, X)_0$ and $F(\boldsymbol{Z}_{q_j}, X)_0$ and write them $Y_i$, $Y_j$, $\tilde{Y}_i$ and $\tilde{Y}_j$ respectively. Clearly

$$\dim_C N(F_0, Y_i)=n_i+1\,,\ \dim_C N(F_0, Y_j)=n_j+1\,.$$

Furthermore, by Lemma 4.2, we have

$$\begin{aligned} N(F_0, Y_i)\cap N(F_0, \tilde{Y}_i) &= N(F_0, Y_j)\cap N(F_0, \tilde{Y}_j)\\ &= N(F_0, Y_i)\cap N(F_0, Y_j)=F_0 \end{aligned}$$

and

$$\dim_C N(F_0, \tilde{Y}_i)\cap N(F_0, \tilde{Y}_j)=n_k+1\,.$$

These facts mean that $N(F_0, X)$ decomposes into

$$N(F_0, X)=N(F_0, Y_i)\oplus N(F_0, Y_j)\oplus N(F_0, \tilde{Y}_i)\cap N(F_0, \tilde{Y}_j).$$

Therefore, if we repeat the same discussion as Case I.1, our lemma is proved in Case I.2.

For Case I.3 and Case I.4, we have only to repeat the similar process to the above cases. We shall omit the details. q. e. d.

LEMMA 4.6. *If Case* I *occurs, then the $S^1$-action is of the linear type.*

PROOF. The linearity at $F_0$ implies that the other normal bundles $N(F_\nu, X)$ decompose into the forms

$$N(F_\nu, X)=N(|a_\nu|)\oplus\sum N_\nu(w_\nu) \tag{4.3}$$

where $\dim_C N(|a_\nu|)=n_0+1$. For the investigation of the remaining weights $\{w_\nu\}$ at $F_\nu$, we shall consider the following three cases.

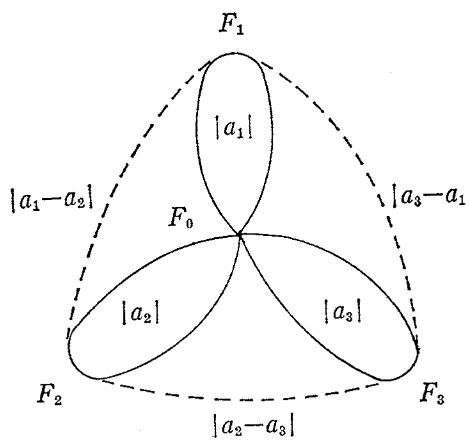
Let us put $V=\{|a_1-a_2|, |a_2-a_3|, |a_3-a_1|\}$.

Case I.A. There exist exactly two elements among $V$ which equal 1.

Case I.B. There exists exactly one element among $V$ which equals 1.

Case I.C. None of $V$ equals 1.

These three cases cover all the cases which really happen, because $\{a_\nu\}$ are mutually distinct.




First we shall deal with Case I.A. We can assume that $|a_2-a_3|=|a_3-a_1|=1$, if necessary by permuting the indices of the fixed point set components $F_\nu$. Then $|a_1-a_2|$ equals 2 because $\{a_\nu\}$ are mutually distinct. Therefore, if we consider $F(\boldsymbol{Z}_2, X)$, we easily see that this lemma holds in this case.

Next we shall consider Case I.B. By the same reason as Case I.A, we can assume $|a_2-a_3|=1$. We shall investigate $N(F_1, X)$. Since $|a_2-a_3|=1$, $|a_1-a_2|$ and $|a_3-a_1|$ are mutually coprime. Hence there exist prime powers $r_2$ and $r_3$ satisfying

$$r_2 | a_1 - a_2 ;\ r_2 \nmid a_1 - a_3 ,$$

$$r_3 | a_1 - a_3 ;\ r_3 \nmid a_1 - a_2 .$$

Therefore, if we recall (4.3) and consider $F(\boldsymbol{Z}_{r_2}, X)_1$ and $F(\boldsymbol{Z}_{r_3}, X)_1$, we see that $N(F_1, X)$ decomposes into

$$N(F_1, X) = N_1(|a_1|) \oplus \sum_{r_2 | m,\, m>0} N_1(m) \oplus \sum_{r_3 | w,\, w>0} N_1(w) \tag{4.4}$$

where $\dim_C \sum N_1(m) = n_2 + 1$ and $\dim_C \sum N_1(w) = n_3 + 1$. Moreover, by the same reason as Case I.1 in Lemma 4.5, each $m$ (resp. $w$) divides $a_1 - a_2$ (resp. $a_1 - a_3$) Thus, it follows from Lemma 4.4 that our $S^1$-action is of the linear type at $F_1$ too. This fact and the assumption $|a_2 - a_3| = 1$ imply our lemma in this case.

Case I.C is more complicated than the before cases. In this case, it suffices to prove the linearity at $F_1$ because the proof of the linearity at another $F_\mu$ is the same. We shall distinguish the following three cases with respect to the divisibility relation between $|a_1 - a_2|$ and $|a_1 - a_3|$.

Case I.C.1. The case where $|a_1 - a_2| \nmid |a_1 - a_3|$ and $|a_1 - a_3| \nmid |a_1 - a_2|$.

In this case, the same argument as Case I.B holds and this lemma is true.

Case I.C.2. The case where $|a_1 - a_2| \nmid |a_1 - a_3|$ and $|a_1 - a_3| \,\Big|\, |a_1 - a_2|$ or $|a_1 - a_2| \,\Big|\, |a_1 - a_3|$ and $|a_1 - a_3| \nmid |a_1 - a_2|$.

We shall prove only the former case. The proof of the latter case is the same as the former.

For an arbitrary prime power $s$ dividing $a_1 - a_3$, let us consider $F(\boldsymbol{Z}_s, X)_1$ and write it $Y$. The effectiveness of our $S^1$-action implies that $s$ does not divide $a_1$. Therefore, by Lemma 4.2, $N(F_1, Y)$ coincides with $\sum N_1(m) \oplus \sum N_1(w)$ stated in (4.4). This shows that both $\{m\}$ and $\{w\}$ are divided by $s$. Since $s$ is an arbitrary prime power dividing $a_1 - a_3$, both $\{m\}$ and $\{w\}$ are divided by $a_1 - a_3$.

Now, to determine the remaining local weights $\{m\}$, $\{w\}$ at $F_1$, we need to examine $N(F_2, X)$. Then the global weights at $F_2$ is given by $\{a_2, a_2 - a_1, a_2 - a_3\}$. The condition of Case I.C.2 means that there exists a prime power $q_1$ satisfying

$$q_1 \,\Big|\, |a_2 - a_1| ;\ q_1 \nmid |a_2 - a_3|,\ |a_3 - a_1| .$$

Consider $F(\boldsymbol{Z}_{q_1}, X)_2$ and $F(\boldsymbol{Z}_{q_3}, X)_2$ where $q_3$ is a prime power dividing $a_2 - a_3$. Then we see that, by Lemma 4.2 and (4.3), $N(F_2, X)$ decomposes into

$$N(F_2, X) = N_2(|a_2|) \oplus \sum_{q_1 | m',\, m'>0} N_2(m') \oplus \sum_{\substack{q_3 | w',\, w'>0 \\ q_1 \nmid w'}} N_2(w') \tag{4.5}$$

where $\dim_C \sum N_2(m') = n_1 + 1$ and $\dim_C \sum N_2(w') = n_3 + 1$. Here, it is not difficult to see that each $m'$ (resp. $w'$) divides $a_2 - a_1$ (resp. $a_2 - a_3$). Therefore the linearity at $F_2$ follows from Lemma 4.4.

Thus, if we consider $F(Z_{|a_1-a_2|}; X)$, we easily see that our $S^1$-action is of the linear type at $F_1$ too.

Case I.C.3. The case where $|a_1-a_2|=|a_1-a_3|$.

In this case, the first half in the proof of Case I.C.2 already shows the linearity at $F_1$. q. e. d.

Now we shall investigate Case II. We may assume $a_2|a_1$, if necessary by permuting the indices of the fixed point set components $\{F_\nu\}$. First we shall prove the following.

LEMMA 4.7. *If an $S^1$-action on $X$ satisfies the condition $a_2|a_1$ of Case* II, *then $N(F_0, X)$ decomposes into*

$$N(F_0, X)=N_0(|a_3|)\oplus \sum_{a_2|\tilde{w},\tilde{w}>0} N_0(\tilde{w})$$

*where* $\dim_C N_0(|a_3|)=n_3+1$.

PROOF. Since we assume $a_2|a_1$, the condition of Case II and the effectiveness of our $S^1$-action means $a_3 \nmid a_1, a_2$. Hence there exists a prime power $p_3$ satisfying

$$p_3|a_3;\ p_3 \nmid a_1, a_2.$$

Moreover the effectiveness of our $S^1$-action and the assumption $a_2|a_1$ imply that $a_2$ and $a_3$ are mutually coprime. Therefore, if we consider $F(\boldsymbol{Z}_{p_3}, X)_0$ and $F(\boldsymbol{Z}_{p_2}, X)_0$ where $p_2$ is a prime power dividing $a_2$, then we see that $N(F_0, X)$ decomposes into

$$N(F_0, X)=\sum_{p_3|m,m>0} N_0(m)\oplus \sum_{\substack{p_3\nmid w,w>0\\ p_2|w}} N_0(w)$$

where $\dim_C \sum N_0(m)=n_3+1$.

Here, it is easy to see that all $m$ divide $a_3$. Conversely, we shall observe that $a_3$ divides all $m$. Let $q$ be an arbitrary prime power dividing $a_3$. Notice that $F(\boldsymbol{Z}_w, X)_0$ does not contain $F_3$. In fact, the relation $w \nmid a_3$ holds, because $w$ has a divisor $p_2$ but $a_3$ does not contain it. Therefore, when $q$ divides neither $a_1$ nor $a_2$, it is easy to see that $q$ does not divide any $w$. Hence $q$ must divide every $m$ because $\dim_C N(F_0, F(\boldsymbol{Z}_q, X))=n_3+1$. When $q$ divides either $a_1$ or $a_2$, $\dim_C N(F_0, F(\boldsymbol{Z}_{p_2q}, X))$ equals either $n_1+1$ or $n_2+1$ respectively by Lemma 4.2. This implies that $q$ must divide all $m$, because $p_2$ divides all $w$. Since $q$ is an arbitrary prime power dividing $a_3$, $a_3$ divides all $m$. Thus, the fact $m=|a_3|$ follows.

Next we shall observe that all $w$ are divided by $a_2$. Since $a_3 \nmid a_1, a_2$, $F(\boldsymbol{Z}_{|a_3|}, X)_0$ does not contain neither $F_1$ nor $F_2$. Take an arbitrary prime power $q$ dividing $a_2$ and consider $F(\boldsymbol{Z}_q, X)_0$. The effectiveness of our $S^1$-action implies that $q$ does not divide $a_3$. However, since $\dim_C N(F_0, F(\boldsymbol{Z}_q, X))=n_1+1+n_2+1$, $q$

must divide all $w$. This shows that $a_2$ divides all $w$. q. e. d.

Since $a_2 | a_1$, let us put $a_1 = h a_2$, where $h$ is an integer distinct from 1 because $\{a_\nu\}$ are mutually distinct. We shall distinguish the following three cases concerning the integer $h$:

Case II.A. $h=-1$.

Case II.B. $h=2$.

Case II.C. $h \neq -1, 2$.

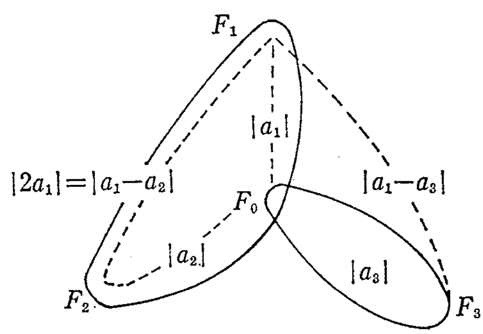
Lemma 4.7 implies that $N(F_1, X)$ decomposes into

$$N(F_1, X) = \sum_{a_2 | m, m>0} N_1(m) \oplus \sum_{\substack{a_2 \nmid w, w>0 \\ w \nmid a_2}} N_1(w), \tag{4.6}$$

where $\dim_C \sum N_1(m) = n_0 + 1 + n_2 + 1$. The fact $w \nmid a_2$ follows from Proposition 1.5 and g. c. d.$(a_2, a_3) = 1$.

*Assertion* 4.8. If our $S^1$-action satisfies Case II.A, it is of the linear type.

PROOF. We note that the assumption $a_1 = -a_2$ implies that the case where $|a_1 - a_3| = |a_2 - a_3| = 1$ does not happen. Therefore, we may assume, if necessary by changing the indices of $F_1$ and $F_2$, that every absolute value of the global weights $\{a_1, a_1 - a_2 = 2a_1, a_1 - a_3\}$ at $F_1$ is greater than 1. We shall distinguish the following two cases moreover.




Case II.A.1. The case where $|a_1 - a_3| \,\big|\, |a_1 - a_2| (= |2a_1|)$.

Since the effectiveness of our $S^1$-action implies g. c. d.$(a_1 - a_3, a_1) = 1$, $|a_1 - a_3|$ must equal 2. Therefore, if we consider $F(\boldsymbol{Z}_2, X)_1$ and use Remark 4.3, the linearity at $F_1$ follows from (4.6). Then, it is easy to see the linearities at the other $F_\nu$.

Case II.A.2. The case where $|a_1 - a_3| \nmid |a_1 - a_2| (= |2a_1|)$.

In this case, the condition of Case II is satisfied at $F_1$. Hence we can apply the $F_1$ version of Lemma 4.7. Therefore, we have $w = |a_1 - a_3|$ for all $w$, that is, the decomposition (4.6) becomes

$$N(F_1, X)=\sum_{a_2|m} N_1(m)\oplus N_1(|a_1-a_3|).$$

Consider $F(\boldsymbol{Z}_q, X)_1$ where $q$ is the maximum power of 2 dividing $2a_1$, so we can see the linearity at $F_1$. Then, it is easy to check the linearity at the other $F_\nu$. q. e. d.

*Assertion* 4.9. If our $S^1$-action satisfies Case II.B, it is of the linear type.

PROOF. When $|a_1-a_3|=|a_2-a_3|=1$, we have $|a_2|=2$, $|a_1|=4$ by easy calculation. Therefore, if we recall (4.6) and consider $F(\boldsymbol{Z}_4, X)_1$, we can see this assertion in this case.

Therefore, there is such $F_\nu$ that the every absolute value of its global weights is greater than 1. Then the proof is parallel to that of Case II.A. q. e. d.

*Assertion* 4.10. If our $S^1$-action satisfies Case II.C, it is of the linear type.

PROOF. We may assume, if necessary by permuting the subindexes of $\{F_\nu\}$, that every absolute value of the global weights $\{a_1, a_1-a_2, a_1-a_3\}=\{ha_2, (h-1)a_2, a_1-a_3\}$ at $F_1$ is greater than 1. The assumption $h\neq -1, 2$ means that integers $ha_2$ and $(h-1)a_2$ do not divide each other. Moreover, the effectiveness of our $S^1$-action implies that

$$(h-1)a_2 \nmid a_1-a_3, \qquad ha_2 \nmid a_1-a_3.$$

Thus, the following three cases can occur with respect to remaining divisibility relations between the global weights at $F_1$.

Case II.C.1. $a_1-a_3 \nmid (h-1)a_2$, $a_1-a_3 \nmid ha_2$.

Case II.C.2. $a_1-a_3 \nmid (h-1)a_2$, $a_1-a_3 | ha_2$.

Case II.C.3. $a_1-a_3 | (h-1)a_2$, $a_1-a_3 \nmid ha_2$.

In Case II.C.1, the condition of Case I is satisfied at $F_1$. Therefore our assertion follows from Lemma 4.6.

In Case II.C.2, the condition of Case II is satisfied at $F_1$. Hence, if we recall (4.6) and use the $F_1$ version of Lemma 4.7, we have

$$N(F_1, X)=N_1(|a_1-a_2|)\oplus \sum_{\substack{a_2|m', m'>0 \\ a_1-a_3|m'}} N_1(m')\oplus \sum_{\substack{a_2\nmid w, w\nmid a_2 \\ a_1-a_3|w}} N_1(w)$$

where $\dim_C N_1(|a_1-a_2|)=n_2+1$ and $\dim_C \sum N_1(m')=n_0+1$.

We shall observe that all $m'$ equal $|a_1|$. Let $Y_1$ and $Y_2$ be $F(\boldsymbol{Z}_{|a_2|}, X)_0$ and $F(\boldsymbol{Z}_{|a_1-a_3|}, X)_1$ respectively. Then the intersection $Y_1\cap Y_2=Y$ is exactly $F(\boldsymbol{Z}_u, X)_0 =F(\boldsymbol{Z}_u, X)_1$ where $u=$l. c. m. $(|a_2|, |a_1-a_3|)$.

Let $r$ be an arbitrary prime power dividing $a_1$. If $r|a_2$ or $r|a_1-a_3$, then $Y$ is clearly fixed pointwise under the restricted $\boldsymbol{Z}_r$-action. Therefore, let us assume $r \nmid a_2, a_1-a_3$, hence $r \nmid a_1-a_2, a_3$. The fact $r\nmid a_3, a_1-a_2$ means $F(\boldsymbol{Z}_r, X)_0\subset Y_1$ and $F(\boldsymbol{Z}_r, X)_1\subset Y_2$, hence

$$F(\boldsymbol{Z}_r, X)_0 \cap F(\boldsymbol{Z}_r, X)_1 \subset Y_1 \cap Y_2 = Y.$$

Here, if we consider the dimensions of these two spaces, we see that they coincide, that is, $Y$ is fixed pointwise under the restricted $\boldsymbol{Z}_r$-action. Thus, it follows that $m'=|a_1|$ for all $m'$.

Hence, the linearity at $F_1$ follows from Proposition 1.4. Then it is easy to see the linearity at the other $F_\nu$.

In Case II.C.3, the argument is parallel to that of Case II.C.2. q. e. d.

Now we shall consider Case III. This case reduces to the previous two cases.

LEMMA 4.11. *Let our $S^1$-action satisfy the condition of Case* III *at some $S^1$-fixed point set component. Then the situation of Case* I *or Case* II *occurs at another $S^1$-fixed point set component.*

PROOF. We may assume that $a_2|a_1$ and $a_3|a_1$, if necessary by permuting the indices of $\{F_\nu\}$. Since $a_2$ and $a_3$ are mutually coprime by the effectiveness of our $S^1$-action on $X$, $a_1$ is written as $ha_2a_3$ where $h$ is an integer.

Now, the global weights at $F_1$ is given by $\{a_1, a_1-a_2, a_1-a_3\}=\{ha_2a_3, (ha_3-1)a_2, (ha_2-1)a_3\}$. Let us distinguish the following three cases:

Case III.1. $|ha_3-1|=|ha_2-1|=1$.

Case III.2. $|ha_3-1|=1,\ |ha_2-1|\neq 1$,

or $|ha_3-1|\neq 1,\ |ha_2-1|=1$.

Case III.3. $|ha_3-1|\neq 1,\ |ha_2-1|\neq 1$.

We note that since $a_2$ and $a_3$ are distinct non-zero integers, the first case does not really happen.

We shall observe that the global weights at $F_1$ satisfy the condition of Case II on both cases in Case III.2. We shall consider only the former case because the proof of the latter case is parallel to that of the former case.

Recall that $|a_3|$ is an integer greater than 1. Hence the solution of the equation $|ha_3-1|=1$ is given by

$$h=1, \quad a_3=2,$$

or

$$h=-1, \quad a_3=-2.$$

Therefore, the global weights at $F_1$ are given by

$$\{2a_2, a_2, 2(a_2-1)\} \quad \text{if} \quad h=1,$$

$$\{2a_2, a_2, 2(a_2+1)\} \quad \text{if} \quad h=-1.$$

It is easy to check that these global weights satisfy the condition of Case II.

In Case III.3, we assert that the global weights at $F_1$ satisfy the condition of Case I at $F_1$. In fact, if we note that, by the condition of Case III.3, each absolute value of the global weights at $F_1$ is greater than 1, and that $a_2$ and $a_3$ are mutually coprime, then we see this assertion. q. e. d.

## 5. Proof of Theorem 0.2

In this section we shall prove Theorem 0.2 stated in Introduction. We continue to assume that $F(S^1, X)$ consists of four connected components. By Proposition 4.1 and taking a suitable lifting $y$, we have only to consider the case where the set $\{a_i\}$ coincides with $\{0, a, 1, a+1\}$ for some positive integer $a$. Moreover we can assume

$$a_0=0\,,\quad a_1=a\,,\quad a_2=1\,,\quad a_3=a+1\,,$$

if necessary by permuting the indices of $\{F_i\}$.

We shall compute the equivariant Euler class of $N(F_0, X)$ in the different two ways. In the one way, we already know

$$e^{S^1}(N(F_0, X))=(x_0-a\alpha)^{n_1+1}(x_0-\alpha)^{n_2+1}(x_0-(a+1)\alpha)^{n_3+1} \tag{5.1}$$

by the proof of Lemma 2.4. In the other way, considering $F(\boldsymbol{Z}_r, X)_0$ for various prime powers $r$, we see

$$N(F_0, X)\,|\,p=n_1t^{-a}+t^{p_1}+\cdots+t^{p_s}+bt+n_2t^{-(a+1)}+t^{q_1}+\cdots+t^{q_t}\,,$$

where $p\in F_0$, and

$$e^{S^1}(N(F_0, X))=\frac{(x_0-a\alpha)^{n_1+1}}{a\alpha}(p_1\alpha)\cdots(p_s\alpha)\prod_{j=1}^{b}(\alpha+\xi_j) \times\frac{(x_0-(a+1)\alpha)^{n_3+1}}{(a+1)\alpha}(q_1\alpha)\cdots(q_t\alpha) \tag{5.2}$$

by Corollary 3.10, where $c(N_0(1))=\prod_{j=1}^{b}(1+\xi_j)$, $|p_1\times\cdots\times p_s|=a$ and $|q_1\times\cdots\times q_t|=a+1$.

Comparing the coefficients between (5.1) and (5.2) in each power of $\alpha$, we obtain

$$\alpha^{s+t-2}\prod(\alpha+\xi_j)=(x_0-\alpha)^{n_2+1}\,. \tag{5.3}$$

LEMMA 5.1. *If $n_0\geqq n_2+1$, then $s=t=1$, that is, our $S^1$-action is of the linear type at $F_0$. If $n_0=n_2$, then $s+t\leqq 3$.*

PROOF. If $n_0\geqq n_2+1$, then $x_0^{n_2+1}$ is a non-zero element. Hence, it follows from (5.3) that $s+t-2=0$, that is $s=t=1$.

If $n_0=n_2$, then $x_0^{n_2}$ is non-zero. This and the identity (5.3) means our lemma.

q. e. d.

LEMMA 5.2. *If $n_0\neq n_2$, then our $S^1$-action is of the linear type.*

PROOF. Since the same discussion as $F_0$ holds for $F_2$, we may assume $n_0>n_2$. Then, by Lemma 5.1, our $S^1$-action is of the linear type at $F_0$.

*Assertion* 5.3. $p_1^{S^1}(X)=(n_0+1)y^2+(n_1+1)(y-a\alpha)^2+(n_2+1)(y-\alpha)^2$

$$+(n_3+1)(y-(a+1)\alpha)^2+hy^2$$

where $h$ is an integer.

PROOF OF *Assertion* 5.3. Since the set $\{y^2, \alpha y, \alpha^2\}$ forms an additive base of $H_{S^1}^4(X, \boldsymbol{Z})$ by Proposition 1.2, $p_1^{S^1}(X)$ is written uniquely as

$$p_1^{S^1}(X)=(n_0+1)y^2+(n_1+1)(y-a\alpha)^2+(n_2+1)(y-\alpha)^2$$
$$+(n_3+1)(y-(a+1)\alpha)^2+h_1y^2+h_2\alpha y+h_3\alpha^2$$

where $\{h_j\}$ are integers. Hence we have

$$\tilde{j}_0^*p_1^{S^1}(X)=(n_0+1)x_0^2+(n_1+1)(x_0-a\alpha)^2+(n_2+1)(x_0-\alpha)^2$$
$$+(n_3+1)(x_0-(a+1)\alpha)^2+h_1x_0^2+h_2\alpha x_0+h_3\alpha^2, \tag{5.4}$$

where $\tilde{j}_0: F_0\rightarrow X$ is the inclusion.

On the other hand, since there is a natural bundle isomorphism $\tilde{j}_0^*TX\cong TF_0 \oplus N(F_0, X)$, we have

$$\tilde{j}_0^*p_1^{S^1}(X)=p_1^{S^1}(F_0)+p_1^{S^1}(N(F_0, X)). \tag{5.5}$$

Here, since our $S^1$-action is of the linear type at $F_0$, we have

$$p_1^{S^1}(N(F_0, X))=(n_0+1)x_0^2+(n_1+1)(x_0-a\alpha)^2+(n_2+1)(x_0-\alpha)^2$$
$$+(n_3+1)(x_0-(a+1)\alpha)^2 \tag{5.6}$$

by Lemma 2.8.

Therefore, if we compare (5.4) with (5.5) using (5.6), then we have

$$p_1^{S^1}(F_0)=h_1x_0^2+h_2\alpha x_0+h_3\alpha^2.$$

We remark that $p_1^{S^1}(F_0)$ must belong to the subgroup $H^4(F_0; \boldsymbol{Z})\otimes H^0(BS^1; \boldsymbol{Z})$ and that we have $n_0\geqq 1$ by the assumption $n_0>n_2$. Hence $h_2$ and $h_3$ must be zero. This proves Assertion 5.3.

Now we return to the proof of Lemma 5.2. We shall consider the following two cases.

Case I. The case where $n_1=n_3$.

When $n_2$ is distinct from $n_1=n_3$, we shall consider $F(\boldsymbol{Z}_p, X)_2$ and $F(\boldsymbol{Z}_q, X)_2$

where $p$ and $q$ are prime numbers dividing $a$ and $a+1$ respectively, and apply Proposition 3.3 to each fixed point set. Then we see that our $S^1$-action is of the linear type at $F_2$. The linearity at $F_0$ and $F_2$ implies our lemma.

Thus we assume $n_1=n_2=n_3$. Then, the following three types at $F_2$ together with the linear type can happen, by Lemma 5.1, with regard to a decomposition of $N(F_2, X)$.

Let $p_2$ be a point of $F_2$ and $t$ the 1-dimensional standard $S^1$-module.

Type 1. $\quad N(F_2, X)|p_2=(n_0-1)t+n_1t^{a-1}+n_3t^a+t^{w_1}+t^{w_2}+t^{m_1}+t^{m_2}$,

where $w_1$ and $w_2$ (resp. $m_1$ and $m_2$) divide $a-1$ (resp. $a$), and $w_1w_2=a-1$ (resp. $m_1m_2=a$).

Type 2. $\quad N(F_2, X)|p_2=n_0t+n_2t^{a-1}+(n_3+1)t^a+t^{w_1}+t^{w_2}$,

where $w_1$ and $w_2$ divide $a-1$, and $w_1w_2=a-1$.

Type 3. $\quad N(F_2, X)|p_2=n_0t+(n_2+1)t^{a-1}+n_3t^a+t^{m_1}+t^{m_2}$,

where $m_1$ and $m_2$ divide $a$, and $m_1m_2=a$.

We shall show that none of these cases really happen. In Type 1, the restriction of $p_1^{S^1}(X)$ to a point $p_2$ is clearly given by

$$\{n_1(a-1)^2+n_3a^2+w_1^2+w_2^2+m_1^2+m_2^2+n_0-1\}\alpha^2 .$$

On the other hand we can deduce its alternative representation from Assertion 5.3. Equating these values, we get

$$a^2+(a-1)^2+h+2=w_1^2+w_2^2+m_1^2+m_2^2 . \tag{5.7}$$

Next, since our $S^1$-action is of the linear type at $F_0$ and $N(F_2, X)$ has the above decomposition, $N(F_1, X)$ has a representation

$$N(F_1, X)|p_1=(n_0+1)t^a+n_2t^{a-1}+n_3t+t^{w_1}+t^{w_2}$$

where $p_1\in F_1$. Therefore, by the same argument as $N(F_2, X)$, we get

$$(a-1)^2+1+ha^2=w_1^2+w_2^2 . \tag{5.8}$$

Furthermore, if we repeat the same process for $N(F_3, X)$, we get

$$a^2+1+h(a+1)^2=m_1^2+m_2^2 . \tag{5.9}$$

By easy calculation using (5.7), (5.8) and (5.9), we obtain

$$a=0 \text{ or } -1.$$

However, this contradicts our assumption that $a$ is a positive integer.

In Type 2 and Type 3, we can deduce a contradiction in a similar way as

Type 1.

Case II. The case where $n_1 \neq n_3$.

We may assume $n_1 > n_3$. Then our $S^1$-action is also of the linear type at $F_1$ by Lemma 5.1. Hence $H(F_2, X)$ has a representation

$$N(F_2, X)|p_2 = (n_0+1-k)t + (n_1+1)t^{a-1} + n_3 t^a + \sum_{i=1}^{k} t^{m_i}$$

where $\prod m_i = a$. Therefore, if we consider $p_1^{S^1}(X)$ and repeat the same argument as Case I for $N(F_2, X)$, then we get

$$\sum m_i^2 = a^2 + k - 1 .$$

On the other hand, an easy calculation shows that the equality

$$\sum m_i^2 < a^2 + k - 1$$

holds for $k \geqq 2$. Thus $k$ must be 1, and Lemma 5.2 follows. q. e. d.

LEMMA 5.4. *Unless $n_0 = n_1 = n_2 = n_3$, our $S^1$-action is of the linear type.*

PROOF. First we notice that the same argument as Lemma 5.2 holds in the case where $n_1 \neq n_3$. Therefore, unless $n_0 = n_2$ and $n_1 = n_3$, our $S^1$-action is of the linear type. Hence we assume $n_0 = n_2$ and $n_1 = n_3$. When $n_0 \neq n_1$, we consider $F(\boldsymbol{Z}_p, X)_0$ and $F(\boldsymbol{Z}_q, X)_0$, where $p$ and $q$ are prime numbers dividing $a$ and $a+1$ respectively, and apply Proposition 3.3 to each fixed point set. Then the linearity at $F_0$ follows. The linearity at the other $S^1$-fixed point set components is similarly proved. q. e. d.

By this lemma, we assume $n_0 = n_1 = n_2 = n_3$ and put it $r$ in the following. Then, if we exclude the linear type, the circumstances of $N(F_i, X)$ are restricted to the following two types by Lemma 5.1.

Type I.

$$N(F_0, X)|p_0 = rt + rt^a + (r+1)t^{a+1} + t^p + t^q ,$$

$$N(F_1, X)|p_1 = rt + (r+1)t^{a-1} + rt^a + t^p + t^q ,$$

$$N(F_2, X)|p_2 = rt + (r+1)t^{a-1} + rt^a + t^{p'} + t^{q'} ,$$

$$N(F_3, X)|p_3 = rt + rt^a + (r+1)t^{a+1} + t^{p'} + t^{q'} ,$$

where $p$ and $q$ (resp. $p'$ and $q'$) are mutually coprime positive integers distinct from 1, and $pq = a$ (resp. $p'q' = a$).

Type II.

$$N(F_0, X)|p_0 = rt + (r+1)t^a + rt^{a+1} + t^u + t^v ,$$

$$N(F_1, X)|p_1 = rt + rt^{a-1} + (r+1)t^a + t^{u'} + t^{v'} ,$$

(5.10)

$$N(F_2, X)|p_2 = rt + rt^{a-1} + (r+1)t^a + t^{u'} + t^{v'} ,$$

$$N(F_3, X)|p_3=rt+(r+1)t^a+rt^{a+1}+t^u+t^v,$$

where $u$ and $v$ (resp. $u'$ and $v'$) are mutually coprime positive integers distinct from 1, and $uv=a+1$ (resp. $u'v'=a-1$).

LEMMA 5.5. *In Type* I, *the set* $\{p, q\}$ *coincides with* $\{p', q'\}$, *that is, our* $S^1$-*action is of the Petrie type.*

PROOF. Let $\varphi_i: p_i \to X$ be the inclusion for each $i$. Then, using the above representation of $N(F_i, X)|p_i$, we can compute $\varphi_i^* p_1^{S^1}(X)$. On the other hand, we put $p_1^{S^1}(X)=Ay^2+By\alpha+C\alpha^2$ where $A$, $B$ and $C$ are integers, and restrict it to each $p_i$. Equating these two values for each $i$, we obtain four equations. Solve these equations using the relation $pq=p'q'=a$, so we deduce this lemma.

q. e. d.

Next, let us investigate Type II. The assumption that $a$ is a positive integer shows $a\geqq 2$ because $a\neq 1$ by Proposition 1.3. If $a=2$, then it is easy to see that our $S^1$-action is of the linear type. Therefore we assume $a\geqq 3$.

LEMMA 5.6. *Type* II *does not really happen.*

PROOF. Consider the normal bundle of $F(\boldsymbol{Z}_a, X)$ in $X$. Then, we see that the relations

$$u'=\pm u+ca$$

$$v'=\pm v+da$$

hold where $c$ and $d$ are integers.

Case 1. The case where $u'=u+ca$ and $v'=v+da$.

Consider the multiplication $u'v'$ and use the fact $uv=a+1$ and $u'v'=a-1$, so we get

$$(cv+du)a+cda^2=-2.$$

This shows that $a$ divides 2, but it contradicts the assumption $a\geqq 3$.

Case 2. The case where $u'=-u+ca$ and $v'=-v+da$.

This case is also excluded by the same argument as Case 1.

Case 3. The case where $u'=u+ca$ and $v'=-v+da$.

Let us consider the multiplication $uv'$. Then, using the fact $uv=a+1$, we get

$$uv'=u(-v+da)$$

$$=-1+(ud+1)a.$$

Since $u$ and $v'$ are assumed to be positive integers, this equation means $d\geqq 0$.

Next consider the multiplication $u'v'$. Then, using the facts $uv=a+1$ and $u'v'=a-1$, we get

$$du+cv'=2.$$

Since $u$ and $v'$ are assumed to be positive integers greater than 1 and $d\geqq 0$, the solution of this equation is uniquely given by $c=1$, $v'=2$ and $d=0$. Hence, if we note the relation $u'v'=a-1$, only the following case remains:

$$u=\frac{1}{2}(a+1),\quad u'=\frac{1}{2}(a-1),\quad v=v'=2\quad (a:\text{odd}). \tag{5.11}$$

*Assertion* 5.7. If $r\geqq 1$, then the case (5.11) does not really happen.

PROOF. Let us put $p_1^{S^1}(X)=Ay^2+B\alpha y+\alpha^2$. Then, if we restrict it to $F_0$ and $F_3$ respectively and use (5.10) and Corollary 3.10, we obtain

$$C=2(r+1)(a^2+a+1)+u^2+v^2-(a+1)^2-1 \tag{5.12}$$

by the similar argument to Assertion 5.3.

Similarly, if we restrict $p_1^{S^1}(X)$ to $F_1$ and $F_2$, we obtain

$$C=2(r+1)(a^2+a+1)+u'^2+v'^2-(a-1)^2-1. \tag{5.13}$$

Therefore, using the facts $uv=a+1$ and $u'v'=a-1$, a contradiction is induced from (5.12) and (5.13). q. e. d.

Thus, let us assume $r=0$. In this case, since $X$ is a cohomology $CP^3$ with a non-trivial $S^1$-action, we have $p_1(X)=4x^2$ by [3] and [17]. On the other hand, we can show $p_1(X)=x^2$ by calculating $p_1^{S^1}(X)$ similarly to Assertion 5.7 using (5.10). Hence this case is also excluded.

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(Received May 19, 1980)

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