Reduction of a positive operator in a separable simplex space

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§ 1. Introduction.

Recently Niiro, Sawashima and Miyajima have obtained elegant results about the reduction theory of a positive, uniformly ergodic operator in an arbitrary Banach lattice [9, 11], unifying their former individual results [6, 7, 8, 10]. In this paper, we shall investigate the reduction of a positive operator in a simplex space. A simplex space is the predual of a Banach lattice of type L [3]. An AM space is a special case of a simplex space.

The author has obtained the decomposition of a positive, strongly ergodic operator in a simplex space E with an order unit into its irreducible components $\{T_\lambda\}_{\lambda\in\Lambda}$, which are in one-to-one correspondence with the set Λ of extreme points of the set $\{\phi\in E'\,;\ \phi\geqq 0,\ \|\phi\|=1,\ T'\phi=\phi\}$ [13]. In § 2, we show that the similar decomposition is obtained even if a simplex space E doesn't contain an order unit. In § 3, we are concerned with the following problem. Let Ω be the set of extreme points of the set $\{z\in E'''\,;\ z\geqq 0,\ \|z\|=1,\ T'''z=z\}$ and Γ is the unit circle in C. Then we have $\sigma(T)\cap\Gamma=\sigma(T'')\cap\Gamma=(\bigcup_{\omega\in\Omega}\sigma(T''_\omega))^-\cap\Gamma$ by the result of [10] since the second dual of a simplex space is an AM space. There is an injective map $\tau:\Lambda\to\Omega$ satisfying $\sigma(T_\lambda)\cap\Gamma=\sigma(T''_{\tau(\lambda)})\cap\Gamma$ and $\tau(\Lambda)$ is a proper subset of Ω, so it is clear that $\sigma(T)\cap\Gamma\supseteqq(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$. Thus our problem is: whether the equality $\sigma(T)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$ holds. In case of a separable simplex space, we solve the problem affirmatively by using the results [14] about the absolute value of an element of a simplex space over the complex field.

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§ 2. Decomposition.

Let E be a simplex space, i.e. an ordered Banach space whose dual is a Banach lattice of type L and $T\in\mathfrak{L}(E)$ be a positive, strongly ergodic operator with the spectral radius $r(T)=1$. We denote by P the limit operator of $M_n=\dfrac{I+T+\cdots+T^{n-1}}{n}$ as $n\to\infty$. Then P is a nonzero, positive projection with $r(P)=1$ and the range space PE is the eigenspace of T for the eigenvalue 1.

Let X be the set $\{x\in E';x\geqq 0,\ \|x\|\leqq 1\}$ endowed with the weak*-topology. Since X is a simplex [3], there exists a unique maximal probability measure μ_x on X with resultant x for each $x\in X$. Let $\partial_e X$ be the set of extreme points of X and $\overline{\partial_e X}$ be the weak*-closure of $\partial_e X$. Then by the well known theorem [3, 4], a simplex space is isometrically isomorphic to $A_0(X)$; the ordered Banach space of continuous affine functions on X vanishing at 0. Moreover it is isometrically isomorphic to the space $\{f\in C(\overline{\partial_e X});\ f(x)=\mu_x(f)$ for all $x\in\overline{\partial_e X}$ and $f(0)=0\}$.

A linear subspace I of E is said to be an *ideal* if it has the properties;

(i) $0\leqq x\leqq y\in I$ implies $x\in I$.

(ii) If $x\in I$, then there is some $y\in I$ with $y\geqq x,\ -x$.

A convex subset F of X is said to be a *face* if $x,\ y\in X$ and $\alpha x+(1-\alpha)y\in F$ imply $x,\ y\in F$ whenever $0<\alpha<1$.

Let I be a closed ideal of E. Put $F=\{x\in X;\ f(x)=0$ for any $f\in I\}$. Then F is a closed face of X and $I=\{f\in E;\ f=0$ on $F\}$. Moreover, the quotient space E/I is isometrically isomorphic to $A_0(F)$ and also to the space $\{f\in C(F);\ f=g|F$ for some $g\in E\}$, where $g|F$ is the restriction of g to F.

Following the method of Miyajima [7], put

$$p(x)=\sup\{Pf(x);\ f\in S_+\},$$

where S_+ is the positive portion of the unit ball in E. Then p is a positive lower semi-continuous function satisfying $p(x)\leqq\|P\|$ for any $x\in X$. Moreover p has the following property.

LEMMA 1. *If $f\in E$ and $f\leqq p$, then $Pf\leqq p$.*

PROOF. Let ε be an arbitrary positive number. Then for every $x\in X$ there exists a function $f_x\in S_+$ for which $f\leqq Pf_x+\varepsilon$ holds in a neighborhood U_x of x. Since X is compact, it is covered by a finite union of such U_x's, say $X=\bigcup_{i=1}^{n}U_{x_i}$. Put $g(x)=\max_{1\leqq i\leqq n}\{f_{x_i}(x)\}$. Then g is a convex, continuous function on X with $g(0)=0$ and $|g(x)|\leqq 1$ with any $x\in X$. Let $h(x)$ be equal to 1 for any nonzero $x\in X$ and $h(0)=0$. Then h is a lower semi-continuous, concave function on X, satisfying $h\geqq g$. By [2, Theorem 28.6], there exists $\phi\in A_0(X)$ such that $g\leqq\phi\leqq h$.

Since f_{x_i} and ϕ are elements of E satisfying $f_{x_i} \leqq \phi$ and P is positive, we have $Pf_{x_i} \leqq P\phi$. So $f \leqq P\phi + \varepsilon$ holds on X. In the same way, put

$$g'(x) = \max\{f(x) - P\phi(x),\ 0\} \quad \text{and} \quad h'(x) = \varepsilon$$

for nonzero $x \in X$ and $h'(0) = 0$. Then $g'(x)$ is a continuous, convex function on X with $g(0) = 0$ and h' is a lower semi-continuous concave function on X such that $g' \leqq h'$. Then there exists $\phi' \in A_0(X)$ such that $g' \leqq \phi' \leqq h'$ i.e. $\|\phi'\| \leqq \varepsilon$. Since f and ϕ' are elements of E and ϕ is an element of S_+, we have

$$f \leqq P\phi + g' \leqq P\phi + \phi' \quad \text{and}$$
$$Pf \leqq P\phi + P\phi' \leqq p + \|P\| \cdot \varepsilon.$$

By the arbitrariness of ε, this proves the lemma. //

A new norm on PE is defined through the function p as follows. If we denote $\inf\{c\,;\ -c \cdot p \leqq f \leqq c \cdot p\}$ by $\|f\|_0$, it is easy to see that $\|f\|_0$ really defines a norm on PE which is equivalent to the original norm induced from that on E. Hereafter, whenever the space PE is concerned, the norm on PE should be considered to be $\|\ \|_0$. The following proposition is easily proved.

PROPOSITION 1. *Equipped with the norm $\|\ \|_0$ and the order induced from that in E, PE is a simplex space.*

We get the following proposition in a similar way to [7, Proposition 4].

PROPOSITION 2. *$(PE)'$ is isometrically isomorphic to $P'E'$ as a Banach lattice.*

By using the function $p(x)$, we obtain the similar results to those of [13] where we have treated simplex spaces with an order unit. We outline these results with some notations.

Let Y be the set $\{x \in E'\,;\ x \geqq 0,\ T'x = x,\ \|x\| \leqq 1\}$, and Λ be the set of all nonzero extreme points of Y. Then Y is identified with the positive portion of the unit ball in $(PE)'$ by Proposition 2.

PROPOSITION 3. *An element $x \in Y$ belongs to Λ if and only if $\|x\| = 1$ and for any $f, g \in PE$, there exists $h \in PE$ such that $h \geqq f$, g and $h(x) = f(x) \vee g(x)$.*

For $\lambda \in \Lambda$, let
$I_\lambda = \{f \in E\,;\ h \geqq f,\ -f$ and $\lambda(h) = 0$ for some $h \in E\}$,
$X_\lambda = \{x \in X\,;\ f(x) = 0$ for any $f \in I_\lambda\}$,
$S_\lambda = \{x \in \overline{\partial_e X}\,;\ f(x) = 0$ for any $f \in I_\lambda\}$,
$Z_\lambda = \{x \in X\,;\ P'\varepsilon_x = p(x) \cdot \lambda\}$

and $N=\{x\in X;\ p(x)=0\}$.
Then we have the following propositions.

PROPOSITION 4. *$X_\lambda\subset Z_\lambda$ i.e.*
$P'\varepsilon_x=p(x)\cdot\lambda$ holds for any $\lambda\in\Lambda$ and $x\in X_\lambda$.

PROPOSITION 5. *I_λ is a T-invariant closed ideal of E. X_λ is a T'-invariant closed face of X.*
$I_\lambda=\{f\in E;\ f=0$ on $X_\lambda\}=\{f\in E;\ f=0$ on $S_\lambda\}$.
$S_\lambda=X_\lambda\cap\overline{\partial_e X}$ and $\partial_e X_\lambda=X_\lambda\cap\partial_e X$.
Moreover,
E/I_λ is isometrically isomorphic to $A_0(X_\lambda)$.

By the above proposition, T and P naturally induce operators U_λ and Q_λ in E/I_λ, respectively. Namely U_λ [resp. Q_λ] is defined as follows; $U_\lambda(\pi_\lambda(f))=\pi_\lambda(Tf)$ [resp. $Q_\lambda(\pi_\lambda(f))=\pi_\lambda(Pf)$] for $f\in E$, where π_λ is the natural mapping of E onto E/I_λ. Then U_λ is a positive operator in E/I_λ and is also strongly ergodic with the limit operator Q_λ.

PROPOSITION 6. *Let $K_\lambda=\{f\in A_0(X_\lambda);\ f=0$ on $X_\lambda\cap N\}$. Then K_λ is the smallest nonzero U_λ-invariant closed ideal in E/I_λ.*

We denote the restriction of U_λ [resp. Q_λ] to K_λ by T_λ [resp. P_λ]. Then we have

THEOREM 1. *T_λ is an irreducible (i.e. having no nonzero proper T-invariant closed ideal), positive, strongly ergodic operator with the limit operator P_λ.*

§ 3. Spectral properties on the unit circle.

In this section, we consider the relation between the spectrum of T and those of $\{T_\lambda\}_{\lambda\in\Lambda}$ or $\{U_\lambda\}_{\lambda\in\Lambda}$ obtained in § 2.

Hereafter, let T be a positive operator in a simplex space E with $r(T)=1$, and assume that 1 be a pole of $R(\alpha, T)$ of order 1 and r be a positive number such that $\{\alpha;\ 0<|\alpha-1|<r\}\subset\rho(T)$. Then the residual operator P of $R(\alpha, T)$ at 1 is also positive, since $\lim_{\alpha\downarrow 1}(\alpha-1)R(\alpha, T)=P$ and $R(\alpha, T)$ is positive for any $\alpha>1$. Moreover T is uniformly ergodic, i.e. M_n converges uniformly to the residual operator P [5, Theorem 6]. Then we can apply the results in § 2 for T. We shall use the same notations defined in the various stages of decomposition, such as Λ, U_λ and T_λ.

Let ρ_∞ denote the unbounded connected component of the resolvent set of T.

Then we have the following proposition by [10, Lemma 2].

PROPOSITION 7. *For any* $\lambda \in \Lambda$, *we have*

$$\rho_\infty(T_\lambda) \supset \rho_\infty(U_\lambda) \supset \rho_\infty(T)$$

and, if $\alpha \in \rho_\infty(T)$, *then* $\|R(\alpha, T_\lambda)\| \leqq \|R(\alpha, T)\|$.

The above proposition implies the following

PROPOSITION 8. *Let* Γ *denote the unit circle, i.e.* $\Gamma = \{\alpha ; |\alpha| = 1\}$. *Then*

$$\sigma(T) \cap \Gamma \supset \Big(\bigcup_{\lambda \in \Lambda} \sigma(U_\lambda)\Big)^- \cap \Gamma \supset \Big(\bigcup_{\lambda \in \Lambda} \sigma(T_\lambda)\Big)^- \cap \Gamma .$$

In case of an AM space, if P is strictly positive (i.e. $P|f| = 0$ implies $f = 0$ for f in the AM space), we have

$$\sup_{\lambda \in \Lambda} \|R(\alpha, T_\lambda)\| = \|R(\alpha, T)\| , \tag{$*$}$$

which plays an important role in proving the inverse inclusion in Proposition 8. The above relation $(*)$ is obtained from $\partial_e X = \overline{\bigcup_{\lambda \in \Lambda} S_\lambda}$ in case of an AM space with an order unit and from $\overline{\partial_e X} = \bigcup_{\lambda \in \Lambda} (Z_\lambda \cap \overline{\partial_e X})$ in case of a general AM space.

But as for a simplex space E, $\partial_e X$ is not always compact even if E has an order unit and the weak*-closure of the face $\bigcup_{\lambda \in \Lambda} X_\lambda$ is not necessarily a face. So the relation $(*)$ is not evident even if we suppose P is strictly positive (i.e. $Ph = 0$ implies $h = 0$ for $h \geqq 0$ in E). The following example shows a case where T is positive, Markov and uniformly ergodic and P is strictly positive, but $\overline{\bigcup_{\lambda \in \Lambda} S_\lambda} \subsetneqq \overline{\partial_e X}$.

EXAMPLE. Let E and $T \in \mathfrak{L}(E)$ be as follows;

$$E = \left\{ f \in C([0, 3]) ; \begin{array}{l} f(1) = 2\displaystyle\int_0^1 x \cdot f(x) dx \\ f(2) = \displaystyle\int_1^2 (x-1) \cdot f(x) dx + \int_2^3 (3-x) \cdot f(x) dx \end{array} \right\}$$

$$Tf(x) = \begin{cases} (1/2)\{f(1+x) + f(3-x)\} & 0 \leqq x \leqq 1 \\ (x-1) \cdot f(x) + (2-x) \cdot f(x+1) & 1 \leqq x \leqq 2 \\ (x-2) \cdot f(x-1) + (3-x) \cdot f(x) & 2 \leqq x \leqq 3 . \end{cases}$$

Then E is a simplex space with an order unit and T is positive, Markov and uniformly ergodic. And we have

$$PE=\left\{\begin{array}{lll} & f(x)=(1/2)\{f(1+x)+f(3-x)\} & 0\leqq x\leqq 1 \\ f\in C([0,3]) ; & f(x)=f(x+1) & 0\leqq x\leqq 2 \\ & f(1)=2\int_0^1 x\cdot f(x)dx=\int_1^2 f(x)dx & \end{array}\right\}$$

$$\cong\{f\in C([1,2]) ;\ f(1)=\int_1^2 f(x)dx=f(2)\}$$

and

$$\overline{\bigcup_{\lambda\in\Lambda} S_\lambda}=[1, 3]\subsetneqq[0, 3]=\overline{\partial_e X}.$$

Next, we shall show the inverse inclusion in Proposition 8 in case of a separable simplex space. Hereafter we assume in addition E is separable. We shall show some propositions by using the results of [14]. Let $\tilde{E}$ be the complexification of E and $|\tilde{f}|$ be the absolute value in E'' of $\tilde{f}\in\tilde{E}$ defined in [14]. By Theorem 4 in [14], we have

$$|\tilde{f}|(x)=\inf\{h(x) ;\ h\in E,\ h\geqq|\tilde{f}| \text{ in } E''\}$$

as a function on X. Using this fact, we have

PROPOSITION 9. *$P''|\tilde{f}|$ is an upper semi-continuous function on Y endowed with $\sigma((PE)', PE)$-topology and satisfies the barycentric calculus (i.e. if $y\in Y$ and ν is a probability measure on Y with resultant y, then $\nu(P''|\tilde{f}|)=P''|\tilde{f}|(y)$).*

PROOF. For $y\in Y$, we have

$$\begin{aligned} P''|\tilde{f}|(y)=|\tilde{f}|(y)&=\inf\{h(y) ;\ h\in E,\ h\geqq|\tilde{f}| \text{ in } E''\} \\ &=\inf\{Ph(y) ;\ h\in E,\ h\geqq|\tilde{f}| \text{ in } E''\}. \end{aligned}$$

Therefore, being considered as a function on Y, $P''|\tilde{f}|$ is an upper semi-continuous function on Y with $\sigma((PE)', PE)$-topology and so satisfies the barycentric calculus by [1, Theorem I. 2.6]. //

PROPOSITION 10. i) *If $\tilde{f}\in\tilde{E}$ and $|\tilde{f}(x)|\leqq M$ for all $x\in\bigcup_{\lambda\in\Lambda} S_\lambda$, then $\|P''|\tilde{f}|\|\leqq M$.*

ii) *If $\sup_{\lambda\in\Lambda}\|R(\alpha, U_\lambda)\|\leqq M_1$, then $\|P''|R(\alpha, T)f|\|\leqq M_1$ for any $f\in E$ with $\|f\|\leqq 1$.*

PROOF. i) $\lambda\in\Lambda$ can be considered as an element of X by Proposition 2. Then there exists a maximal probability measure μ_λ on X with resultant λ supported by $\partial_e X\cap S_\lambda=\partial_e S_\lambda$. By Corollary 2 to Theorem 2 in the author's preceding paper [14], $|\tilde{f}(x)|=|\tilde{f}|(x)$ holds for $x\in\partial_e X$. Since $|\tilde{f}|$ satisfies the barycentric calculus by Theorem 3 in [14],

$$|\tilde{f}|(\lambda)=\mu_\lambda(|\tilde{f}|)=\int_{\partial_e S_\lambda}|\tilde{f}|\,d\mu_\lambda\leqq M\int_{\partial_e S_\lambda}d\mu_\lambda=M\,.$$

Therefore we have $|P''|\tilde{f}|(\lambda)|\leqq M$ for any $\lambda\in\Lambda$. Considering $P''|\tilde{f}|$ as a function on Y, we see that $P''|\tilde{f}|$ satisfies the barycentric calculus by Proposition 9. So, for any $y\in Y$,

$$P''|\tilde{f}|(y)=\nu_y(P''|\tilde{f}|)=\int_{\Lambda\cup\{0\}}P''|\tilde{f}|\,d\nu_y\leqq M\,,$$

where ν_y is the maximal probability measure on Y with resultant y supported by $\Lambda\cup\{0\}$. Therefore,

$$\|P''|\tilde{f}|\|=\sup_{y\in Y}|P''|\tilde{f}|(y)|\leqq M\,.$$

ii) Suppose that $\sup_{\lambda\in\Lambda}\|R(\alpha, U_\lambda)\|\leqq M_1$. Then for any $x\in S_\lambda$ we have by Proposition 5,

$$\begin{aligned}|R(\alpha, T)f(x)|&\leqq\|\pi_\lambda(R(\alpha, T)f)\|=\|R(\alpha, U_\lambda)\pi_\lambda(f)\|\\&\leqq\|R(\alpha, U_\lambda)\|\,\|f\|\,,\end{aligned}$$

where π_λ is defined in §2. Therefore, for $f\in E$ with $\|f\|\leqq 1$, it follows that $R(\alpha, T)f\in\tilde{E}$ and $|R(\alpha, T)f(x)|\leqq M_1$ for all $x\in\bigcup_{\lambda\in\Lambda}S_\lambda$. By applying the preceding result, we have $\|P''|R(\alpha, T)f|\|\leqq M_1$. //

THEOREM 2. *Let T be a uniformly ergodic positive operator with $r(T)=1$ in a separable simplex space E, Λ be the set of all nonzero extreme points of the set $\{x\in E'\,;\ x\geqq 0,\ T'x=x,\ \|x\|\leqq 1\}$ and I_λ be the set $\{f\in E\,;\ h\geqq f,\ -f$ and $\lambda(h)=0$ for some $h\in E\}$ for any $\lambda\in\Lambda$. Then there exists $M\geqq 0$ such that*

$$\|R(\alpha, T)\|\leqq 2\sup_{\lambda\in\Lambda}\|R(\alpha, U_\lambda)\|+M\quad\textit{for any } \alpha,\ |\alpha|>1,$$

where U_λ is the induced operator in E/I_λ from T.

PROOF. For α, $|\alpha|>1$ and $f\in E$, we have

$$R(\alpha, T)f=\frac{T^n}{\alpha^n}R(\alpha, T)f+\sum_{i=1}^{n}\frac{T^{i-1}}{\alpha^i}f\quad\text{for } n\in\boldsymbol{N}$$

by the relation $(\alpha-T)\cdot R(\alpha, T)=I$. Since $R(\alpha, T)f$ is an element of the complexification $\tilde{E}$ of E, we consider the absolute value in E''. Then we have

$$|T''\tilde{f}|\leqq T''|\tilde{f}|\quad\text{for } \tilde{f}\in\tilde{E}$$

as T is positive. Since E'' has an order unit 1, we have

$$|R(\alpha, T)f|\leqq|T^nR(\alpha, T)f|+nK_n\|f\|\cdot 1$$

$$\leqq T''^{n}|R(\alpha,\ T)f|+nK_n\|f\|\cdot 1,$$

where $K_n=\sup_{0\leq i\leq n-1}\|T^i\|$. Therefore

$$|R(\alpha,\ T)f|\leqq M_n''|R(\alpha,\ T)f|+\frac{n-1}{2}K_n\|f\|\cdot 1,$$

where $M_n''=\dfrac{I+T''+\cdots+T''^{n-1}}{n}$. Using the relation $\|\tilde{g}\|=\||\tilde{g}|\|$ for $\tilde{g}\in\tilde{E}$, we have

$$\|R(\alpha,\ T)f\|\leqq\|M_n''|R(\alpha,\ T)f|\|+\frac{n-1}{2}K_n\|f\|.$$

Since M_n'' converges uniformly to P'', there exists $j\in N$ such that

$$\|M_j''-P''\|<\frac{1}{2}.$$

By Proposition 10, we have

$$\|R(\alpha,\ T)f\|\leqq\|P''|R(\alpha,\ T)f|\|+\frac{1}{2}\|R(\alpha,\ T)f\|+\frac{j-1}{2}K_j\|f\|$$

$$\leqq\sup_{\lambda\in\Lambda}\|R(\alpha,\ U_\lambda)\|\cdot\|f\|+\frac{1}{2}\|R(\alpha,\ T)\|\cdot\|f\|+\frac{j-1}{2}K_j\|f\|.$$

By putting $M=(j-1)K_j$, we have the desired result. //

PROPOSITION 11. *Let $\alpha_0\in\Gamma$ satisfy the following condition; $\alpha_0\in\rho(U_\lambda)$ for any $\lambda\in\Lambda$ and $\sup_{\lambda\in\Lambda}\|R(\alpha_0,\ U_\lambda)\|<\infty$. Then $\alpha_0\in\rho(T)$.*

PROOF. By the assumption and Lemma 3 in [10], there exists a positive number d such that $\sup_{\lambda\in\Lambda}\|R(\alpha,\ U_\lambda)\|$ is bounded in the set $\{\alpha;\ |\alpha-\alpha_0|<d\}$. By Theorem 2, $\|R(\alpha,\ T)\|$ is bounded in the set $\{\alpha;\ |\alpha-\alpha_0|<d,\ |\alpha|>1\}$. Hence $\alpha_0\in\rho(T)$. //

Let $\{\lambda_n\}$ be an arbitrary chosen sequence of elements of Λ. Denote U_{λ_n}, Q_{λ_n} and E/I_{λ_n} simply by U_n, Q_n and E_n respectively. Let $\mathfrak{m}=\{\{f_n\}\ ;\ f_n\in E_n,\ \sup_n\|f_n\|<\infty\}$. With linear structure and order defined coordinatewise and norm defined by $\|\{f_n\}\|=\sup_n\|f_n\|$, $\mathfrak{m}$ is a simplex space. Operators $\hat{U}$ and $\hat{Q}$ are defined by $\hat{U}\{f_n\}=\{U_nf_n\}$ and $\hat{Q}\{f_n\}=\{Q_nf_n\}$. Let $\mathfrak{U}$ be an arbitrary fixed ultrafilter on N containing no finite set. Put

$$J_{\mathfrak{U}}=\{\{f_n\}\ ;\ h_n\geqq f_n,\ -f_n \text{ and } \mathfrak{U}-\lim\|Q_nh_n\|=0 \text{ for some } \{h_n\}\in\mathfrak{m}\},$$

where $\mathfrak{U}-\lim$ is the ultrafilter limit with respect to $\mathfrak{U}$. Let $\grave{E}$ be the factor space

$\mathfrak{m}/J_{\mathfrak{U}}$. Since $J_{\mathfrak{U}}$ is easily seen to be $\hat{U}$-invariant, the operators $\hat{U}$ and $\hat{Q}$ induce operators in $\grave{E}$ which are denoted by $\grave{U}$ and $\grave{Q}$ respectively. Then we get the following proposition in the similar way to Lemma 4 in [6].

PROPOSITION 12. *$\grave{U}$ is a uniformly ergodic positive operator in $\grave{E}$ and $\grave{Q}$ is the residual operator of $R(\alpha, \grave{U})$ at 1. If J is a closed $\grave{U}$-invariant ideal containing $\grave{Q}\grave{E}$, the operator $\grave{U}|_J$ has the same spectral properties on the set $\{\alpha; |\alpha|>1-r\}$ as those $\grave{U}$ has.*

Lemma 5 in [10] plays an essential role in proving the reduction theory in case of a Banach lattice, but it can't be applied to the case of a simplex space E, since the absolute value doesn't exist in E. So we prepare the following lemma instead of Lemma 5 in [10].

LEMMA 2. *Let $\{U_n\}$ be defined above and b be a positive number such that*

$$\sup_{\alpha>1}\|R(\alpha, U_n)(I-Q_n)\|\leqq b\,.$$

(The existence of such b is guaranteed by the assumption that 1 is a pole of $R(\alpha, T)$ of order 1 and U_n is induced from T.) Then the relations

$$\|f_n\|=1,\ f_n\in E_n,\ |\alpha_0|=1 \quad and \quad \|U_nf_n-\alpha_0f_n\|<\frac{1}{16b} \tag{$**$}$$

imply $\{f_n\}\notin J_{\mathfrak{U}}$.

PROOF. The absolute value of an element of E_n does not exist in E_n but in the second dual E_n'' [14]. So we consider the second dual operator U_n'' in E_n'', which is an AM space. Then U_n'' is a positive operator with $r(U_n'')=\|U_n''\|=1$ and 1 is a pole of $R(\alpha, U_n'')$ with the residual operator Q_n'', where $Q_n''|_{E_n}=Q_n$. Moreover $\sup_{\alpha>1}\|R(\alpha, U_n'')(I-Q_n'')\|\leqq b$ holds. So by virtue of the assumption $(**)$, we can apply Lemma 5 in [10], and get $\|Q_n''|f_n|\|\geqq\frac{1}{2}$ since $|f_n|$ exists in E_n''. Consider $I=\{\{\phi_n\};\ \mathfrak{U}-\lim\|Q_n''|\phi_n|\|=0\}$. Then $I\cap\mathfrak{m}\supset J_{\mathfrak{U}}$. For if $\{g_n\}\in J_{\mathfrak{U}}$, there exists $\{h_n\}$ such that

$$h_n\geqq g_n,\ -g_n,\quad \mathfrak{U}-\lim\|Q_n''h_n\|=0\,.$$

Since $Q_n''|g_n|\leqq Q_n''h_n$, we have $\mathfrak{U}-\lim\|Q_n''|g_n|\|=0$, which implies $\{g_n\}\in I\cap\mathfrak{m}$. By the definition of I, $\|Q_n''|f_n|\|\geqq\frac{1}{2}$ implies $\{f_n\}\in I\cap\mathfrak{m}$. Therefore, $\{f_n\}\notin J_{\mathfrak{U}}$. //

The following lemma may be shown along the same line as the proof of [12, Theorem 1].

LEMMA 3. *Let E be a simplex space, $T\in L(E)$ be a positive irreducible oper-*

ator such that $r(T)=1$ and 1 be a pole of $R(\alpha, T)$. Let r be a positive number such that

$$\{\alpha\,;\ 0<|\alpha-1|<r\}\subset\rho(T)$$

and α_0 be in $\sigma(T)\cap\Gamma$. Then

$$\{\alpha\,;\ 0<|\alpha-\alpha_0|<r\}\subset\rho(T).$$

We are now prepared to give the main result, which can be proved by using above lemmas and propositions along the same line as the proof of [6, Theorem 8]. We will sketch the proof for completeness.

PROPOSITION 13. *Let T be a uniformly ergodic positive operator with $r(T)=1$ in a separable simplex space E and Λ be the set of all nonzero extreme points of the set $\{x\in E'\,;\ x\geqq 0,\ \|x\|\leqq 1,\ T'x=x\}$ and I_λ be the set $\{f\in E\,;\ h\geqq f,\ -f$ and $\lambda(h)=0$ for some $h\in E\}$ for any $\lambda\in\Lambda$. Then*

$$\sigma(T)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(U_\lambda))^-\cap\Gamma,$$

where U_λ is the induced operator in E/I_λ from T.

PROOF. Since the inclusion

$$\sigma(T)\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(U_\lambda))^-\cap\Gamma$$

is proved in Proposition 8, it suffices to show the inverse inclusion which is equivalent to

$$\rho(T)\supset(\bigcap_{\lambda\in\Lambda}\rho(U_\lambda))^\circ\cap\Gamma.$$

Let α_0 be in $(\bigcap_{\lambda\in\Lambda}\rho(U_\lambda))^\circ\cap\Gamma$. By Proposition 11, it is sufficient to show that the assumption of unboundedness of the set $\{\|R(\alpha_0, U_\lambda)\|\,;\ \lambda\in\Lambda\}$ yields a contradiction.

The first step; Let r and b be positive numbers satisfying

$$\{\alpha\,;\ |\alpha-\alpha_0|<r\}\subset\bigcap_{\lambda\in\Lambda}\rho(U_\lambda)$$

$$\{\alpha\,;\ 0<|\alpha-1|<r\}\subset\rho(T)$$

and

$$\sup_{\alpha>1}\|R(\alpha, T)(I-P)\|\leqq b.$$

Let s be a positive number less than r and $\frac{1}{2b}$. Then by the same way as in the first step of the proof of [10, Theorem 6], there exists an α_1 and a sequence $\{\lambda_n\}$ of elements of Λ such that $|\alpha_1-\alpha_0|<s$ and $\|R(\alpha_0, U_n)\|>n$, $\|R(\alpha_1, U_n)\|>n$

hold for any n.

The second step; From the sequence $\{U_n\}$ obtained in the first step, we construct a new simplex space $\check{E}$ and a positive operator $\check{T}$ in $\check{E}$ following the method described above. Let $\check{J}$ denote the closed ideal in $\check{E}$ generated by the eigenspace of $\check{T}$ for the eigenvalue 1. Then $\check{J}$ is the minimal $\check{T}$-invariant closed ideal, hence the restriction of $\check{T}$ to $\check{J}$, which is denoted by $\check{T}|_{\check{J}}$, is an irreducible positive operator having the following properties; $R(\alpha, \check{T}|_{\check{J}})$ has a simple pole at $\alpha=1$, and $\{\alpha; 0<|\alpha-1|<r\}\subset\rho(\check{T}|_{\check{J}})$.

The third step; We can show that α_0 and α_1 belong to $\sigma(T)$ by the same way as in the third step of the proof of [10, Theorem 6] by using Lemma 2 instead of Lemma 5 in [10].

The fourth step; Applying Proposition 12 to the results of the third step, we have $\alpha_0, \alpha_1\in\sigma(\check{T}|_{\check{J}})$. This contradicts Lemma 3 since $\rho(\check{T}|_{\check{J}})\supset\{\alpha; 0<|\alpha-1|<r\}$ and $s<r$. //

The operator U_λ in the above proposition is not always irreducible, but the operator T_λ defined before Theorem 1 is irreducible by Theorem 1. Since T_λ is the restriction of U_λ to the closed U_λ-invariant ideal generated by the eigenspace of U_λ for the eigenvalue 1 (this is clear from the definition of K_λ in Proposition 6), we get $\sigma(T_\lambda)\cap\Gamma=\sigma(U_\lambda)\cap\Gamma$ in the same way as the proof of [10, Lemma 8]. Using this fact and Proposition 13, we get the following theorem.

THEOREM 3. *Let T be a uniformly ergodic positive operator with $r(T)=1$ in a separable simplex space E and Λ be the set of all nonzero extreme points of the set $\{x\in E'; x\geqq 0, \|x\|\leqq 1, T'x=x\}$. Then*

$$\sigma(T)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^{-}\cap\Gamma.$$

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