Topological entropy and periodic points of maps of the circle

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R. Bowen and J. Franks [BF] gave lower bounds for the topological entropy and the number of periodic points of a continuous map of the interval with a periodic point of period divisible by an odd integer $\geqq 3$. The aim of this paper is to prove similar results for maps of the circle.

For the definition and properties of the topological entropy $h(f)$ of a map f, see [AKM], [DGS], [W].

For a map f, x is said to be a periodic point of f of period n if $f^n x = x$ and $f^i x \neq x$ for $i=0, 1, \cdots, n-1$. In the sequel $p_k(f)$ denotes the number of periodic points of f of period k.

For a positive integer n, let σ_n denote the unique positive root of the equation $t^n - t^{n-1} - 1 = 0$.

THEOREM 1. *Let f be a continuous map of the circle into itself with mapping degree d.*

(1) *In the case that $d=0$ or -1, assume that f has a periodic point of period n, an odd integer $\geqq 3$, then we have*

(a) $h(f) \geqq \log \sigma_n$, *and*

(b) $\liminf_k \frac{1}{k} \log p_k(f) \geqq \log \sigma_n$.

(2) *In the case that $d=1$, assume that f has both a fixed point and a periodic point of period n, an odd prime integer, then we have the same inequalities as in* (1).

(3) *In the case that $|d| \geqq 2$, we have*

(a) $h(f) \geqq \log |d|$, *and*

(b) $\liminf_k \frac{1}{k} \log p_k(f) \geqq \log |d|$.

REMARKS.

(1) Note that $\sigma_n > \sqrt[n]{2} > 1$. In particular $\log \sigma_n > 0$.

(2) In (2) of Theorem 1, the existence of a fixed point is essential. For instance, the $\frac{1}{3}$-rotation of the circle has periodic points of period 3, but its

topological entropy is zero and it has no periodic points of period $n\neq 3$.

(3) (a) of (3) is the simplest case of a theorem of A. Manning [M]. But our proof is different from that in [M].

Making use of the formula $h(f^m)=m\cdot h(f)$ $(m\geqq 0)$ and Theorem 1, we obtain

COROLLARY 2. *If a continuous map f of the circle has a fixed point and $h(f)=0$, then the period of any periodic point of f is a power of 2.*

Although the method of the proof of Theorem 1 is similar to that in [BF], we need more precise considerations.

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§ 1. Matrix representation of a periodic orbit.

Throughout this section we assume that a continuous map f of the circle has a periodic point of period n, an odd integer $\geqq 3$. In this section, we represent the orbit by a matrix and estimate the topological entropy and the number of periodic points.

Let $T=\{x_1, x_2, \cdots, x_n\}$ be the periodic orbit, ordered counter-clockwise on the circle as in Figure 1. Let $I_1=[x_n, x_1]$, $I_2=[x_1, x_2]$, $\cdots$, $I_n=[x_{n-1}, x_n]$, and give the counter-clockwise orientation on each I_i. Then the set of intervals $\{I_1, I_2, \cdots, I_n\}$ can be viewed as a base for $H_1(S^1, T; \boldsymbol{R})\cong \boldsymbol{R}^n$. Let $A=(a_{ij})_{i,j=1,2,\cdots,n}$ be the representation matrix of the induced map $f_*: H_1(S^1, T; \boldsymbol{R})\to H_1(S^1, T; \boldsymbol{R})$ with respect to the base $\{I_1, I_2, \cdots, I_n\}$; that is,

$$f_*(I_i)=\sum_{j=1}^{n} a_{ij}I_j \quad (i=1, 2, \cdots, n).$$

LEMMA 1.
$$\begin{aligned}\det(A-tI)&=(d-t)(1-t^n)/(1-t)\\&=(d-t)(1+t+t^2+\cdots+t^{n-1}).\end{aligned}$$

PROOF. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & H_1(S^1;\boldsymbol{R}) & \to & H_1(S^1,T;\boldsymbol{R}) & \to & H_0(T;\boldsymbol{R}) & \to & H_0(S^1;\boldsymbol{R}) & \to 0\\
& \downarrow D=f_* & & \downarrow A=f_* & & \downarrow B=f_* & & \downarrow C=f_* & \\
0 \to & H_1(S^1;\boldsymbol{R}) & \to & H_1(S^1,T;\boldsymbol{R}) & \to & H_0(T;\boldsymbol{R}) & \to & H_0(S^1;\boldsymbol{R}) & \to 0\\
& \wr\| & & \wr\| & & \wr\| & & \wr\| & \\
& \boldsymbol{R} & & \boldsymbol{R}^n & & \boldsymbol{R}^n & & \boldsymbol{R} &
\end{array}$$

where each row is a part of the homology exact sequence of the pair (S^1, T), and D, A, B, and C are the representation matrices of the maps between the homology groups induced by f. By the exactness, we have

$$\det(A-tI)=\det(D-tI)\cdot\det(B-tI)/\det(C-tI)\,.$$

Since D, B, and C are the 1×1-matrix (d), a cyclic permutation matrix, and the 1×1-matrix (1) respectively, we have $\det(D-tI)=d-t$, $\det(B-tI)=1-t^n$, and $\det(C-tI)=1-t$. This completes the proof. □

LEMMA 2. $\sum_{i=1}^{n} a_{ij}=d$ *for each* $j=1, 2, \cdots, n$.

PROOF. Immediate from $f_*(I_1+I_2+\cdots+I_n)=d\cdot(I_1+I_2+\cdots+I_n)$. □

We say that a finite sequence $\underline{I}=(i_0, i_1, \cdots, i_k)$ $(1\leqq i_r\leqq n)$ is admissible if $a_{i_r i_{r+1}}\neq 0$ for each r, and we call k the length of $\underline{I}$. We say that $\underline{I}$ is periodic if $i_0=i_k$.

For an admissible sequence $\underline{I}=(i_0, i_1, \cdots, i_k)$, we define a family of closed subintervals

$$F(\underline{I}\,;\ \alpha_1, \alpha_2, \cdots, \alpha_k)$$

$(\alpha_r=1, 2, \cdots, |a_{i_{r-1}i_r}|\,;\ r=1, 2, \cdots, k)$ by induction on the length of the sequences. For any sequence (i) of length 0 $(1\leqq i\leqq n)$, define $F((i))=I_i$. Consider an admissible sequence (i, j) of length 1 $(a_{ij}\neq 0)$. Then $f(I_i)$ covers I_j at least $|a_{ij}|$ times in one direction. So we choose $|a_{ij}|$ closed subintervals $F((i, j)\,;\ \alpha)$ $(1\leqq\alpha\leqq|a_{ij}|)$ of $F((i))$ so that each $f_{ij}^{(\alpha)}=f|_{F((i,j);\alpha)}$ satisfies $f_{ij}^{(\alpha)}(\operatorname{int} F((i, j)\,;\ \alpha))=\operatorname{int} F(j)$ and $f_{ij}^{(\alpha)}$ has local degree $+1$ (resp. -1) if $a_{ij}>0$ (resp. $a_{ij}<0$). Inductively, for an admissible sequence $(i_0, i_1, \cdots, i_k)$ of length k, we choose closed subintervals $F((i_0, i_1, \cdots, i_k)\,;\ \alpha_1, \cdots, \alpha_k)$ $(1\leqq\alpha_k\leqq|a_{i_{k-1}i_k}|)$ of $F((i_0, i_1, \cdots, i_{k-1})\,;\ \alpha_1, \cdots, \alpha_{k-1})$ so that

$$f_{i_0 i_1}^{(\alpha_1)}(\operatorname{int} F((i_0, i_1, \cdots, i_k)\,;\ \alpha_1, \cdots, \alpha_k))$$
$$=\operatorname{int} F((i_1, i_2, \cdots, i_k)\,;\ \alpha_2, \cdots, \alpha_k)$$

for each α_k.

For admissible sequences $\underline{I}=(i_0, i_1, \cdots, i_k)$ and $\underline{J}=(j_0, j_1, \cdots, j_k)$ of length k, $F(\underline{I}\,;\ \alpha_1, \alpha_2, \cdots, \alpha_k)\neq F(\underline{J}\,;\ \beta_1, \beta_2, \cdots, \beta_k)$ implies $\operatorname{int} F(\underline{I}\,;\ \alpha_1, \alpha_2, \cdots, \alpha_k)$ and $\operatorname{int} F(\underline{J}\,;\ \beta_1, \beta_2, \cdots, \beta_k)$ are disjoint. Especially for a sequence $\underline{I}=(i_0, i_1, \cdots, i_k)$, $F(\underline{I}\,;\alpha_1, \cdots, \alpha_k)$ and $F(\underline{I}\,;\beta_1, \cdots, \beta_k)$ are disjoint if $(\alpha_1, \alpha_2, \cdots, \alpha_k)\neq(\beta_1, \beta_2, \cdots, \beta_k)$.

For an admissible and periodic sequence $\underline{I}=(i_0, i_1, \cdots, i_k)$, $f^k|_{F(\underline{I};\alpha_1,\cdots,\alpha_k)}$ is a surjective map from the subinterval $F(\underline{I}\,;\ \alpha_1, \cdots, \alpha_k)$ of I_{i_0} onto $I_{i_k}=I_{i_0}$. Thus there exists a point $p(\underline{I}\,;\ \alpha_1, \cdots, \alpha_k)$ in $F(\underline{I}\,;\ \alpha_1, \cdots, \alpha_k)$ fixed by f^k. Note that $p(\underline{I}\,;\ \alpha_1, \cdots, \alpha_k)\neq p(\underline{I}\,;\ \beta_1, \cdots, \beta_k)$ if $(\alpha_1, \cdots, \alpha_k)\neq(\beta_1, \cdots, \beta_k)$.

Now we apply the analogous argument to that in [BF], and we obtain the following lemma.

LEMMA 3. *For any positive integer* N,

$$h(f)\geqq\limsup_{k}\frac{1}{Nk}\log\ \mathrm{Tr}|A|^{Nk}-\frac{1}{N}\log 3\,,$$

where $|A|=(|a_{ij}|)$.

We use some properties of non-negative matrices without proof. For further details and proofs, see Chapter XIII of [G].

A non-negative matrix B is called irreducible if for any i, j there exists $r>0$ such that the (i, j)-component of the r^{th} power of B is positive; $(B^r)_{ij}>0$. In this case r can be chosen as follows:

$$r\leqq m-1 \quad \text{if } i\neq j$$
$$r\leqq m \quad \text{if } i=j\,,$$

where m is the degree of the minimal polynomial of B.

Consider the matrix A obtained from the periodic orbit. By reordering the base $\{I_1, I_2, \cdots, I_n\}$, we put A into the form

$$\begin{pmatrix} A_1 & 0 & 0 & \cdot & 0 \\ * & A_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ * & * & * & * & A_s \end{pmatrix}, \tag{$*$}$$

where each $|A_q|$ is irreducible. We call each $|A_q|$ an irreducible factor of $|A|$.

Now we apply the analogous argument to that in [BF], and we obtain the following lemma.

LEMMA 4. For any irreducible factor $|A_q|$ of $|A|$, we have

$$p_k(f)\geqq\mathrm{Tr}|A_q|^k-2n-\sum_{\substack{m|k\\ m<k}}\mathrm{Tr}|A_q|^m\,.$$

§ 2. Proof of Theorem 1 (1).

In this section, we assume the mapping degree d of f is 0 or -1, and f has a periodic point of period n, an odd integer $\geqq 3$. Let A be the matrix obtained as in § 1 from the periodic orbit of period n. Using the form $(*)$ of A, we have

$$\det(A-tI)=\prod_{i=1}^{s} p_i(t),$$

where $p_i(t)=\det(A_i-tI)$.

LEMMA 5. There exists a factor $p_q(t)$ $(q=1, 2, \cdots, s)$ of $\det(A-tI)$ such that the degree r of $p_q(t)$ is not less than two, and the coefficient of t^{r-1} in $p_q(t)$ is non-zero; i.e. $\mathrm{Tr}\, A_q\neq 0$.

PROOF. This follows from the fact that the coefficient of t^{n-1} in $\det(A-tI)$ is non-zero, and the polynomial $1+t+t^2+\cdots+t^{n-1}$ has no linear divisors over $\boldsymbol{R}$ when n is odd. $\square$

In terms of $p_q(t)$ and A_q corresponding to it as in Lemma 5, we have

LEMMA 6. $$\limsup_k \sqrt[k]{\mathrm{Tr}\,|A_q|^k}\geqq\sigma_n .$$

PROOF. We define the graph G_q of $|A_q|$ as follows: G_q is a 1-complex which consists of r vertices $1, 2, \cdots, r$ and $|(A_q)_{ij}|$ directed edges $i\xrightarrow{\alpha} j$ from i to j $(i, j=1, 2, \cdots, r;\ \alpha=1, 2, \cdots, |(A_q)_{ij}|)$. For a path $i_0\xrightarrow{\alpha_1} i_1\xrightarrow{\alpha_2}\cdots\xrightarrow{\alpha_k} i_k$ in G_q, we call k its length. A path $i_0\xrightarrow{\alpha_1} i_1\xrightarrow{\alpha_2}\cdots\xrightarrow{\alpha_k} i_k$ is said to be a circuit if $i_0=i_k$.

Since $\mathrm{Tr}\, A_q\neq 0$, some $(A_q)_{ii}$ $(1\leqq i\leqq r)$ is not zero. We may assume $(A_q)_{11}\neq 0$ by reordering the base. Let C_1 be the circuit in G_q consisting of only one edge $1\xrightarrow{\alpha} 1$. We choose a circuit C_2 in G_q through the vertex 1, different from C_1, with length $a\leqq r$. Such a circuit C_2 exists. In fact, since $|A_q|$ is irreducible, there exists $i\neq 1$ such that $(A_q)_{1i}\neq 0$; that is, 1 and i are joined by an edge. On the other hand, there exists a path from i to 1 of length $\leqq r-1$. By joining the edge and the path just chosen, we obtain a desired circuit C_2.

Then the path $C_{j_1}C_{j_2}\cdots C_{j_k}$ $(j_i=1 \text{ or } 2)$ which is a combination of C_1's and C_2's is a path from 1 to 1. Let S_i be the number of distinct paths of the form $C_{j_1}C_{j_2}\cdots C_{j_k}$ of length i. Then S_i satisfies

$$\sum_{i=1}^{\infty} S_i t^i=\sum_{k=1}^{\infty}(t+t^a)^k$$
$$=(t+t^a)/(1-t-t^a).$$

Let $g(t)=1-t-t^a$. Then $g(0)=1>0$ and $g(1/\sigma_n)\leqq 0$. Hence the radius of convergence of $\sum_i S_i t^i$ is at most $1/\sigma_n$; that is, $\limsup_i \sqrt[i]{S_i}\geqq\sigma_n$. On the other hand, it is obvious that $\mathrm{Tr}\,|A_q|^i\geqq S_i$. This completes the proof. $\square$

LEMMA 7. (Perron-Frobenius Theorem). *An irreducible non-negative matrix* $B=(b_{ij})_{i,j=1,2,\cdots,n}$ *always has a positive characteristic value* $\rho(B)$, *called the Frobenius root of* B, *which is a simple root of the characteristic equation, and the absolute values of all the other characteristic values of* B *do not exceed* $\rho(B)$.

If B *has* h *characteristic values* $\lambda_1=\rho(B)$, $\lambda_2, \cdots, \lambda_h$ *with absolute value* $\rho(B)$, *then all of them are simple and they are the roots of the equation;*

$$\lambda^h-\rho(B)^h=0\,.$$

Moreover, if $h>1$, *by reordering a base, we can put* B *into the following cyclic form;*

$$\begin{pmatrix} 0 & B_1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & B_2 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & B_{h-1} \\ B_h & 0 & 0 & \cdot & 0 & 0 \end{pmatrix},$$

where the blocks along the main diagonal are square matrices.

For the proof, see p. 53 of [G].

We call an irreducible non-negative matrix primitive if $h=1$ in Lemma 7.

Note that, for an irreducible matrix B, $\operatorname{Tr} B>0$ implies that B is primitive. In fact, if B is not primitive, B can be put into the cyclic form as in Lemma 7 and then $\operatorname{Tr} B=0$.

Now we consider $|A_q|$ with $\operatorname{Tr}|A_q|>0$. Since such $|A_q|$ is primitive, we have

$$\frac{1}{2}\lambda^k<\operatorname{Tr}|A_q|^k\leqq r\cdot\lambda^k, \tag{**}$$

for any large k, where $\lambda=\rho(|A_q|)$. Thus, by Lemma 6, we have for any N,

$$\limsup_k \sqrt[Nk]{\operatorname{Tr}|A_q|^{Nk}}=\limsup_k \sqrt[k]{\operatorname{Tr}|A_q|^k}=\lambda\geqq\sigma_n\,.$$

By Lemma 3, we have

$$\begin{aligned} h(f) &\geqq \limsup_k \frac{1}{Nk}\log\ \operatorname{Tr}|A|^{Nk}-\frac{1}{N}\log 3 \\ &\geqq \limsup_k \frac{1}{Nk}\log\ \operatorname{Tr}|A_q|^{Nk}-\frac{1}{N}\log 3 \\ &=\log\lambda-\frac{1}{N}\log 3\,. \end{aligned}$$

Since N is arbitrary,

$$h(f) \geqq \log \lambda \geqq \log \sigma_n .$$

On the other hand, by Lemma 4 and (**), we have

$$\begin{aligned} p_k(f) &\geqq \frac{1}{2}\lambda^k - 2n - r\sum_{\substack{m \mid k \\ m<k}} \lambda^m \\ &\geqq \frac{1}{2}\lambda^k - 2n - r\sum_{m \leqq [k/2]} \lambda^m \\ &\geqq \frac{1}{2}\lambda^k - 2n - r(\lambda^{[k/2]+1})/(\lambda-1) . \end{aligned}$$

Thus we obtain

$$\liminf_k \frac{1}{k} \log p_k(f) \geqq \log \lambda \geqq \log \sigma_n .$$

This completes the proof of Theorem 1 (1).

§ 3. Proof of Theorem 1 (2).

For the case $d \neq 0, -1$, we need the following lemma.

LEMMA 8. Assume that f is with mapping degree d, and for an odd prime integer n, f has a periodic point of period n. Then the matrix $|A|$, obtained from this periodic orbit as in § 1, has at most two irreducible factors.

PROOF. Recall that every A_i $(i=1, 2, \cdots, s)$ in the form (*) is a matrix with integer coefficients. Thus every $p_i(t) = \det(A_i - tI)$ is a polynomial with integer coefficients. Since n is an odd prime integer, the polynomial $1+t+t^2+\cdots+t^{n-1}$ is irreducible over $\boldsymbol{Q}$. Hence, if $|A|$ is not irreducible, $\det(A-tI) = (d-t)(1+t+t^2+\cdots+t^{n-1})$ has only two divisors corresponding to (*). This completes the proof. □

We prove Theorem 1 (2). We assume $d=1$ and f has both a fixed point and a periodic point of period n, an odd prime integer. Let A be the matrix obtained from this periodic orbit.

If $|A|$ is not irreducible, applying the same argument as in § 2 to the primitive irreducible factor of $|A|$ corresponding to $1+t+t^2+\cdots+t^{n-1}$, we obtain the desired results.

If $|A|$ is irreducible and $a_{ii} \neq 0$ for some i $(1 \leqq i \leqq n)$, then $|A|$ itself is primitive, and we obtain the desired results as in § 2.

Now we assume that $|A|$ is irreducible and $a_{ii} = 0$ for every $i = 1, 2, \cdots, n$. Let $T = \{x_1, x_2, \cdots, x_n\}$ be the periodic orbit, ordered counter-clockwise on the circle. We may assume the fixed point, say x_0, lies between x_n and x_1. Let $J_0=[x_n, x_0]$, $J_1=[x_0, x_1]$, and $J_i=I_i=[x_{i-1}, x_i]$ $(i=2, 3, \cdots, n)$, and give the counter-clockwise orientation to each J_i. Note that $J_0\cup J_1=I_1=[x_n, x_1]$.

We can choose a universal covering $\pi: \boldsymbol{R}\to S^1=\boldsymbol{R}/\boldsymbol{Z}$, a lift $\tilde{f}: \boldsymbol{R}\to\boldsymbol{R}$ of f and the points $0<z_1<z_2<\cdots<z_n<1$ such that

$$\pi(0)=x_0,$$

$$\pi(z_i)=x_i \ (i=1, 2, \cdots, n), \quad \text{and}$$

$$0<z_1<\tilde{f}(z_1)<1.$$

Since $0\in\boldsymbol{R}$ corresponds to the fixed point of f, $\tilde{f}(0)=k$ is an integer.

We consider the graph of the map $y=\tilde{f}(x)$. Since $\tilde{f}(z_1)$ is one of $z_2, z_3, \cdots, z_n$, the point $(z_1, \tilde{f}(z_1))$ is one of the points indicated by starts in Figure 2. Since $a_{ii}=0$ for all i, the segment joining two points $(z_1, \tilde{f}(z_1))$ and $(z_2, \tilde{f}(z_2))$ does not cut across the area B, where

$$B=\{(x, y)\in\boldsymbol{R}^2;\ z_1+r\leqq y\leqq z_2+r,\ r\in\boldsymbol{Z}\}.$$

Thus the point $(z_2, \tilde{f}(z_2))$ is one of the points indicated by stars on the line $x=z_2$ in Figure 3; that is, $\tilde{f}(z_2)$ is one of $z_3, z_4, \cdots, z_1+1$. Inductively it is proved that each point $(z_i, \tilde{f}(z_i))$ is one of the points indicated by stars on the line $x=z_i$ in Figure 4 $(i=1, 2, \cdots, n)$. In particular $\tilde{f}(z_n)$ is one of $z_1+1, z_2+1, \cdots, z_{n-1}+1$.

Let $\underline{T}=\{x_0, x_1, \cdots, x_n\}$. With respect to the base $\{J_0, J_1, \cdots, J_n\}$ of $H_1(S^1, \underline{T}; \boldsymbol{R})\cong\boldsymbol{R}^{n+1}$, the induced map $f_*: H_1(S^1, \underline{T}; \boldsymbol{R})\to H_1(S^1, \underline{T}; \boldsymbol{R})$ is represented as an $(n+1)\times(n+1)$-matrix $\underline{A}=(\underline{a}_{ij})_{i,j=0,1,\cdots,n}$:

$$f_*(J_i)=\sum_{j=0}^{n}\underline{a}_{ij}J_j \qquad (i=0, 1, \cdots, n).$$

Assume $\tilde{f}(0)=k\leqq 0$. Then $f(J_1)$ covers each of J_1 and J_2. Thus in the graph $\underline{G}$ of $\underline{A}$, there exist edges $1\xrightarrow{\alpha}1$ and $1\xrightarrow{\beta}2$. Let C_1 be a circuit consisting of only one edge $1\xrightarrow{\alpha}1$. Since $|A|$ is irreducible, there exists an admissible sequence $(i_0, i_1, \cdots, i_r)$ of length $\leqq n-1$ such that $i_0=2$ and $i_r=1$. This implies that, in $\underline{G}$, there exists a path from 2 to 1 of length $\leqq n-1$. (Recall that $J_0+J_1=I_1$ in $H_1(S^1, \underline{T}; \boldsymbol{R})$.) Joining the edge $1\xrightarrow{\beta}2$ and the path from 2 to 1, we have a circuit C_2 through 1 in $\underline{G}$, different from C_1, and of length $\leqq n$. Let $|A'|$ be the irreducible factor of $|\underline{A}|$, whose graph contains both C_1 and C_2. Note that $|A'|$ is primitive, since $\mathrm{Tr}|A'|\geqq|a_{11}|>0$. By the same argument as in §2, we have the desired results.

Assume $\tilde{f}(0)=k\geqq 1$, then $f(J_0)$ covers each of J_n and J_0. Considering these intervals, we obtain the desired results just as above.

The proof of Theorem 1 (2) is completed.

§4. Proof of Theorem 1 (3).

For the proof of Theorem 1 (3), we first state a proposition on the existence of periodic points.

PROPOSITION 9. Assume that $|d|\geqq 2$. Then for any odd prime integer n, f has at least $|d(d^{n-1}-1)|$ periodic points of period n.

PROOF. Since $|d|\geqq 2$, f has a fixed point. Hence we can choose a lift $\tilde{f}: \boldsymbol{R}\to\boldsymbol{R}$ of f such that $\tilde{f}(0)=0$.

Since the mapping degree of f is d, $\tilde{f}(x+1)=\tilde{f}(x)+d$ for any $x\in\boldsymbol{R}$. Thus, if $\alpha\in\boldsymbol{R}$ satisfies $\tilde{f}(\alpha)=\alpha+k$ for some $k\in\boldsymbol{Z}$, then we have inductively

$$\tilde{f}^n(\alpha)=\alpha+k(d^n-1)/(d-1) \qquad (n\geqq 1). \tag{***}$$

Consider the case $d\geqq 2$. For each $i=1, 2, \cdots, d-1$, let α_i be a point in $[0, 1)$ such that $\tilde{f}(\alpha_i)=\alpha_i+(i-1)$ and $\tilde{f}(x)>x+(i-1)$ for any $x\in(\alpha_i, 1)$, and let β_i be a point in $(\alpha_i, 1]$ such that $\tilde{f}(\beta_i)=\beta_i+i$ and $\tilde{f}(x)<x+i$ for any $x\in(\alpha_i, \beta_i)$. (See Figure 5.) Then f has no fixed points in each (α_i, β_i). By (***) we have

$$\tilde{f}^n(\beta_i)-\tilde{f}^n(\alpha_i)=(\beta_i-\alpha_i)+(d^n-1)/(d-1)>(d^n-1)/(d-1).$$

Thus for each $j=1, 2, \cdots, (d^n-1)/(d-1)-1$, there exists a point γ_{ij} in (α_i, β_i) such that $\tilde{f}^n(\gamma_{ij})=\gamma_{ij}+(i-1)(d^n-1)/(d-1)+j$. (See Figure 6.) That is, there exist at least $(d^n-1)/(d-1)-1$ points in (α_i, β_i) which are fixed by f^n. Since n is prime and the points γ_{ij} are not fixed by f, these are periodic of period n. Thus we have at least $((d^n-1)/(d-1)-1)(d-1)=d(d^{n-1}-1)$ periodic points of period n.

In the case that $d\leqq -2$, a similar argument leads us to the desired result. □

Now we prove Theorem 1 (3). Since there is a periodic point of period n, an odd prime integer $\geqq 3$, by Proposition 9, we have a matrix A defined in §1 from this periodic orbit.

If $|A|$ is irreducible, $\mathrm{Tr}\, A=\pm(d-1)\neq 0$ implies that $|A|$ is primitive. Moreover we have $\rho(|A|)\geqq|d|$. This inequality is proved as follows: For any non-negative irreducible matrix $B=(b_{ij})_{i,j=1,2,\cdots,n}$, we have

$$\rho(B)\geqq \min_{1\leqq i\leqq n}\Big(\sum_{j=1}^{n} b_{ij}\Big).$$

(See p. 65 of [G].) Thus by Lemma 2, we obtain

$$\begin{aligned}\rho(|A|)&=\rho({}^t|A|)\\&\geqq \min_{1\leqq j\leqq n}\left(\sum_{i=1}^{n}|a_{ij}|\right)\\&\geqq \min_{1\leqq j\leqq n}\left|\sum_{i=1}^{n}a_{ij}\right|\\&=|d|.\end{aligned}$$

If $|A|$ is not irreducible, by Lemma 8, A can be put into the form

$$\begin{bmatrix} A_1 & 0 \\ * & A_2 \end{bmatrix},$$

where each $|A_i|$ is irreducible. Then either $\det(A_1-tI)$ or $\det(A_2-tI)$ is $d-t$, that is, either $|A_1|$ or $|A_2|$ is the 1×1-matrix $(|d|)$, which is, of course, primitive and whose Frobenius root is equal to $|d|$.

Now, using $|A|$ (if $|A|$ is irreducible) or $|A_i|=(|d|)$ (otherwise), we complete the proof of Theorem 1 (3) just in the same manner as in § 2.

We have completed the proof of Theorem 1.

References

[AKM] Adler, R. L., Konheim, A. G. and M. H. McAndrew, Topological Entropy, Trans. Amer. Math. Soc. **114** (1965), 309-319.

[BF] Bowen, R. and J. Franks, The Periodic Points of Maps of the Disk and the Interval, Topology **15** (1976), 337-342.

[DGS] Denker, M., Grillenberger, C. and K. Sigmund, Ergodic Theory on Compact Spaces, Lecture Notes in Math. **527**, Springer, 1976.

[G] F. R. Gantmacher, The Theory of Matrices, vol. II, Chelsea, 1959.

[M] A. Manning, Topological Entropy and the First Homology Group, Dynamical Systems—Warwick 1974, Lecture Notes in Math. **468**, Springer, 1975, 185-190.

[W] P. Walters, Ergodic Theory—Introductory Lectures, Lecture Notes in Math. **458**, Springer, 1975.

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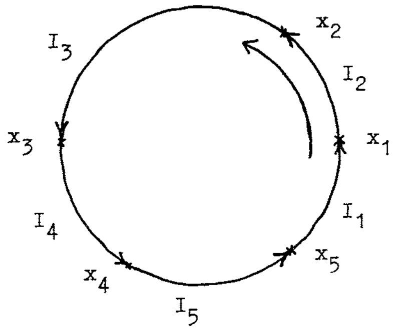


Figure 1

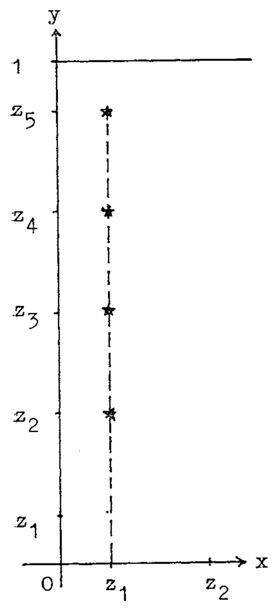


Figure 2

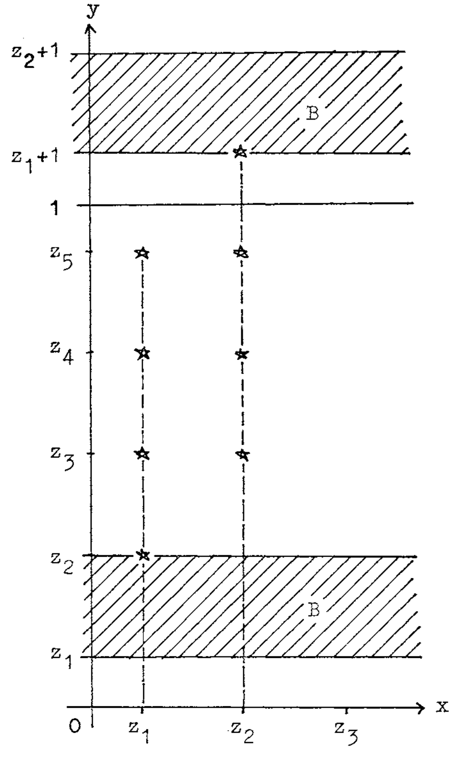


Figure 3

Figures 1, 2, 3 and 4 are written in the case that $n=5$.

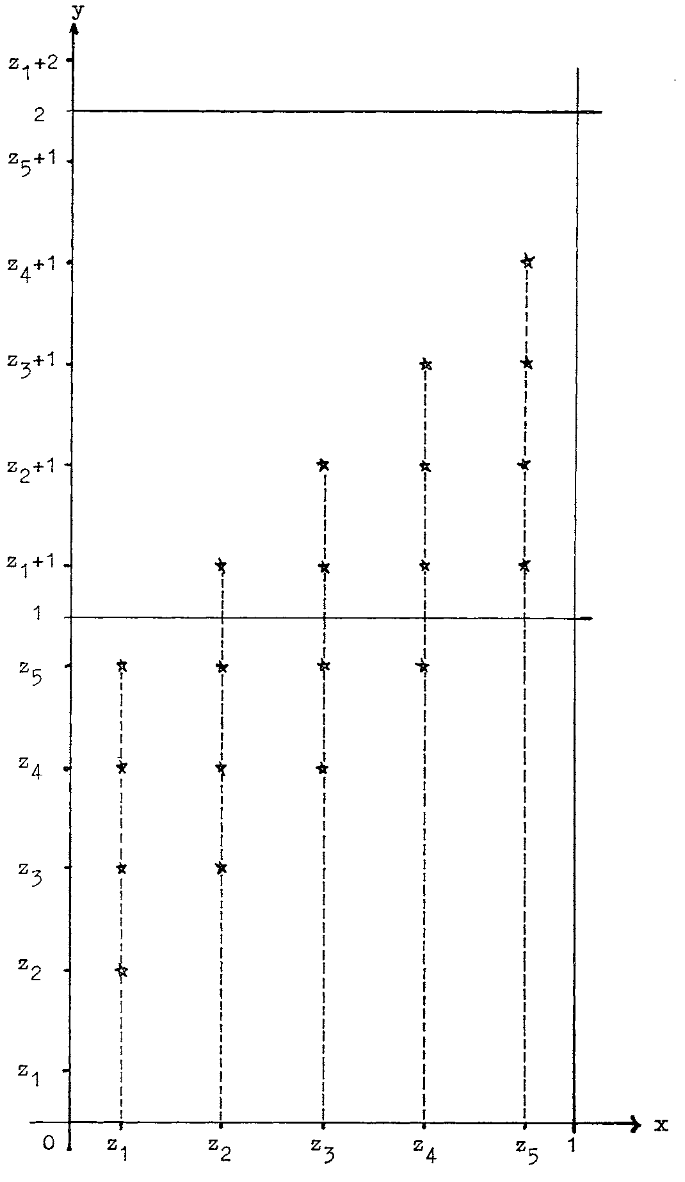
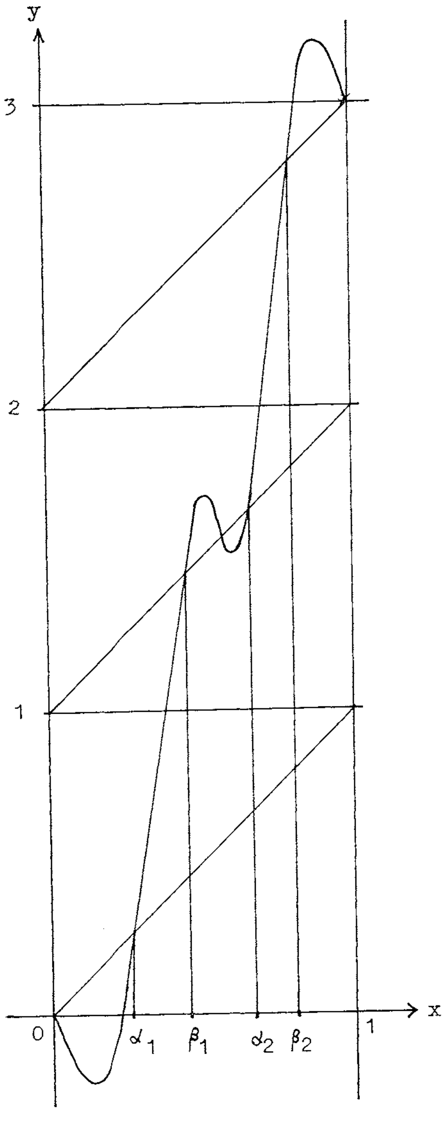
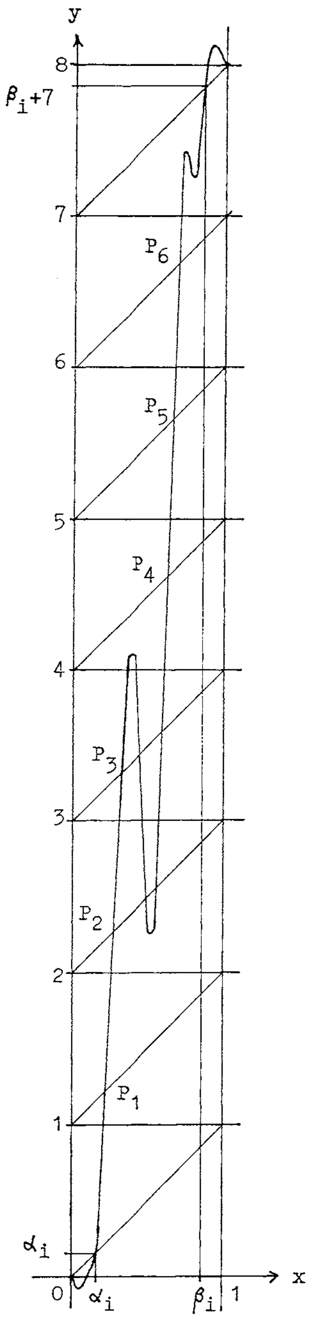


Figure 4



$y = \tilde{f}(x)$ in the case that $d = 3$.

Figure 5



$y = \tilde{f}^n(x)$ in the case that $d = 2,\ n = 3,\ i = 1.$

P_j denotes the point $(\gamma_{ij}, \tilde{f}^n(\gamma_{ij})) = (\gamma_{ij}, \gamma_{ij}+j).$

Figure 6