On Fujita's strong cancellation theorem for the affine plane[*)]

By Tatsuji KAMBAYASHI

Dedicated to the memory of Aldo Andreotti

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Introduction

The so-called Zariski Problem asks if an algebraic surface X satisfying $X \times A^1 \cong A^3$ must be an A^2. (Here, A^n for any n stands for an affine n-space.) M. Miyanishi had set up a program for an affirmative solution of this problem and had carried out most of it in collaboration with T. Sugie (cf. [6], [7]). In early 1979, T. Fujita resolved the last difficulty in Miyanishi's program and thereby completed a proof of the affirmative answer to the problem (cf. [1]). In their joint solution, Iitaka's theory of logarithmic Kodaira dimensions (cf. [3], [4]) (which draws upon Hironaka's resolution of singularities [11]) played a significant part. For this and other reasons, in the cited papers [6], [7] and [1] the ground field was always assumed to be of characteristic zero. However, it soon became apparent to several of us, notably to M. P. Murthy and P. Russell, that only the low-dimensional, obviously characteristic-free portion of Iitaka's theory was indispensable and that all of the arguments previously employed by Fujita, Miyanishi and Sugie are valid in positive characteristics. (For a detailed analysis, see 4.3 and 3.4 below.) Coupled with the absence of nontrivial separable forms of A^2 (cf. [5]), we do have, therefore, Fujita-Miyanishi-Sugie's

CANCELLATION THEOREM. *Let X, Y be algebraic varieties defined over a perfect field k, and suppose that X is two-dimensional and $X \times_k Y \cong$ (k-isomorphic to) A^n for some n. Then, $X \cong A^2$.* (See 4.3 below for a proof.)

Now, in the decisive paper [1], Fujita further indicates a topological proof of what we herein call

(SCT) STRONG CANCELLATION THEOREM. *Let X be an affine variety and suppose that $X \times Y \cong A^2 \times Y$ for some algebraic variety Y. Then, $X \cong A^2$.*

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Here, the ground field is of necessity the complex-number field. See [1; § 3, especially (3.3)].

Our aim in the present paper is to algebraicize the proof of this result (SCT) so as to extend it over perfect ground fields k, with the additional assumption that Y be "resoluble", or "admit a smooth completion". (For a precise definition, see § 1 below.) Again by [5] we may just as well *assume k to be algebraically closed of arbitrary characteristic,* and we shall do so henceforth. It turns out that in order to prove (SCT) we must extend Iitaka's theory at least to resoluble varieties when $\mathrm{char}(k)>0$ and prove the cancellation invariance of the logarithmic invariants. This essential step is taken in § 1. In § 2 we give a simple, geometric proof to Miyanishi's characterization theorem for $\boldsymbol{A}^2$ (cf. [6]) utilizing another recent result of his found in [8]. Here again, the assumption of $\mathrm{char}(k)=0$ originally made in [6] is unnecessary. In § 3 we prove that our surface X is an $\boldsymbol{A}^2$ by reduction to the characterization theorem just mentioned. The point is to show several key invariants are well-behaved in our particular cancellation situation, even if some of those may not be cancellation invariants in general. In § 4 we offer some more information about the whole affair.

We deal only with algebraic varieties, i. e., separated integral schemes of finite type over k. Notations are standard and similar to those in [1], [6], [7].

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1. Iitaka's logarithmic invariants.

Taking off somewhat from Iitaka, we call (V, D, X) a *nonsingular triple* if V is an algebraic variety smooth but not necessarily complete over k, D is a reduced effective divisor on V whose singularities are at worst of normal crossing type, and X is the complement of $\mathrm{Supp}(D)$ in V. We often write $X=V-D$

by slight abuse of notation. Such a triple is said to be *complete* if V is complete over k. By a *morphism of nonsingular triples* $(V_1, D_1, X_1)\rightarrow(V_2, D_2, X_2)$ is meant a morphism $f\colon V_1\rightarrow V_2$ of k-schemes such that $f(X_1)\subseteq X_2$. Then, $f^{-1}(D_2)\subseteq D_1$, clearly.

Given a nonsingular triple (V, D, X), *the sheaf of germs of m-ple logarithmic q-forms along D*, denoted by $\Omega^q(\log D)^{\otimes m}$, is defined in the usual way (cf. Iitaka [3]). It is a locally free sheaf of $\mathcal{O}_V$-modules.

1.1. THEOREM. *Let $f\colon(V_1, D_1, X_1)\rightarrow(V_2, D_2, X_2)$ be a morphism of nonsingular triples. For $i=1, 2$, let $\mathcal{O}_i$ be the structure sheaf of V_i and let $\mathcal{T}_i:=\Omega^q(\log D_i)^{\otimes m}$.*

(a) *If $f\colon V_1\rightarrow V_2$ is dominant and generically separable, then there is a natural sheaf map $f^*(\mathcal{T}_2)\rightarrow\mathcal{T}_1$ which is a monomorphism of $\mathcal{O}_1$-modules. This map induces, for each open set $U\subseteq V_2$, a k-linear injective map of sections $\mathcal{T}_2(U)\hookrightarrow\mathcal{T}_1(f^{-1}U)$.*

(b) *If f is proper and birational, and the restriction of f to X_1 is proper also, then the map $\mathcal{T}_2(U)\hookrightarrow\mathcal{T}_1(f^{-1}U)$ just above is surjective as well.*

PROOF. (a) Let, for $i=1, 2$, $\mathcal{K}_i$ and $\Omega^q(*V_i)^{\otimes m}$ be respectively the constant sheaves of rational functions and m-ple rational q-forms on V_i. The assumption implies that there is a natural monomorphism $\mathcal{K}_1\underset{\mathcal{K}_2}{\otimes}\Omega^q(*V_2)^{\otimes m}\rightarrow\Omega^q(*V_1)^{\otimes m}$. But, $\Omega^q(*V_2)^{\otimes m}\cong\mathcal{K}_2\underset{\mathcal{O}_2}{\otimes}\mathcal{T}_2$, because $\mathcal{T}_2$ is locally free of rank equal to that of $\Omega^q(*V_2)^{\otimes m}$ over $\mathcal{K}_2$. So, a monomorphism

$$\mathcal{K}_1\otimes\mathcal{T}_2=\mathcal{K}_1\underset{\mathcal{O}_2}{\otimes}\Omega^q(\log D_2)^{\otimes m}\rightarrow\Omega^q(*V_1)^{\otimes m}$$

is obtained. This induces, by restriction, a monomorphism $f^*\mathcal{T}_2=\mathcal{O}_1\underset{\mathcal{O}_2}{\otimes}\mathcal{T}_2\rightarrow\Omega^q(*V_1)^{\otimes m}$. We shall verify, stalk by stalk, that the image under this mapping falls within the subsheaf $\mathcal{T}_1=\Omega^q(\log D_1)^{\otimes m}$ of $\Omega^q(*V_1)^{\otimes m}$. Enough to do this for $m=1$ and $q=1$. Let $f(P_1)=P_2$ with closed points P_i on V_i $(i=1, 2)$, and let $z_1, \cdots, z_r, w_1, \cdots, w_{n-r}$ be a set of local parameters for V_1 at P_1 such that $(z_1\cdots z_r=0)$ gives D_1 at P_1 locally. Then, if one of the components of D_2 passes through P_2 and is defined there by $(y=0)$, we may write $y=uz_1^{a(1)}\cdots z_r^{a(r)}$ with u a local unit at P_1 and $a_1, \cdots, a_r$ non-negative integers, because $f^{-1}D_2\subseteq D_1$.1) Consequently, the image of $1\otimes(dy/y)$ under the monomorphism in question is

$$\frac{d(uz_1^{a(1)}\cdots z_r^{a(r)})}{uz_1^{a(1)}\cdots z_r^{a(r)}}=\frac{du}{u}+\sum_{i=1}^{r}a_i\frac{dz_i}{z_i}$$

which is certainly logarithmic along D_1. Thus, a monomorphism $f^*\mathcal{T}_2\rightarrow\mathcal{T}_1$ is obtained. Next, let $U\subseteq V_2$ be any open set, and consider the natural injection

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1) In this paper, for typographical reasons, a_i, m_j, α_t, etc., become respectively $a(i)$, $m(j)$, $\alpha(t)$, etc., when used as superfix or suffix.

$(f^*\mathscr{F}_2)(f^{-1}U)\hookrightarrow\mathscr{F}_1(f^{-1}U)$ preceded by the composition of the maps

$$\mathscr{F}_2(U)\xrightarrow{\phi}(f_*f^*\mathscr{F}_2)(U)\xrightarrow{\psi}(f^*\mathscr{F}_2)(f^{-1}U).$$

In $f^*\mathscr{F}_2=\mathcal{O}_1\otimes\mathscr{F}_2$, $\mathscr{F}_2$ is locally free and $\mathcal{O}_1$ is torsion-free, each as an $\mathcal{O}_2$-module. So the map ϕ is injective. But the map ψ is bijective by the definition of f_*. Hence $\mathscr{F}_2(U)\hookrightarrow(f^*\mathscr{F}_2)(f^{-1}U)\hookrightarrow\mathscr{F}_1(f^{-1}U)$, as desired.

(b) Assume now f to be birational and proper on V_1 and X_1. So, $f^{-1}D_2=D_1$. Let V_2° be an open subset of V_2 on which f^{-1} is defined. Then we have an open immersion $g: V_2^\circ\to V_1$ such that $f\circ g=$the identity map on V_2°. We may suppose $V_2-V_2^\circ$ has only components of codimension >1. Set $D_2^\circ:=D_2\cap V_2^\circ$, $X_2^\circ:=X_2\cap V_2^\circ$. Then $g(X_2^\circ)\subseteq X_1$ because $f^{-1}(D_2)=D_1$. Consequently, the arguments of part (a) can be applied to $g:(V_2^\circ, D_2^\circ, X_2^\circ)\to(V_1, D_1, X_1)$ and gives a monomorphism $g^*\mathscr{F}_1\to\mathscr{F}_2^\circ:=$(the restriction of $\mathscr{F}_2$ on V_2°). Thus, for any open $U\subseteq V_2$, one gets a k-linear injective map $\mathscr{F}_1(f^{-1}U)\hookrightarrow\mathscr{F}_2^\circ(g^{-1}f^{-1}U)=\mathscr{F}_2^\circ(U\cap V_2^\circ)=\mathscr{F}_2(U\cap V_2^\circ)$. But, since $\mathscr{F}_2$ is locally free, and since $\operatorname{codim}(V_2-V_2^\circ)>1$ and V_2 is normal, we know $\mathscr{F}_2(U\cap V_2^\circ)=\Gamma(U\cap V_2^\circ, \mathscr{F}_2)=\Gamma(U, \mathscr{F}_2)=\mathscr{F}_2(U)$. So, an injection $\mathscr{F}_1(f^{-1}U)\hookrightarrow\mathscr{F}_2(U)$ is obtained. In combination with $\mathscr{F}_2(U)\hookrightarrow\mathscr{F}_1(f^{-1}U)$ of part (a), this gives an isomorphism $\mathscr{F}_2(U)\cong\mathscr{F}_1(f^{-1}U)$ as claimed. Q. E. D.

Let Y be an algebraic variety defined over k, not necessarily smooth or complete over k. By virtue of Nagata's Completion Theorem [10] one can embed Y into a complete variety $\bar{Y}$ over k as an open subset. *Assume* that Hironaka's Main Theorems about resolution of singularities [11] hold for $\bar{Y}$ and $\bar{Y}-Y$. (This is always the case if $\operatorname{char}(k)=0$ or if $\dim Y\leqq 2$; also if $\dim Y=3$ and $\operatorname{char}(k)\neq 2$, 3 or 5 by [0].) Then one can construct a complete nonsingular triple (V, D, X) together with a proper birational morphism $X\to Y$. When that is so, we shall call Y a *resoluble* variety and (V, D, X) *a smooth completion of* Y (cf. Iitaka [3, 4]). For such a variety Y and for any given finite sequence $M=(m_1, \cdots, m_N)$ of nonnegative integers m_i, *the logarithmic M-invariant of* Y is defined and denoted by

$$P_M(Y):=h^\circ(\Omega^1(\log D)^{\otimes m(1)}\underset{\mathcal{O}_V}{\otimes}\cdots\otimes\Omega^N(\log D)^{\otimes m(N)}) \tag{1}$$

where h° (locally free sheaf $\mathcal{S}$) stands for the k-vector space dimension of the global sections to $\mathcal{S}$. The results in 1.1 above guarantee the legitimacy of this definition. To wit:

1.2. COROLLARY. *The logarithmic M-invariant of a resoluble algebraic variety is independent of the choice of a smooth completion, and is stable under proper birational morphisms among such varieties.*

The proof of this corollary is essentially due to David Wright, who astutely remarked to the author that in Theorem 1.1 the varieties V_1, V_2 need not be supposed complete over k. His observation enabled us to dispense with the "common domination" argument used in the characteristic zero case [3].

PROOF OF 1.2 (D. Wright). Suppose given a resoluble variety Y with smooth completions (V_1, D_1, X_1), (V_2, D_2, X_2) and proper birational morphisms $g_1 : X_1 \to Y$, $g_2 : X_2 \to Y$. Then, there arises a unique birational map $\phi : V_1 \cdots \to V_2$ whose restriction to X_1 equals $g_2^{-1} \circ g_1$ (considered as a birational map). Let V_1° be the maximal open subset on which ϕ is regular; then each irreducible component of $V_1 - V_1^\circ$ has codimension >1. Put $X_1^\circ := X_1 \cap V_1^\circ$ and $D_1^\circ := V_1^\circ - X_1^\circ$. It is immediately seen that $(V_1^\circ, D_1^\circ, X_1^\circ)$ is a nonsingular triple and $f := \phi|_{V_1^\circ} : V_1^\circ \to V_2$ defines a birational morphism of nonsingular triples $(V_1^\circ, D_1^\circ, X_1^\circ) \to (V_2, D_2, X_2)$. Now, for $i=1, 2$ and $M=(m_1, \cdots, m_N)$ given, let $\mathcal{S}_i$ be the sheaf of differential M-forms on V_i logarithmic along D_i such as on the right-hand side of (1), and let $\mathcal{S}_1^\circ$ be the restriction of $\mathcal{S}_1$ on V_1°. Then, $\mathcal{S}_1^\circ$ is the sheaf of differential M-forms on V_1° logarithmic along D_1°, and one may apply 1.1 (a) to the morphism f and $\mathcal{S}_1^\circ$, $\mathcal{S}_2$ so as to conclude $h^\circ(\mathcal{S}_2) \leqq h^\circ(\mathcal{S}_1^\circ)$. But then, since $\operatorname{codim}(V_1 - V_1^\circ) > 1$ and $\mathcal{S}_1$, $\mathcal{S}_1^\circ$ are locally free, $h^\circ(\mathcal{S}_1^\circ) = h^\circ(\mathcal{S}_1)$ exactly as in the proof of 1.1 (b). It follows that $h^\circ(\mathcal{S}_2) \leqq h^\circ(\mathcal{S}_1)$. The preceding reasoning remains valid when (V_1, D_1, X_1) and (V_2, D_2, X_2) are interchanged. So, $h^\circ(\mathcal{S}_1) \leqq h^\circ(\mathcal{S}_2)$, and hence $h^\circ(\mathcal{S}_1) = h^\circ(\mathcal{S}_2)$. We can thus define this last number to be $P_M(Y)$.

As for the second half of the assertion, it is reduced to what was proved just now. Namely, if $f : Y_1 \to Y_2$ is a proper birational morphism and (V_1, D_1, X_1), (V_2, D_2, X_2) are smooth completions of Y_1, Y_2, respectively, then (V_1, D_1, X_1) may be considered a smooth completion of Y_2 as well through the composed proper birational morphism $X_1 \to Y_1 \to Y_2$, so that all of the forgoing arguments apply here around $Y = Y_2$. Q. E. D.

Of particular use for us among these invariants are the *logarithmic plurigenera* $P_m(Y)$ *of* Y defined for all natural numbers m as follows:

$$P_m(Y) := h^\circ(\Omega^n(\log D)^{\otimes m}) = P_{0,\cdots,0,m}(Y),$$

where $n = \dim Y$. The *logarithmic Kodaira dimension* $\kappa(Y)$ *of* Y is defined as being equal to d $(=-\infty, 0, 1, 2, \cdots)$ if $P_m(Y)$ has the same order as m^d as an arithmetic function of m.

1.3. COROLLARY. *For the affine space* $\boldsymbol{A}^N$ *of arbitrary dimension* N *and for any* $M=(m_1, \cdots, m_N)$, $P_M(\boldsymbol{A}^N)=0$. *In particular,* $\kappa(\boldsymbol{A}^N) = -\infty$.

A proof easily obtains by calculations relative to the standard embedding $\boldsymbol{A}^N \hookrightarrow \boldsymbol{P}^N =$ the n-dimensional projective space, these being justified by 1.2.

An invariant $I(-)$ attached to certain varieties over k is said to be *a cancellation invariant*[2] if $X\times_k Z\cong Y\times_k Z$ implies $I(X)=I(Y)$ provided $I(X)$, $I(Y)$ and $I(Z)$ are all defined. We shall show that the logarithmic M-invariants are cancellation invariants of resoluble varieties.

Let us first recall these facts: Let V be a smooth algebraic variety over k, and $\mathcal{F}$, $\mathcal{G}$ locally free sheaves of $\mathcal{O}_V$-modules of finite rank. Then, for any $q>0$,

$$\Lambda^q(\mathcal{F}\oplus\mathcal{G})\cong\sum_{i=0}^{q}(\Lambda^{q-i}\mathcal{F})\underset{\mathcal{O}_V}{\otimes}(\Lambda^i\mathcal{G}) \tag{2}$$

where Λ^r denotes the r-th exterior power and $\sum$ a direct sum of $\mathcal{O}_V$-modules. As for symmetric tensor products,

$$S(\mathcal{F}\oplus\mathcal{G})\cong S(\mathcal{F})\underset{\mathcal{O}_V}{\otimes}S(\mathcal{G}) \tag{3}$$

holds, where S denotes the symmetric $\mathcal{O}_V$-algebra operation. (For proofs, one might look up, for instance, *Algèbre Multilinéaire* by N. Bourbaki and conduct suitable sheafifications.)

1.4. LEMMA. *Let (V, D, X) and (W, E, Y) be non-singular triples, and put $H:=V\times E+D\times W$, a divisor on $V\times_k W$. Then,*

$$\Omega^q(\log H)^{\otimes m}\cong\sum_{\alpha}\Big[\bigotimes_{i=0}^{q}(\Omega^i(\log D)^{\otimes\alpha(i)}\underset{k}{\otimes}\Omega^{q-i}(\log E)^{\otimes\alpha(i)})\Big]$$

where the direct sum $\sum$ of $(\mathcal{O}_V\underset{k}{\otimes}\mathcal{O}_W)$-modules is taken over all $(q+1)$-tuples $\alpha=(\alpha_0,\cdots,\alpha_q)$ of non-negative integers α_i such that $|\alpha|=\alpha_0+\cdots+\alpha_q=m$.

PROOF. It is apparent that $(V\times_k W, H, X\times_k Y)$ is a nonsingular triple. So, we are allowed to compute as follows: First,

$$\Omega^1(\log H)\cong(\mathcal{O}_V\underset{k}{\otimes}\Omega^1(\log E))\oplus(\Omega^1(\log D)\underset{k}{\otimes}\mathcal{O}_W)$$

is clearly true. By applying (2) to this, we get

$$\begin{aligned}\Omega^q(\log H)&\cong\sum_{i=0}^{q}[(\mathcal{O}_V\underset{k}{\otimes}\Omega^{q-i}(\log E))\otimes(\Omega^i(\log D)\underset{k}{\otimes}\mathcal{O}_W)]\\ &\cong\sum_{i=0}^{q}[\Omega^i(\log D)\underset{k}{\otimes}\Omega^{q-i}(\log E)]\,.\end{aligned} \tag{4}$$

Next take the m-th symmetric tensor power of either end of (4), taking into account the formula (3). Then, the formula for $\Omega^q(\log H)^{\otimes m}$ as asserted above readily follows. Q. E. D.

1.5. COROLLARY. *Let (V, D, X), (W, E, Y) be smooth completions of resoluble*

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[2] This notion is due to Takao Fujita.

varieties X', Y', respectively. The notations being otherwise the same as in 1.4, *the following equalities hold:*

(a) $h^\circ(\Omega^q(\log H))=\sum_{i=0}^{q} h^\circ(\Omega^i(\log D))\cdot h^\circ(\Omega^{q-i}(\log E))$.

(b) $h^\circ(\Omega^q(\log H)^{\otimes m})=\sum_{|\alpha|=m} P_{\alpha(1),\cdots,\alpha(q)}(X')\cdot P_{\alpha(q-1),\cdots,\alpha(0)}(Y')$.

(c) $P_m(X'\times_k Y')=P_m(X')\cdot P_m(Y')$.

Proof is easy and omitted.

1.6. THEOREM. *Let X_1, X_2 and Y be resoluble algebraic varieties defined over k, and assume that $X_1\times_k Y\cong X_2\times_k Y$. Let $N:=\dim X_1=\dim X_2$. Then, for every $M=(m_1, \cdots, m_N)$, $P_M(X_1)=P_M(X_2)$ holds.*

PROOF. Let (V_i, D_i, X'_i) and (W, E, Y') be smooth completions of X_i $(i=1, 2)$ and Y, respectively. Then, for $i=1, 2$, $(V_i\times_k W, V_i\times E+D_i\times W, X'_i\times_k Y')$ is a smooth completion of $X_i\times_k Y$, and these non-singular triples share all logarithmic M-invariants by virtue of 1.2. We may therefore suppose from the beginning that X_1, X_2 and Y are smooth over k, so that $X'_i=X_i$ for $i=1, 2$ and $Y'=Y$.

Now, for finite sequences $M=(m_1, \cdots, m_N)$, $M'=(m'_1, \cdots, m'_N)$, $\cdots$ of non-negative integers m_i, m'_j, $\cdots$, we introduce a lexicographic order. Namely, $M'<M$, *by definition*, if and only if $m'_i=m_i$ for all $i >$ some t, and $m'_t<m_t$. We also define $|M|:=m_1+\cdots+m_N$ for all M. Putting $H_i:=V_i\times E+D_i\times W$ for $i=1, 2$, we now compute $\Omega^M(\log H_i):=\Omega^1(\log H_i)^{\otimes m(1)}\otimes\cdots\otimes\Omega^N(\log H_i)^{\otimes m(N)}$ by means of 1.4. It is then not hard to verify that

$$\Omega^M(\log H_i)\cong\left(\Omega^M(\log D_i)\underset{k}{\otimes}\mathcal{O}_W\right)\oplus\left(\Sigma\,\Omega^{M'}(\log D_i)\underset{k}{\otimes}\Omega^{M''}(\log E)\right),$$

where (i) the direct sum Σ is over those M' which satisfy $M'<M$ and $|M'|\leqq|M|$, and (ii) M'' are sequences well-determined by those M' (but the actual make-up of M'' is immaterial). As a consequence,

$$P_M(X_i\times_k Y)=P_M(X_i)+\sum_{M'}(P_{M'}(X_i)\cdot P_{M''}(Y)) \tag{5}$$

for $i=1, 2$. Therefore, when M is given, one can perform an induction on the finite, totally-ordered set $\{M': M'<M \text{ and } |M'|\leqq|M|\}$ so as to deduce $P_M(X_1)=P_M(X_2)$, given that the left-hand side of (5) is equal for $i=1$ and $i=2$ and that $P_{M'}(X_1)=P_{M'}(X_2)$ for all M' with $M'<M$, $|M'|\leqq|M|$. Q. E. D.

2. Miyanishi's characterization of the affine plane.

The ground field k is again algebraically closed of arbitrary characteristic. For any ring R, the multiplicative group of the units of R will be denoted by R^*. Our aim is to prove

2.1. THEOREM. *Let* $X=\mathrm{Spec}\,A$ *be an affine surface smooth over* k. *Suppose that* A *is a factorial domain* (viz. *a* UFD), *that* $A^*=k^*$ *and that* X *contains a cylinderlike open set* $U\cong C\times_k \boldsymbol{A}^1$. *Then* X *is isomorphic to the affine plane* $\boldsymbol{A}^2$ *over* k.

Originally, this was proved by Miyanishi in [6] in 1975 with a somewhat different formulation and with the additional assumptions of X being rational and the characteristic $\mathrm{char}(k)$ of k being zero. We offer a simple, algebro-geometric proof of this theorem in the present form, making use of a very recent result of Miyanishi [8]. Meanwhile, Richard G. Swan has obtained a purely algebraic proof of Theorem 2.1 without even assuming the smoothness of X. (His proof is unpublished.) We have borrowed Swan's argument to show the curve C above to be rational. But, otherwise, our proof is shorter and different in essential points than Swan's.

PROOF OF 2.1. First note that (X is normal and affine)$\Rightarrow$(U is normal and incomplete over k)$\Rightarrow$(C is k-smooth and affine). Likewise, the assumption of $\mathrm{Pic}(X)=0$ implies $\mathrm{Pic}(U)=\mathrm{Pic}(C)=0$. Then C must be a rational curve. Indeed: let $\tilde{C}$ be a complete normal model of the function field $k(C)$ of C. Then C is contained in $\tilde{C}$ as an open subscheme, and there arises a natural surjective homomorphism $\mathrm{Pic}(\tilde{C})\to\mathrm{Pic}(C)$ with a finitely-generated kernel. So, if $\tilde{C}$ were not rational, $\mathrm{Pic}(C)$ would not be zero. It follows that C, $\tilde{C}$, U, and X are all rational varieties. Now, since U is affine, its complement $X-U$ is pure of codimension one and supports a divisor on X. But X is factorial, so $X-U=\mathrm{Supp}(s)$ for some principal divisor (s) with $s\in A$. Therefore, putting $C=\mathrm{Spec}\,B$ with a k-algebra B, one can write

$$A_s=A[s^{-1}]=B[t] \text{ (within the field of quotients of } A)$$

where t is transcendental over B. Since s is a unit in A_s, it is one in $B[t]$ as well, so that $s\in B$. Thus $s\in A\cap B$. This permits us to substitute $s^n t$ for t freely without altering the above relation $A_s=B[t]$, n being an integer. Hence we may, and shall, suppose $t\in A$ from the beginning.

Consider now the situation

$$\begin{tikzcd}
U\cong C\times_k \boldsymbol{A}^1 \arrow[r, hook] \arrow[d, "\pi"'] & X=\mathrm{Spec}\,A \arrow[d, dashed, "\tilde{\pi}"] \\
C=\mathrm{Spec}\,B \arrow[r, hook] & \tilde{C}\cong\boldsymbol{P}^1=\text{the projective line}
\end{tikzcd}$$

where π is the projection map to C and $\tilde{\pi}$ is the rational map determined by π. We claim that $\tilde{\pi}$ is regular on X, i.e., $\tilde{\pi}$ is a morphism. Proof: $s\in B$ may be viewed as a morphism from $C=\operatorname{Spec} B$ to $\boldsymbol{A}^1$, extending to a rational map $\tilde{s}: \tilde{C} \dashrightarrow \boldsymbol{A}^1$. But $s \in A \cap k(\tilde{C})$ also, whence $s=\tilde{s}\circ\tilde{\pi}: X \to \boldsymbol{A}^1$ is regular on X. Therefore, $\tilde{\pi}$ could not possibly have a point of indeterminacy anywhere on X. The claim is now good. Furthermore, the image $\tilde{\pi}(X)$ of $\tilde{\pi}$ cannot be all of $\tilde{C}$ because, if $\tilde{\pi}(X)=\tilde{C}$, $\tilde{s}$ would have a pole on $\tilde{C} \cong \boldsymbol{P}^1$ and s could not then be everywhere regular on X. It follows that $C' := \tilde{\pi}(X)$ is an affine normal curve and one can write $C'=\operatorname{Spec} D$ with $D \subseteq A$. But D is a rational, finitely-generated k-domain of dimension one and $D^*=k^*$. Consequently,

$$D=k[u] \text{ with } u \in A \text{ transcendental over } k.$$

We have thus extended the trivial $\boldsymbol{A}^1$-bundle $\pi: U=C\times_k \boldsymbol{A}^1 \to C$ to a fibration $\tilde{\pi}: X \to C' \cong \boldsymbol{A}^1$ whose general fibers are k-isomorphic to $\boldsymbol{A}^1$.

At this point, we invoke a recent result of Miyanishi which describes singular fibers of this type of fibration [8; Lemma 1.1].[3] According to Miyanishi's lemma, a singular fiber of $\tilde{\pi}$ with e distinct components contributes $e-1$ toward the rank of $\operatorname{Pic}(X)\otimes\boldsymbol{Q}$, $\boldsymbol{Q}$=the rational numbers. As $\operatorname{Pic}(X)=0$ in our case, the only possible singular fibers are of the type $\tilde{\pi}^*(P)=dL$ where $P\in C'$, $d>1$ and $L\cong\boldsymbol{A}^1$ contained in X. Take one such fiber. One can suppose without loss of generality that P is given by $(u=0)$ on $C'=\operatorname{Spec} D=\operatorname{Spec}(k[u])$. Then, since X is factorial, $L=(z)$=a principal divisor on X with $z\in A$ and $(u)=d(z)=(z^d)$ as divisors on X. So, $u=vz^d$ for some $v\in A^*=k^*$. We can then assume $v=1$, and hence $u=z^d$. That being so, let $c\in k$ be arbitrarily taken and consider the fiber above $(u=c)$. It is given on X by the principal divisor $(u-c)=(z^d-c)=(z^d-b^d)$ where $b\in k$ is a d-th root of c. Then, in case $p:=\operatorname{char}(k)>0$ and d is divisible by p, the general fibers of $\tilde{\pi}$ would be non-reduced; and in all other cases the general fibers would have several (>1) components. Since the general fibers of $\tilde{\pi}$ are known to be $\boldsymbol{A}^1$, this is contradictory in either case, and no singular fibers can therefore occur in $\tilde{\pi}: X\to C'\cong\boldsymbol{A}^1$. Thus, X is an $\boldsymbol{A}^1$-bundle over $\boldsymbol{A}^1$, which has to be trivial. We conclude that $X\cong\boldsymbol{A}^1\times_k\boldsymbol{A}^1\cong\boldsymbol{A}^2$. Q. E. D.

3. Proof of the main result.

We are now in a position to prove the following " Strong Cancellation Theorem " over an algebraically closed ground field k of arbitrary characteristic. The theorem was first proved by Fujita [1; Cor. (3.3)] when k=the complex number field.

3.1. THEOREM. *Let $X=\operatorname{Spec} A$ be an affine algebraic variety and Y a resoluble algebraic variety, both defined over k. Assume that there exists a k-isomorphism*

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[3] This is the only place in the proof where the k-smoothness of X is used.

$X \times_k Y \cong A^2 \times_k Y$, *where* A^2 *denotes the affine plane over* k. *Then*, $X \cong A^2$.

Our proof of this theorem consists in showing that X satisfies all of the conditions of Theorem 2.1. It will be given in the rest of this section as 3.2 through 3.7.

3.2. *X is a smooth surface.* For any variety let us denote by $\dim(-)$ and $\delta(-)$, respectively, its dimension and the number of irreducible components of its singular locus. Then, as Fujita has remarked, these are obvious cancellation invariants because $\dim(Z \times_k W) = \dim(Z) + \dim(W)$ and $\delta(Z \times_k W) = \delta(Z) + \delta(W)$. In our situation of 3.1, this implies $\dim(X) = \dim(A^2) = 2$ and $\delta(X) = \delta(A^2) = 0$.

3.3. *X is rational.* For any resoluble variety let $q(-)$ stand for the dimension of the Albanese variety of its smooth completion. Again, as is well-known, $q(Z \times_k W) = q(Z) + q(W)$ for any resoluble varieties Z and W, so that $q(-)$ is a cancellation invariant. In our situation of 3.1 this means $q(X) = q(A^2) = 0$. On the other hand, as shown in 1.6, the logarithmic invariants are cancellation invariants. So, X and A^2 share all logarithmic invariants. Thus, by 1.3,

$$P_M(X) = 0 \text{ for any } M = (m_1, m_2)\,;\text{ and } \kappa(X) = -\infty\,. \qquad (6)$$

Therefore, if (S, D, X) is any smooth completion of X and K_S is a canonical divisor on S, we have

$$|m(K_S + D)| = \varnothing \text{ for all } m > 0\,. \qquad (7)$$

In particular, the geometric genus p_g and the bigenus P_2 of S both vanish. Then, the arithmetic genus $p_a = p_g - \dim H^1(S, \mathcal{O}_S) = p_g - q = 0 - 0 = 0$. (Note that $p_g = 0$ assures the equality $q = \dim H^1(S, \mathcal{O}_S)$.) Now Castelnuovo's Criterion $P_2 = p_a = 0$ [13] applies to the smooth complete surface S and tells us that S and, hence, X are rational varieties.

3.4. *X contains a cylinderlike open set.* It is in fact the main result of the two papers combined, Miyanishi-Sugie [7] and Fujita [1], that a smooth affine rational surface, whose logarithmic Kodaira dimension equals $-\infty$, contains a cylinderlike open set. In those papers the ground field k is assumed to be of characteristic zero. However, once a smooth complete rational surface S and a reduced effective divisor D of simple normal crossing type together satisfy the condition (7) and $S - D$ is affine, all the arguments in [7] and [1] proceed in the framework of the well-established theory of rational surfaces, in which none of the pathologies of nonzero characteristics occur. In particular, principal machineries used by the three authors, such as the Riemann-Roch Theorem and the classification of relatively minimal rational surfaces, are completely characteristic-blind.

After specific checks of all relevant arguments in [7] and [1] we are assured of the existence of a cylinderlike open set contained in $X=S-D$ in all characteristics.

3.5. *A has no nontrivial units.* Let A^1_* denote the punctured affine line: $A^1_*:=\mathrm{Spec}\ k[T, T^{-1}]=A^1-\{0\}$. Then, the nontrivial units A^*-k^* are in a natural one-to-one correspondence with the dominant morphisms $X\to A^1_*$. If we had one such morphism which were also generically separable, then by 1.1 (a)

$$P_{1,0}(X)=h^\circ(\Omega^1(\log D))\geqq h^\circ(\Omega^1(\log(A^1-A^1_*)))=1$$

would follow, whereas $P_{1,0}(X)=0$ as seen in (6), whence a contradiction. So, if $\mathrm{char}(k)=0$, there is nothing more to prove. Suppose, therefore, $\mathrm{char}(k)=p>0$, and let $f\in A^*-k^*$. Call K the field of quotients of A. Then the *perfect* field $L:=\bigcap_{i=1}^{i=\infty} K^{p^i}$ sits between k and its finitely generated extension K, so that L, too, is finitely generated over k. Such L can be perfect only if algebraic over k, but k is algebraically closed. Hence $L=k$. Now take the p^s-th root u of f within K, $s\geqq 0$, providing that the p^{s+1}-th root of f does not exist inside K. Then, since $u\in K$ is integral over A which is normal, $u\in A$ and $u^{-1}\in A$ also. We now have a unit $u\in A^*-k^*$ with $u\notin K^p$. If $t_1, \cdots, t_r$ form a separating transcendence base for K/k, one can write $du=g_1dt_1+\cdots+g_rdt_r$ with, say, $g_1\neq 0$ because $du\neq 0$. Then, by exchanging dt_1 with du_1, we get a K-base $\{du, dt_2, \cdots, dt_r\}$ of the space of differential 1-forms $\Omega^1(K/k)$. Consequently, $\{u, t_2, \cdots, t_r\}$ is a separating transcendence base for K/k, and K is separably generated over $k(u)$. This means that the dominant morphism $X\to A^1_*$ corresponding to u is generically separable, and we are done.

3.6. *A is a factorial domain.* Toward proving this, we first make some general remarks. Let Z be an algebraic variety smooth and complete over k, and let $W\subseteq Z$ be an open subvariety. Then, there is a natural surjective homomorphism $\mathrm{Pic}(Z)\to\mathrm{Pic}(W)$. Let P_Z be the group of k-rational points of the Picard variety of Z, and let NS_Z be the Néron-Severi group of Z. As is well-known, P_Z is divisible and NS_Z is finitely-generated. So, the natural exact sequence $0\to P_Z\to\mathrm{Pic}(Z)\to NS_Z\to 0$ splits, and $\mathrm{Pic}(Z)\cong P_Z\oplus NS_Z$. It follows that $\mathrm{Pic}(W)$, too, is a direct sum of its subgroup of divisible elements and a finitely generated subgroup. One other remark: For any smooth algebraic varieties W_1 and W_2, there is a natural injective homomorphism $\mathrm{Pic}(W_1)\oplus\mathrm{Pic}(W_2)\to\mathrm{Pic}(W_1\times_k W_2)$ which splits (so $\mathrm{Pic}(W_1)\oplus\mathrm{Pic}(W_2)$ is a direct summand of $\mathrm{Pic}(W_1\times_k W_2)$). PROOF: Fix closed points $P_1\in W_1$, $P_2\in W_2$ and make identifications $i_1: W_1=W_1\times P_2\hookrightarrow W_1\times W_2$ and $i_2: W_2=P_1\times W_2\hookrightarrow W_1\times W_2$ *via* obvious maps. Then, in relation to the projection maps p_1, p_2: $W_1\times W_2\to W_1, W_2$, the equalities $p_ti_t=$(the identity map on W_t) hold for $t=1, 2$. Now, for any pair of line bundles L_1, L_2 on W_1, W_2, respectively, $(L_1, L_2)\mapsto p_1^*L_1+p_2^*L_2\in\mathrm{Pic}(W_1\times W_2)$ gives an injection, and $M\mapsto(i_1^*M, i_2^*M)\in\mathrm{Pic}(W_1)\oplus\mathrm{Pic}(W_2)$ provides a retraction, because $(i_1^*p_1^*L_1+i_1^*p_2^*L_2, i_2^*p_1^*L_1+i_2^*p_2^*L_2)=((p_1i_1)^*L_1, (p_2i_2)^*L_2)=(L_1, L_2)$. Done.

We now return to our situation $X\times Y\cong \boldsymbol{A}^2\times Y$. Let N be the singular locus of Y. Then, the given isomorphism induces isomorphisms $X\times N\cong \boldsymbol{A}^2\times N$ and $X\times(Y-N)\cong \boldsymbol{A}^2\times(Y-N)$. So, by replacing Y by $Y-N$, we shall suppose that the original Y is already non-singular. We may no longer consider Y resoluble, but at least Y is an open subvariety of a complete smooth variety. We apply the two preceding remarks: write $\mathrm{Pic}(Y)=F\oplus G$ with the divisible part $G\subseteq\mathrm{Pic}(Y)$ and a finitely-generated subgroup $F\subseteq\mathrm{Pic}(Y)$. Since $\mathrm{Pic}(X\times Y)\cong\mathrm{Pic}(\boldsymbol{A}^2\times Y)\cong\mathrm{Pic}(Y)$, we see that $\mathrm{Pic}(X)\oplus F\oplus G$ is a direct summand of $F\oplus G$. But, $\mathrm{Pic}(X)$ is finitely generated because the rational variety S has a trivial Picard variety (see 3.3). Then, after factoring out the divisible parts of both sides one must conclude that $\mathrm{Pic}(X)\oplus F$ is a direct summand of F. This could occur only if $\mathrm{Pic}(X)=\{0\}$.

3.7. *Conclusion of the proof.* We have only to apply the results of 3.2, 3.4, 3.5 and 3.6 to Miyanishi's characterization theorem 2.1 of the affine plane. Then, $X\cong \boldsymbol{A}^2$ follows. Q. E. D.

3.8. COROLLARY. *Let X be an algebraic variety over k, and assume that $X\times_k \boldsymbol{A}^n\cong \boldsymbol{A}^{n+2}$ for some integer $n>0$. Then, $X\cong \boldsymbol{A}^2$.*

This obvious corollary to Theorem 3.1 is the original cancellation theorem of Fujita-Miyanishi-Sugie (cf. [7], [1]). Notice that X need not be assumed affine because that follows from the assumption by way of Künneth formula.

4. Comments and discussions.

4.1. The key result of Miyanishi-Sugie [7] and Fujita [1] is the theorem that *a smooth affine rational surface X with $\kappa(X)=-\infty$ contains a cylinderlike open set.* As observed in 3.4 above, the ground field here may have an arbitrary characteristic. Peter Russell [12] has strengthened this theorem while considerably simplifying its proof. He has shown that the assumption of X being "affine" may be replaced by "connected at infinity" without affecting the conclusion of the theorem. Here, the latter phrase means there exists a smooth completion (S, D, X) of X such that D is connected. Russell's result seems best possible in this direction, as Sugie has produced a counter-example to the theorem without the assumption of connectivity at infinity.

4.2. For non-rational surfaces, the existence theorem for cylinderlike open sets ("affine-ruled" in [12]) is a bit stronger: *If S is a non-rational algebraic surface smooth and complete over k, and if D is a reduced effective divisor on S such that $|m(D+K_S)|=\emptyset$ for all $m>0$ (K_S being a canonical divisor on S), then $S-D$ contains a cylinderlike open set.* This result is due to Miyanishi and Sugie [7; Theorems 2.1 and 2.2]. These authors again assume char$(k)=0$. But this is unnecessary because the assumption implies $|mK_S|=\emptyset$, or plurigenera $P_m=0$, for all $m>0$. In particular $p_g=P_2=0$ (cf. 3.3 above). Hence, the dimension q of the Albanese variety is nonzero and equals $\dim_k H^1(S, \mathcal{O}_S)$, and the relevant arguments in [7], including Lemma 1.2 of Kodaira quoted and used there, are valid without further ado in any characteristic. As a consequence, *any nonrational smooth surface Y with $\kappa(Y)=-\infty$ contains a cylinderlike open set.* Proof is easy: Take a smooth completion (S, D, X) of Y with proper birational morphism $f: X\to Y$. Let $C\times A^1$ be a cylinderlike open set in X. Then, no curve on X with a positive intersection with the fibers $P\times A^1$ ($P\in C$) could be contracted by f to a point on Y, because such a curve would have a positive geometric genus. So, only some fibers of $C\times A^1$ are contracted by f, and $f(C\times A^1)$, minus some points, would be a cylinderlike open set in Y.

4.3. *The original cancellation theorem* (3.8) was obtained above as a corollary to the stronger Theorem 3.1. However, *just to prove* 3.8 *in arbitrary characteristics is straightforward and does not require any reshaping of Iitaka theory* as done in § 1 above. Actually, we can prove even a stronger assertion: $X\times Y\cong A^n$, $\dim X=2\Rightarrow X\cong A^2$. In fact, from the assumption follows swiftly that X is a smooth, affine, factorial surface devoid of nonconstant invertible regular functions. Therefore, in order to apply Theorem 2.1 to X, all that remains to be established is $\kappa(X)=-\infty$ in view of 3.4. (It is obvious that Iitaka theory extends immediately to curves and surfaces even if char$(k)>0$.) To prove $\kappa(X)=-\infty$, M. Pavaman Murthy has offered the following clever remark: There is a dominant, generically separable morphism $A^n\to X$ obtained by composing the given isomorphism with the projection $X\times Y\to X$. By repeatedly taking generic hyperplanes (using Noether's normalization) one can cut down on the dimensions until a dominant, generically separable morphism $A^2\to X$ is obtained. Then, by 1.1 (a) which requires no new proof in char$(k)>0$ cases, we get $0=P_m(A^2)\geqq P_m(X)$ for all m, whence $\kappa(X)=-\infty$. (Also, by Lüroth's Theorem [13], we see that X is rational.) In Russell's paper [12] an essentially same argument is given for the same purpose.

4.4. We do not know if the Picard group Pic$(-)$ is a cancellation invariant, even though in 3.6 above it behaved like one because of special circumstances.

4.5. After writing up all the foregoing paragraphs, we became aware of the following facts: Let X and Y be normal algebraic varieties over k. Then, the groups of invertible regular functions $\mathcal{U}(X):=\Gamma(X, \mathcal{O}_X^*)$ and $\mathcal{U}(Y):=\Gamma(Y, \mathcal{O}_Y^*)$ are each isomorphic to a direct sum of k^* and a free abelian group of finite rank (describable explicitly). Furthermore, a natural isomorphism

$$\mathcal{U}(X\times_k Y)/k^*\cong(\mathcal{U}(X)/k^*)\oplus(\mathcal{U}(Y)/k^*) \tag{8}$$

exists. The first assertion is in fact true without the normality assumption and is proven in § 1 of H. Sumihiro's paper, " Equivariant completion " (*J. Math. Kyoto Univ.* **14** (1974), 1-28). The second assertion (8) we actually knew before, though in a slightly different form. Its proof is easy and is similar to that of the second remark of 3.6 above. We are indebted to Spencer Bloch for suggesting it. From these it follows immediately that *the functor* $\mathcal{U}(-)/k^*$ *is a cancellation invariant on normal algebraic varieties.* This fact, then, would provide an alternate proof of 3.5. Finally, we acknowledge T. Fujita's recent letter to us which is partially responsible for the remarks in this paragraph.

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Department of Mathematics
Northern Illinois University
De Kalb, IL 60115
U. S. A.