Theory of logarithmic differential forms and logarithmic vector fields

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This note was planned originally as a part of a forthcoming paper [7]. Since the things we are treating here are relatively independent, we publish this part separately.

The main purpose of this note is to develop a general study of logarithmic forms and logarithmic vector fields along a divisor D in a complex manifold S for the application of the study of Gauss-Manin connection in [7]. (For the definition of them, see (1.1), (1.2) and (1.4).) In the case when D is a union of smooth subvarieties, which are normally crossing, there have already existed such notions. (For instance see P. Deligne [2] or M. Katz [3].) But our interest and treatment is a bit different from them, so that we are going to develop a general theory for a *singular and non-normal crossing divisor D.*

Such logarithmic forms appear naturally as coefficients of Gauss-Manin connections of certain family $X \to S$. Also the modules of logarithmic vector fields Der_S $(\log D)$ will be identified with the modules of a certain relative de Rham cohomology group $H^{\cdot}_{DR}(\Omega^{\cdot}_{X/S})$ of the family $X \to S$, so that the Gauss-Manin connection becomes an affine connection with a logarithmic pole (see [7]). These facts are the motivations for the study in this note.

On the other hand the general theory which we develop in this note seems to have some other applications for the study of reflexion groups (see [8]) and for the study of arrangements of hyperplanes in $\boldsymbol{R}^n$ (see H. Terao [9]). It might also be interesting to see that some phenomena indicate that logarithmic forms and logarithmic vector fields reflect some topological properties of the complement $S-D$ (see (1.11), (2.12), (2.13)).

In § 1, we give definitions for $\Omega^q_S(\log D)$ and Der_S $(\log D)$ and then give a simple criterion for those modules to be $\mathcal{O}_S$-free ((1.8)). In § 2, we study the residues of logarithmic forms. We give a criterion for the residues to be holomorphic on the normalization $\tilde{D}$ of D. In § 3 we introduce the concepts of logarithmic stratification of S and the logarithmic characteristic variety $L(\log D)$ in the cotangent bundle T^*S

of S. Then we give a simple correspondence between holonomic strata of S and holonomic components of $L(\log D)$ ((3.17)). In § 4, we study the discriminant D_W of a finite reflexion group W.

Some results of this note were already announced in [4] § 1. Since some proofs lacked there and the last theorem in § 1 was incorrect (see § 3), we repeat them with complete proofs.

§ 1. Logarithmic forms and logarithmic vector fields

(1.1) Let U be a domain of $\boldsymbol{C}^n$, and $D\subset U$ be a hypersurface of U defined by an equation $h(z)=0$, where h is holomorphic on U. Let ω be a meromorphic q-form on U, which may have poles only along D. Then the following four conditions for ω are equivalent:

i) $h\omega$ and $hd\omega$ are holomorphic on U.

ii) $h\omega$ and $dh\wedge\omega$ are holomorphic on U.

iii) There exists a holomorphic function $g(z)$ and a holomorphic (q-1)-form ξ and a holomorphic q-form η on U, such that:

  a) $\dim_{\boldsymbol{C}} D\cap\{z\in U:\ g(z)=0\}\leq n-2$,

  b) $g\omega=\dfrac{dh}{h}\wedge\xi+\eta$.

iv) There exists an $(n-2)$-dimensional analytic set $A\subset D$ such that the germ of ω at any point $p\in D-A$ belongs to $\dfrac{dh}{h}\wedge\Omega^{q-1}_{U,p}+\Omega^{q}_{U,p}$, where $\Omega^{q}_{U,p}$ denotes the module of germs of holomorphic q-forms on U at p.

PROOF. The equivalence of i) and ii) is evident from the formula, $hd\omega+dh\wedge\omega=d(h\omega)$.

ii)$\Rightarrow$iii). Let us present the form ω by $\dfrac{\sum_I a_I(z)dz_I}{h}$ where $I=(i_1,\cdots,i_q)$, $1\leq i_1,\cdots,i_q\leq n$ is a multi-index and $dz_I=dz_{i_1}\wedge\cdots\wedge dz_{i_q}\cdot a_I(z)$ is a holomorphic function on U and $\{a_I(z)\}$ is skew symmetric with respect to the index I.

Since $dh\wedge\sum_I a_I(z)dz_I$ is divisible by $h(z)$, there exists a skew-symmetric system $\{b_J(z)\}_J$ of holomorphic functions on U with a multi-index $J=(j_1,\cdots,j_{q+1})$ such that we have an equality

$$\sum_{\{j,i_1,\cdots,i_q\}=\{J\}}\operatorname{sgn}\begin{pmatrix} J \\ j,i_1,\cdots,i_q\end{pmatrix}\frac{\partial h}{\partial z_j}a_I(z)=h(z)b_J(z)$$

for any J. Here $\{J\}$ means the set of numbers $\{j_1,\cdots,j_{q+1}\}$ by forgetting the order of them. Then we calculate

$$\frac{\partial h}{\partial z_j}\omega=\frac{dh}{h}\wedge\sum_{I'}a_{jI'}dz_{I'}+\frac{1}{q!}\sum_{I}b_{jI}dz_I,$$

where I' is a multi-index $(i_1,\cdots,i_{q-1})$. For a suitable j, we put $g=\frac{\partial h}{\partial z_j}$, $\xi=\sum_{I'}a_{jI'}dz_{I'}$ and $\eta=(q!)^{-1}\sum_{I}b_{jI}dz_I$.

iii)⇒iv). Put $A=D\cap\{z\in U:\ g(z)=0\}$.

iv)⇒ii). Let ω be a q-form with the condition iv). Then clearly $h\omega$ and $dh\wedge\omega$ are holomorphic on $U-A$. Since $\mathrm{codim}_U(A)\geq 2$, by an extension theorem of Riemann, $h\omega$ and $dh\wedge\omega$ are holomorphic on U. q. e. d.

(1.2) Definition. A meromorphic q-form on U is called *a q-form with logarithmic pole along D* or *logarithmic q-form,* if it satisfies the equivalent conditions of (1.1).

Let S be an n-dimensional complex manifold and D be a hypersurface of D. Let $h_p=0$ be a reduced equation for D, locally at $p\in D$. A meromorphic q-form is logarithmic along D at p, if $h_p\omega$ and $h_p d\omega$ are holomorphic. We denote

$$\Omega^q_{S,p}(\log D):=\{\text{germ of logarithmic } q\text{-form at } p\},$$

$$\Omega^q_S(\log D):=\bigcup_{p\in S}\Omega^q_{S,p}(\log D).$$

(1.3) By using the equivalence of (1.1), we easily conclude,

i) $\Omega^q_S(\log D)$, $q=0,1,\cdots,n$, are coherent $\mathcal{O}_S$-modules,

ii) $\bigoplus_{q=0}^{n}\Omega^q_S(\log D)$ is an $\mathcal{O}_S$-exterior algebra,

iii) $\bigoplus_{q=0}^{n}\Omega^q_S(\log D)$ is closed under the exterior differentiation.

Note that, by definition, $\Omega^0_{S,p}(\log D)=\Omega^0_{S,p}$ and $\Omega^n_{S,p}(\log D)=\frac{1}{h_p}\Omega^n_{S,p}$.

Let $\omega_I,\cdots,\omega_n$ be n elements of $\Omega^1_{S,p}(\log D)$, then the edge product $\omega_1\wedge\cdots\wedge\omega_n$ has a local presentation $a(z)\frac{dz_0\wedge\cdots\wedge dz_n}{h_0}$ for a certain holomorphic function $a(z)$ ($\because$ ii)).

(1.4) Definition. Let S and $D\subset S$ be as (1.2). A holomorphic vector field δ on S is *logarithmic* if it satisfies the following equivalent conditions:

i) For any smooth point p of D, the tangent vector δ(p) of p is tangent to D,

ii) For any point p of D, the derivation δh_p of the local equation for D belongs to the ideal $(h_p)\mathcal{O}_{S,p}$.

We denote

$$\mathrm{Der}_{S,p}(\log D)=\{\delta:\ \text{germ of a holomorphic vector field on } S \text{ at } p \text{ such that } \delta(h_p)\in(h_p)\},$$

$$\mathrm{Der}_S(\log D) = \bigcup_{p\in S} \mathrm{Der}_{S,p}(\log D).$$

(1.5) By definition we get:

i) $\mathrm{Der}_S(\log D)$ is a coherent $\mathcal{O}_S$-submodule of Der_S, where Der_S is the sheaf of holomorphic vector fields on S,

ii) $\mathrm{Der}_S(\log D)$ is closed under the bracket product $[\cdot,\cdot]$,

iii) Let $\delta^1,\cdots,\delta^n$ be any n-elements of $\mathrm{Der}_{S,p}(\log D)$.

Put $\delta^1\wedge\cdots\wedge\delta^n = v(z)\dfrac{\partial}{\partial z_1}\wedge\cdots\wedge\dfrac{\partial}{\partial z_n}$.

Then $v(z)$ belongs to the ideal $(h_p)\mathcal{O}_{S,p}$. ($\because$ At any smooth point $p\in D$, $\delta^1(p),\cdots,\delta^n(p)$ are tangent to the $(n-1)$-dimensional space D, and therefore they are linearly dependent.)

(1.6) Let

$$\mathrm{Der}_{S,p}\times\Omega^q_{S,p}\ni(\delta,\ \omega)\longrightarrow\delta\cdot\omega\in\Omega^{q-1}_{S,p},\ q=1,\cdots,n,$$

and

$$\mathrm{Der}_{S,p}\times\Omega^q_{S,p}\ni(\delta,\ \omega)\longrightarrow L_\delta(\omega)\in\Omega^q_{S,p},\ q=1,\cdots,n,$$

be the notions for the inner product between forms and vector fields and for the Lie derivative.

LEMMA. i) *The above notions of the inner product and the Lie derivatives are extended to*:

$$\mathrm{Der}_{S,p}(\log D)\times\Omega^q_{S,p}(\log D)\ni(\delta,\ \omega)\longrightarrow\delta\cdot\omega\in\Omega^{q-1}_{S,p}(\log D),$$
$$\mathrm{Der}_{S,p}(\log D)\times\Omega^q_{S,p}(\log D)\ni(\delta,\ \omega)\longrightarrow L_\delta(\omega)\in\Omega^q_{S,p}(\log D).$$

ii) *By the inner product*

$$\mathrm{Der}_{S,p}(\log D)\times\Omega^1_{S,p}(\log D)\ni(\delta,\ \omega)\longrightarrow\delta\cdot\omega\in\mathcal{O}_{S,p},$$

each is the dual $\mathcal{O}_{S,p}$-module of the other.

PROOF. i) Take $\delta\in\mathrm{Der}_{S,p}(\log D)$ and $\omega\in\Omega^1_{S,p}(\log D)$. Let us present ω in the form of (1.1) iii): $g\omega=\dfrac{dh}{h}\wedge\xi+\eta$, where ξ and η are holomorphic forms and g is a non-zero divisor on D. Then

$$g(\delta\cdot\omega)=\delta\cdot g\omega=\delta\left(\frac{dh}{h}\right)\wedge\xi-\frac{dh}{h}\wedge\delta\cdot\xi+\delta\cdot\eta,$$

where $\delta\left(\dfrac{dh}{h}\right)=\dfrac{\delta h}{h}$, $\delta\cdot\xi$ and $\delta\cdot\eta$ are holomorphic. A priori $\delta\cdot\omega$ may have poles only along D, and $\delta\cdot\omega$ have the presentation as in (1.1) iii). Thus $\delta\cdot\omega$ is logarithmic. Since

$$L_{\delta}(\omega)=d(\delta\cdot\omega)-\delta\cdot d\omega,$$

$L_{\delta}(\omega)$ is also logarithmic.

ii) Let us denote by $\mathcal{F}^*$ the dual $\mathcal{O}_{S,p}$-module of an $\mathcal{O}_{S,p}$-module $\mathcal{F}$.

By taking the dual of the following sequence of inclusions

$$\Omega^1_{S,p}\subset\Omega^1_{S,p}(\log\ D)\subset\frac{1}{h}\Omega^1_{S,p},$$

we obtain:

$$\mathrm{Der}_{S,p}\supset(\Omega^1_{S,p}(\log\ D))^*\supset h\mathrm{Der}_{S,p}.$$

By the above inclusion, we regard an element δ of $\Omega^1_{S,p}(\log\ D)^*$ as a holomorphic vector field. Since $\delta\cdot(dh/h)=(\delta h)/h$ is holomorphic, $\delta h=h(\delta\cdot(dh/h))\in(h)$ and thus δ belongs to $\mathrm{Der}_{S,p}(\log\ D)$.

Conversely, by taking the dual of the inclusions

$$h\mathrm{Der}_{S,p}\subset\mathrm{Der}_{S,p}(\log D)\subset\mathrm{Der}_{S,p},$$

we obtain

$$\frac{1}{h}\Omega^1_{S,p}\supset(\mathrm{Der}_{S,p}(\log D))^*\supset\Omega^1_{S,p}.$$

Thus we regard an element ω of $(\mathrm{Der}_{S,p}(\log D))^*$ as a differential form of the type $\left(\sum_{i=1}^{n} a_i(z)dz_i\right)/h$. Since $\delta_{ij}=\dfrac{\partial h}{\partial z_i}\dfrac{\partial}{\partial z_j}-\dfrac{\partial h}{\partial z_j}\dfrac{\partial}{\partial z_i}$ belongs to $\mathrm{Der}_{S,p}(\log\ D)$,

$$\delta_{ij}\cdot\omega=\left(a_j\frac{\partial h}{\partial z_i}-a_i\frac{\partial h}{\partial z_j}\right)/h$$

must be holomorphic for $1\leq i,j\leq n$. Then

$$dh\wedge\omega=\left(\sum_{ij}\left(a_j\frac{\partial h}{\partial z_i}-a_i\frac{\partial h}{\partial z_j}\right)dz_i\wedge dz_j\right)/h$$

is holomorphic and $\omega\in\Omega^1_{S,p}(\log\ D)$. q. e. d.

(1.7) COROLLARY. *$\Omega^1_{S,p}(\log\ D)$ and $\mathrm{Der}_{S,p}(\log\ D)$ are reflexive $\mathcal{O}_{S,p}$-modules. Especially when $\dim_C S=2$, then $\Omega^1_S(\log\ D)$ and $\mathrm{Der}_S(\log\ D)$ are locally free $\mathcal{O}_S$-*modules.

(1.8) In general $\Omega^1_{S,p}(\log\ D)$ and $\mathrm{Der}_{S,p}(\log\ D)$ are not locally free module. We give a simple criterion for them to be $\mathcal{O}_S$-free.

THEOREM. i) *$\Omega^1_{S,p}(\log D)$ is $\mathcal{O}_{S,p}$-free if and only if $\overset{n}{\wedge}\Omega^1_{S,p}(\log D)=\Omega^n_{S,p}(\log D)$, i.e., if there exist n-elements $\omega_1,\cdots,\omega_n\in\Omega^1_{S,p}(\log D)$ such that*

$$\omega_1\wedge\cdots\wedge\omega_n=\text{unit}\frac{dz_1\wedge\cdots\wedge dz_n}{h}.$$

Then the set of forms $\{\omega_1,\cdots,\omega_n\}$ makes a system of $\mathcal{O}_{S,p}$-free basis for $\Omega^1_{S,p}(\log D)$. Moreover, we have

$$\Omega^q_{S,p}(\log D)=\sum_{i,1\cdots,i_q}\mathcal{O}_{S,p}\omega_{i_1}\wedge\cdots\wedge\omega_{i_q}$$
$$\textit{for } q=1,\cdots,n.$$

ii) *$\mathrm{Der}_{S,p}(\log D)$ is $\mathcal{O}_{S,p}$-free if and only if there exist n elements $\delta^1,\cdots,\delta^n\in\mathrm{Der}_{S,p}(\log D)$ with $\delta^i=\sum_{j=1}^n a^i_j(z)\frac{\partial}{\partial z_j}$, $i=1,\cdots,n$, such that the determinant $\det(a^i_j(z))_{i,j=1,\cdots,n}$ is a unit multiple of h_p. Then the vector fields $\delta_1,\cdots,\delta^n$ is a system of free basis for $\mathrm{Der}_{S,p}(\log D)$.*

PROOF. i) Suppose that $\Omega^1_{S,p}(\log D)$ is $\mathcal{O}_{S,p}$-free. Since $\Omega^1_S(\log D)$ is $\mathcal{O}_S$-coherent ((1.3) i)), there is a small neighborhood U of p in S, such that $\Omega^1_S(\log D)|_U$ is $\mathcal{O}_S$-free. Let $\{\omega_1,\cdots,\omega_n\}$ be a system of free basis on U. Now let us put $\omega_1\wedge\cdots\wedge\omega_n=a(z)\times\frac{dz_1\wedge\cdots\wedge dz_n}{h}$. Then by (1.3), $a(z)$ is a holomorphic function on U. Since $dz_1,\cdot,dz_n$ is a free basis of $\Omega^1_{S,p'}(\log D)$ for $p'\in S-D$, $\omega_1\wedge\cdots\wedge\omega_n$ is a unit multiple of $dz_1\wedge\cdots\wedge dz_n$ at $p'\in S-D$. Thus $a(z)$ does not vanish on $S-D$. Let $p'\in D$ be a smooth point of D. Suppose for instance $\frac{\partial h}{\partial z_j}(p')\neq 0$. Then by (1.1) iii), we may choose a system of free basis of $\Omega^1_{S,p'}(\log D)$ by $\left\{\frac{dh}{h}, dz_1,\cdots,\widehat{dz_j},\cdots,dz_n\right\}$. Thus $\omega_1\wedge\cdots\wedge\omega_n$ must be a unit multiple of $\frac{dh}{h}\wedge dz_1\wedge\cdots\wedge\widehat{dz_j}\wedge dz_n=\frac{\partial h}{\partial z_j}\frac{dz_1\wedge\cdots\wedge dz_n}{h}$, hence $a(z)$ does not vanish at a smooth point p' of D. Thus the zero-locus of $a(z)$ has codimension 2, which implies that $a(z)$ does not vanish on U.

Conversely suppose $\omega_1,\cdots,\omega_n\in\Omega^1_{S,p}(\log D)$ such that

$$\omega_1\wedge\cdots\wedge\omega_n=\frac{dz_1\wedge\cdots\wedge dz_n}{h}.$$

For any $\omega\in\Omega^q_{S,p}(\log D)$, let us put

$$\omega\wedge\omega_{i_1}\wedge\ldots\wedge\omega_{i_{n-q}}=a_{i_1}\cdots_{i_{n-q}}\frac{dz_1\wedge\cdots\wedge dz_n}{h},$$
$$\text{for}\quad 1\leq i_1,\cdots,i_{n-q}\leq n$$

where $a_{i_1\cdots i_{n-q}}$ are some holomorphic functions at p. Then the difference

$$\omega'=\omega-\sum \operatorname{sgn}\begin{pmatrix}1,\cdots,n \\ i_1,\cdots,i_{n-q}, j_1,\cdots,j_q\end{pmatrix} a_{i_1\cdots i_{n-q}}\omega_{j_1}\wedge\cdots\wedge\omega_{j_q}$$

satisfies:

$$\omega'\wedge\omega_{i_1}\wedge\cdots\wedge\omega_{i_{n-q}}=0 \text{ for } 1\leq i_1,\cdots,i_{n-q}\leq n.$$

Thus ω' is a meromorphic q-form which is zero on $S-D$, hence $\omega'\equiv 0$.

ii) Suppose that $\mathrm{Der}_{S,p}(\log D)$ is $\mathcal{O}_{S,p}$-free. Since $\mathrm{Der}_S(\log D)$ is coherent ((1.5) i)), there exists a small neighborhood U of p so that $\mathrm{Der}_S(\log D)|_U$ is a free $\mathcal{O}_S$-module. Let $\{\delta^1,\cdots,\delta^n\}$ be a system of free basis of it. Put $\delta^i=\sum_{j=1}^n a_j^i(z)\frac{\partial}{\partial z_j}$ and $\det(a_j^i(z))=$ $v\cdot h$, where v is a holomorphic function on U ((1.5) iii)). Since $\frac{\partial}{\partial z_1},\cdots,\frac{\partial}{\partial z_n}$ is a free base system for $p'\notin U-D$, v does not vanish on $U-D$. At a smooth point p' of D, we may choose a system of free basis $\left\{z_1\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right\}$, where $z_1=0$ is a local equation for D. Thus v does not vanish on a smooth point of D, hence v does not vanish anywhere on U. Conversely let $\delta^i=\sum_{i=1}^n a_j^i(z)\frac{\partial}{\partial z_j}$, $i=1,\cdots,n$, be logarithmic vector fields at p such that $\det(a_j^i)_{i,j=1,\cdots,n}=h$. For any $\delta=\sum_{j=1}^n b_j\frac{\partial}{\partial z_j}\in\mathrm{Der}_{S,p}(\log D)$, let us put (cf. (1.5) iii))

$$\hat{i}\begin{vmatrix} a_1^1\cdots a_n^1 \\ a_1^i\cdots a_n^i \\ a_1^n\cdots a_n^n \\ b_1\cdots b_n\end{vmatrix}=c_ih, \qquad i=1,\cdots,n.$$

Then one may check $\delta=\sum_{i=1}^n(-1)^{n-i}c_i\delta_i$. q. e. d.

(1.9) The following gives an another criterion.

LEMMA. *Let* $\delta^i=\sum_{i=1}^n a_j^i(z)\frac{\partial}{\partial z_j}$, $i=1,\cdots,n$, *be a system of holomorphic vector fields at* p *such that*

i) $[\delta^i, \delta^k]\in\sum_{k=1}^n\mathcal{O}_{S,p}\delta^k$ *for* $i,j=1,\cdots,n$,

ii) $\det(a_j^i)=h$ *defines a reduced hypersurface* D.

Then for $D=\{h(z)=0\}$, $\delta^1,\cdots,\delta^n$ *belong to* $\mathrm{Der}_{S,p}(\log D)$, *and hence* $\{\delta^1,\cdots,\delta^n\}$ *is a system of free basis of* $\mathrm{Der}_{S,p}(\log D)$.

PROOF. Let $(b_j^i(z))_{i,j=1,\cdots,n}$ be the adjoint matrix of $(a_j^i(z))_{i,j=1,\cdots,n}$. Put

$\omega_j=\frac{\sum_{i=1}^{n} b_j^i dz_i}{h}$, $j=1,\cdots,n$. Clearly $\delta^i\cdot\omega_j=\delta_{ij}$, $i,j=1,\cdots,n$. Since any $r\times r$ principal minor of $(b_j^i)_{i,j=1,\cdots,n}$ is divisible by h^{r-1}, $h\omega_{j_1}\wedge\cdots\wedge\omega_{j_r}$ is holomorphic for $1\leq j_1,\cdots,j_r\leq n$. The fact that the system $\{\delta^1,\cdots,\delta^n\}$ is involutive implies

$$d\omega_k=\sum_{1\leq i,j\leq n}\mathcal{O}_{S,p}\omega_i\wedge\omega_j,\qquad k=1,\cdots,n.$$

This implies especially that $hd\omega_k$ is holomorphic and therefore $\omega_k\in\Omega^1_{S,p}(\log D)$, $k=1,\cdots,n$. By the definition of ω_k, $k=1,\cdots,n$,

$$\omega_1\wedge\cdots\wedge\omega_n=\frac{\det(b_j^i)dz_1\wedge\cdots\wedge dz_n}{h^n}=\frac{dz_1\wedge\cdots\wedge dz_n}{h},$$

hence by (1. 8) i), $\{\omega_1,\cdots,\omega_n\}$ is a system of free basis of $\Omega^1_{S,p}(\log D)$. Then $\{\delta^1,\cdots,\delta^n\}$ is a dual system of free basis of $\mathrm{Der}_{S,p}(\log D)$. q. e. d.

(1.10) Suppose that $\mathrm{Der}_{S,p}(\log D)$ and $\Omega^1_{S,p}(\log D)$ are $\mathcal{O}_{S,p}$-free and let $\{\delta^1,\cdots,\delta^n\}$ and $\{\omega_1,\cdots,\omega_n\}$ be systems of dual basis of them. Then:

i) For a germ of holomorphic function f at p, the differential is given by

$$df=\sum_{i=1}^{n}(\delta^i f)\omega_i.$$

ii) For n germs $f_1,\cdots,f_n$ of holomorphic functions at p,

$$\frac{\partial(f_1,\cdots,f_n)}{\partial(z_1,\cdots,z_n)}=\det(\delta^i f_j)_{i,j=1,\cdots,n}\cdot\frac{\omega_1\wedge\cdots\wedge\omega_n}{dz_1\wedge\cdots\wedge dz_n}.$$

(1.11) EXAMPLE. Let σ_i, $i\in J$ be a finite system of real linear forms on a real Euclidean space E of dimension n, and let H_i, $i\in J$ be the associated system of real hyperplanes in E.

Let S be the symmetric algebra of E with the natural grading $\bigoplus_{d=0}^{\infty}S^d$. (i. e. S may be considered as an $\boldsymbol{R}$-algebra generated by the real dual space E^* of E).

Put $\Delta=\prod_{i\in J}\sigma_i\in S$, which is a polynomial defining a divisor $\bigcup_{i\in J}H_i$ in E. Put

$$\mathrm{Der}_E:=\{\delta\in\mathrm{Hom}_{\boldsymbol{R}}(S,S):\ \delta(PQ)=(\delta P)Q+P(\delta Q)\ \text{for}\ P,Q\in S\}$$

$$\mathrm{Der}_E(\log\Delta):=\{\delta\in\mathrm{Der}_E:\ \delta\cdot\Delta\in\Delta S\}.$$

Clearly Der_E and $\mathrm{Der}_E(\log\Delta)$ are finite S-modules of rank n.

An element $\delta\in\mathrm{Der}_E$ is called homogeneous of degree d if $\delta S^k\subset S^{k+d}$ for $k=0,1,$

$\cdots$.

THEOREM. i) $\mathrm{Der}_E(\log \Delta)$ *is an* S*-free module if and only if there exist homogeneous elements* $\delta^1, \cdots, \delta^n \in \mathrm{Der}_E(\log \Delta)$, *such that*

a) $\delta^1, \cdots, \delta^n$ *are* S*-linearly independent,*

b) $\sum_{i=1}^{n} (\deg \delta^i + 1) = \deg \Delta$.

ii) *In the above case, we have an inequality,*

$$\prod_{i=1}^{n} (\deg \delta^i + 2) \geq \sharp\{\textit{connected component of } E - \bigcup_{i \in J} H_i\}.$$

PROOF. Since i) is a simple corollary of theorem (1.8), we prove only ii).

We may assume that σ_i, $i \in J$ span the space E^*; otherwise we reduce the situation for a lower dimensional E.

Consider a derivation $\delta^1 = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$ and call it the Euler derivations. δ' is characterized by the property

$$\delta' P = kP \quad \text{for} \quad P \in S^k \quad k = 0, 1, 2, \cdots,$$

and therefore δ' does not depend on the coordinate by its definition. Note that the Euler derivation lies in $\mathrm{Der}_E(\log \Delta)$.

Let us choose and fix a positive definite symmetric two-form $I \in S^2$ on E. Since $\delta' I = 2I$, the vector field δ' and the sphere defined by $I = c = \text{const.} \neq 0$ is transversal. Thus one may decompose $\delta \in \mathrm{Der}_E$ into $\delta = \delta_1 + \delta_2$, where δ_1 is tangent to $I = c$ and δ_2 is parallel to δ^1. Then $\delta I = \delta_1 I + \delta_2 I = \delta_2 I$ and hence $\delta_2 = \frac{\delta I}{2I} \delta'$. Thus "the spherical component" of δ is given by

$$I\delta_1 = I\delta - \frac{\delta I}{2} \delta'.$$

The sphere $\{I = c > 0\}$ minus H_i, $i \in J$, is decomposed into a union of open cells $\bigcup_{l \in M} X_l$ of dimension $n-1$. Now for any $\delta \in \mathrm{Der}_E(\log \Delta)$, "the spherical component" $I\delta - \frac{\delta I}{2} \delta^1$ may be considered as a vector field on the sphere which preserves the cell decomposition.

Now let $\{\delta^1, \cdots, \delta^n\}$ be a system of homogeneous S-free basis of $\mathrm{Der}_E(\log \Delta)$ as in Theorem i). Without loss of generality, we may assume that δ^1 is the Euler derivation. Then "the spherical components" of $\delta^2, \cdots, \delta^n$,

$$\xi^i = I\delta^i - \frac{\delta^i I}{2} \delta^1, \qquad i = 2, \cdots, n, \tag{1}$$

span the tangent space of each cell X_l, $l \in M$.

LEMMA. *In each chamber of* $E - \bigcup_{i \in J} H_i$, *the surface defined by* $\Delta = c =$ *constant* $\neq 0$, *is concave.*

PROOF. Take a linear coordinate system $z_1, \cdots, z_n$ of E and present $\sigma_i = \sum_{j=1}^{n} a_i^j z_j$, $i \in J$ for some $a_i^j \in \boldsymbol{R}$. Then

$$\frac{\partial^2}{\partial z_j \partial z_k} \log \Delta = -\sum_{i \in J} \frac{a_i^j a_i^k}{\sigma_i^2} \qquad \text{for} \quad j, k = 1, \cdots, n,$$

and hence

$$\sum_{j,k} \xi_j \xi_k \frac{\partial^2}{\partial z_j \partial z_k} \log \Delta = -\sum_{i \in J} \frac{\left(\sum_{j=1}^{n} a_i^j \xi_j\right)^2}{\sigma_i^2} < 0 \quad \text{for } \xi \in \boldsymbol{R}^n,\ \xi \neq 0.$$

q. e. d.

Since a sphere $I = c$ is convex, the above lemma implies that the restriction $\Delta|_{X_l}$ has just one critical point on X_l for $l \in M$.

By using the basis (1) of the tangent space of X_l, $\xi_i \Delta = 0$, $i = 2, \cdots, n$ has just one common zero in X_l, for $l \in M$. Since $\delta^i \Delta$ is divisible by Δ (by the definition of $\mathrm{Der}_E(\log \Delta)$),

$$\frac{\xi_i \Delta}{\Delta} = I \frac{\delta^i \Delta}{\Delta} - \frac{\deg \Delta}{2} \delta^i I, \quad i = 2, \cdots, n,$$

are homogeneous polynomials of degree $(\deg \delta^i + 2)$.
Thus we obtain an inequality

$$\frac{1}{2} \#\{\text{connected component of } E - \bigcup_{i \in J} H_i\} \leq \prod_{i=2}^{n} \deg \frac{\xi_i \Delta}{\Delta} = \prod_{i=2}^{n} (\deg \delta^i + 2)$$

End of the proof of the theorem.

(1.12) In [9] H. Terao has studied some numerical criterion for $\mathrm{Der}_E(\log \Delta)$ to be free.

All examples show that:

i) If $\mathrm{Der}_E(\log \Delta)$ is free, then one has the equality, $\prod_{i=1}^{n} (\deg \delta_i + 2) = \#\{$connected of $E - \bigcup_{i \in J} H_i\}$,

ii) If $\mathrm{Der}_E(\log \Delta)$ is free, then the complementary space $E^c - \bigcup_{i \in J} H_i^c$ of the complexifications of E and H_i, $i \in J$, is a $K(\pi, 1)$ space.

H. Terao gave also several counter examples for the converse of ii).

§ 2. Residue of logarithmic forms

(2.1) Let (S, D) be as in § 1 (1.2), a pair of n-dimensional complex manifold and its divisor. Let π: $\tilde{D} \rightarrow D$ be the normalization of D. Denote by $\mathcal{O}_D$ and $\mathcal{M}_D$ (resp. $\mathcal{O}_{\tilde{D}}$ and $\mathcal{M}_{\tilde{D}}$) the sheaf of germs of holomorphic or meromorphic functions on D (resp. $\tilde{D}$). Denote by $\Omega^q_{\tilde{D}}$ (resp. $\Omega^q_{\tilde{D}}$) the sheaf of germs of holomorphic q-forms on D (resp. $\tilde{D}$). (i. e. $\Omega^q_{D,p}=\Omega^q_{S,p}/h_p\Omega^q_{S,p}+dh_p\wedge\Omega^{q-1}_{S,p}$ and $\mathcal{M}_D\otimes\Omega^q_D=\pi_*(\mathcal{M}_{\tilde{D}}\otimes\Omega^q_{\tilde{D}})$).

(2.2) Definition. *The residue morphism* res. is a sheaf homomorphism res.: $\Omega^q_S(\log D) \rightarrow \mathcal{M}_D \underset{\mathcal{O}_D}{\otimes} \Omega^{q-1}_D=\pi_*(\mathcal{M}_{\tilde{D}} \underset{\mathcal{O}_{\tilde{D}}}{\otimes} \Omega^{q-1}_{\tilde{D}})$, which is defined as follows:
For any $\omega \in \Omega^q_{S,p}(\log D)$, let us take a presentation ω as in § 1. (1.1) iii) b)

$$g\omega=\frac{dh}{h}\wedge\xi+\eta.$$

Then res. ω is presented by $\dfrac{1}{g}\xi$ in $\mathcal{M}_D\otimes\Omega^q_D$.

To show the well-definedness of res. (ω), we need a lemma.

(2.3) Lemma. *Let h be a germ of holomorphic function at p, which is not a constant. Then there exists an integer ρ such that for any $\omega \in \Omega^q_{S,p}$ satisfying*

$$\omega\wedge dh=0, \quad \left(\frac{\partial h}{\partial z_i}\right)^{\rho}\omega \in dh\wedge\Omega^{q-1}_{S,p}, \ q=1, \cdots, n, \ i=1, \cdots, n.$$

Proof. This is a direct corollary of a lemma in [5].

(2.4) *The well-definedness of* res. (ω).

Let ω has two different presentations:

$$g_i\omega=\frac{dh}{h}\wedge\xi_i+\eta_i \quad i=1, 2.$$

Then $\dfrac{dh}{h}\wedge(g_2\xi_1-g_1\xi_2)=(g_1\eta_2-g_2\eta_2)$ and hence $(g_1\eta_2-g_2\eta_2)\wedge dh=0$. By the use of Lemma (2.3), $\left(\dfrac{\partial h}{\partial z_i}\right)^{\rho}(g_1\eta_2-g_2\eta_1)=dh\wedge\zeta$ for some $\zeta \in \Omega^{q-1}_{S,p}$. i. e.

$$dh\wedge\left((g_2\xi_1-g_1\xi_2)\left(\frac{\partial h}{\partial z_i}\right)^{\rho}-h\zeta\right)=0.$$

Again by the Lemma (2.3), there exists $\sigma \in \Omega^{q-2}_{S,p}$ so that

$$(g_2\xi_1-g_1\xi_2)\left(\frac{\partial h}{\partial z_i}\right)^{2\rho}=h\zeta\left(\frac{\partial h}{\partial z_i}\right)^{\rho}+dh\wedge\sigma.$$

(In case $q=1$, $\sigma=0$). Thus the class of $(g_2\xi_1-g_1\xi_2)\left(\frac{\partial h}{\partial z_i}\right)^{2\rho}$ in $\Omega^1_{D,p}$ is zero. For a suitable i, $\left(\frac{\partial h}{\partial z_i}\right)^{2\rho}$ is non-zero divisor of $\mathcal{O}_{D,p}$. Thus $\frac{1}{g_1}\xi_1$ and $\frac{1}{g_2}\xi_2$ define the same class in $\mathcal{M}_{D,p}\underset{=D,p}{\otimes}\Omega^1_{D,p}$.

(2.5) For an $\omega\in\Omega^q_{S,p}(\log D)$, res. $\omega=0$ if and only if ω is holomorphic. ($\because$ res. $\omega=0\Longleftrightarrow g\omega\in\Omega^q_{S,p}\Longleftrightarrow\omega\in\Omega^q_{S,p}$).

Thus we get an exact sequence

$$0\longrightarrow\Omega^q_S\longrightarrow\Omega^q_S(\log D)\xrightarrow{\text{res.}}\pi_*(\mathcal{M}_D\otimes\Omega^{q-1}_{\tilde{D}}).$$

(2.6) The following diagram is commutative:

$$\begin{array}{ccc}\Omega^q_S(\log D) & \xrightarrow{\text{res.}} & \pi_*(\mathcal{M}_{\tilde{D}}\otimes\Omega^{q-1}_{\tilde{D}}) \\ \downarrow d & & \downarrow d \\ \Omega^{q+1}_S(\log D) & \xrightarrow{\text{res.}} & \pi_*(\mathcal{M}_{\tilde{D}}\otimes\Omega^q_{\tilde{D}}).\end{array}$$

(2.7) res. $\Omega^q_S(\log D)$ defines a $\mathcal{O}_{\tilde{D}}$-coherent submodule of $\mathcal{M}_{\tilde{D}}\otimes\Omega^{q-1}_{\tilde{D}}$.

PROOF. Let $h(z)=0$ be a local equation for $D\cap U$ in an open set $U\subset S$. Then

$$\frac{\partial h}{\partial z_i}\cdot\text{res. }\Omega^q_S(\log D)|_U\subset\Omega^{q-1}_{\tilde{D}}|_{D\cap U}.$$

Together with the fact that $\Omega^q_S(\log D)$ is an $\mathcal{O}_S$-coherent module and that res. is a homomorphism $\mathcal{O}_S\longrightarrow\mathcal{O}_D\subset\mathcal{O}_{\tilde{D}}$, we obtain that res. $\Omega^q_S(\log D)|_U$ is $\mathcal{O}_{\tilde{D}}$-coherent.

q. e. d.

(2.8) Note that in general res. $\Omega^q_S(\log D)$ does not consists of holomorphic forms on $\tilde{D}$ but of meromorphic forms. This fact makes a rather big difference in contrast with the theory due to Deligne [1]. Now let us show:

LEMMA. res. $\Omega^1_S(\log D)$ *contains* $\pi_*\mathcal{O}_{\tilde{D}}$.

PROOF. Let $\alpha\in\pi_{*,p}\mathcal{O}_{\tilde{D}}$. Since $\frac{\partial h}{\partial z_i}$ is an universal denominator, $\frac{\partial h}{\partial z_i}\alpha$ belongs to $\mathcal{O}_{D,p}$. Let $a_i\in\mathcal{O}_{S,p}$ be a representative of $\frac{\partial h}{\partial z_i}\alpha$. Then $\frac{\partial h}{\partial z_j}a_i-\frac{\partial h}{\partial z_i}a_j=b_{ij}h\in\mathcal{I}_{D,p}$ for some $b_{ij}\in\mathcal{O}_{S,p}$. Now let us define

$$\omega=\frac{\sum_{i=1}^{n} a_i dz_i}{h}.$$

Then $\frac{\partial h}{\partial z_j}\omega=a_j\frac{dh}{h}+\sum_{i=1}^{n} b_{ij}dz_i$, and hence

$$\omega\in\Omega^1_{S,p}(\log D) \text{ and } \operatorname{res}.\omega=\left.\frac{a_j}{\frac{\partial h}{\partial z_j}}\right|_D=\alpha. \qquad \text{q. e. d.}$$

(2.9) THEOREM. *Let S, D be as (2.1). Let $(D,p)=(D_1,p)\cup\cdots\cup(D_m,p)$ be the local irreducible decomposition of D at a point $p\in D$, and let $h=h_1\cdots h_m$ be the corresponding decomposition of the equation. Then the following conditions are equivalent:*

i) $\Omega^1_{S,p}(\log D)=\sum_{i=1}^{m}\mathcal{O}_{S,p}\frac{dh_i}{h_i}+\Omega^1_{S,p}$,

ii) $\Omega^1_{S,p}(\log D)$ *is generated by closed forms,*

iii) $\operatorname{res}.\Omega^1_{S,p}(\log D)=\bigoplus_{i=1}^{m}\mathcal{O}_{D_i,p}$,

iv) a) D_i *is normal* (i. e. $\dim_C \operatorname{Sing} D_i\leq n-3$) *for* $i=1,\cdots,m$,

b) $D_i \pitchfork D_j$ (i. e. *outside of an* $(n-3)$*-dimensional subset of* D, D_i *and* D_j *is normal crossing*) *for* $i\neq j$, $i, j=1,\cdots,m$,

c) $\dim_C D_i\cap D_j\cap D_k\leq n-3$ *for* $i\neq j\neq k\neq i$, $i, j, k=1,\cdots,m$.

PROOF. Obviously i) implies ii), since $d\left(\frac{dh_i}{h_i}\right)=0$. Let $\omega\in\Omega^1_{S,p}(\log D)$, such as $d\omega=0$. Then because of the commutativity of (2.6), the residue of ω on each branch D_i is constant $c_i\in\boldsymbol{C}$. Thus by the exactness of (2.5), there exists some $\xi\in\Omega^1_{S^n,0}$, such that $\omega=\sum_{i=1}^{m}c_i\frac{dh_i}{h_i}+\xi$. Hence $\omega\in\sum_{i=1}^{m}\mathcal{O}_{S,p}\frac{dh_i}{h_i}+\Omega^1_{S^n,0}$. Thus ii) implies i). The equivalence of i) and iii) is a direct corollary of (2.5). Suppose iii). Then together with the lemma (2.8), we get $\bigoplus_{i=1}^{m}\mathcal{O}_{D_{i,p}}\supseteq\operatorname{res}.\Omega^1_{S,p}(\log D)\supseteq\bigoplus_{i=1}^{m}\mathcal{O}_{\tilde{D}_{i,p}}$. Especially $\mathcal{O}_{D_{i,p}}=\mathcal{O}_{\tilde{D}_{i,p}}$ $i=1,\cdots,m$, which implies D_i $i=1,\cdots,m$ is normal. Thus D_i is smooth outside of an $(n-3)$-dimensional set. Suppose that two components D_i and D_j are not transversal. This means that D_i and D_j are tangent along an $(n-2)$-dimensional set V through p. Since D_i and D_j are smooth at a general point of V, we may choose a local coordinate $(y_1,\cdots,y_n)$ of S at q of a general point of V so that $(D_i,q)=\{y_1=0\}$ $(D_j,q)=\{y_1+y_2^m=0\}$ for some $m\geq 2$. Then $\omega=\frac{y_2dy_1-my_1dy_2}{y_1(y_1+y_2^m)}$ belongs to $\Omega^1_{S,q}(\log D)$

and res. $\omega|_{D_{i,q}}=y_2^{1-m}$ which is meromorphic and has a pole along $D_i\cap D_j$. Since res. $\Omega_S^1(\log D)$ is coherent, the condition iii) implies that res. $\Omega_S'(\log D)=\bigoplus_{i=1}^{m}\mathcal{O}_{D_i}$ in a neighborhood of p. On the other hand we can choose the point q close to p, which is a contradiction. Hence D_i and D_j are transversal. Suppose that there exist three irreducible components D_i, D_j, D_k so that $\dim_C D_i\cap D_j\cap D_k=n-2$. At a general point of $D_i\cap D_j\cap D_k$, D_i, D_j and D_k are smooth and any two of them are normal crossing. Hence we can find a local coordinate $(y_1,\cdots,y_n)$ of S at a general point q of $D_i\cap D_j\cap D_k$, such that $(D_i, q)=\{y_1=0\}$, $D_j=\{y_2=0\}$, $D_k=\{y_1-y_2=0\}$. Then $\omega=\frac{1}{y_1-y_2}\left(\frac{dy_1}{y_1}-\frac{dy_2}{y_2}\right)$ belongs to $\Omega^1_{S,q}(\log D)$ and res. $\omega|_{D_i,q}=-\frac{1}{y_2}$ which has a pole along $D_i\cap D_j\cap D_k$. Since we may choose q close enough to p, this contradicts the assumption iii). Thus for any three different components D_i, D_j, D_k, $\dim_C D_i\cap D_j\cap D_k\leq n-3$. q. e. d.

(2.10) *Note.* Let B be a small ball in S centered at p. Then the fundamental group $\pi_1(B-D)$ does not depend on B for sufficiently small ball B. We shall call it the local fundamental group of the complement of D at p and denote it by $\pi_{\mathrm{loc},p}(S-D)$. In the case of (2.9) iv), one can show that $\pi_{\mathrm{loc},p}(S-D)$ is free abelian group of rank m, and hence the local homology group is a free abelian group of rank m. Then one may regard that $\frac{dh_i}{h_i}$, $i=1,\cdots,m$, are dual basis of it.

(2.11) Before going to a general problem when res. $\Omega_S^1(\log D)$ is holomorphic, let us study the case when dim $S=n=2$.

THEOREM. *Let $h(x,y)\in\mathcal{O}_{C^2,0}$ is a reduced function and let $C=\{h(x,y)=0\}$ be a germ of a reduced plane curve at $0\in\boldsymbol{C}^2$. Then* res. $\Omega^1_{C^2,0}(\log C)$ *is holomorphic on the normalization $\tilde{C}$, if and only if C is a smooth curve or a normal crossing of two smooth curves.*

PROOF. Since $\dim S=2$, $\Omega^1_{C^2,0}(\log C)$ is $\mathcal{O}_{C^2,0}$-free ($\because$ (1.7)), let ω_1 and ω_2 be free generators of $\Omega^1_{C^2,0}(\log C)$. Let us present $\frac{dh}{h}\in\Omega^1_{C^2,0}(\log C)$ by $\frac{dh}{h}=a\omega_1+b\omega_2$ for some $a,b\in\mathcal{O}_{C^2,0}$. Then $1=a\Big|_D$ res. $\omega_1+b\Big|_D$ res. ω_2. Since res. ω_1, res. $\omega_2\in\pi_*(\mathcal{O}_{\tilde{C}})$, either $a\Big|_D$ or $b\Big|_D$ is a unit. Thus one of $a(0)$ or $b(0)$ is non-zero and therefore we may choose $\frac{dh}{h}=a\omega_1+b\omega_2$ as a part of a free base system of $\Omega^1_{C^2,0}(\log D)$. Since $\mathrm{Der}_{C^2,0}(\log C)$ is the dual module of $\Omega^1_{C^2,0}(\log D)$ ($\because$ (1.6)) there exists a logarithmic vector field $X\in\mathrm{Der}_{C^2,0}(\log D)$ such that $\langle X\cdot\frac{dh}{h}\rangle=1$, i. e., $Xh=h$. Since

h is reduced, C may have only an isolated singularity at $0 \in \boldsymbol{C}^2$. Thus if C is not smooth at 0, h defines a quasi-homogeneous isolated singularity at 0. Then due to a theorem ([4]), one can find a holomorphic local coordinate transformation at $0 \in \boldsymbol{C}^2$, so that after the change of the coordinate h becomes a polynomial of the following type: There exist rational numbers $r, s \in \boldsymbol{Q}$ such that $0<r\leq \frac{1}{2}$, $0<s\leq \frac{1}{2}$ and $rx\frac{\partial h}{\partial x}+sy\frac{\partial h}{\partial y}=h$. Then one may check that $\frac{dh}{h}$ and $\frac{sydx-rxdy}{h}$ are a free base system of $\Omega^1_{\boldsymbol{C}^2,0}(\log C)$ $\left(\because \frac{sydx-rxdy}{h}\wedge\frac{dh}{h}=\frac{dx\wedge dy}{h}\right.$ and (1.8) i)).

The residue of $\frac{sydx-rxdy}{h}$ is equal to $\left.\frac{sy}{\frac{\partial h}{\partial x}}\right|_D=\left.\frac{-rx}{\frac{\partial h}{\partial y}}\right|_D$.

Now let d be a common denominator of r and s and put $d\cdot r=p$, $d\cdot s=q$. For a point $(x_0, y_0)\neq(0,0)$ in C, $\boldsymbol{C}\ni t\longmapsto(t^p x_0,\ t^q y_0)\in C$ is a parametric presentation of a branch C_0 of C containing (x_0, y_0). (We don't care if the presentation is a branched covering of the normalization of C_0.) Suppose $\frac{\partial h}{\partial x}(x_0, y_0)\neq 0$. Then over the branch C_0, the residue of $\frac{sydx-rxdy}{h}$ is equal to

$$\frac{st^q y_0}{\frac{\partial h}{\partial x}(t^p x_0, t^q y_0)}=\frac{sy_0}{\frac{\partial h}{\partial x}(x_0, y_0)}t^{p+q-d}.$$

Since the residue must be holomorphic on $\tilde{C}$, the degree $p+q-d\geq 0$ i. e. $r+s-1\geq 0$. This is possible only when $r=s=\frac{1}{2}$. This means that h is a homogeneous polynomial of degree 2. Then after a linear change of the coordinates, h becomes xy or x^2. Since h is reduced, we have the only possibility $h=xy$. q. e. d.

(2.12) *Note.* For a local plane curve C at $0\in\boldsymbol{C}^2$, let us take a small ball neighborhood B of $0\in\boldsymbol{C}^2$. Then the fundamental group $\pi_1(B-C)$ does not depend on the radius of B for small B. The local fundamental group $\pi_1(B-C)$ is abelian if and only if $(C,0)$ is smooth or normal crossing. Hence $\pi_1(B-C)$ is abelian if and only if res. $\Omega^1_{C,0}(\log C)=\pi_{*,0}\mathcal{O}_{\tilde{C}}$.

(2.13) LEMMA. *Let S and D are as in* (2.1). *At a point $p\in D$ the following condition* i) *implies the condition* ii) *and the condition* ii) *implies the condition* iii).

i) *The local fundamental group $\pi_{\mathrm{loc},q}(S-D)$ for q in a neighborhood of p is abelian.*

ii) *There exists an $(n-3)$-dimensional analytic set A so that $D-A$ has only* *normal crossing singularity in a neighborhood of p.*

iii) res. $\Omega^1_{S,p}(\log D)=\pi_*(\mathcal{O}_{\tilde{D}})_p$.

PROOF. i)$\Longrightarrow$ii). Suppose that there exists an $(n-2)$-dimensional branch V of Sing D, along which the singularity of D is not normal crossing. We may consider D to be a family of germs of plane curves along V. Along the general point q of V, the family is topologically trivial. Hence the local fundamental group $\pi_{\mathrm{loc},q}(S-D)$ is isomorphic to the local fundamental group of the complement of a plane curve, which is isomorphic to the generic member, which is singular with non-normal crossing singularity. Thus $\pi_{\mathrm{loc},q}(S-D)$ is not abelian and the condition i) is not satisfied.

ii)$\Longrightarrow$iii). Suppose $D-A$ has only normal crossing singularities. Then any point $q\in D-A$ satisfies the condition (2.9) iv). For $\omega\in\Omega^1_{S,p}(\log D)$, res. ω is holomorphic on $\widetilde{D-A}$. Since $\mathrm{codim}_D(\tilde{D}-\widetilde{D-A})=2$, res. ω is holomorphic on the normal space $\tilde{D}$.

(2.14) *Note.* All known examples show that the converse of (2.13) is true. It is quite interesting to ask whether the conditions i), ii), iii) of (2.13) are equivalent.

We can reduce the question to the three dimensional case.

Question. Let D be a hypersurface in $\boldsymbol{C}^3$ near the origin and suppose $0\in D$. If $D-\{0\}$ has at most normal crossing singularities, then is the local fundamental group $\pi_{\mathrm{loc},0}(\boldsymbol{C}^3-D)$ abelian?

§ 3. Logarithmic stratification and logarithmic variety

In this paragraph we introduce logarithmic stratification of S and study the characteristic variety $L_S(\log D)$ associated with $\mathrm{Der}_S(\log D)$.

(3.1) Let S and D be as in (1.2) and (2.1), an n-dimensional complex manifold and a divisor of it. $\mathrm{Der}_S(\log D)$ is the module of logarithmic vector fields associated with (S, D) (1.4). For any point p, let us denote by $\mathrm{Der}_S(\log D)$ (p), the linear subspace of the tangent space $T_{S,p}$ of S at p which consists of the vectors $\delta(p)$ of the values of the vector field δ of $\mathrm{Der}_{S,p}(\log D)$ at p.

(3.2) LEMMA. I. *Let S and D be as (3.1). There exists uniquely a stratification $\{D_\alpha, \alpha\in I\}$ of S with the following properties:*

i) *Stratum D_α, $\alpha\in I$ is a smooth connected immersed submanifold of S. S is a disjoint union $\bigcup_{\alpha\in I} D_\alpha$ of the strata.*

ii) *Let $p\in S$ belong to a stratum D_α. Then the tangent space $T_{D_\alpha,p}$ of D_α at p*

coincides with the subspace $\mathrm{Der}_S(\log \mathrm{D})\ (p) \subset T_{S,p}$.

II. *The stratification above satisfies automatically the following "frontier condition".*

iii) *If* $D_\alpha \cap \bar{D}_\beta \neq \emptyset$ *for some* $\alpha, \beta \in I$ *and* $\alpha \neq \beta$, *then* $D_\alpha \subset \partial D_\beta$.

PROOF. I. Since $\mathrm{Der}_S(\log D)$ is coherent, there exist a neighborhood U of p so that $\mathrm{Der}_S(\log D)|_U$ has finite generators $\delta^1, \cdots, \delta^m$ as $\mathcal{O}_S$-module. Since $\mathrm{Der}_S(\log D)$ is closed under the bracket product, $\{\delta^1, \cdots, \delta^m\}$ consists an involutive system of vectorfields on U (i. e. $[\delta^i, \delta^j] \in \sum_{k=1}^{m} \Gamma(U, \mathcal{O}_S)\delta^k$, $i, j=1, \cdots, m,$). Then U can be decomposed to maximal integral submanifolds. Since this decomposition has the property stated in (3.2) ii), it does not depend on the choise of the generators $\delta^1, \cdots, \delta^m$. Let U' be an open neighborhood of $p' \in S$ with the decomposition as above. By the uniqueness of the integral submanifold, the decompositions coincide in $U \cap U'$ The collection of global connected components of such integral submanifold satisfies the desired property of I.

II. Suppose $D_\alpha \cap \bar{D}_\beta \neq \emptyset$. Since $D_\alpha \cap \bar{D}_\beta$ is a closed subset of D_α with the induced topology from S, it is also closed with the manifold topology of D_α. Let us show that $D_\alpha \cap \bar{D}_\beta$ is an open subset of D_α with the manifold topology of it. For a point $p \in D_\alpha \cap \bar{D}_\beta$, choose logarithmic vectorfields $\delta^1, \cdots, \delta^k$ near p in S, such that $\delta^1(p), \cdots, \delta^k(p)$ span the tangent space $T_{D_\alpha,p}$ of D_α at p. Let us consider the integral $\exp\left(\sum_{i=1}^{k} c_i\delta^i\right)$, which is a local homeomorphism of S into S near p. For small $\varepsilon > 0$, $\exp\left(\sum_{i=1}^{k} c_i\delta^i\right)(p)$, $|c_1|^2 + \cdots + |c_k|^2 < \varepsilon$, fills a manifold neighborhood of D_α at p. Let $\{p_n\}_{n=1,2,\ldots}$ be a sequence of points of D_β, which converges to p. For any $(c_1, \cdots, c_k)$, there exists $N \in \boldsymbol{N}$ such that $\{p_n\}_{n \geq N}$ lies in the defining domain of $\exp(\sum c_i\delta^i)$, so that the sequence $\left\{\exp\left(\sum_{i=1}^{k} c_i\delta^i\right)(p_n)\right\}_{n \geq N}$ converges to $\exp\left(\sum_{i=1}^{k} c_i\delta^i\right)(p)$. Since $\sum_{i=1}^{k} c_i\delta^i$ is logarithmic, $\exp\left(\sum_{i=1}^{k} c_i\delta^i\right)(p_n)$ lies in D_β. This proves that $D_\alpha \cap \bar{D}_\beta$ contains a manifold neighborhood of p in D_α. Since D_α is connected, $D_\alpha \subset \bar{D}_\beta$. If $\alpha \neq \beta$, then $D_\alpha \cap D_\beta = \emptyset$ and therefore $D_\alpha \subset \bar{D}_\beta - D_\beta = \partial D_\beta$. q. e. d.

(3.3) DEFINITION. The stratification $\{D_\alpha,\ \alpha \in I\}$ of (3.2) will be called the *logarithmic stratification of* S. A stratum D_α is called a *logarithmic stratum.*

Note that the manifold topology of D_α does not coincide with the topology induced from S in general (cf. (3.12) ii)).

(3.4) By definition, the following lemma is almost trivial.

LEMMA.

i) *Let δ be any logarithmic vector field defined on a certain open set S. Then the integral* $\exp(t\delta)$ *which is a local homeomorphism on a certain set, brings a point of a stratum to a point in the same stratum.*

ii) *Suppose S is paracompact. Any two points of one stratum can be conbined by a finite number of transformation of type* $\exp(t\delta)$ *succesively.*

iii) *Connected components of $S-D$ and $D-\operatorname{Sing} D$ are logarithmic strata.*

(3.5) The following lemma is simple but useful.

LEMMA. *Let $h(x_1, \cdots, x_n, y_1, \cdots, y_m)$ be a holomorphic function defined in a neighborhood of 0 in $\boldsymbol{C}^n \times \boldsymbol{C}^m$. The following condition* A) *(resp.* (A′)) *is equivalent to the condition* B) *(resp.* B′)).

A) $\frac{\partial h}{\partial y_1}, \cdots, \frac{\partial h}{\partial y_m}$ *are contained in an ideal of* $\mathcal{O}_{C^{n+m},0}$ *generated by*

$$\left\{\frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_n}\right\},$$

i. e., $\left(\frac{\partial h}{\partial y_1}, \cdots, \frac{\partial h}{\partial y_m}\right) \subset \left(\frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_n}\right)$.

A′) $\left(\frac{\partial h}{\partial y_1}, \cdots, \frac{\partial h}{\partial y_m}\right) \subset \left(h, \frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_m}\right)$.

B) *There exists a local biholomorphic map*

$$\varphi: (\boldsymbol{C}^n \times \boldsymbol{C}^m, 0) \longrightarrow (\boldsymbol{C}^n \times \boldsymbol{C}^m, 0),$$

such that i) $\varphi(x, y) = (\varphi_1(x, y), \cdots, \varphi_n(x, y), y)$ *and* $\varphi(x, 0) = (x, 0)$,

ii) $h(\varphi(x, y)) = h(x, 0)$.

B′) *There exists a local biholomorphic map*

$$\varphi: (\boldsymbol{C}^n \times \boldsymbol{C}^m, 0) \longrightarrow (\boldsymbol{C}^n \times \boldsymbol{C}^m, 0)$$

and a holomorphic function $v(x, y)$ defined near 0,

such that i) $\varphi(x, y) = (\varphi_1(x, y), \cdots, \varphi_n(x, y), y)$ *and* $\varphi(x, 0) = (x, 0)$, $v(x, 0) \equiv 1$,

ii) $h(\varphi(x, y)) = v(x, y) h(x, 0)$.

PROOF. B) (resp. B′)) $\Longrightarrow$ A) (resp. A′)):
Take the partial derivative by y_i of the equation ii) of B) (resp. B′)). Together with i), we obtain

$$\sum_{j=1}^{n}\frac{\partial h}{\partial x_j}(\varphi(x,y))\frac{\partial\varphi_j}{\partial y_i}+\frac{\partial h}{\partial y_i}(\varphi(x,y))=\begin{cases}0 & \text{B)}\\ \left(\dfrac{\partial v}{\partial y_i}\Big/v\right)h(\varphi(x,y)). & \text{B}')\end{cases}$$

Since $\varphi(x,y)$ is locally a biholomorphic map, we may substitute (x, y) by $\varphi^{-1}(x,y)$, and we obtain A) (resp. A′)).

A) (resp. A′))$\Longrightarrow$B) (resp. B′)):

Let us construct inductively a sequence of local biholomorphic mappings

$$\varphi^{(k)}:\ (\boldsymbol{C}^n\times\boldsymbol{C}^m,\ 0)\longrightarrow(\boldsymbol{C}^n\times\boldsymbol{C}^m,\ 0),\quad k=0,1,\cdots,m,$$

(resp., and a sequence of holomorphic function $v^{(k)}(x, y)$ with $v^{(k)}(x,0)=1$, $k=0,1,\cdots,m$) with the following properties:

i) $\varphi^{(k)}(x,y)=(\varphi_1^{(k)}(x,y),\cdots,\varphi_n^{(k)}(x,y),\ y_1,\cdots,y_m)$
such that
$\varphi_i^{(k)}(x,y_1,\cdots,y_{m-k},\ 0)=x_i,\quad i=1,\cdots,n.$

ii) $h(\varphi^{(k)}(x,y))=h(x,y_1,\cdots,y_{m-k},\ 0)$
(resp. $h(\varphi^{(k)}(x,y))=v^{(k)}(x,y)\cdot h(x,y_1,\cdots,y_{m-k},\ 0)$).

For $k=0$, we take $\varphi^{(k)}$ to be identity and $v^{(0)}$ to be a constant 1.

Suppose the construction of $\varphi^{(k)}$ and $v^{(k)}$ is done for $k<m$. From the condition A) (resp. A′)), there is a relation:

$$*)\qquad \frac{\partial}{\partial y_{m-k}}h(x,y)+\sum_{j=1}^{n}\frac{\partial h}{\partial x_j}(x,y)g_j(x,y)=\begin{cases}0 & \text{A)}\\ u(x,y)h(x,y). & \text{A}')\end{cases}$$

Let us consider the integral ξ of the vector field

$$\frac{\partial}{\partial y_{m-k}}+\sum_{j=1}^{n}g_j(x,y_1,\cdots,y_{m-k},\ 0,\cdots,0)\frac{\partial}{\partial x_j},$$

which is a local biholomorphic map with a parameter t

$$\xi:\ (\boldsymbol{C}^n\times\boldsymbol{C}^m+\boldsymbol{C},0)\longrightarrow(\boldsymbol{C}^n\times\boldsymbol{C}^m,0)$$
$$\xi(x,y,t)=(\xi_1(x,y_1,\cdots,y_{m-k},t),\cdots,\xi_n(x,y_1,\cdots,y_{m-k},t),\ y_1,\cdots,y_{m-k-1},y_{m-k}+t,y_{m-k+1},\cdots,y_m)$$

with $\xi_i(x,y_1,\cdots,y_{m-k},0)=x_i,\ i=1,\cdots,n.$

From $*$) we obtain

$$**)\qquad \mathrm{h}(\xi(x_1,y_1,\cdots,y_{m-k},0,t))=\begin{cases}h(x,y_1,\cdots,y_{m-k},0) & \text{A)}\\ \exp(tu(x,y_1,\cdots,y_{m-k},0))h(x,y_1,\cdots,y_{m-k},0). & \text{A)}'\end{cases}$$

Substitute $y_{m-k}=0$ and $t=y_{m-k}$, in $**)$, and denote

$$\begin{aligned}&\eta(x, y_1, \cdots, y_{m-k})\\&=(\xi_1(x, y_1, \cdots, y_{m-k-1}, 0, y_{m-k}), \cdots, \xi_n(x, y_1, \cdots, y_{m-k-1}, 0, y_{m-k})).\end{aligned}$$

Then $**)$ becomes:

$$\begin{aligned}&h(\eta(x, y_1, \cdots, y_{m-k}),\ y_1, \cdots, y_{m-k}, 0)\\&=\begin{cases}h(x, y_1, \cdots, y_{m-k-1}, 0) & \text{A)}\\ \exp(y_{m-k}u(x, y_1, \cdots, y_{m-k-1}, 0))h(x, y_1, \cdots, y_{m-k-1}, 0). & \text{A}')\end{cases}\end{aligned}$$

Put

$$\begin{aligned}\varphi^{(k+1)}(x, y)&=\varphi^{(k)}(\eta(x, y_1, \cdots, y_{m-k}), y)\\&=(\varphi_1^{(k)}(\eta(x, y_1, \cdots, y_{m-k}), y),\\&\qquad\cdots, \varphi_n^{(k)}(\eta(x, y_1, \cdots, y_{m-k}), y), y).\end{aligned}$$

Then

i) $\quad \varphi_i^{(k+1)}(x, y_1, \cdots, y_{m-k-1}, 0)$

$=\varphi_i^{(k)}(\eta(x, y_1, \cdots, y_{m-k-1}, 0), y_1, \cdots, y_{m-k-1}, 0)$

$=\eta_i(x, y_1, \cdots, y_{m-k-1}, 0)$

$=\xi_i(x, y_1, \cdots, y_{m-k-1}, 0, 0)=x_i, \quad i=1, \cdots, n.$

ii) $\quad h(\varphi^{(k+1)}(x, y))$

$=h(\eta(x, y_1, \cdots, y_{m-k}), y_1, \cdots, y_{m-k}, 0)$

$=h(x, y_1, \cdots, y_{m-k-1}, 0)$ A)

or

$=v^{(k)}(\eta(x, y), y)h(\eta(x, y), y_1, \cdots, y_{m-k}, 0)$

$=v^{(k)}(\eta, y)\exp(y_{m-k}u(x, y_1, \cdots, y_{m-k-1}, 0))h(x, y_1, \cdots, y_{m-k-1}, 0)$ A$'$)

q. e. d.

(3.6) Let D_α be a logarithmic stratum of (S, D) of dimension m, $\dim_C D_\alpha=m$. At each point $p\in D_\alpha$, there exist a neighborhood U of p, a local coordinate system $z_1, \cdots, z_n$, and an equation $h=0$ of $D\cap U$ such that

$$h(z)=h(z_1, \cdots, z_{n-m}, 0, \cdots, 0),$$

and

$$D_\alpha\cap U=\{z_1=\cdots=z_{n-m}=0\}.$$

PROOF. Since $\dim_C D_\alpha=m$, we can find logarithmic vector fields $\delta^i=\sum_{j=1}^n g_j^i(z)\frac{\partial}{\partial z^j}$, $i=1, \cdots, m$, which are defined in a neighborhood of p, and $\delta^1(p), \cdots, \delta^m(p)$ are $\boldsymbol{C}$-linearly independent. By suitable change of indices of the coordinate z, we may assume $G=(g_j^i(z))_{i=1,\cdots,m,\ j=n-m+1,\cdots,n}$ is an invertible matrix in a neighborhood of p. By multiplying the inverse matrix of G, we may assume that:

$$\delta^i=\frac{\partial}{\partial z_{n-m+i}}+\sum_{j=1}^{n-m} g_j^i(z)\frac{\partial}{\partial z_j},\quad i=1,\cdots,m.$$

Thus $\dfrac{\partial h}{\partial z_{n-m+i}}\in\sum_{j=1}^{n-m}\mathcal{O}_{C^n,p}\dfrac{\partial h}{\partial z_j}+\mathcal{O}_{C^n,p}h,\ i=1,\cdots,m$ for a generator h of $\mathcal{J}_D$. Applying Lemma (3.5), one can find a local coordinate transformation φ at p such that for $h^*=h\circ\varphi$, we obtain

$$h^*(z_1,\cdots,z_n)=\text{unit}\cdot h^*(z_1,\cdots,z_{n-m},0,\cdots,0).$$

Thus $h^*(z_1,\cdots,z_{n-m},0,\cdots,0)=0$ is the desired equation for D at p. q. e. d.

(3.7) Let D_α be a logarithmic stratum of (S,D). Since D is analytically trivial along D_α, the multiplicity of D is constant along D_α, which we shall denote $\text{mult.}_{D_\alpha}D$.

Suppose $\text{Der}_S(\log D)$ is $\mathcal{O}_S$-free. Then for any logarithmic stratum D_α, there is an inequality,

$$\dim S\leq\dim D_\alpha+\text{mult.}_{D_\alpha}D.$$

PROOF. For any point $p\in D_\alpha$, take a local coordinate system $z_1,\cdots,z_n$ as in (3.6). Then we may choose $\mathcal{O}_{S,p}$-free basis $\delta^1,\cdots,\delta^n$ of $\text{Der}_{S,p}(\log D)$ of the form,

$$\begin{cases}\delta^i=\displaystyle\sum_{j=1}^{n-m}g_j^i(z)\frac{\partial}{\partial z_j} & i=1,\cdots,n-m,\\[2ex] \delta^i=\dfrac{\partial}{\partial z_i} & i=n-m+1,\cdots,n,\end{cases}$$

where $g_j^i(z)$ is a holomorphic function such that $g_j^i|_{D_\alpha}=0$ for $i,j=1,\cdots,n-m$.

Then $\det(g_j^i)_{i,j=1,\cdots,n-m}$ is a defining equation of D ((1.8) ii)) and the multiplicity of $\det(g_j^i)$ is at least $n-m$ along D_α. q. e. d.

(3.8) DEFINITION. i) A point $a\in S$ is called *holonomic*, if there exists an open neighborhood U of p such that U intersects with only a finite number of logarithemic strata.

ii) A logarithmic stratum D_α is called *holonomic*, if there exists an open neighborhood U of D_α, such that U intersects with only a finite number of logarithmic strata.

(3.9) Let us denote $I_p=\{\beta\in I\colon \bar{D}_\beta\ni p\}$ for a point $p\in S$ and $I_\alpha=\{\beta\in I\colon \bar{D}_\beta\supset D_\alpha\}$ for a stratum D_α.

PROPOSITION. *A point $p\in S$ (resp. a stratum D_α) is holonomic, if and only if I_p (resp. I_α) is a finite set and $\bigcup_{\beta\in I_p} D_\beta$ (resp. $\bigcup_{\beta\in I_\alpha} D_\beta$) contains an open neighborhood of p (resp. D_α).*

PROOF. Let U be an open neighborhood of p (resp. D_α) such that $I_U=\{\beta: D_\beta \cap U\neq\emptyset\}$ is finite. Since I_p (resp. I_α) is a subset of I_U, it is a finite set. Then $\bigcup_{\beta\in I_p} D_\beta \supseteq U-\bigcup_{\beta\in I_U-I_p} D_\beta \supseteq U-\bigcup_{\beta\in I_U-I_p}\bar{D}_\beta$ (resp. $\bigcup_{\beta\in I_\alpha} D_\beta \supset U-\bigcup_{\beta\in I-I_\alpha}\bar{D}_\beta$) contains an open neighborhood of p (resp. D_α). The converse of the proposition is trivial by definition. q. e. d.

Note that only the finiteness of I_p or I_α does not imply the holonomicity.

(3.10) PROPOSITION. i) *A stratum D_α is holonomic if it contains a holonomic point. Every point of a holonomic stratum is holonomic.*
ii) *Let D_α be holonomic. Then any stratum D_β with $D_\alpha\subset\bar{D}_\beta$ is holonomic.*
iii) *$\bigcup_{\beta\in I_\alpha} D_\beta$ is an open neighborhood of D_α if D_α is holonomic.*

PROOF. i) Let $p\in D_\alpha$. Because of the frontier condition ((3.2) iii)), $I_p=I_\alpha$. Let us consider the set $\bigcup_{\beta\in I_\alpha} D_\beta$. Since $\bigcup_{\beta\in I_\alpha} D_\beta$ is invariant by the local transformation of type $\exp(t\delta)$ of S (cf. (3.4)), it contains an open neighborhood of a point $p\in D_\alpha$ if and only if it contains an open neighborhood of D_α.
ii) Let D_α be holonomic. Then by definition there exists a neighborhood U of D_α which consists of holonomic points. If $\bar{D}_\beta\supset D_\alpha$ then $D_\beta\cap U\neq\emptyset$, and hence D_β contains a holonomic point. Then by (3.10.i)) D_β is holonomic.
iii) For any $\beta\in I_\alpha$ (i. e. $\bar{D}_\beta\supset D_\alpha$), I_β is contained in I_α. Since D_β is holonomic, $\bigcup_{\gamma\in I_\beta} D_\gamma$ contains an open neighborhood of D_β, thus $\bigcup_{\beta\in I_\alpha} D_\beta$ contains a neighborhood of D_β for any $\beta\in I_\alpha$. q. e. d.

(3.11) Let $S^{(1)},\cdots,S^{(k)}$ are manifolds and $D^{(i)}\subset S^{(i)}$, $i=1,\cdots,k$, are divisors of them. Let $D_\alpha^{(i)}, \alpha\in I^{(i)}$, $i=1,\cdots,k$, be the logarithmic stratification of $(S^{(i)}, D^{(i)})$. Now let us put $S=S^{(1)}\times\cdots\times S^{(k)}$, $D=\bigcup_{i=1}^{k} S^{(1)}\times\cdots\times D^{(i)}\times\cdots\times S^{(k)}$. Then the logarithmic stratification for the pair (S, D) is given by $D_{\alpha_1}^{(1)}\times\cdots\times D_{\alpha_k}^{(k)}$, $D_{\alpha_2}^{(i)}\times\cdots\times D_{\alpha_k}^{(l)}$, $(\alpha_1,\cdots,\alpha_k)\in I^{(1)}\times\cdots\times I^{(k)}$. A stratum $D_{\alpha_1}^{(1)}\times\cdots\times D_{\alpha_k}^{(k)}$ is holonomic if and only if $D_{\alpha_i}^{(i)}$, $i=1,\cdots,k$, are holonomic.

(3.12) DEFINITION. For an integer $r\geq 0$, let us put

$$A_r=\{p\in S:\ \mathrm{rank}_C\ \mathrm{Der}_S(\log D)(p)\leq r\}.$$

From the definition we know directly the following properties:

i) A_r is a closed analytic subset of S. ($\because$ $\mathrm{Der}_S(\log D)$ is coherent.)

ii) $A_r = \bigcup_{\dim D_\alpha \le r} D_\alpha$.

iii) For any point $p \in A_r - A_{r-1}$, $\dim_p(A_r - A_{r-1}) = r$.

iv) Let $\{p_m\}_{m=1,\cdots}$ be a sequence of S which converges to a point $p_0 = \lim_{m\to\infty} p_m \in S$. Suppose $p_m \in D_{\alpha_m}$ for $m=0,1,2,\cdots$. Then for almost all m, $\dim D_{\alpha_m} \ge \dim D_{\alpha_0}$.

(3.13) LEMMA. i) *Let p be a point of a stratum D_α. Then p is holonomic if and only if*

$$\dim_p A_r \le r \quad \text{for} \quad \dim D_\alpha \le r \le n.$$

ii) *If D_α is holonomic, the manifold topology of D_α coincides with the induced topology from S.*

iii) *The property "holonomic" is a local property in the following sense:*

Let S' be an open set of S and put $D' = D \cap S'$. Then a point $p \in S'$ is holonomic with respect to the logarithmic stratification of D', if and only if $p \in S$ is holonomic with respect to the logarithmic stratification of D.

PROOF. Let $p \in D_\alpha$ be holonomic, and $U := \bigcup_{\beta \in I_\alpha} D_\beta$ be an open neighborhood of D_α ((3.10) iii)). Then $A_r \cap U$ is a closed subvariety of U. For any smooth point $q \in A_r \cap U$ and a small neighborhood $U' \subset U$ of q, we have a presentation,

$$A_r \cap U' = \bigcup_{\substack{\beta \in I_\alpha \\ \dim D_\beta \le r}} (D_\beta \cap U').$$

Since I_α is a finite set, the right hand side of the presentation is at most countable union of immersed submanifolds of dimension $\le r$. By a property of the dimension, such union cannot be a manifold of dimension $> r$, and therefore the presentation becomes a closed variety of dimension $\le r$.

Conversely, suppose that there exists a neighborhood U of p, so that $\dim(A_r \cap U) \le r$ for $\dim D_\alpha \le r \le n$. We assume also $A_{\dim D_\alpha - 1} \cap U = \emptyset$.

Put

$$I_r := \{\beta \in I\colon\ D_\beta \cap U = \emptyset,\ \dim D_\beta = r\}.$$

Then we have a presentation

$$(A_r - A_{r-1}) \cap U = \bigcup_{\beta \in I_r} U \cap D_\beta.$$

The left hand side of the presentation is either a void set or a closed r-dimensional subvariety of $U - A_{r-1}$, whose closure in U is $A_r \cap U$. Thus $(A_r - A_{r-1}) \cap U$ is a finite union $A^{(1)}\cup\cdots\cup A^{(m)}$ of irreducible components in $U-A_{r-1}$ (by shrinking U if needed). Let D_1 be a component of $D_\beta\cap U$ for $\beta\in I_r$. Since D_1 is an r-dimensional manifold in $(A_r-A_{r-1})\cap U$, it is contained in an irreducible component $A^{(i)}$, $1\leq i\leq m$. If $A^{(i)}\neq D_1$, any point $q\in A^{(i)}\cap\partial D_1\neq\emptyset$ should be contained in a certain $D_\gamma\cap U$ for $\gamma\in I_r$, $\gamma\neq\beta$. Thus $D_\gamma\cap\partial D_\beta\ni q$ and hence $D_\gamma\subset\partial D_\beta$ ((3.2) iii)). But this is impossible since $D_\gamma\cap U$ and $D_\beta\cap U$ are r-dimensional subvarieties contained in A_r. Thus $D_1=A^{(i)}$. This implies especially $\sharp I_r\leq m\leq\infty$. q.e.d.

(3.14) EXAMPLE. Let H_i, $i\in I$ be a system of hyperplanes in a complex Euclidean space E. Suppose that the system is locally finite in a domain $S\subset E$. Put $D=\bigcup_{i\in I} H_i\cap S$.

Then

i) For any finite subset $I'\subset I$, a connected component of $(\bigcap_{i\in I'} H_i-\bigcup_{j\in I-I'} H_j)\cap S$ is a logarithmic stratum of (S, D).

ii) Any stratum of (S, D) is holonomic.

For the proof we have only to show the following lemma.

LEMMA. *Let the notations be as above. Let $F=(\bigcap_{i\in I'} H_i-\bigcup_{j\in I-I'} H_j)\cap S$ be a k-dimensional face. At any point $p\in F$, the logarithmic tangent space* $\mathrm{Der}_S(\log D)(p)$ $\subset T_{S,p}$ *is identified with the k-dimensional tangent space $T_{F,p}(\simeq\bigcap_{i\in I'} H_i)$.*

PROOF. By a translation, we assume that p is the origin $O\in E$. Let H_i, $i\in I'\subset I$ be the system of all hyperplanes of I which pass through the origin, and σ_i, $i\in I'$, be a linear form defining the hyperplane H_i. For any $\delta\in\mathrm{Der}_{S,O}(\log D)$, $\delta\left(\prod_{i\in I'}\sigma_i\right)$ is divisible by $\prod_{i\in I'}\sigma_i$. Then $\delta\sigma_i$ is divisible by σ_i and hence $\delta(0)$ is tangent to H_i, $i\in I'$.

Conversely, let $x_1,\cdots,x_n$ be a linear coordinate system of E, such that $\bigcap_{i\in I'} H_i=\{x_{k+1}=\cdots=x_n=0\}$. Then any σ_i, $i\in I'$, is a linear combination of $x_{k+1},\cdots,x_n$, and therefore $\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_k}\in\mathrm{Der}_{S,0}(\log D)$. Clearly $\left\{\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_k}\right\}$ spans the tangent space of $\bigcap_{i\in I'} H_i$. q.e.d.

(3.15) Now we define logarithmic characteristic subvariety of the cotangent bundle T_S^* of S and study the relationship with the logarithmic stratification.

DEFINITION. A *logarithmic characteristic subvariety* $L_S(\log D)$ is a closed subvariety T_S^* defined as follows.

Let U be an open set of S, where $\mathrm{Der}_S(\log D)|_U$ is generated by a finite number of vector fields $\delta^1, \cdots, \delta^k \in \Gamma(U, \mathrm{Der}_S(\log D))$. Then

$$*) \qquad L_S(\log D) \cap T_S^*|_U = \{(x, \xi) \in T_S^*|_U : \sigma(\delta^i)(x, \xi) = 0, i = 1, \cdots, k\}$$

where $\sigma(\delta^i)$ is a linear form on T_S^* associated with the vector fields $\delta^i, i=1, \cdots, k$.

The variety $L_S(\log D)$ is well-defined if we have shown that the right hand side of $*)$ is independent of the choice of the generator system $\delta^1, \cdots, \delta^k$, which is almost trivial.

(3.16) For a stratum D_α, denote by N^*D_α the conormal bundle of D_α which is an n-dimensional submanifold of T^*S. Then

$$L_S(\log D) = \bigcup_{\alpha \in I} N^*D_\alpha.$$

PROOF. For a point $p \in D_\alpha$, the co-vector $\xi \in T_{S,p}^*$ belongs to $L_S(\log D)$

$\Longleftrightarrow$ $\langle \delta(p), \xi \rangle = 0$ for all $\delta \in \mathrm{Der}_{S,p}(\log D)$.

$\Longleftrightarrow$ ξ is orthogonal to $\mathrm{Der}_{S,p}(\log D)(p) = T_{D_\alpha,p}$.

$\Longleftrightarrow$ $\xi \in (N^*D_\alpha)_p$.

(3.17) Note that the decomposition of (3.16) is not an irreducible decomposition. Let us take an irreducible decomposition of $L_S(\log D)$:

$$L_S(\log D) = \bigcup_{\beta \in J} L_\beta.$$

Because of (3.16), $\dim L_\beta \geq n$ for all $\beta \in J$.

DEFINITION. An irreducible component L_β of $L_S(\log D)$ is *holonomic* if $\dim L_\beta = n$.

(3.18) PROPOSITION. *Consider a correspondence* $D_\alpha \longmapsto \overline{N^*D_\alpha}$ *(the closure of the conormal bundle of* D_α *in* T^*S*). Then this induces a one-to-one correspondence* $\{$*holonomic stratum of* $(S, D)\} \longrightarrow \{$*holonomic component of* $L_S(\log D)\}$.

PROOF. Firstly let us show that the correspondence is well defined. Let D_α be a holonomic stratum, then $U = \bigcup_{\beta \in I_\alpha} D_\beta$ is an open neighborhood of D_α which is dense in S. Then

$$L_S(\log D) \cap T_S^*|_U = \bigcup_{\beta \in I_\alpha} N^*D_\beta.$$

Since I_α is a finite set and D_β, $\beta \in I_\alpha$, is locally closed submanifold of U ((3.13) ii)), the $L_S(\log D) \cap T_S^*|_U$ is an n-dimensional variety. Hence especially N^*D_α is an irreducible component of $L_S(\log D) \cap T_S^*|_U$.

Now let us show the correspondence is injective. Suppose $\overline{N^*D_\alpha} = \overline{N^*D_\beta}$ for two holonomic strata D_α and D_β. Since N^*D_α and N^*D_β are dense and open in their closures, $N^*D_\alpha \cap N^*D_\beta \neq \emptyset$. Then $D_\alpha \cap D_\beta \neq \emptyset$ and $\alpha=\beta$.

(3.19) EXAMPLES. Let us give some examples, which illustrate the meaning of the logarithmic stratifications.

1. Let $(X, 0) \to (S, 0)$ be a universal unfolding of a germ f of a holomorphic function which has an isolated critical point at the origin. Let $D \subset S$ be the discriminant locus of the mapping $X \to S$, which is a hypersurface of S.

Then we have the following statements ([7]):

i) $\mathrm{Der}_S(\log D)$ *and* $\Omega_S^1(\log D)$ *are locally free* $\mathcal{O}_S$*-modules.*

ii) *Let* D_α *be a logarithmic stratum of* (S, D). *For any point* $t_0 \in D_\alpha$, *there exists a neighborhood* U *of* t_0 *in* S *and holomorphic mappings* ϕ_i: $D_\alpha \cap U \to X$, $i=1, \cdots, k$, *for a non-negative integer* k *such that*

a) ϕ_i, $i=1, \cdots, k$ *are sections of* $X \to S$.

b) *For any* $t \in D_\alpha \cap U$, *the set* $\{\phi_1(t), \cdots, \phi_k(t)\}$ *is just the singularities of the fiber* X_t *of* $X \to S$ *at* t.

c) *Families* $(X_t, \phi_i(t))_{t \in D_\alpha \cap U}$, $i=1, \cdots, k$, *are analytically trivial.*

The subvariety $D_\alpha \cap U$ *is characterized by the maximal subvariety of* U *over which there exist sections with the above properties.*

As corollaries to ii),

iii) *A point* $t \in S$ *is holonomic if and only if the fiber* X_t *over* t *has only simple singularities.*

iv) *For any logarithmic stratum* D_α *of* (S, D), *there is an equality,*

$$\dim D_\alpha + \mathrm{mult.}_{D_\alpha} D - \dim S = \sum_{i=1}^{k} (\mu_i - \tau_i),$$

where $\mu_i = \dim \mathcal{O}_{X, \phi_i(t)} \Big/ \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$

and $\tau_i = \dim \mathcal{O}_{X, \phi_i(t)} \Big/ \left(f - t, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$, $i=1, \cdots, k$, *for* $t \in D_\alpha$.

2. Let E be a complexification of a real vector space on which a finite group W generated by reflexions is acting. The quotient variety $\Omega := E/W$ is smooth. Moreover one may regard Ω as a complex vector space. Let $D \subset \Omega$ be the ramification divisor of $E \to \Omega$.

Then we have the following statements ([8]):

i) $\mathrm{Der}_\Omega(\log D)$ *and* $\Omega_\Omega^1(\log D)$ *are* $\mathcal{O}_\Omega$*-free modules.*

ii) *Let* E_α, $\alpha \in I$ *be the facet decomposition of* E *as in the example* (3.14), *where* H_i, $i \in \Sigma$ *is the set of complexified reflexion hyperplanes. Then any* E_α *is in-*

variant under the action of W so that $E_\alpha/W \subset \Omega$ $\alpha \in I$ is the logarithmic stratification of (Ω, D).

As corollaries to i) and ii),

iii) *Any logarithmic stratum of (Ω, D) is holonomic.*

iv) ***For** any logarithmic stratum D_α of (Ω, D), there is an equality*

$$\dim D_\alpha + \text{mult.}_{D_\alpha} D = \dim \Omega.$$

Addendum

Recently H. Terao has succeeded in proving the equality of (i) of (1.12) (See [9], [10]).

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