Arrangements of hyperplanes and their freeness I

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(Communicated by N. Iwahori)

§ 0. Introduction

Let V be an n-dimensional $(n\geq 2)$ real vector space throughout this paper. We can naturally assume V as an affine space over $\boldsymbol{R}$.

DEFINITION 0.1. An *arrangement* (in V) is a non-void finite family of $(n-1)$-dimensional real vector subspaces of V.

Let X be an arrangement. We are interested in the singularity of $\bigcup_{L\in X} L$ at the origin. By using the terminology of the theory of the differential forms with logarithmic pole ([7]), we will define a special class of the singularities. The arrangements which have such singularities at the origin are called to be free. Ring-theoretically the free arrangement is the arrangement whose singular locus is Cohen-Macaulay.

Let X be a free arrangement and H a hyperplane through the origin with $H\in X$ (or $H\notin X$), then our two main theorems determine the condition of H such that $X\setminus\{H\}$ (resp. $X\cup\{H\}$) is also free. The two theorems, Removal Theorem and Addition Theorem, lead us to an interesting method constructing free arrangements. The free arrangements constructed by the method are characterized only combinatorially, not algebraically. This fact may urge us to the unsettled conjecture that the freeness of the arrangement is a combinatorial property.

The simplicial arrangements studied by B. Grünbaum [4] and P. Deligne [3] are usually free arrangements. So K. Saito [6] conjectured that the free arrangements can be characterized by a certain topological condition which is satisfied by the simplicial arrangements. In § 2, we may partially answer his conjecture.

The author wishes to thank K. Saito for his helpful advices.

§ 1. Arrangement

Fix an affine coordinate of V. Then there is a one-to-one correspondence between the set of the squarefree non-zero polynomials which are the products of linear polynomials of n-variables up to constant and the set of the arrangements in V: any arrangement is defined by an equation $\prod_{i=1}^{m} l_i=0$, where l_i's are distinct (up to constant) non-zero linear polynomials of n-variables, and vice versa.

The number m is called the *order* of the arrangement. Denote the order of an arrangement X by $|X|$.

The following definitions give some simplest arrangements:

DEFINITION 1.1. An arrangement defined by $\prod_{i=1}^{m} l_i=0$ in V is called *hyperbolic* if each l_i belongs to the ideal (in the coordinate ring of V) generated by $\{l_1, l_2\}$.

DEFINITION 1.2. An arrangement is called *of fiber type* if it is defined by an equation $\prod_{i=1}^{m} l_i \cdot \prod_{j=1}^{k} l'_j=0$ in V, where

(1) an arrangement defined by $\prod_{i=1}^{m} l_i=0$ is hyperbolic,

(2) for any $k_1, k_2 \in \{1, \cdots, k\}$, there exists $i_0 \in \{1, \cdots, m\}$ such that l_{i_0} is an element of the ideal generated by l'_{k_1} and l'_{k_2}.

DEFINITION 1.3. An arrangement X in V is called *simplicial* if the connected components of $V \setminus \bigcup_{L \in X} L$ are open simplicial cones.

REMARK. The simplicial arrangements in $\boldsymbol{R}^3$ have been studied by B. Grünbaum. He listed up all simplicial arrangements whose orders are less than 38 up to the combinatorial equivalence. (In detail see [4].)

These two examples of arrangements have a common topological property: let X be an arrangement defined by $\prod_{i=1}^{m} l_i=0$ in V, then X_C stands for the family of (n−1)-dimensional complex subspaces of $V_C = V \otimes_{\boldsymbol{R}} \boldsymbol{C}$ defined by $\prod_{i=1}^{m} l_i=0$, then we have

THEOREM 1.4. *If X is simplicial or of fiber type, then $V_C \setminus \bigcup_{L \in X_C} L$ is a $K(\pi, 1)$-space* (see [3]).

Some topologists conjecture that these two examples give all the arrangements satisfying that $V_C \setminus \bigcup_{L \in X_C} L$ is $K(\pi, 1)$, when V is 3-dimensional. In §2, we shall treat the locally freeness of some coherent sheaf which has seemed to have relations with the $K(\pi, 1)$-spaces.

§2. Freeness of Arrangement

We shall define in 2.1 *the sheaf of germs of the rational differential forms with logarithmic pole* in order to define the freeness of an arrangement. Let M

be an algebraic manifold and D a reduced divisor in M. Let x be a point of M and Q_x a local defining equation of D at x.

DEFINITION 2.1. Define

$$\Omega_M^q(\log D)_x := \{\text{the germ at } x \text{ of the rational } q\text{-differential form } \omega;\ Q_x\omega \text{ and } Q_x(d\omega) \text{ are both regular near } x\}$$

for any $q \geq 0$. We can introduce a natural structure of algebraic coherent sheaf to $\bigcup_{x\in M} \Omega_M^q(\log D)_x$ and denote this sheaf by $\Omega_M^q(\log D)$.

This definition was introduced in [6]. If D is of normal crossing, this sheaf is a locally free sheaf ([2]). For a general D, $\Omega_M^1(\log D)_x$ is not necessarily a free $\mathcal{O}_{M,x}$-module. When is $\Omega_M^1(\log D)_x$ a free $\mathcal{O}_{M,x}$-module? In [6], K. Saito conjectured that $\Omega_M^1(\log D)_x$ is free if and only if there exists a neighborhood U_x of x such that $(X\setminus D)\cap U_x$ is $K(\pi, 1)$.

A reduced divisor D in M is called to be *free* at $x\in M$ if the stalk at x of $\Omega_M^1(\log D)$, $\Omega_M^1(\log D)_x$, is a free $\mathcal{O}_{M,x}$-module.

We need one more equivalent condition of the freeness of D at x.

DEFINITION 2.2. Define

$$\mathrm{Der}_M(\log D)_x := \{\text{the germ at } x \text{ of the regular vector field } \theta; \theta\cdot Q_x \in Q_x\cdot \mathcal{O}_{M,x}\}.$$

We can introduce a natural structure of an algebraic coherent sheaf to $\bigcup_{x\in M}\mathrm{Der}_M(\log D)_x$ and denote this sheaf by $\mathrm{Der}_M(\log D)$.

PROPOSITION 2.3. *A reduced divisor D is free at x if and only if* $\mathrm{Der}_M(\log D)_x$ *is a free $\mathcal{O}_{M,x}$-module.*

The proof is an obvious conclusion from the "duality" between $\Omega_M^1(\log D)_x$ and $\mathrm{Der}_M(\log D)_x$ (see [5]).

Let X be an arrangement in V and $Q=\prod_{i=1}^m l_i$ be a defining equation of X. Consider a reduced divisor $D_X = V(Q) = \bigcup_{L\in X} L$. We say that X is *free* at x if D_X is free at x.

Denote $\mathrm{Der}_V(\log D_X)_0$ and $\mathcal{O}_{V,0}$ by $D(X)$ and $\mathcal{O}$ respectively, where 0 stands for the origin of V.

If X_1 and X_2 are two arrangements such that $X_1\cap X_2=\emptyset$ and $X=X_1\cup X_2$. Then $D(X)=D(X_1)\cap D(X_2)$ because

$$\theta\cdot Q=\theta\cdot(Q_1Q_2)=Q_1(\theta\cdot Q_2)+Q_2(\theta\cdot Q_1)\ ,$$

where Q_1, Q_2, and Q are the defining equations of X_1, X_2, and X respectively. Thus it is obvious that $D(Y)$ contains $D(X)$ for any subset Y of X.

EXAMPLE. Let X be a hyperbolic arrangement defined by $\prod_{i=1}^{m} l_i=0$. Take an affine coordinate system $(l_1, l_2, y_3, \cdots, y_n)$ of V such that each l_i belongs to the ideal generated by $\{l_1, l_2\}$. Then a subset

$$\left\{l_1\frac{\partial}{\partial l_1}+l_2\frac{\partial}{\partial l_2}+\sum_{i=3}^{n} y_i\frac{\partial}{\partial y_i},\ l_2\cdots l_m\frac{\partial}{\partial l_2},\ \frac{\partial}{\partial y_3},\cdots,\frac{\partial}{\partial y_n}\right\}$$

of $D(X)$ becomes a system of free bases of $D(X)$. This means that X is free at the origin. Notice that every arrangement is hyperbolic thus free at the origin if $\dim V$ is two.

PROPOSITION 2.4. *An arrangement X is free at the origin of V if and only if $\mathcal{O}/\mathfrak{a}$ is 0 or $(n-2)$-dimensional Cohen-Macaulay, where $\mathfrak{a}$ is the Jacobian ideal (the ideal of $\mathcal{O}$ generated by all partial derivatives of Q).*

PROOF. Let $(x_1, \cdots, x_n)$ be the affine coordinate of V. Then any element of $D(X)$ can be locally described as $\sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i}$ with $f_i\in\mathcal{O}$ $(i=1, \cdots, n)$. Define the Euler vector field θ_0 by $\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}$, and θ_0 belongs to $D(X)$ because $\theta_0\cdot Q=(\deg Q)Q$.

Moreover we define a submodule

$$\mathrm{Ann}\,(X):=\{\theta\in D(X);\ \theta\cdot Q=0\}$$

of $D(X)$. Then we have a decomposition

$$D(X)=\mathcal{O}\cdot\theta_0\oplus\mathrm{Ann}\,(X)$$

because $\theta-\frac{\theta\cdot Q}{Q}\frac{1}{\deg Q}\cdot\theta_0\in\mathrm{Ann}\,(X)$. Since any projective module over a local ring is free, $D(X)$ is free if and only if $\mathrm{Ann}\,(X)$ is free.

We have an exact sequence

$$0\longrightarrow\mathrm{Ann}\,(X)\xrightarrow[\alpha]{}\mathcal{O}^n\xrightarrow[\beta]{}\mathcal{O}\xrightarrow[\gamma]{}\mathcal{O}/\mathfrak{a}\longrightarrow 0\ ,$$

where $\alpha\left(\sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i}\right)=\begin{bmatrix}f_1\\ \vdots\\ f_n\end{bmatrix}$ for $\sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i}\in\mathrm{Ann}\,(X)$,

$$\beta\left(\begin{bmatrix}g_1\\ \vdots\\ g_n\end{bmatrix}\right)=\sum_{i=1}^{n} g_i\frac{\partial Q}{\partial x_i}\qquad\text{for}\quad\begin{bmatrix}g_1\\ \vdots\\ g_n\end{bmatrix}\in\mathcal{O}^n\ ,$$

and γ is the natural projection. Thus Ann (X) is free if and only if the homological dimension of $\mathcal{O}/\mathfrak{a}$ is less than three.

Recall the Auslander-Buchsbaum equality

$$\text{depth}_{\mathcal{O}}(\mathcal{O}/\mathfrak{a})+\text{homolog.dim}_{\mathcal{O}}(\mathcal{O}/\mathfrak{a})=\text{Krull dim } \mathcal{O}=n\ ,$$

if $\mathcal{O}/\mathfrak{a}=0$.

On the other hand, we have Krull dim $(\mathcal{O}/\mathfrak{a})\leq n-2$, then

$$\begin{aligned} &\text{homolog.dim}_{\mathcal{O}}(\mathcal{O}/\mathfrak{a})\leq 2 \\ \Longleftrightarrow\ &\text{depth}_{\mathcal{O}}(\mathcal{O}/\mathfrak{a})\geq n-2\geq \text{Krull dim } (\mathcal{O}/\mathfrak{a}) \\ \Longleftrightarrow\ &\text{depth}_{\mathcal{O}}(\mathcal{O}/\mathfrak{a})=\text{Krull dim } (\mathcal{O}/\mathfrak{a})=n-2 \\ \Longleftrightarrow\ &\mathcal{O}/\mathfrak{a} \text{ is Cohen-Macaulay of dimension } (n-2)\ , \end{aligned}$$

if $\mathcal{O}/\mathfrak{a}\neq 0$. When $\mathcal{O}/\mathfrak{a}=0$, Ann (X) is obviously free. Thus we have proved 2.4.
Q.E.D.

PROPOSITION 2.5. *Let X be an arrangement in V. Then X is free at the origin of V if and only if it is free at every point of V.*

PROOF. Assume that $D(X)=\text{Der}_V(\log D_X)_0$ is free, then there exists an open neighborhood U of the origin such that $D_X\cap U$ is free at any point of U, because $\text{Der}_V(\log D_X)$ is coherent. Recall that X has $\boldsymbol{R}^*$-action in V because X is defined by a homogeneous polynomial. Thus we know that $\text{Der}_V(\log D_X)_y$ is a free $\mathcal{O}_{V,y}$-module for any $y\in V$. Q.E.D.

From now on we shall fix an arrangement X defined by Q in V and a hyperplane H through the origin which is not an element of X. Denote an arrangement $X\cup\{H\}$ by X'. Let h be a linear polynomial which defines H (i.e. $H=V(h)$), then X' is defined by hQ.

PROPOSITION 2.6. *Assume that X is free. Then an arrangement X' is free if and only if $\mathcal{O}/\mathfrak{a}(H)$ is 0 or $(n-2)$-dimensional Cohen-Macaulay, where*

$$\mathfrak{a}(H):=\{\theta\cdot h\in\mathcal{O};\ \theta\in D(X)\}\ .$$

PROOF. Consider the following two exact sequences:

$$(\alpha):\quad 0\longrightarrow D(X')\longrightarrow D(X)\longrightarrow D(X)/D(X')\longrightarrow 0\ ,$$

$$(\beta):\quad 0\longrightarrow D(X)/D(X')\xrightarrow{\ \cdot h\ }\mathcal{O}/h\mathcal{O}\longrightarrow\mathcal{O}/\mathfrak{a}(H)\longrightarrow 0\ ,$$

where $\cdot h$ stands for the map sending $\bar{\theta} \in D(X)/D(X')$ to $\overline{\theta \cdot h} \in \mathcal{O}/h\mathcal{O}$. In fact if $\theta \in D(X)$ and $\theta \cdot h \in h\mathcal{O}$, then $\theta \cdot (hQ) = h(\theta \cdot Q) + Q(\theta \cdot h) \in (hQ)\mathcal{O}$, thus $\theta \in D(X')$. By using elementary homological algebra we know that

$$\begin{aligned} & D(X') \text{ is free} \\ \Longleftrightarrow\ & \text{homolog. dim}_{\mathcal{O}}\,(D(X)/D(X')) \leq 1 \\ \Longleftrightarrow\ & \text{homolog. dim}_{\mathcal{O}}\,(\mathcal{O}/\mathfrak{a}(H)) \leq 2 \\ \Longleftrightarrow\ & \text{depth}_{\mathcal{O}}\,(\mathcal{O}/\mathfrak{a}(H)) \geq n-2\,, \end{aligned}$$

because of (α), (β), and the Auslander-Buchsbaum equality if $\mathcal{O}/\mathfrak{a}(H) \neq 0$.

If $\mathcal{O}/\mathfrak{a}(H) = 0$, it is obvious that $D(X')$ is free.

On the other hand, $\dim V(\mathfrak{a}(H)) \leq n-2$ as we will show in the next Proposition 2.9. This implies that Krull $\dim(\mathcal{O}/\mathfrak{a}(H)) \leq n-2$. Thus $D(X')$ is free if and only if

$$\text{depth}_{\mathcal{O}/\mathfrak{a}(H)}(\mathcal{O}/\mathfrak{a}(H)) = \text{depth}_{\mathcal{O}}(\mathcal{O}/\mathfrak{a}(H)) = \text{Krull}\dim(\mathcal{O}/\mathfrak{a}(H)) = n-2\,.$$

Q.E.D.

We need some more definitions before stating 2.9:

DEFINITION 2.7. Let A be an $(n-2)$-dimensional real vector subspace of V. Define the *multiplicity at* A of X by the number of the hyperplanes (of X) through A. It will be denoted by $\mu(A; X)$.

Put $X(A) := \{L \in X;\ L \supset A\}$, and it is obvious that $X(A)$ is $\emptyset$ or a hyperbolic arrangement whose order is $\mu(A; X)$.

Denote a defining equation of $X(A)$ by $Q(A)$. (If $X(A) = \emptyset$, put $Q(A) = 1$.)

DEFINITION 2.8. Take a subset Y of X satisfying the equality

$$\mu(H \cap L;\ Y) = \mu(H \cap L;\ X) - 1$$

for any $L \in X$. Denote a defining equation of Y by $\mathfrak{b}(H; X)$.

Such a subset Y cannot be uniquely determined, but an ideal $(h, \mathfrak{b}(H; X))$ of $\mathcal{O}$ is uniquely determined.

PROPOSITION 2.9. *The ideal* $\mathfrak{a}(H)$ *is contained in another ideal* $(h, \mathfrak{b}(H; X))$ *and these two ideals have the common radical. Moreover* $\mathcal{O}/\mathfrak{a}(H)$ *is* 0 *or* $(n-2)$-*dimensional Cohen-Macaulay if and only if* $\mathfrak{a}(H) = (h, \mathfrak{b}(H; X))$.

PROOF. We prove this by six steps:

Step 1: Assume that $X'=X\cup\{H\}$ is hyperbolic defined by $\prod_{i=1}^{m} l_i=0$ with $h=l_m$. Then we have

$$(h, b(H; X))=(l_m, l_2\cdots l_{m-1}) .$$

On the other hand, as we knew in the previous Example, $D(X)$ is spanned by the set

$$\left\{l_1\frac{\partial}{\partial l_1}+l_2\frac{\partial}{\partial l_2}+y_3\frac{\partial}{\partial y_3}+\cdots+y_n\frac{\partial}{\partial y_n},\ l_2\cdots l_{m-1}\frac{\partial}{\partial l_2},\ \frac{\partial}{\partial y_3},\cdots,\frac{\partial}{\partial y_n}\right\} .$$

This implies that 2.9 is true if X' is hyperbolic, so we shall assume that $n\geq 3$ from now on.

Step 2: Under the same assumptions as in 2.9, define

$$\mathcal{A}(H):=\{\text{the } (n-2)\text{-dimensional vector subspace } A \text{ of } V;\ A\subset H \text{ and } \mu(A; X)\geq 2\} .$$

Recall the definition 2.7 of $X(A)$. It is obvious that $X(A)\cap X(A')=\emptyset$ if A and A' are distinct elements of $\mathcal{A}(H)$. Moreover each $X(A)$ is a hyperbolic arrangement. Since X contains $X(A)$, $D(X)$ is contained in $D(X(A))$. Thus we have

$$\mathfrak{a}(H)=D(X)\cdot h\subset \bigcap_{A\in\mathcal{A}(H)} D(X(A))\cdot h=\bigcap_{A\in\mathcal{A}(H)}(h, b(H; X(A)))=(h, b(H; X))$$

because of the result of Step 1.

Step 3: Let $\mathfrak{b}(H)$ be the homogeneous ideal (of the coordinate ring of V) generated by

$$\left\{\frac{Q}{Q(A)};\ A\notin\mathcal{A}(H)\right\}\cup\{h\} .$$

For any $(n-2)$-dimensional vector subspace $A\notin\mathcal{A}(H)$, take a system of affine coordinate $(x_1,\cdots,x_n)$ of V such that $A\subset V(x_1)\cap V(x_2)$. Then we can assume that $h=a_1x_1+\cdots+a_nx_n$ with some $a_i\neq 0$ $(i\geq 3)$. A vector field $\frac{Q}{Q(A)}\frac{\partial}{\partial x_i}$ belongs to

$$D(X\setminus X(A))\cap D(X(A))=D(X) ,$$

and thus $\frac{Q}{Q(A)}$ belongs to $D(X)\cdot h=\mathfrak{a}(H)$. Moreover the Euler vector field is in $D(X)$, so h belongs to $\mathfrak{a}(H)$. Thus we have $\mathfrak{b}(H)\subset\mathfrak{a}(H)$.

Step 4: Assume that X is not hyperbolic. The zero points set of $(h, b(H; X))$, $V(h, b(H; X))$, is $\bigcup_{A\in\mathcal{A}(H)} A$. Take a point p of H outside $V(h, b(H; X))$. If $p\notin X$, then $p \notin V(\mathfrak{b}(H))$. If $p \in X$, then there exists a unique $(n-1)$-dimensional real vector subspace $L \in X$ such that p is on L. Since X is not hyperbolic, there is an $(n-2)$-dimensional real vector subspace A such that $\mu(A; X) \geq 2$, $p \notin A$, and thus $A \notin \mathcal{A}(H)$. And we know that $\frac{Q}{Q(A)}$ does not vanish at p, which means that $p \notin V(\mathfrak{b}(H))$. This proves that $V(\mathfrak{b}(H)) \subset V(h, b(H; X))$. The converse inclusion is obvious, so we have

$$V(\mathfrak{b}(H)) = V(h, b(H; X)) .$$

Thus two ideals $\mathfrak{b}(H)$ and $(h, b(H; X))$ have the common radical. On the other hand we have inclusions

$$\mathfrak{b}(H) \subset \mathfrak{a}(H) \subset (h, b(H; X)) ,$$

which imply that $\mathfrak{a}(H)$ and $(h, b(H; X))$ have the common radical.

Step 5: Define $\{A_1, \cdots, A_r\} = \mathcal{A}(H)$ and $\mu(i) = \mu(A_i; X)$ $(i=1, \cdots, r)$. Take a subset $\{H_1, \cdots, H_r\}$ of X such that $A_i \subset H_i$ $(i=1, \cdots, r)$. Denote a defining equation of H_i by h_i, then one choice of $b(H; X)$ is $\prod_{i=1}^{r} h_i^{\mu(i)-1}$. Apply Noether's primary decomposition theorem, and we have

$$(h, b(H; X)) = \bigcap_{i=1}^{r} (h, h_i^{\mu(i)-1}) .$$

This implies that the height of all prime divisors of the ideal $(h, b(H; X))$ is two unless $(h, b(H; X)) = \mathcal{O}$. Thus we know that $\mathcal{O}/\mathfrak{a}(H)$ is $(n-2)$-dimensional Cohen-Macaulay or 0 if $\mathfrak{a}(H) = (h, b(H; X))$.

Step 6: Assume that $\mathcal{O}/\mathfrak{a}(H)$ is $(n-2)$-dimensional Cohen-Macaulay. (If $\mathcal{O}/\mathfrak{a}(H) = 0$, then $\mathfrak{a}(H) = (h, b(H; X)) = \mathcal{O}$.) Apply again Noether's primary decomposition theorem, and we have

$$\mathfrak{a}(H) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r ,$$

where $V(\mathfrak{q}_i) = A_i$ $(i=1, \cdots, r)$. Notice that $\sqrt{\mathfrak{q}_i} = (h, h_i)$.

For any A_i, choose an $(n-2)$-dimensional real vector subspace A such that $A \subset H_i$, $\mu(A; X) \geq 2$, and $A \notin \mathcal{A}(H)$. Then $\frac{Q}{Q(A)}$ can be divided by $\frac{Q(A_i)}{h_i}$. The quotient $\frac{Qh_i}{Q(A)Q(A_i)}$ does not vanish at A_i. This implies that $\frac{Qh_i}{Q(A)Q(A_i)}$ does not belong to $\sqrt{\mathfrak{q}_i}$ and that $\frac{Q}{Q(A)} \in \mathfrak{b}(H) \subset \mathfrak{a}(H) \subset \mathfrak{q}_i$. Thus we have $\frac{Q(A_i)}{h_i} \in \mathfrak{q}_i$ because $\mathfrak{q}_i$ is primary. Since

$$(h, h_i^{\mu(i)-1}) = \left(h, \frac{Q(A_i)}{h_i}\right) \subset \mathfrak{q}_i ,$$

we deduce that

$$(h, \mathfrak{b}(H; X)) = \bigcap_{i=1}^{r} (h, h_i^{\mu(i)-1}) \subset \bigcap_{i=1}^{r} \mathfrak{q}_i = \mathfrak{a}(H) . \qquad \text{Q.E.D.}$$

For each arrangement we shall attach two sequences of natural numbers. At first we state the following

DEFINITION 2.10. Let θ be an element of $D(X)$. By using the affine coordinate $(x_1, \cdots, x_n)$ of V, θ can be described as

$$\theta = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \quad \text{with } f_1, \cdots, f_n \in \mathcal{O} .$$

We say that θ is *homogeneous of degree* m if each f_i is homogeneous polynomial of degree m with respect to $(x_1, \cdots, x_n)$.

DEFINITION 2.11. Let $(\theta_1, \theta_2, \cdots)$ be a sequence of homogeneous elements of $D(X)$. The sequence $(\theta_1, \theta_2, \cdots)$ is *regular* if

$$\deg \theta_i = \text{Min}\,\{\deg \theta;\ \theta \text{ is a homogeneous element of } D(X) \text{ and } \theta \notin \mathcal{O}\theta_1 + \cdots + \mathcal{O}\theta_{i-1}\}$$

for any $i=1, 2, \cdots$. (When $i=1$, define $\mathcal{O}\theta_1 + \cdots + \mathcal{O}\theta_{i-1} = 0$.)

Then there exists a maximal regular sequence $(\eta_1, \eta_2, \cdots)$ composing of homogeneous elements of $D(X)$. The increasing sequence of numbers $(\deg \eta_1, \deg \eta_2, \cdots)$ is called the *generator sequence* of X.

The generator sequence of X is well-defined in the next sense:

PROPOSITION 2.12. *The generator sequence does not depend on the choice of the maximal regular sequence of $D(X)$, and is a finite sequence.*

PROOF. The latter part of this proposition follows from the fact that $D(X)$ is a Noetherian module over $\mathcal{O}$.

As for the former part, let M_i be the submodule of $D(X)$ generated by $\{\theta \in D(X);\ \deg \theta \le i\}$. Then the integer i appears $\dim_R(M_i/M_{i-1})$-times in the generator sequence. Q.E.D.

Next we will define the other sequence of the natural numbers attached to an arrangement X in V.

DEFINITION 2.13. Define θ_i inductively by the equalities

$$(*)\quad \deg\theta_i = \mathrm{Min}\,\{\deg\theta;\ \theta \text{ is a homogeneous element of } D(X),\ \{\theta_1, \theta_2, \cdots, \theta_{i-1}, \theta\} \text{ is } \mathcal{O}\text{-independent}\}$$

for $i=1, 2, \cdots$. The maximal increasing sequence $(\deg\theta_1, \deg\theta_2, \cdots)$ is called the *structure sequence* of X.

The structure sequence is also well-defined:

PROPOSITION 2.14. *The structure sequence does not depend on the choices of* $(\theta_1, \theta_2, \cdots)$ *satisfying the condition* $(*)$, *and is the sequence of the increasing* n *natural numbers.*

PROOF. Let $(\theta_1, \theta_2, \cdots)$ and $(\theta'_1, \theta'_2, \cdots)$ be two sequences in $D(X)$ satisfying $(*)$. At first $\deg\theta_1=\deg\theta'_1$ by the very definition of θ_1 and θ'_1. Next assume that $\deg\theta_j=\deg\theta'_j$ $(j=1, \cdots, i)$ and $\deg\theta_{i+1}>\deg\theta'_{i+1}$. Then $\{\theta_1, \theta_2, \cdots, \theta_i, \theta'_{i+1}\}$ is an $\mathcal{O}$-dependent subset of $D(X)$, thus

$$f_{i+1}\theta'_{i+1} \in \mathcal{O}\theta_1 + \cdots + \mathcal{O}\theta_i$$

for some $f_{i+1} \in \mathcal{O}\setminus\{0\}$. By the same reason, $\{\theta_1, \theta_2, \cdots, \theta_i, \theta'_1\}$, $\{\theta_1, \theta_2, \cdots, \theta_i, \theta'_2\}, \cdots$, and $\{\theta_1, \theta_2, \cdots, \theta_i, \theta'_i\}$ are all $\mathcal{O}$-dependent subsets of $D(X)$, thus we have

$$f_j\theta'_j \in \mathcal{O}\theta_1 + \cdots + \mathcal{O}\theta_i$$

for some $f_j \in \mathcal{O}\setminus\{0\}$ $(j=1, \cdots, i)$. These facts imply that

$$\mathcal{O}f_1\theta'_1 + \mathcal{O}f_2\theta'_2 + \cdots + \mathcal{O}f_{i+1}\theta'_{i+1} \subset \mathcal{O}\theta_i + \cdots + \mathcal{O}\theta_i\,,$$

but the left handside is a free $\mathcal{O}$-module of rank $i+1$, while the right handside is a free $\mathcal{O}$-module of rank i. This is a contradiction.

As for the latter part of 2.14, we can easily see that $\{\theta_1, \theta_2, \cdots\}$ is a maximal $\mathcal{O}$-independent subset of $D(X)$. On the other hand the rank of the $\mathcal{O}$-module $D(X)$ equals to n. Q.E.D.

Then the free arrangement can be characterized as follows:

PROPOSITION 2.15 (K. Saito). *Let* $(d_1, \cdots, d_n)$ *be the structure sequence of an arrangement* X. *Then we have* $d_1+\cdots+d_n \geq |X|$. *Then* X *is free if and only if* $d_1+\cdots+d_n=|X|$.

The proof can be seen in [5].

DEFINITION 2.16. The *restriction* of X to H is the arrangement $\bigcup_{L\in X}\{H\cap L\}$

in the $(n-1)$-dimensional real vector space H. Denote this arrangement in H by X_H.

DEFINITION 2.17. The *chambers* of an arrangement X are the connected components of $V\setminus \bigcup_{L\in X} L$. Denote the number of the chambers of X by $f(X)$.

We need three more lemmas before Removal Theorem and Addition Theorem.

LEMMA 2.18. *We have*

$$f(X)+f(X_H)=f(X') ,$$

and

$$\sigma(H;\, X)+|X_H|=|X| , \quad \textit{where} \quad \sigma(H;\, X)=\deg b(H;\, X) .$$

PROOF. Since every chamber of an arrangement is convex, $f(X')-f(X)$ equals to the number of the chambers (of X) which H intersects. This implies the first equality.

Recall that $\sigma(H;\, X)=\sum_{A\in\mathcal{A}(H)}(\mu(A;\, X)-1)$, and we immediately have the second equality. Q.E.D.

The following lemma is obvious:

LEMMA 2.19. *Let θ be an element of $D(X')$. Then the restriction $\theta(H)$ of θ to H belongs to $D(X_H)$.*

LEMMA 2.20. *Assume that X_H is free with its structure sequence $(d_1, \cdots, d_{n-1})$. Let $(\theta_1, \cdots, \theta_k)$ $(k\leq n-1)$ be a sequence composed of homogeneous elements of $D(X')$ and $d_i=\deg\theta_i$ $(i=1, \cdots, k)$. Then the following two conditions are equivalent:*

(1) *a sequence $(\theta_1(H), \cdots, \theta_k(H))$ is regular in $D(X_H)$,*

(2) $\theta_i-\sum_{j=1}^{i-1} f_j\theta_j \notin h\cdot D(X)$ *for any $i=1, \cdots, k$, and any $f_1, \cdots, f_{i-1}\in\mathcal{O}$.*

PROOF. A vector field $\theta_i-\sum_{j=1}^{i-1} f_j\theta_j$ belongs to $h\cdot D(X)$ if and only if $\theta_i(H)=\sum_{j=1}^{i-1}\bar{f}_j\theta_j(H)$ in $D(X_H)$, where each $\bar{f}_j$ is the image of the natural projection of $\mathcal{O}$ onto $\mathcal{O}/h\mathcal{O}$. Notice that $\deg\theta_i(H)=\deg\theta_i=d_i$ for $i=1, \cdots, k$. These facts prove 2.20 in the light of the regularity. Q.E.D.

REMOVAL THEOREM. *Assume that X' is free with the structure sequence $(d_1, \cdots, d_n)$, then the following two conditions are equivalent:*

(1) *X is free,*

(2) *X_H is free with the structure sequence $(d_1, \cdots, \hat{d}_i, \cdots, d_n)$ for some i $(1 \leq i \leq n)$, where $\hat{d}_i$ implies the lack of d_i.*

Moreover when both of these two conditions hold, the structure sequence of X is $(d_1, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_n)$.

PROOF. (1)$\Rightarrow$(2): Assume that X is free, then $\mathfrak{a}(H)=(h, b(H; X))$ because of 2.9. So there exists a homogeneous element η of $D(X)$ such that $\eta \cdot h = b(H; X)$. Let $\{\theta_1, \cdots, \theta_n\}$ be a system of homogeneous basis of $D(X)$ such that $\deg \theta_1 \leq \deg \theta_2 \leq \cdots \leq \deg \theta_n$. Notice that $(\theta_1 \cdot h, \cdots, \theta_n \cdot h) = (h, b(H; X))$. If $\theta_j \cdot h = g_j h + k_j b(H; X)$ with $g_j, k_j \in \mathcal{O}$ and $j \in \{1, \cdots, n\}$, then $(\theta_j - k_j \eta) \cdot h = g_j h$, thus $\theta_j - k_j \eta \in D(X')$. At last we can choose a system of homogeneous basis $\{\theta_1, \cdots, \theta_n\}$ of $D(X)$, where $\theta_i = \eta$ for some i $(1 \leq i \leq n)$ and $\theta_j \in D(X')$ for any $j \neq i$. By applying the criterion of 2.15, we know that a subset

$$\{\theta_1, \cdots, \theta_{i-1}, h\theta_i, \theta_{i+1}, \cdots, \theta_n\}$$

of $D(X')$ gives a system of homogeneous basis of $D(X')$. Thus we know that the structure sequence of X is $(d_1, d_2, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_n)$.

Consider a sequence

$$B = (\theta_1(H), \cdots, \theta_{i-1}(H), \theta_{i+1}(H), \cdots, \theta_n(H))$$

composed of homogeneous elements of $D(X_H)$.

Then B is $\mathcal{O}/h\mathcal{O}$-independent. In fact if

$$\sum_{j \neq i} \bar{f}_j \theta_j(H) = 0 \in D(X_H) ,$$

then $\sum_{j \neq i} f_j \theta_j \in h \cdot D(X)$, where $f_j \in \mathcal{O}$ $(j \neq i)$. Since $\{\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_n\}$ is a part of a system of basis of $D(X)$, we have $f_j \in h\mathcal{O}$ and $\bar{f}_j = 0$ in $\mathcal{O}/h\mathcal{O}$ for $j \neq i$.

On the other hand, we have

$$\begin{aligned} \sum_{j \neq i} \deg \theta_j(H) &= \sum_{j \neq i} \deg \theta_j = \left(\sum_{j=1}^{n} \deg \theta_j \right) - \deg \eta = |X| - \deg b(H; X) \\ &= |X| - \sigma(H; X) = |X_H| , \end{aligned}$$

because of 2.15 and 2.18. This implies that B is the structure sequence of X_H, and X_H is free in the light of 2.15.

(2)$\Rightarrow$(1): Let $\{\theta_1, \cdots, \theta_n\}$ be a system of homogeneous basis of $D(X')$ with $\deg \theta_i = d_i$ $(i = 1, \cdots, n)$. We can assume that $d_i < d_{i+1}$ or $i = n$.

Assume that X is not free, then

$$\theta_i - \sum_{j<i} f_j \theta_j \notin h \cdot D(X)$$

for any $i=1, \cdots, n$ and any $f_1, \cdots, f_{i-1} \in \mathcal{O}$. In fact if $\theta_i - \sum_{j<i} f_j \theta_j \in h \cdot D(X)$, then a subset

$$\left\{\theta_1, \cdots, \theta_{i-1}, \frac{\theta_i - \sum_{j<i} f_j \theta_j}{h}, \theta_{i+1}, \cdots, \theta_n\right\}$$

of $D(X)$ is $\mathcal{O}$-independent and

$$\begin{aligned} &\deg \theta_1 + \cdots + \deg \theta_{i-1} + \deg \frac{\theta_i - \sum_{j<i} f_j \theta_j}{h} + \deg \theta_{i+1} + \cdots + \deg \theta_n \\ &= \left(\sum_{i=1}^{n} \deg \theta_i\right) - 1 = |X'| - 1 = |X| . \end{aligned}$$

Thus X is free because of 2.15, which contradicts our assumption.

By applying 2.20, we know that a sequence $(\theta_1(H), \cdots, \theta_{i-1}(H))$ is regular and that

$$\theta_i(H) \notin (\mathcal{O}/h\mathcal{O})\theta_1(H) + \cdots + (\mathcal{O}/h\mathcal{O})\theta_{i-1}(H) .$$

Since the generator sequence of X_H is $(d_1, \cdots, \hat{d}_i, \cdots, d_n)$, we have $d_{i+1} \leq \deg \theta_i(H) = d_i$ and $i<n$. This contradicts our assumptions. Q.E.D.

Next we shall prove

ADDITION THEOREM. *Assume that X is free with the structure sequence $(d_1, \cdots, d_n)$, then the following two conditions are equivalent:*

(1) *X' is free,*

(2) *X_H is free with the structure sequence $(d_1, \cdots, \hat{d}_i, \cdots, d_n)$ for some i $(1 \leq i \leq n)$.*

Moreover when both of these two conditions hold, the structure sequence of X' is $(d_1, \cdots, d_{i-1}, d_i+1, d_{i+1}, \cdots, d_n)$.

PROOF. (1)$\Rightarrow$(2): Since both X' and X are free, X' has a sequence $(d_1, \cdots, d_{i-1}, d_i+1, d_{i+1}, \cdots, d_n)$ as its structure sequence for some i $(1 \leq i \leq n)$ because of Removal Theorem. And apply Removal Theorem again.

(2)$\Rightarrow$(1): We can assume that $d_i < d_{i+1}$ or $i=n$. By 2.18, we have

$$\sigma(H; X) = |X| - |X_H| = \left(\sum_{i=1}^{n} d_i\right) - (d_1 + \cdots + \hat{d}_i + \cdots + d_n) = d_i .$$

If there exists an element φ of $D(X)$ such that $\varphi\cdot h \notin h\mathcal{O}$ and $\deg \varphi = d_i$, put $\varphi\cdot h = gh + c\,b(H; X)$ with $g \in \mathcal{O}$ and $c \in \boldsymbol{R}\setminus\{0\}$. Then we have

$$\frac{1}{c}(\varphi - g\theta_0)\cdot h = \frac{1}{c}(\varphi\cdot h - g(\theta_0\cdot h)) = b(H; X)\,,$$

where θ_0 is the Euler vector field. Thus X' is free because of 2.9.

Assume that X′ is not free, and any element (of $D(X)$) whose degree is no more than d_i belongs to $D(X')$. Let $\{\theta_1, \cdots, \theta_n\}$ be a system of basis of $D(X)$ with $\deg \theta_i = d_i$ for any $i=1, \cdots, n$. Since $\{\theta_1, \cdots, \theta_i\} \subset D(X')$, $\{\theta_1(H), \cdots, \theta_i(H)\} \subset D(X_H)$.

On the other hand, $\theta_k - \sum_{j<k} f_j\theta_j \notin h\cdot D(X)$ for any $k=1, \cdots, i$ and any $f_1, \cdots, f_k \in \mathcal{O}$. (In fact if $\theta_k - \sum_{j<k} f_j\theta_j$ belongs to $h\cdot D(X)$, then $\dfrac{\theta_k - \sum_{j<k} f_j\theta_j}{h}$ belongs to $D(X)$. This contradicts the definition of θ_k.) By applying 2.20, we know that $(\theta_1(H), \cdots, \theta_{i-1}(H))$ is a regular sequence and that

$$\theta_i(H) \notin (\mathcal{O}/h\mathcal{O})\theta_1(H) + \cdots + (\mathcal{O}/h\mathcal{O})\theta_{i-1}(H)\,.$$

Since the generator sequence of X_H is $(d_1, \cdots, \hat{d}_i, \cdots, d_n)$, we have $d_{i+1} \leq \deg \theta_i(H) = d_i$ and $i<n$. This contradicts our assumptions. Q.E.D.

THEOREM 2.21. *An arrangement of fiber type is free.*

PROOF. Let X be an arrangement of fiber type defined by an equation $\prod_{i=1}^{m} l_i \cdot \prod_{j=1}^{k} l'_j = 0$ as in 1.2. An arrangement defined by $\prod_{i=1}^{m} l_i = 0$ is hyperbolic with its structure sequence $(0, \cdots, 0, 1, m-1)$. By applying Addition Theorem inductively, we have $(0, \cdots, 0, 1, k, m-1)$ as the structure sequence of X. Q.E.D.

In case that $n=3$, we have

REMOVAL THEOREM ($n=3$). *Assume the same assumptions as in the general removal theorem and put $n=3$. Then the following two conditions are equivalent:*

(1) *X is free,*

(2) *$\sigma(H; X) = d_1-1, d_2-1,$ or d_3-1.*

Moreover when $\sigma(H; X) = d_1-1, d_2-1,$ or d_3-1, the structure sequence of X is (d_1-1, d_2, d_3), (d_1, d_2-1, d_3) or (d_1, d_2, d_3-1) respectively.

ADDITION THEOREM ($n=3$). *Assume the same assumptions as in the general addition theorem and put $n=3$. Then the following two conditions are equivalent:*

(1) *X' is free,*

(2) $\sigma(H; X)=d_1, d_2,$ *or* d_3.

Moreover when $\sigma(H; X)=d_1, d_2,$ *or* d_3, *the structure sequence of* X' *is* (d_1+1, d_2, d_3), (d_1, d_2+1, d_3), *or* (d_1, d_2, d_3+1) *respectively.*

The proofs are easily derived from the fact that any arrangement in 2-dimensional real vector space is free.

These two criteria are very useful for determining whether a given arrangement is free or not. In fact we can determine all (except one — $A_2(37)$ (see [4]) —) free simplicial arrangements whose orders are less than 38 (see the Table 1). Consequently we have found some simplicial arrangements which are not free. They give the counterexamples of the "only if" part of the conjecture which was proposed by K. Saito ([6]).

EXAMPLE. Put $V=\boldsymbol{R}^3$ and $Q=xyz(x-y)(x-z)(y-z)(x+y-z)$. This arrangement is called $A_1(7)$ in [4].

At first the arrangement X_0 defined by $Q_0=xz(x-z)$ is hyperbolic, then free with its structure sequence $(0, 1, 2)$.

Put $H_1=V(y)$ and $H_2=V(y-z)$. Define $X_1=X_0\cup\{H_1\}$, then $\sigma(H_1; X_0)=0$. By applying Addition Theorem, we know that X_1 is free with its structure sequence $(1, 1, 2)$.

Next define $X_2=X_1\cup\{H_2\}$, then $\sigma(H_2; X_1)=1$. By applying Addition Theorem again, we know that X_2 is free with its structure sequence $(1, 2, 2)$.

Thus we can inductively apply Addition Theorem and deduce that $A_1(7)$ is free and that its structure sequence is $(1, 3, 3)$.

When one is staring at Table 1, one may be aware of the equality

$$f(X)=(d_1+1)(d_2+1)(d_3+1)$$

if X is free. We can prove this equality when X belongs to the following class:

DEFINITION 2.22. A free arrangement X in the three-dimensional real vector space V is called to be *inductive* if there exist the free arrangements $X_1, \cdots, X_m$ such that:

$$X=X_0\supset X_1\supset X_2\supset\cdots\supset X_m,$$

$$|X_i|+1=|X_{i-1}| \quad \text{for } i=1, \cdots, m, \text{ and } |X_m|=1.$$

For example, the arrangement of fiber type is inductive.

Table 1. All simplicial free arrangements in the three-dimensional real vector space whose order are less than 38

$\|X\|$	Name of X	$f(X)$	Structure sequence of X
3	$A_0(3)$	8	(1, 1, 1)
4	$A_0(4)$	12	(1, 1, 2)
5	$A_0(5)$	16	(1, 1, 3)
6	$A_0(6)$	20	(1, 1, 4)
	$A_1(6)$	24	(1, 2, 3)
7	$A_0(7)$	24	(1, 1, 5)
	$A_1(7)$	32	(1, 3, 3)
8	$A_0(8)$	28	(1, 1, 6)
	$A_1(8)$	40	(1, 3, 4)
9	$A_0(9)$	32	(1, 1, 7)
	$A_1(9)$	48	(1, 3, 5)
10	$A_0(10)$	36	(1, 1, 8)
	$A_1(10)$	60	(1, 4, 5)
	$A_2(10)$	60	(1, 4, 5)
	$A_3(10)$	60	(1, 4, 5)
11	$A_0(11)$	40	(1, 1, 9)
	$A_1(11)$	72	(1, 5, 5)
12	$A_0(12)$	44	(1, 1, 10)
	$A_1(12)$	84	(1, 5, 6)
	$A_2(12)$	84	(1, 5, 6)
	$A_3(12)$	84	(1, 5, 6)
13	$A_0(13)$	48	(1, 1, 11)
	$A_1(13)$	96	(1, 5, 7)
	$A_2(13)$	96	(1, 5, 7)
	$A_3(13)$	96	(1, 5, 7)
14	$A_0(14)$	52	(1, 1, 12)
	$A_1(14)$	112	(1, 6, 7)
	$A_2(14)$	112	(1, 6, 7)
	$A_4(14)$	112	(1, 6, 7)
15	$A_0(15)$	56	(1, 1, 13)
	$A_1(15)$	120	(1, 5, 9)
15	$A_2(15)$	128	(1, 7, 7)
	$A_4(15)$	128	(1, 7, 7)
16	$A_0(16)$	60	(1, 1, 14)
	$A_1(16)$	144	(1, 7, 8)
	$A_2(16)$	144	(1, 7, 8)
	$A_3(16)$	144	(1, 7, 8)
	$A_4(16)$	140	(1, 6, 9)
	$A_5(16)$	144	(1, 7, 8)
	$A_6(16)$	144	(1, 7, 8)
17	$A_0(17)$	64	(1, 1, 15)
	$A_1(17)$	160	(1, 7, 9)
	$A_2(17)$	160	(1, 7, 9)
	$A_3(17)$	160	(1, 7, 9)
	$A_4(17)$	160	(1, 7, 9)
	$A_5(17)$	160	(1, 7, 9)
	$A_7(17)$	160	(1, 7, 9)
18	$A_0(18)$	68	(1, 1, 16)
	$A_1(18)$	180	(1, 8, 9)
	$A_3(18)$	180	(1, 8, 9)
	$A_4(18)$	180	(1, 8, 9)
	$A_5(18)$	180	(1, 8, 9)
	$A_7(18)$	180	(1, 8, 9)
19	$A_0(19)$	72	(1, 1, 17)
	$A_1(19)$	192	(1, 7, 11)
	$A_2(19)$	200	(1, 9, 9)
	$A_3(19)$	192	(1, 7, 11)
	$A_4(19)$	200	(1, 9, 9)
	$A_5(19)$	200	(1, 9, 9)
	$A_6(19)$	200	(1, 9, 9)
20	$A_0(20)$	76	(1, 1, 18)
	$A_1(20)$	220	(1, 9, 10)
	$A_2(20)$	220	(1, 9, 10)
	$A_3(20)$	220	(1, 9, 10)
	$A_4(20)$	220	(1, 9, 10)

(to be continued)

Table 1 (Continued)

$\lvert X \rvert$	Name of X	$f(X)$	Structure sequence of X
20	$A_5(20)$	216	(1, 8, 11)
21	$A_0(21)$	80	(1, 1, 19)
	$A_1(21)$	240	(1, 9, 11)
	$A_2(21)$	240	(1, 9, 11)
	$A_3(21)$	240	(1, 9, 11)
	$A_4(21)$	240	(1, 9, 11)
	$A_5(21)$	240	(1, 9, 11)
22	$A_0(22)$	84	(1, 1, 20)
	$A_1(22)$	264	(1, 10, 11)
	$A_3(22)$	264	(1, 10, 11)
	$A_4(22)$	264	(1, 10, 11)
23	$A_0(23)$	88	(1, 1, 21)
24	$A_0(24)$	92	(1, 1, 22)
	$A_1(24)$	312	(1, 11, 12)
25	$A_0(25)$	96	(1, 1, 23)
	$A_1(25)$	336	(1, 11, 13)
	$A_2(25)$	336	(1, 11, 13)
	$A_4(25)$	336	(1, 11, 13)
	$A_5(25)$	320	(1, 9, 15)
	$A_6(25)$	336	(1, 11, 13)
	$A_7(25)$	336	(1, 11, 13)
26	$A_0(26)$	100	(1, 1, 24)
	$A_1(26)$	364	(1, 12, 13)
	$A_3(26)$	364	(1, 12, 13)
	$A_4(26)$	364	(1, 12, 13)
27	$A_0(27)$	104	(1, 1, 25)
	$A_2(27)$	392	(1, 13, 13)
	$A_3(27)$	392	(1, 13, 13)
	$A_4(27)$	392	(1, 13, 13)

$\lvert X \rvert$	Name of X	$f(X)$	Structure sequence of X
28	$A_0(28)$	108	(1, 1, 26)
	$A_1(28)$	420	(1, 13, 14)
	$A_4(28)$	420	(1, 13, 14)
	$A_5(28)$	420	(1, 13, 14)
	$A_6(28)$	420	(1, 13, 14)
29	$A_0(29)$	112	(1, 1, 27)
	$A_1(29)$	448	(1, 13, 15)
	$A_3(29)$	448	(1, 13, 15)
	$A_4(29)$	448	(1, 13, 15)
	$A_5(29)$	448	(1, 13, 15)
30	$A_0(30)$	116	(1, 1, 28)
	$A_1(30)$	480	(1, 14, 15)
	$A_3(30)$	476	(1, 13, 16)
31	$A_0(31)$	120	(1, 1, 29)
	$A_1(31)$	480	(1, 11, 19)
	$A_2(31)$	504	(1, 13, 17)
	$A_3(31)$	504	(1, 13, 17)
32	$A_0(32)$	124	(1, 1, 30)
	$A_1(32)$	544	(1, 15, 16)
33	$A_0(33)$	128	(1, 1, 31)
	$A_1(33)$	576	(1, 15, 17)
34	$A_0(34)$	132	(1, 1, 32)
	$A_1(34)$	612	(1, 16, 17)
	$A_2(34)$	612	(1, 16, 17)
35	$A_0(35)$	136	(1, 1, 33)
36	$A_0(36)$	140	(1, 1, 34)
	$A_1(36)$	684	(1, 17, 18)
37	$A_0(37)$	144	(1, 1, 35)
	$A_1(37)$	720	(1, 17, 19)
	$A_3(37)$	720	(1, 17, 19)

(We do not know whether $A_2(37)$ is free or not.)
(The names of the simplicial arrangements above are due to B. Grünbaum ([4]).)

PROPOSITION 2.23. *If X is an inductive free arrangement with its structure sequence (d_1, d_2, d_3), then $f(X)=(d_1+1)(d_2+1)(d_3+1)$.*

PROOF. When $|X_m|=1$, the structure sequence of X_m is $(0, 0, 1)$ and $f(X_m)=2$. Thus the equality holds. Assume that the equality holds in the case of X_{k+1} $(k=0, \cdots, m-1)$. Let (d_1, d_2, d_3) be the structure sequence of X_{k+1}. We divide our situation into four cases:

$$\begin{aligned}
&\text{Case 1:} \quad d_1=0\,,\\
&\text{Case 2:} \quad d_1=1<d_2\,,\\
&\text{Case 3:} \quad d_1=d_2=1\,,\\
&\text{Case 4:} \quad d_1\geq 2\,.
\end{aligned}$$

Because the Euler operator belongs to $D(X_{k+1})$, Case 4 cannot happen.

In Case 1, we can assume that $\frac{\partial}{\partial z}$ belongs to $D(X_{k+1})$ for a suitable coordinate (x, y, z). Thus we know that X_{k+1} is hyperbolic and that $(d_1, d_2, d_3)=(0, 1, |X_{k+1}|-1)$. Put $H=X_k\setminus X_{k+1}$, then $\sigma(H; X_{k+1})=0$ or $|X_{k+1}|-1$.

Case 1.1: If $\sigma(H; X_{k+1})=0$, then the structure sequence of X_k is $(1, 1, |X_{k+1}|-1)$. Thus we have

$$\begin{aligned}
f(X_k)&=f(X_{k+1})+f((X_{k+1})_H)\\
&=2|X_{k+1}|+2|X_{k+1}|\\
&=4|X_{k+1}|\\
&=(1+1)(1+1)\{(|X_{k+1}|-1)+1\}\,.
\end{aligned}$$

Case 1.2: If $\sigma(H; X_{k+1})=|X_{k+1}|-1$, then the structure sequence of X_k is $(0, 1, |X_{k+1}|)$. Thus we have

$$\begin{aligned}
f(X_k)&=f(X_{k+1})+f((X_{k+1})_H)\\
&=2|X_{k+1}|+2\\
&=(0+1)(1+1)(|X_{k+1}|+1)\,.
\end{aligned}$$

In Case 2, if $\sigma(H; X_{k+1})=1$, then the structure sequence of X_k does not contain 1. This is contradiction.

Thus in Case 2 or 3 we can assume that $d_1=1$ and $\sigma(H; X_{k+1})=d_2$ or d_3. Notice that

$$f((X_{k+1})_H)=2(|X_{k+1}|-\sigma(H; X_{k+1}))\,.$$

For example, assume that $\sigma(H; X_{k+1})=d_2$. Then we have

$$
\begin{aligned}
f(X_k) &= f(X_{k+1}) + f((X_{k+1})_H) \\
&= (1+1)(d_2+1)(d_3+1) + 2\{(1+d_2+d_3) - d_2\} \\
&= (1+1)(d_2+2)(d_3+1) .
\end{aligned}
$$

This proves 2.23 because the structure sequence of X_k is $(1, d_2+1, d_3)$.

Almost all free arrangements seem to be inductive so far as we know. But we have only one (by now) free arrangement which is not inductive (see Figure 1).

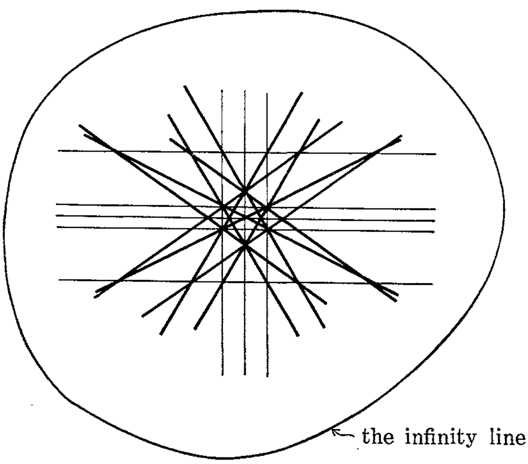

Figure 1
(This figure is in $P^2(\boldsymbol{R})$ via the natural projection of $\boldsymbol{R}^3 \setminus \{0, 0, 0\}$ onto $P^2(\boldsymbol{R})$.)

REMARK. Let W be a finite subgroup of $\boldsymbol{GL}(n; \boldsymbol{R})$ generated by reflexions. Then the set of the invariant hyperplanes with respect to the reflexions in W makes an arrangement X in $\boldsymbol{R}^n$. In [6], K. Saito proved that such an arrangement is free. Moreover its structure sequence $(d_1, \cdots, d_n)$ coincides with the sequence of the power indices of X. In this case the equality

$$f(X) = \sharp W = \prod_{i=1}^{n} (1+d_i)$$

is well-known (for example see [1]).

ADDENDUM. (13 June 1980) Recently we have succeeded in proving an equality

$$f(X) = \prod_{i=1}^{n} (1+d_i)$$

for any free arrangement X with its structure sequence $(d_1, \cdots, d_n)$ (see [8] when $n=3$ and [9] for general n).

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(Received February 13, 1979)

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