Arrangements of hyperplanes and their freeness II
—the Coxeter equality—

By Hiroaki TERAO

(Communicated by N. Iwahori)

§ 1. Introduction

This note is a sequel to [5].

Let V be an n-dimensional ($n \geq 2$) real vector space. We can naturally assume V as an affine space over $\boldsymbol{R}$.

Let X be an arrangement in V, i.e., a finite family of $(n-1)$-dimensional real vector subspaces of V.

Fix an affine coordinate $(x_1, \cdots, x_n)$ of V. Let $Q \in \boldsymbol{R}[x_1, \cdots, x_n]$ be a square-free non-zero polynomial which is a product of linear polynomials such that Q defines X.

Consider a reduced divisor $D = V(Q)$ of a complex affine manifold $V_{\boldsymbol{C}} = V \underset{\boldsymbol{R}}{\otimes} \boldsymbol{C}$. We say that X is free if $\Omega^1_{V_{\boldsymbol{C}}}(\log D)_{V_{\boldsymbol{C}},0}$ is a free $\mathcal{O}_{V_{\boldsymbol{C}},0}$-module. Another $\mathcal{O}_{V_{\boldsymbol{C}},0}$-module

$$D(X) = \mathrm{Der}_{V_{\boldsymbol{C}}}(\log D)_{V_{\boldsymbol{C}},0} = \{\text{the germ at } 0 \text{ of the regular vector field } \theta;\ \theta \cdot Q \in Q \cdot \mathcal{O}_{V_{\boldsymbol{C}},0}\}$$

is free if and only if X is free ([4]).

We introduce a graduation into $D(X)$ by

$$D(X)_\nu = \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \in D(X);\ \deg f_i = \nu \quad \text{or} \quad f_i = 0 \ (i = 1, \cdots, n) \right\}.$$

If $\theta \in D(X)_\nu$, we say that $\deg \theta = \nu$. Similarly we denote the subset of elements of degree ν of a homogeneous ideal I by I_ν in this note, e.g., $(x_1, \cdots, x_n)_\nu$ is the set of the zero polynomial and homogeneous polynomials of degree ν.

When X is free, let $\{\theta_0, \theta_1, \theta_2, \theta_3, \cdots, \theta_{n-1}\}$ be a system of basis of $D(X)$. Put $d_i = \deg \theta_i$ and assume that $d_0 \leq d_1 \leq d_2 \leq \cdots \leq d_{n-1}$. Then the sequence $(d_0, d_1, d_2, \cdots, d_{n-1})$, called the structure sequence of X, depends only on X ([5]).

Let

$$f_{n-1} = \#\{\text{connected component of } V \setminus \bigcup_{H \in X} H\}.$$

Then the equality

$$f_{n-1}=\prod_{i=0}^{n-1}(d_i+1)$$

is called the Coxeter equality. In this note we shall prove the Coxeter equality when $n=3$.

REMARK. Assume that an inner product is defined on V. Let G be a finite subgroup of $GL(V)$ generated by orthogonal reflexions. Let

$X=\{H;\ H$ is a hyperplane which is invariant under some orthogonal reflexion in $G\}$.

Then X is an arrangement in V.

On the other hand, G naturally acts on $\boldsymbol{C}[x_1, \cdots, x_n]$ and there exist G-invariant polynomials $g_1, \cdots, g_n$ such that $\boldsymbol{C}[x_1, \cdots, x_n]^G=\boldsymbol{C}[g_1, \cdots, g_n]$ with $\deg(g_1)\leq \cdots \leq \deg(g_n)$.

Then H. S. M. Coxeter proved that

$$f_{n-1}=\sharp G=\prod_{i=1}^{n}(\deg(g_i)+1) \quad ([1]).$$

In [3], K. Saito showed that such an arrangement X is free and $(\deg(g_1), \cdots, \deg(g_n))$ is its structure sequence.

Thus what we will show is that there is a class of arrangements, a set of free arrangements, which are not necessarily invariant under some finite group $G\subset GL(V)$, but on which the Coxeter equality holds when $n=3$.

§2. Some Definitions

Let X be an arrangement in a 3-dimensional real vector space V. Let $H\in X$ and $\dot{X}=X\setminus\{H\}$. Let h be a defining equation of H. Denote $\mathcal{O}_{V_C,0}$ by $\mathcal{O}$. Put $m=\sharp X$.

Define $X_H=\{H\cap K\}_{K\in X}$, then X_H is also an arrangement in a 2-dimensional real vector space H. Let $k=\sharp X_H$. The arrangement X_H is free and its structure sequence is $(1, k-1)$. Moreover if a subset $\{\theta_0, \theta_1\}$ of $D(X_H)$ is $\mathcal{O}/(h)$-independent with $\deg\theta_0=1, \deg\theta_1=k-1$, then $\{\theta_0, \theta_1\}$ is a system of free basis of $D(X_H)$ ([5]).

Let π: $D(X)\to D(X_H)$ be the restriction of vector fields. Then π preserves the degrees: $\pi(D(X)_\nu)\subset D(X_H)_\nu$ $(\nu\in \boldsymbol{Z})$. Define $\pi_\nu=\pi|_{D(X)_\nu}$, and

$$\ker\pi_\nu=\{h\cdot\theta;\ \theta\in D(\dot{X})_{\nu-1}\}$$

$$=\mathrm{im}\,(h\cdot :\ D(\dot{X})_{\nu-1} \longrightarrow D(X)_{\nu}),$$

where $h\cdot$ is the mapping which sends $\theta \in D(\dot{X})$ to $h\cdot\theta \in D(X)$. Thus the sequence

$$0 \longrightarrow D(\dot{X})_{\nu-1} \xrightarrow{h\cdot} D(X)_{\nu} \longrightarrow D(X_H)_{\nu}$$

is exact.

Let $\{\theta_0, \cdots, \theta_s\}$ be a system of minimal generators of $D(X)$ with $d_i = \deg \theta_i$, $d_0 \leq d_1 \leq \cdots \leq d_s$. The sequence $(d_0, d_1, \cdots, d_s)$ is called the generator sequence of X. It is known that ([5]):

$$\begin{array}{ccc} X \text{ is free} & \Longleftrightarrow & d_0+d_1+d_2=m \\ & \searrow \quad \nearrow & \\ & s=2. & \end{array}$$

Define a mapping

$$\varphi:\ \mathcal{O}^{s+1} \longrightarrow D(X)$$

by

$$\varphi:\ \begin{bmatrix} f_0 \\ \vdots \\ f_s \end{bmatrix} \in \mathcal{O}^{s+1} \longmapsto \sum_{i=0}^{s} f_i \cdot \theta_i \in D(X).$$

Then φ is surjective and $\ker \varphi$ is a free $\mathcal{O}$-module of dimension $(s-2)$ because of the following

LEMMA. homolog. dim $D(X) \leq 1$.

PROOF. We have an exact sequence

$$0 \longrightarrow D(X) \longrightarrow D(\dot{X}) \xrightarrow{\cdot h} \mathcal{O}/(h) \longrightarrow \mathcal{O}/\mathfrak{a}(H) \longrightarrow 0,$$

where $\cdot h$ stands for the map sending $\theta \in D(\dot{X})$ to $\overline{\theta\cdot h} \in \mathcal{O}/(h)$ and $\mathfrak{a}(H) = \{\theta\cdot h \in \mathcal{O};\ \theta \in D(\dot{X})\}$. By inductively applying elementary theory of the homological algebra, we have the assertion.

Let

$$\left\{ \begin{bmatrix} f_0^{(3)} \\ \vdots \\ f_s^{(3)} \end{bmatrix}, \cdots, \begin{bmatrix} f_0^{(s)} \\ \vdots \\ f_s^{(s)} \end{bmatrix} \right\} \subset \mathcal{O}^{s+1}$$

basis of $\ker \varphi$ with $e_i = \deg f_j^{(i)} \theta_j$ $(3 \leq i \leq s,\ 0 \leq j \leq s)$, $e_3 \leq \cdots \leq e_s$. Then the sequence $(e_3, \cdots, e_s)$ depends only on X.

The sequence $(d_0, \cdots, d_s;\ e_3, \cdots, e_s)$ is called the *characteristic sequence* of X.

(If X is free, the generator sequence coincides with the structure sequence.)

Another definition of $(e_3, \cdots, e_s)$ is as follows:

$$e_i = \mathrm{Min}\left\{ d \in \boldsymbol{Z};\ \begin{bmatrix} f_0 \\ \vdots \\ f_s \end{bmatrix} \in \ker\varphi,\quad f_i \neq 0,\quad f_{i+1} = \cdots = f_s = 0, \right.$$

$$\left. \text{and } d = \deg(f_j \cdot \theta_j)\ \ (0 \leq j \leq i) \right\}.$$

Thus we have $d_i < e_i$ $(i=3, \cdots, s)$.

In [5], we proved that $d_0=0$ or 1. If $d_0=0$, X is free and its characteristic sequence is $(0, 1, m-1)$.

§ 3. Proof of the Coxeter equality

In this section we prove the Coxeter equality for a free arrangement X in a 3-dimensional real vector space V. More generally we have the generalized Coxeter equality for any arrangement in V.

At first we need

PROPOSITION 1. *The restriction of vector fields*

$$\pi_\nu:\ D(X)_\nu \longrightarrow D(X_H)_\nu$$

is surjective for $\nu \geq m-2$, *that is, the sequence*

$$0 \longrightarrow D(\dot{X})_{\nu-1} \xrightarrow{h\cdot} D(X)_\nu \longrightarrow D(X_H)_\nu \longrightarrow 0$$

is exact.

PROOF. It suffices to prove this proposition when $\nu = m-2$.

Let $\{l_1, \cdots, l_k\}$ be the set of intersection lines on H, i.e., $\{l_1, \cdots, l_k\} = \bigcup_{K \in \dot{X}} \{K \cap H\}$. Assume that $l_i \neq l_j$ if $i \neq j$. Let μ_i be the multiplicity at l_i, i.e., $\mu_i = \sharp\{K \in X;\ l_i \subset K\}$. Take $H_i \in \dot{X}$ such that $l_i \subset H_i$ $(i=1, \cdots, k)$. For each l_i, define an element $\mathfrak{X}_i$ of $D(X)_{m-\mu_i}$ as follows:

By an appropriate coordinate change we can assume that the affine coordinate of V is (x, y, z) and that H and H_i are defined by $z=0$ and $y=0$ respectively. Fix this coordinate, and $l_i = \{y=z=0\}$. Put an arrangement $X_i = \{K \in X;\ K \not\supset l_i\}$. Let Q_i be defining equation of X_i. Then $\mathfrak{X}_i$ is given by $Q_i \dfrac{\partial}{\partial x}$.

From now on fix a coordinate such that $H = \{z=0\}$ and $H_1 = \{h_1 = y = 0\}$. Let

h_i be a defining equation of H_i $(i=2, \cdots, k)$. Thus we can identify an element of $D(X_H)_\nu$ with an element of type $f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}$, $f, g\in(x, y)_\nu$. Put $\mathfrak{Y}_0=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$, and $\mathfrak{Y}_i=\frac{f_i}{q_i}\pi(\mathfrak{X}_i)$, where $q_i=Q_i(x, y, 0)$ and $f_i=\prod_{j\neq i}h_j(x, y, 0)\in \boldsymbol{C}[x, y]$ $(i=1, \cdots, k)$. Then $\mathfrak{Y}_i\in D(X_H)$ $(i=0, \cdots, k)$, $\deg\mathfrak{Y}_i=k-1$, and $\{\mathfrak{Y}_0, \mathfrak{Y}_i\}$ is independent over $\mathcal{O}$ $(i=1, \cdots, k)$. Thus $\{\mathfrak{Y}_0, \mathfrak{Y}_i\}$ is a system of free basis of $D(X_H)$ $(i=1, \cdots, k)$. Therefore there exist $\alpha_2, \cdots, \alpha_k\in \boldsymbol{C}^*$ and $\lambda_2, \cdots, \lambda_k\in \boldsymbol{C}[x, y]$ such that

$$\mathfrak{Y}_1=\alpha_i\mathfrak{Y}_i+\lambda_i\mathfrak{Y}_0 \quad (i=2, \cdots, k).$$

Thus we have

$$\begin{aligned}\left(\frac{q_i}{f_i}\right)\mathfrak{Y}_1&=\alpha_i\left(\frac{q_i}{f_i}\right)\mathfrak{Y}_i+\lambda_i\left(\frac{q_i}{f_i}\right)\mathfrak{Y}_0\\&=\alpha_i\pi(\mathfrak{X}_i)+\lambda_i\left(\frac{q_i}{f_i}\right)\pi(\mathfrak{X}_0)\in \operatorname{im}\pi,\end{aligned}$$

where $\mathfrak{X}_0=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$.

Then $\left(\frac{q_1}{f_1}, \cdots, \frac{q_k}{f_k}\right)\mathfrak{Y}_1\subset \operatorname{im}\pi$.

On the other hand

$$D(X_H)_{m-2}=(x, y)_{m-3}\mathfrak{Y}_0+(x, y)_{m-k-1}\mathfrak{Y}_1,$$

thus it suffices to show that

$$(x, y)_{m-k-1}=\left(\frac{q_1}{f_1}, \cdots, \frac{q_k}{f_k}\right)_{m-k-1}.$$

Noticing that $\frac{q_i}{f_i}=\prod_{j\neq i}h_j(x, y, 0)^{\mu_j-2}$ $(i=1, \cdots, k)$ and $m-k-1=\sum_{j=1}^{k}(\mu_j-2)$, we know that the proof of Proposition 1 reduces to the following

LEMMA. *Let $k\geq 2$. Let l_i be a linear form of two variables x, y $(i=1, \cdots, k)$. Assume that $\{l_i, l_j\}$ $(i\neq j)$ makes a system of basis of a two-dimensional vector space $\boldsymbol{C}x+\boldsymbol{C}y$. Let ν_j $(j=1, \cdots, k)$ be non-negative integers and $\nu=\sum_{j=1}^{k}\nu_j$.*

Then

$$(x, y)_\nu=\left(\prod_{j\neq 1}l_j^{\nu_j}, \cdots, \prod_{j\neq k}l_j^{\nu_j}\right)_\nu.$$

PROOF. We shall prove this lemma by an induction on k.

If $k=2$, consider an exact sequence

$$(x, y)_{\nu_1-1}\oplus(x, y)_{\nu_2-1}\xrightarrow{\ \phi\ }(l_2^{\nu_2}, l_1^{\nu_1})_{\nu-1}\longrightarrow 0,$$

where $\phi((f, g))=fl_2^{\nu_2}+gl_1^{\nu_1}$ $(f\in(x, y)_{\nu_1-1}, g\in(x, y)_{\nu_2-1})$. Since l_1 and l_2 are coprime,

$\ker\phi=0$. Thus

$$\dim_C(l_2^{\nu_2}, l_1^{\nu_1})_{\nu-1}=\dim_C(x, y)_{\nu_1-1}+\dim_C(x, y)_{\nu_2-1}$$
$$=\nu_1+\nu_2=\dim(x, y)_{\nu-1},$$

and $(l_2^{\nu_2}, l_1^{\nu_1})_{\nu-1}=(x, y)_{\nu-1}$.

When $k>2$, put

$$V=\left(\prod_{j\neq 1} l_j^{\nu_j}, \cdots, \prod_{j\neq k} l_j^{\nu_j}\right)_\nu.$$

Then we have

$$V\supset l_1^{\nu_1}\left(\prod_{j\neq 1,2} l_j^{\nu_j}, \cdots, \prod_{j\neq 1,k} l_j^{\nu_j}\right)_{\nu_2+\cdots+\nu_k}=l_1^{\nu_1}(x, y)_{\nu_2+\cdots+\nu_k},$$

because of the assumption of the induction. Similarly we have

$$V\supset l_2^{\nu_2}(x, y)_{\nu_1+\nu_3+\cdots+\nu_k}.$$

For any $f\in(x, y)_\nu$, we have

$$f\in l_1^{\nu_1}(x, y)_{\nu_2+\cdots+\nu_k}+l_2^{\nu_2}(x, y)_{\nu_1+\nu_3+\cdots+\nu_k}\subset V.$$

This completes the proof.

In the proof above, we have already proved

COROLLARY. *$D(X)$ is generated by the set of vector fields whose degrees are less than m, i.e., $d_i<m$ $(i=0, \cdots, s)$, where $(d_0, \cdots, d_s;\ e_3, \cdots, e_s)$ is the characteristic sequence of X.*

PROPOSITION 2.

$$\dim D(X)_\nu=(\nu+1)(\nu+2)-\frac{f_2}{2}+\frac{(\nu-m+1)(\nu-m+2)}{2}+1$$

for any $\nu\geq m-2$.

PROOF. In the light of Proposition 1, we have

$$\dim D(X)_\nu-\dim D(\dot{X})_{\nu-1}=\dim D(X_H)_\nu.$$

Since $D(X_H)_\nu=(x, y)_{\nu-1}\mathfrak{Y}_0+(x, y)_{\nu-k+1}\mathfrak{Y}_1$, $\dim D(X_H)_\nu=\nu+(\nu-k+2)=2\nu-k+2$. On the other hand, we have $f_2-\dot{f}_2=2k$, where

$$\dot{f}_2=\#\{\text{connected component of } V\setminus\bigcup_{H\in\dot{X}} H\}.$$

Put

$$\dim D(X)_\nu=(\nu+1)(\nu+2)-\frac{f_2}{2}+C+1.$$

If $\#X=2$, then the characteristic sequence is $(0, 1, 1)$, we have

$$\dim D(X)_{\nu-m+2}=\frac{(\nu-m+3)(\nu-m+4)}{2}+\frac{(\nu-m+2)(\nu-m+3)}{2}+\frac{(\nu-m+2)(\nu-m+3)}{2}$$
$$=(\nu-m+3)(\nu-m+4)-2+C+1,$$

then

$$C=\frac{(\nu-m+1)(\nu-m+2)}{2}. \qquad \text{Q.E.D.}$$

So far we have completed the preparations for proving the following

THEOREM 1. *Let* $(d_0, d_1, d_2, \cdots, d_s;\ e_3, \cdots, e_s)$ *be the characteristic sequence of* X. *Then we have*

(1) $m=\sum_{i=0}^{s} d_i-\sum_{i=3}^{s} e_i,$

(2) $f_2=m(m+1)-\sum_{i=0}^{s} d_i(d_i+1)+\sum_{i=3}^{s} e_i(e_i+1)+2.$

PROOF. By the definition of the characteristic sequence, we have

(A) $\dim D(X)_\nu=\sum_{i=0}^{s}\frac{(\nu-d_i+1)(\nu-d_i+2)}{2}-\sum_{i=3}^{s}\frac{(\nu-e_i+1)(\nu-e_i+2)}{2}$

for sufficiently large ν.

On the other hand, we have proved

(B) $\dim D(X)_\nu=(\nu+1)(\nu+2)-\frac{f_2}{2}+\frac{(\nu-m+1)(\nu-m+2)}{2}+1$

for any $\nu\geq m-2$ in Proposition 2.

Comparing the coefficient of ν of (A) with that of (B), we have

$$-\sum_{i=0}^{s}\frac{(2d_i-3)}{2}+\sum_{i=3}^{s}\frac{(2e_i-3)}{2}=3-\frac{2m-3}{2}.$$

By comparing also the constant term of (A) with that of (B), we have

$$\sum_{i=0}^{s}\frac{(d_i-1)(d_i-2)}{2}-\sum_{i=3}^{s}\frac{(e_i-1)(e_i-2)}{2}=3-\frac{f_2}{2}+\frac{(m-1)(m-2)}{2}.$$

Thus we have Theorem 1 by modifying these equalities above.

THEOREM 2 (The generalized Coxeter equality). *Let* $\alpha_i=e_i-d_i>0$ $(i=3, \cdots, s)$. *Then*

$$f_2=(d_0+1)(d_1+1)(d_2+1)-2\sum_{i=3}^{s}\Big(\sum_{j<i}\alpha_j+m-d_i\Big)\alpha_i-2(d_0+d_1+d_2\sum_{i=3}^{s}\alpha_i.$$

PROOF. By Theorem 1 (1), (2),

$$\frac{f_2}{2}=\frac{(\sum d_i-\sum e_i)(\sum d_i-\sum e_i+1)}{2}-\sum\frac{d_i(d_i+1)}{2}+\sum\frac{e_i(e_i+1)}{2}+1$$

$$=1+d_0d_1+d_0d_2+d_1d_2-\sum_{i=3}^{s}\left\{\sum_{j<i}(e_j-d_j)+\sum_{i=0}^{s}d_i-\sum_{i=3}^{s}e_i-d_i\right\}$$
$$-(d_0+d_1+d_2)\sum_{i=3}^{s}(e_i-d_i).$$

If $d_0=1$, we have

$$1+d_0d_1+d_0d_2+d_1d_2=1+d_1+d_2+d_1d_2=\frac{(d_0+1)(d_1+1)(d_2+1)}{2}.$$

If $d_0=0$, the characteristic sequence of X is $(0, 1, m-1)$. Thus we have

$$1+d_0d_1+d_0d_2+d_1d_2=m=\frac{(d_0+1)(d_1+1)(d_2+1)}{2}. \qquad \text{Q.E.D.}$$

This leads us to the following

THEOREM 3. *The following two conditions are equivalent*:

(1) X is free,

(2) $f_2=(d_0+1)(d_1+1)(d_2+1)$.

PROOF. Since $m-d_i>0$ and $\alpha_i>0$ $(i=3, \cdots, s)$, the condition (2) is equivalent to $s=2$, thus to the condition (1). Q.E.D.

Finally it is natural to ask the following

QUESTION. *Is the Coxeter equality true for an arrangement in n-dimensional real vector space*?

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(Received July 10, 1979)

Department of Mathematics
Division of Natural Science
International Christian University
Osawa, Mitaka, Tokyo
181 Japan