A reduction of Banach lattices and its application to positive ergodic operators

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§ 1. Introduction

A positive operator T on a Banach lattice E is called irreducible if it has no non-trivial closed T-invariant order ideal (a subspace I of E is called an order ideal, or simply an ideal, if it satisfies the condition that $|y|\leqq|x|$ and $x \in I$ imply $y \in I$). Such an operator may be considered to be fundamental among positive operators, and it is of interest to know to what extent one can construct general positive operators from the irreducible ones. As to this problem, Sawashima and Niiro [7] showed that a strongly ergodic positive contraction T on $E=C(X)$ (X being a compact Hausdorff space) can be decomposed into irreducible operators (an operator T is said to be strongly ergodic if $(1/N)\sum_{n=1}^{N} T^n$ converges strongly as $N\to\infty$). They also investigated the relation between the spectrum of T and those of its irreducible "components" using their former results about the irreducible operators [6]. Recently the author found that the similar results hold for positive ergodic operators on AM- or AL-spaces without the condition of contractivity ([3], [4] and [5]).

The purpose of this paper is to unify the methods adopted in the previous papers and to generalize the results so far obtained.

In § 2, apart from the reduction of operators, a reduction of a Banach lattice with respect to a certain subspace is discussed.

In § 3, the results in § 2 are applied to obtain a reduction of positive ergodic operators.

§ 2. A reduction of Banach lattices

In this section, E denotes an arbitrarily fixed Banach lattice, and F denotes a closed subspace of E which is a lattice with respect to the order induced by that in E. We fix a positive element e of F, and denote by E_e the order ideal in E generated by e, i.e., $E_e=\{x \in E;\ |x|\leqq c\cdot e \text{ for some } c \in \boldsymbol{R}\}$.

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On the space E_e, a new norm $\|\cdot\|_e$ is introduced by defining $\|x\|_e=\inf\{c;\ |x|\leqq c\cdot e\}$ for $x\in E_e$. Let F_e be $F\cap E_e$. Then E_e and F_e are AM-spaces with respect to the norm $\|\cdot\|_e$ and the order induced by that in E. Hence by the well-known representation theorem of Kakutani [1], there exists a compact Hausdorff space Ω [resp. Λ] for which E_e [resp. F_e] is isometrically isomorphic to $C(\Omega)$ [resp. $C(\Lambda)$] as a Banach lattice. Hereafter the isomorphic image of an element $x\in E_e$ [resp. $y\in F_e$] in $C(\Omega)$ [resp. $C(\Lambda)$] is denoted by $\tilde{x}$ [resp. $\hat{y}$].

In the sequel, it will be shown that there exists a Banach lattice E_λ which is naturally associated with $\lambda\in\Lambda$. In the first place, we fix an arbitrary element $\lambda\in\Lambda$ and let A denote any neighbourhood of λ in Λ. Then the set $\{x\in E;\ |x|\leqq y$ for some $y\in F_e$ satisfying $\hat{y}|_A=0\}$ is an order ideal in E, hence its closure is a closed order ideal, which will be denoted by I_A. We note the following fact about the ideal I_A.

LEMMA 1. *I_A does not contain e.*

PROOF. Since Λ is a compact Hausdorff space, there exists a function $f\in C(\Lambda)$ such that $0\leqq f\leqq 1$, $f(\lambda)=1$ and $\operatorname{supp}(f)\subset A$. Then there exists $y_0\in F_e$ for which $\hat{y}_0=f$. Let $x\in E$ and $y\in F_e$ satisfy $|x|\leqq|y|$ and $\hat{y}|_A=0$. Then

$$|e-x|\geqq|e-|x||\geqq(e-|x|)\vee 0\geqq(e-y)\vee 0\geqq y_0\,.$$

Hence $\|e-x\|\geqq\|y_0\|$, which implies $e\notin I_A$.//

Thus we can define a semi-norm $\|\cdot\|_A$ on E by assigning for each $x\in X$ the positive number

$$\|x\|_A=\frac{\|x+I_A\|}{\|e+I_A\|}\,,$$

where

$$\|x+I_A\|=\inf\{\|x+y\|;\ y\in I_A\}\,.$$

Since I_A is an order ideal, $|x|\leqq|y|$ implies $\|x\|_A\leqq\|y\|_A$, i.e., $\|\cdot\|_A$ is a lattice semi-norm on E. As usual, we consider the set $\mathfrak{N}_\lambda$ of all neighbourhoods of λ as a directed set according to the inclusion relation. Then we can define the limit superior of the net $\{\|x\|_A\}_{A\in\mathfrak{N}_\lambda}$ for each $x\in X$ in the standard manner, though the value ∞ must be admitted as the limit superior. We denote this limit superior for $x\in X$ by $\|x\|_\lambda$, and the set $\{x\in E;\ \|x\|_\lambda<\infty\}$ is denoted by E_λ^{00}.

PROPOSITION 1. *E_λ^{00} is an order ideal in E on which $\|\cdot\|_\lambda$ is a lattice semi-norm, and the completion of $E_\lambda^{00}/\{x\in E;\ \|x\|_\lambda=0\}$ with respect to the quotient norm of $\|\cdot\|_\lambda$ is a Banach lattice.*

PROOF. Since $\|\cdot\|_A$ is a lattice semi-norm on E for any neighbourhood A of λ, it is clear that $\|\cdot\|_\lambda$ has the following properties:

i) $x, y \in E$ $|x| \leqq |y|$ imply $\|x\|_\lambda \leqq \|y\|_\lambda$,

ii) $\|x+y\|_\lambda \leqq \|x\|_\lambda + \|y\|_\lambda$ holds for any $x, y \in E$,

iii) $\|kx\|_\lambda = |k| \cdot \|x\|_\lambda$ holds for any $k \in \boldsymbol{C}$ and $x \in E$,

where the addition and multiplication including $+\infty$ are defined conventionally. Hence E_λ^{00} is clearly an order ideal in E and $\|\cdot\|_\lambda$ is a lattice semi-norm in E_λ^{00}. This implies that the set $\{x \in E;\ \|x\|_\lambda = 0\}$ is an order ideal in E_λ^{00}, which is closed with respect to the topology defined by the semi-norm $\|\cdot\|_\lambda$. Hence the last assertion in the proposition holds.//

We denote the Banach lattice described in the above proposition by E_λ^0, and we also use the symbol $\|\cdot\|_\lambda$ to denote the norm in E_λ^0, expecting that this convention will lead to no serious confusion. The canonical image of $x \in E_\lambda^{00}$ in E_λ^0 will be written as $[x]_\lambda$.

The following relation holds between E_e and E_λ^{00}.

LEMMA 2. *$E_e \subset E_\lambda^{00}$ and $\|x\|_\lambda \leqq \|x\|_e$ holds for $x \in E_e$.*

PROOF. Let $x \in E_e$ and A be an arbitrary neighbourhood of λ. Then $|x| \leqq \|x\|_e \cdot e$, and this implies $\|x + I_A\| \leqq \|x\|_e \cdot \|e + I_A\|$ since I_A is an order ideal. Therefore $\|x\|_A \leqq \|x\|_e$ for any A, and hence $\|x\|_\lambda \leqq \|x\|_e$.//

The above lemma shows that we may consider the closed order ideal generated by $[e]_\lambda$, which we denote by E_λ. By definition, a positive element x of a Banach lattice E is called a quasi-interior positive element (or non-support element of E_+) if the order ideal E_x is dense in E. Then the following proposition holds.

PROPOSITION 2. *E_λ has a quasi-interior positive element, and the canonical image of F_e in E_λ is one dimensional.*

PROOF. $[e]_\lambda$ is clearly a quasi-interior positive element of E_λ. To prove the rest of the assertion, take an element $x \in F_e$ and an arbitrary positive number ε. Let $A = \{\mu \in \Lambda;\ |\hat{x}(\mu) - \hat{x}(\lambda)| \leqq \varepsilon\}$ and $B = \{\mu \in \Lambda;\ |\hat{x}(\mu) - \hat{x}(\lambda)| < 2\varepsilon\}$. Then A is a closed neighbourhood of λ and $A \subset B$. Since B is open there exists a function $f \in C(\Lambda)$ such that $f = 0$ on A, $f = \hat{x} - \hat{x}(\lambda) \cdot \hat{e}$ on B^c, and $|f(\mu)| \leqq 2\varepsilon$ for $A^c \cap B$. Let $y \in F_e$ be the element which corresponds to f. Then clearly $y \in I_A$ and $|x - \hat{x}(\lambda) \cdot e - y| \leqq 4\varepsilon e$. This implies $\|x - \hat{x}(\lambda) \cdot e\|_\lambda \leqq 4\varepsilon$ since $y \in I_{A'}$ for any neighbourhood A' of λ which is contained in A. Noting that $\varepsilon > 0$ may be chosen arbitrarily small, we conclude $[x]_\lambda = \hat{x}(\lambda)[e]_\lambda$. This completes the proof.//

REMARK. $E_\lambda = E_\lambda^0$ does not hold in general, even if e is a quasi-interior positive element of E.

Now the Banach lattice E_λ is composed for each $\lambda \in \Lambda$ as above, a natural mapping α from $E^{00} = \bigcap_{\lambda \in \Lambda} E_\lambda^{00}$ into $\prod_{\lambda \in \Lambda} E_\lambda^0$ is defined by setting $\alpha(x) = ([x]_\lambda)_{\lambda \in \Lambda}$ for $x \in E^{00}$. Since E^{00} clearly contains E_e, the restriction α_e of α to E_e is well-defined. Before stating the following proposition about the kernel of α_e, we note that the range of a positive projection P satisfies the condition imposed on F at the beginning of this section. In fact, PE is a closed subspace of E and the supremum of $x, y \in PE$ in PE is given by $P(x^\vee y)$, where $x^\vee y$ denotes the supremum of x and y in E. See [7] Proposition 1 or [9] p. 214.

PROPOSITION 3. *If F is the range of a positive projection P, then the kernel of the mapping α_e defined above is contained in the absolute kernel $\{x \in E;\ P|x|=0\}$ of P.*

PROOF. Let $\lambda \in \Lambda$ and let A be a neighbourhood of λ. Then it follows from the definition of I_A that $PI_A \subset I_A$. This implies $\|Px\|_A \leqq \|P\| \cdot \|x\|_A$ for any $x \in E$, and hence $\|Px\|_\lambda \leqq \|P\| \cdot \|x\|_\lambda$. Since $x \in \operatorname{Ker} \alpha_e$ if and only if $x \in E_e$ and $\|x\|_\lambda = 0$ for all $\lambda \in \Lambda$, we see that $x \in \operatorname{Ker} \alpha_e$ is equivalent to $|x| \in \operatorname{Ker} \alpha_e$. Therefore $x \in \operatorname{Ker} \alpha_e$ implies $\|P|x|\|_\lambda = 0$ for any $\lambda \in \Lambda$. On the other hand we know that $\|P|x|\|_\lambda = \widehat{P|x|}(\lambda)$ for $x \in E_e$ from the proof of Proposition 2. Hence $x \in \operatorname{Ker} \alpha_e$ implies $\widehat{P|x|} = 0$, and hence $P|x| = 0$. This is the desired conclusion.//

COROLLARY 1. *If F is the range of a strictly positive projection P (i.e., P is a positive projection such that $P|x|=0$ implies $x=0$), then the mapping α_e is injective.*

COROLLARY 2. *Under the same assumption as in Corollary 1, $x \leqq y$ in E is equivalent to $[x]_\lambda \leqq [y]_\lambda$ in E_λ for any $\lambda \in \Lambda$ whenever x and $y \in E_e$.*

PROOF OF COROLLARY 2. Let x and y be elements of E_e satisfying $[x]_\lambda \leqq [y]_\lambda$ for all $\lambda \in \Lambda$. Then $[(x-y)^\vee 0]_\lambda = ([x]_\lambda - [y]_\lambda)^\vee 0 = 0$ since the canonical mapping $E_e \to E_\lambda$ is a lattice homomorphism. Together with Corollary 1, this implies $(x-y)^\vee 0 = 0$, hence $x \leqq y$. The converse implication is obvious.//

Though the following is a corollary to Proposition 2, it is stated here as Corollary 3 for its similarity to Corollary 2 and for the convenience of later references.

COROLLARY 3. *Without the assumption that F is the range of a positive* *projection P, $x \leqq y$ in E is equivalent to $[x]_\lambda \leqq [y]_\lambda$ for any $\lambda \in \Lambda$ if x and $y \in F_e$.*

PROOF. As shown in the proof of Proposition 2, $[x]_\lambda = \hat{x}(\lambda)[e]_\lambda$ and $[y]_\lambda = \hat{y}(\lambda)[e]_\lambda$ for any $\lambda \in \Lambda$ if x and $y \in F_e$. Therefore $[x]_\lambda \leqq [y]_\lambda$ for any $\lambda \in \Lambda$ is equivalent to $\hat{x} \leqq \hat{y}$, and hence to $x \leqq y$.//

REMARK. The conclusion of Proposition 3 is valid for the mapping α if e is a quasi-interior positive element of E.

If F is a closed sublattice of E, we can get further information about the relation of E_λ and the space Ω (recall that Ω is the topological space such that E_e is isometrically lattice isomorphic to $C(\Omega)$). In this case there exists a continuous surjection $\pi: \Omega \to \Lambda$ which is characterized by the property $\tilde{f}(\omega) = \hat{f}(\pi(\omega))$, where $f \in F_e$ and $\omega \in \Omega$. In the following Lemma 3 and Proposition 4, we assume that F is a closed sublattice of E and π means the above surjection.

LEMMA 3. *Let A be a closed neighbourhood of $\lambda \in \Lambda$ and let $f \in E_e$ be an element such that $\tilde{f} = 0$ on $\pi^{-1}(A)$. Then $f \in I_A$.*

PROOF. Since I_A is an order ideal we may assume $f \geqq 0$ without loss of generality. Let ε be an arbitrary positive number and put $U = \{\omega \in \Omega; \tilde{f}(\omega) < \varepsilon\}$. Then there exists an open set $B \supset A$ such that $\pi^{-1}(B) \subset U$, since $\pi^{-1}(A) = \bigcap_C \pi^{-1}(C)$ where C runs over all open sets containing A. There exists a positive element $g \in F_e$ such that $\hat{g} = 0$ on A and $\hat{g} = \|f\|_e$ on B^c. Then clearly $f \wedge g \in I_A$ and $|f - f \wedge g| \leqq \varepsilon e$. This implies $\|f - f \wedge g\| \leqq \varepsilon \|e\|$, hence $f \in I_A$ since ε is chosen arbitrarily small.//

Let Ω_λ denote the set $\pi^{-1}(\lambda)$. Then we have the following

PROPOSITION 4. *$\{f \in E_e; \tilde{f}|_{\Omega_\lambda} = 0\} \subset \{f \in E_e; \|f\|_\lambda = 0\}$ holds for any $\lambda \in \Lambda$.*

PROOF. Let $f \in E_e$ be such that $\tilde{f}|_{\Omega_\lambda} = 0$ and let ε be a positive number. Then $U = \{\omega \in \Omega; |\tilde{f}(\omega)| < \varepsilon\}$ is an open neighbourhood of Ω_λ, and there exists an open neighbourhood A of λ such that $\pi^{-1}(A) \subset U$ since $\Omega_\lambda = \bigcap_C \pi^{-1}(C)$ where C runs over all open neighbourhoods of λ. Take a closed neighbourhood A' of λ such that $A' \subset A$. Then there exists an element $g \in E_e$ such that $\tilde{g} = 0$ on $\pi^{-1}(A')$, $\tilde{g} = \tilde{f}$ on $\pi^{-1}(A^c)$ and $|\tilde{g}(\omega)| \leqq \varepsilon$ for $\omega \in \pi^{-1}(A \backslash A')$. This element g satisfies $|f - g| \leqq 2\varepsilon e$, and $g \in I_{A'}$ by Lemma 3. This implies $\|f\|_\lambda \leqq 2\varepsilon$, and hence $\|f\|_\lambda = 0$ by the arbitrariness of ε.//

This proposition shows that E_λ is the separated completion of $C(\Omega_\lambda)$ with respect to the semi-norm induced by $\|\cdot\|_\lambda$.

To end up with this section, we note the following fact. If F is the range of a positive projection P, then it is not only a lattice with respect to the induced order but also a Banach lattice for a suitable norm equivalent to the original one (see [9] p. 214). Consequently it makes sense to say that an element is a quasi-interior positive element of F, and the topological space Λ is uniquely determined up to homeomorphism if e is chosen among the quasi-interior positive elements of F (see [9] p. 166).

§ 3. Applications to a reduction of operators

In this section, the symbols used in § 2 indicate the same things, for example, E is a Banach lattice and F is a closed subspace of E which is a lattice with respect to the induced order, and e is a positive element of F. Our first result about the reduction of operators is the following

PROPOSITION 5. *Let $\lambda \in \Lambda$ and let T be a bounded linear operator on E which leaves the order ideal I_A invariant for any neighbourhood A of λ. Then $x \in E_\lambda^{00}$ implies $Tx \in E_\lambda^{00}$ and a bounded linear operator T_λ^0 on E_λ^0 is defined by the following formula:*

$$T_\lambda^0[x]_\lambda = [Tx]_\lambda, \quad x \in E_\lambda^{00}.$$

Moreover T_λ^0 satisfies $\|T_\lambda^0\| \leqq \|T\|$, where $\|T_\lambda^0\|$ or $\|T\|$ denotes the operator norm of the respective operator.

PROOF. By the assumption, $\|Tx+I_A\| \leqq \|T\| \cdot \|x+I_A\|$ holds for any $x \in E$ and any neighbourhood A of λ. This implies $\|Tx\|_\lambda \leqq \|T\| \cdot \|x\|_\lambda$ and hence $Tx \in E_\lambda^{00}$ if $x \in E_\lambda^{00}$. The above inequality shows that T_λ^0 is unambiguously defined and satisfies the inequality about the operator norms.//

The following theorem gives a simple sufficient condition for an operator T to satisfy the requirement described in Proposition 5.

THEOREM 1. *Let T be a linear operator on E such that there exists a positive linear operator S for which $F \subset \{x \in E; |Sx| \leqq c|x| \text{ for some positive number } c\}$ and $|Tx| \leqq S|x|$ hold for any $x \in E$. Then for any $\lambda \in \Lambda$, $TE_\lambda^{00} \subset E_\lambda^{00}$ holds and a bounded linear operator T_λ^0 on E_λ^0 is defined by the following formula as in Proposition 5:*

$$T_\lambda^0[x]_\lambda = [Tx]_\lambda, \quad x \in E_\lambda^{00}.$$

Moreover T_λ^0 leaves E_λ invariant, and the restriction T_λ of the operator T_λ^0 to E_λ satisfies $\|T_\lambda\| \leqq \|T_\lambda^0\| \leqq \|T\|$.

PROOF. Let $\lambda \in \Lambda$ and let A be a neighbourhood of λ. If $x \in E$ and $y \in F_e$

satisfy $\hat{y}|_A=0$ and $|x|\leqq y$, then $|Tx|\leqq S|x|\leqq Sy\leqq cy$ hold for some positive number c. This shows that $x\in I_A$ implies $Tx\in I_A$. Hence T_λ^0 is well-defined as a bounded linear operator on E_λ^0 by Proposition 5. Moreover if $x\in E_e$, then $|Tx|\leqq S|x|\leqq\|x\|_e Se\leqq c\|x\|_e e$ holds for some positive number c. Therefore

$$|T_\lambda^0[x]_\lambda|=|[Tx]_\lambda|=[|Tx|]_\lambda\leqq c\|x\|_e[e]_\lambda .$$

Hence E_λ is T_λ^0-invariant, so the restriction of T_λ^0 to E_λ defines an operator on E_λ. The inequalities about the operator norms are clear.//

It is evident that the set $\mathcal{L}_\lambda$ of all bounded operators on E which meets the condition in Proposition 5 for a fixed $\lambda\in\Lambda$ is a subalgebra of all bounded operators on E. For any $T\in\mathcal{L}_\lambda$ we consider the operator T_λ^0 on E_λ^0 which is defined in Proposition 5, and if T_λ^0 leaves E_λ invariant, its restriction to E_λ is denoted by T_λ. It is clear that the mapping $T\to T_\lambda^0$ from $\mathcal{L}_\lambda$ into the algebra of all bounded operators on E_λ^0 is an algebraic homomorphism of norm 1.

Let $\rho_\infty(S)$ denote the unbounded connected component of the resolvent set $\rho(S)$ of an operator S, and let $R(\alpha, S)$ denote the resolvent $(\alpha-S)^{-1}$ for $\alpha\in\rho(S)$. Then we have the following

PROPOSITION 6. *Let $T\in\mathcal{L}_\lambda$ for a $\lambda\in\Lambda$, and $\alpha\in\rho_\infty(T)$. Then $\alpha\in\rho_\infty(T_\lambda^0)$, $R(\alpha, T)\in\mathcal{L}_\lambda$, and*

$$R(\alpha, T_\lambda^0)=R(\alpha, T)_\lambda^0 .$$

Moreover, if E_λ is T_λ^0-invariant, E_λ is also $R(\alpha, T_\lambda^0)$-invariant, $\alpha\in\rho_\infty(T_\lambda)$ and $R(\alpha, T_\lambda)$ is the restriction of $R(\alpha, T_\lambda^0)$ and hence

$$\|R(\alpha, T_\lambda)\|\leqq\|R(\alpha, T_\lambda^0)\| .$$

PROOF. Let A be an arbitrary neighbourhood of λ. Then by the assumption and Lemma 2 in [7], I_A is $R(\alpha, T)$-invariant; hence $R(\alpha, T)\in\mathcal{L}_\lambda$. Since

$$\begin{aligned}(\alpha-T_\lambda^0)R(\alpha, T)_\lambda^0[x]_\lambda&=(\alpha-T_\lambda^0)[R(\alpha, T)x]_\lambda\\&=[(\alpha-T)R(\alpha, T)x]_\lambda\\&=[x]_\lambda\end{aligned}$$

and similarly $R(\alpha, T_\lambda^0)(\alpha-T_\lambda^0)[x]_\lambda=[x]_\lambda$ for any $x\in E_\lambda^{00}$, we get $\alpha\in\rho_\infty(T_\lambda^0)$ and $R(\alpha, T)_\lambda^0=R(\alpha, T_\lambda^0)$. The last assertion also follows from Lemma 2 in [7].//

COROLLARY. *Let T satisfy the condition in Theorem 1. Then $\rho_\infty(T)\subset\rho_\infty(T_\lambda^0)\subset\rho_\infty(T_\lambda)$ holds for any $\lambda\in\Lambda$, and*

$$\|R(\alpha, T)\|\geqq\|R(\alpha, T_\lambda^0)\|\geqq\|R(\alpha, T_\lambda)\|$$

for any $\alpha \in \rho_\infty(T)$.

PROOF. Clear from Theorem 1, Proposition 6 and Lemma 2 in [7].//

Hereafter we apply our results to a reduction of positive operators. Consider the following conditions on a bounded linear operator T on E:

1) *T is a positive operator on E.*

2) *T is strongly mean ergodic, i.e.,* $(1/N)\sum_{n=1}^{N} T^n$ *converges to an operator P as* $N\to\infty$.

2)′ *T is uniformly mean ergodic, i.e.,* $(1/N)\sum_{n=1}^{N} T^n$ *converges to an operator P in operator norm as* $N\to\infty$.

Under the condition 1), the operator P with the property 2) or 2)′ is a positive projection whose range is the space of fixed elements for T, i.e., $PE=\{x\in E; Tx=x\}$. As remarked after Proposition 2 in § 2, PE is a lattice with respect to the induced order. Hence we may apply the reduction in § 2 where we take PE as F and any positive element of PE as e.

THEOREM 2. *Suppose T is a linear operator on a Banach lattice E satisfying the conditions* 1) *and* 2). *For any positive element e of PE, let Λ, E_λ^{00} and E_λ^0 $(\lambda\in\Lambda)$ denote the spaces which appear in the reduction theory in* § 2 *applied to the triple E, $F=PE$ and e. Then the following mappings*

$$T_\lambda^0[x]_\lambda=[Tx]_\lambda$$
$$P_\lambda^0[x]_\lambda=[Px]_\lambda$$

where $x\in E_\lambda^{00}$, are well-defined as bounded linear operators on E_λ^0, which leave E_λ invariant. Their restrictions to E_λ are denoted by T_λ and P_λ, respectively. Then P_λ^0 [resp. P_λ] is a positive projection which satisfies $T_\lambda^0P_\lambda^0=P_\lambda^0T_\lambda^0=P_\lambda^0$ [resp. $T_\lambda P_\lambda=P_\lambda T_\lambda=P_\lambda$], and the range of P_λ is one dimensional.

Suppose further that T satisfy the condition 2)′. *Then* $(1/N)\sum_{n=1}^{N} T_\lambda^{0n}$ $\Big[$*resp.* $(1/N)\sum_{n=1}^{N} T_\lambda^n\Big]$ *converges to P_λ^0 [resp. P_λ] in operator norm as $N\to\infty$, and hence T_λ^0 [resp. T_λ] also satisfies* 1) *and* 2)′ *for any $\lambda\in\Lambda$.*

PROOF. T and P clearly satisfy the condition in Theorem 1. In fact, $|Tx|\leqq T|x|$ and $|Px|\leqq P|x|$ hold for any $x\in E$, and $|Tx|=|x|=|Px|$ hold for any $x\in PE$. These show that T or P itself can play the role of S for T or P in the condition, respectively. Therefore T_λ^0 and P_λ^0 are well-defined as bounded linear operators on E_λ^0 for any $\lambda\in\Lambda$ by Theorem 1. By the definition, T_λ^0 and P_λ^0 are positive operators and leave E_λ invariant since $Te=Pe=e$. The relation $T_\lambda^0P_\lambda^0=P_\lambda^0T_\lambda^0=P_\lambda^0$ is derived from $TP=PT=P$, and hence $T_\lambda P_\lambda = P_\lambda T_\lambda = P_\lambda$ holds. That the range of P_λ is one dimensional follows from Proposition 2.

To prove that $(1/N)\sum_{n=1}^{N} T_\lambda^{0n}$ $\left[\text{resp. } (1/N)\sum_{n=1}^{N} T_\lambda^n\right]$ converges uniformly to P_λ^0 [resp. P_λ] as $N\to\infty$, we define $T_N=(1/N)\sum_{n=1}^{N} T^n$ and $S_N=T_N+P$ for any natural number N. Then $|(T_N-P)x|\leqq S_N|x|$ holds for any $x\in E$, and $|S_N x|=2|x|$ holds for any $x\in PE$. Hence $\left\|(1/N)\sum_{n=1}^{N} T_\lambda^n - P_\lambda\right\| \leqq \left\|(1/N)\sum_{n=1}^{N} T_\lambda^{0n} - P_\lambda^0\right\| \leqq \|T_N-P\|$ holds for any $\lambda\in\Lambda$ by Theorem 1. This shows that if T satisfies 1) and 2)′, then T_λ^0 [resp. T_λ] also satisfies 1) and 2)′, and the limit of $(1/N)\sum_{n=1}^{N} T_\lambda^{0n}$ $\left[\text{resp. } (1/N)\sum_{n=1}^{N} T_\lambda^n\right]$ is P_λ^0 [resp. P_λ].//

Following the notations in [7], if a linear operator S on a linear space G leaves a subspace H invariant, we denote by S/H the operator on G/H which maps the equivalence class of $x\in G$ mod. H to the equivalence class of Sx mod. H. Then we have the following.

COROLLARY. *Suppose T satisfy 1) and 2)′, $0\leqq e\in PE$, and let Λ, E_λ, etc. be the same as in Theorem 2. Then for any $\lambda\in\Lambda$, the closed order ideal $I_\lambda=\{x\in E;\ P_\lambda|x|=0\}$ is T_λ-invariant and the operator T_λ/I_λ on E_λ/I_λ is an irreducible operator.*

PROOF. I_λ is clearly T_λ-invariant. I_λ is also P_λ-invariant and $(1/N)\sum_{n=1}^{N}(T_\lambda/I_\lambda)^n$ converges to P_λ/I_λ in operator norm as $N\to\infty$, since $(1/N)\sum_{n=1}^{N} T_\lambda^n$ converges to P_λ in operator norm as $N\to\infty$ by Theorem 2. It is easy to see that P_λ/I_λ is a strictly positive projection whose range is the one dimensional subspace spanned by the equivalence class of $[e]_\lambda$ mod. I_λ. All of this implies the irreducibility of T_λ/I_λ.//

Thus we have obtained the irreducible "components" of T which satisfy the conditions 1) and 2)′. Proposition 4 shows that the method used for this purpose is a generalization of that in the previous papers [4], [5] and [7].

As to the spectra of T and T_λ's, we have the following

THEOREM 3. *In the same situation as in Theorem 2, the following relations hold:*

$$\sigma(T)\cap\Gamma\supset\Big(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda)\Big)^-\cap\Gamma\supset\Big(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda/I_\lambda)\Big)^-\cap\Gamma\,,$$

where $\Gamma=\{\alpha\in\boldsymbol{C};\ |\alpha|=1\}$ and $\sigma(T)$ denotes the spectrum of T, and so on. Moreover, if T satisfies 2)′,

$$\Big(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda)\Big)^-\cap\Gamma=\Big(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda/I_\lambda)\Big)^-\cap\Gamma\,.$$

PROOF. The inclusion relations follow immediately from Proposition 6 and Theorem 1, since $\rho(T)\cap\Gamma\subset\rho_\infty(T)$ by the conditions 1) and 2). The last equality follows from Lemma 7 in [7] since the condition 2)′ implies that $\alpha=1$ is a pole of the resolvent $R(\alpha, T)$ of order at most 1 by Theorem 6 in [2].//

From the previous results in [4], [5] and [7], where E is an AM- or AL-space, it is hoped that the inclusion relation

$$\sigma(T)\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$$

in Theorem 3 may be replaced by the equality when the element $e\in PE$ is a quasi-interior positive element of E and T satisfies the conditions 1) and 2)′. Recently I. Sawashima and F. Niiro have shown that this is indeed the case, and their proof will appear in [8].

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