On some estimates for the wave equation in L^p and H^p

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§1. Introduction

A recent work by S. Sjöstrand [9] gives interesting L^p-estimates for the solution of the Cauchy problem for the wave equation:

$$\text{(E)}\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}=\Delta u\,, & t>0,\quad x\in \boldsymbol{R}^n \quad \left(\Delta=\sum_{j=1}^{n}\dfrac{\partial^2}{\partial x_j^2}\right), \\ u(0, x)=f(x)\,, & x\in \boldsymbol{R}^n, \\ \dfrac{\partial u}{\partial t}(0, x)=g(x)\,, & x\in \boldsymbol{R}^n\,. \end{cases}$$

A typical example of the results contained in [9] reads as follows: suppose $f(x)\equiv 0$, then the estimate

$$\|u(t, \cdot)\|_{L^p}\le C_p(t)\|g\|_{L^p}$$

holds if $p_0<p<p_0'$ and does not hold if $p<p_0$ or $p>p_0'$. These results do not contain the case of the critical index p, *i.e.* the case $p=p_0$ or p_0'. In this paper, we shall prove that *the estimates hold for the critical index p as well.*

First, we shall explain the results deduced from the analysis of S. Sjöstrand [9] and the main results of the present paper. We write the solution $u=u(t, x)$ of (E) as

$$u=(U(t)f)(x)+(V(t)g)(x)$$

and regard $U(t)$ and $V(t)$ as operators acting in some function spaces on $\boldsymbol{R}^n$. If $n=1$, $U(t)$ and $V(t)$ are bounded operators in $L^p(\boldsymbol{R})$, $1<p<\infty$, *i.e.*, the following inequalities hold for *all* $p\in(1, \infty)$:

$$(1)\qquad \|U(t)f\|_p\le C_p(t)\|f\|_p\,,$$

$$(2)\qquad \|V(t)f\|_p\le C_p(t)\|f\|_p\,,$$

where

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$$\|f\|_p=\left(\int|f(x)|^p dx\right)^{1/p}.$$

The situation is quite different if $n\geq 2$; we have the inequality (1) only for $p=2$ and the inequality (2) only for *some* $p\in(1,\infty)$. But, instead of the simple inequalities (1) and (2), we have the following estimates, for $n\geq 2$ and $p\neq 2$, if we take k, s, and r sufficiently large:

$$\left\|kt^{-k}\int_0^t (t-s)^{k-1}U(s)f\,ds\right\|_p\leq C_p(t)\|f\|_p\,, \tag{3}$$

$$\left\|kt^{-k}\int_0^t (t-s)^{k-1}V(s)f\,ds\right\|_p\leq C_p(t)\|f\|_p\,, \tag{4}$$

$$\|U(t)f\|_p\leq C_p(t)\|(1-\Delta)^{s/2}f\|_p\,, \tag{5}$$

$$\|V(t)f\|_p\leq C_p(t)\|(1-\Delta)^{r/2}f\|_p\,. \tag{6}$$

Indeed the analysis of S. Sjöstrand [9] shows the following results:

estimate (3) holds if $(n-1)|1/p-1/2|<k$ and does not hold if $(n-1)|1/p-1/2|>k$;
estimate (4) holds if $(n-1)|1/p-1/2|<k+1$ and does not hold if $(n-1)|1/p-1/2|>k+1$;
estimate (5) holds if $(n-1)|1/p-1/2|<s$ and does not hold if $(n-1)|1/p-1/2|>s$;
estimate (6) holds if $(n-1)|1/p-1/2|<r+1$ and does not hold if $(n-1)|1/p-1/2|>r+1$.

The principal aim of the present paper is to show that (i) estimates (3)~(6) hold for the critical index p, *i.e.* (3), (4), (5) or (6) respectively holds even if $(n-1)|1/p-1/2|=k$, $k+1$, s or $r+1$, and (ii) these estimates are extended to the case $0<p\leq 1$ if we replace $L^p(\boldsymbol{R}^n)$ by $H^p(\boldsymbol{R}^n)$ the H^p-space given by Fefferman and Stein [3].

The contents of this paper are as follows. In § 2, we shall reduce the problem to the study of some Fourier multipliers. § 3 is devoted to some preliminary arguments. In § 4, we give our main results.

Throughout this paper, the letter ϕ denotes a fixed smooth function on $\boldsymbol{R}$ such that

$$0\leq\phi(x)\leq 1\,,\quad \phi(x)=0 \text{ if } x\leq 1, \text{ and } \phi(x)=1 \text{ if } x\geq 2$$

and the letter C will denote a positive constant which may be different in each occasion.

§ 2. Reduction of the problem

$H^p = H^p(\boldsymbol{R}^n)$, $0 < p \leq 1$, is defined to be the set of all tempered distributions f such that

$$f^+(x) = \sup_{0<\varepsilon<\infty} |(f*\varphi_\varepsilon)(x)| \in L^p(\boldsymbol{R}^n) ,$$

where φ is some fixed element of $\mathcal{S}(\boldsymbol{R}^n)$ with nonvanishing integral, *i.e.*

$$\int \varphi(x)dx \neq 0 ,$$

and

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$$

and $f*\varphi_\varepsilon$ is the convolution

$$(f*\varphi_\varepsilon)(x) = \langle f, \varphi_\varepsilon(x-\cdot)\rangle .$$

The norm in H^p, $0<p\leq 1$, is defined by

$$\|f\|_{H^p} = \|f^+\|_p , \qquad 0<p\leq 1 .$$

$H^p = H^p(\boldsymbol{R}^n)$, $1<p<\infty$, is defined to be equal to the space L^p;

$$\|f\|_{H^p} = \|f\|_p , \qquad 1<p<\infty .$$

For more details about H^p, see Fefferman-Stein [3] and Latter [5].

For a bounded function m on $\boldsymbol{R}^n$, we consider the *Fourier multiplier transformation* T_m defined by

$$T_m f = \mathcal{F}^{-1}(m(\xi)\hat{f}(\xi)) ,$$

where $\hat{f} = \mathcal{F}f$ is the Fourier transform of f and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform;

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = (2\pi)^{-n/2}\int_{\boldsymbol{R}^n} f(x)e^{-i\xi\cdot x}dx ,$$

$$(\mathcal{F}^{-1}f)(\xi) = (\mathcal{F}f)(-\xi) .$$

We shall study the Fourier multiplier transformations in H^p, $0<p<\infty$. If T_m (originally defined on $L^2\cap H^p$) can be extended to a bounded operator from H^p to H^p, then we say that m is a *Fourier multiplier* for H^p. $\mathcal{M}(H^p)$ denotes the set of all the Fourier multipliers for H^p. The norm $\|m\|_{\mathcal{M}(H^p)}$ is defined to be the operator norm of T_m in H^p;

$$\|m\|_{\mathcal{M}(H^p)} = \sup_{\substack{f \in H^p \\ f \neq 0}} \frac{\|T_m f\|_{H^p}}{\|f\|_{H^p}} .$$

The operators $U(t)$ and $V(t)$ are Fourier multiplier transformations;

$$U(t)=T_m \quad \text{with} \quad m(\xi)=\cos t|\xi| ,$$

$$V(t)=T_m \quad \text{with} \quad m(\xi)=\frac{\sin t|\xi|}{|\xi|} .$$

Also the operators

$$f \longmapsto kt^{-k}\int_0^t (t-s)^{k-1} U(s)f\, ds ,$$

$$f \longmapsto kt^{-k}\int_0^t (t-s)^{k-1} V(s)f\, ds ,$$

$$(1-\Delta)^{s/2} f \longmapsto U(t)f ,$$

and

$$(1-\Delta)^{r/2} f \longmapsto V(t)f$$

are Fourier multiplier transformations T_m with

$$m(\xi)=kt^{-k}\int_0^t (t-s)^{k-1}\cos s|\xi|\, ds ,$$

$$m(\xi)=kt^{-k}\int_0^t (t-s)^{k-1}\frac{\sin s|\xi|}{|\xi|} ds ,$$

$$m(\xi)=(1+|\xi|^2)^{-s/2}\cos t|\xi| ,$$

and

$$m(\xi)=(1+|\xi|^2)^{-r/2}\frac{\sin t|\xi|}{|\xi|}$$

respectively. Thus our problems are to distinguish when these functions are Fourier multipliers for H^p and to obtain the estimates of the $\mathcal{M}(H^p)$-norms of these multipliers which are functions of t. We shall slightly generalize the problems. Let ϕ be a positively homogeneous function of degree 1 which is positive and smooth on $\boldsymbol{R}^n\backslash\{0\}$;

$$\phi(\xi)>0 , \quad \xi \neq 0 ,$$

$$\phi(t\xi)=t\phi(\xi) , \quad t>0 , \quad \xi \neq 0 .$$

Let $t>0$, $k>0$, $s\geq 0$, $r\geq -1$ and set

$$m_{1,k,t}(\xi)=kt^{-k}\int_0^t (t-s)^{k-1}\cos s\phi(\xi) ds ,$$

$$m_{2,k,t}(\xi)=kt^{-k}\int_0^t (t-s)^{k-1}\frac{\sin s\phi(\xi)}{\phi(\xi)}ds\ ,$$

$$m_{3,s,t}(\xi)=(1+|\xi|^2)^{-s/2}\cos t\phi(\xi)$$

and

$$m_{4,r,t}(\xi)=(1+|\xi|^2)^{-r/2}\frac{\sin t\phi(\xi)}{\phi(\xi)}\ .$$

We shall study the problems for these multipliers under certain conditions on ϕ.

We have the following

LEMMA 1. (i) $m_{1,k,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\psi(\phi(\xi))\phi(\xi)^{-k}\cos(\phi(\xi)-\pi k/2)\in\mathcal{M}(H^p)\ .$$

If $m_{1,k,t}\in\mathcal{M}(H^p)$, *then* $\|m_{1,k,t}\|_{\mathcal{M}(H^p)}$ *does not depend on* t.

(ii) $m_{2,k,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\psi(\phi(\xi))\phi(\xi)^{-k-1}\sin(\phi(\xi)-\pi k/2)\in\mathcal{M}(H^p)\ .$$

If $m_{2,k,t}\in\mathcal{M}(H^p)$, *then* $t^{-1}\|m_{2,k,t}\|_{\mathcal{M}(H^p)}$ *does not depend on* t.

(iii) $m_{3,s,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\psi(\phi(\xi))|\xi|^{-s}\cos\phi(\xi)\in\mathcal{M}(H^p)\ .$$

If $m_{3,s,t}\in\mathcal{M}(H^p)$, *then*

$$\|m_{3,s,t}\|_{\mathcal{M}(H^p)}\le C(1+t)^s\ .$$

(iv) $m_{4,r,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\psi(\phi(\xi))|\xi|^{-r-1}\sin\phi(\xi)\in\mathcal{M}(H^p)\ .$$

If $m_{4,r,t}\in\mathcal{M}(H^p)$ *and* $r\ge 0$, *then*

$$\|m_{4,r,t}\|_{\mathcal{M}(H^p)}\le Ct(1+t)^r\ .$$

The proof of this lemma is based on the following results.

THEOREM A (Stein [10], p. 232; Miyachi [8]). *Let* $0<p<\infty$ *and*

$$k=\max\{[n|1/p-1/2|]+1,\ [n/2]+1\}\ .$$

Suppose that $m\in C^k(\boldsymbol{R}^n\backslash\{0\})$ *and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha m(\xi)\right|\le|\xi|^{-|\alpha|}\ ,\quad |\alpha|\le k\ .$$

Then $m\in\mathcal{M}(H^p)$.

PROPOSITION A. (i) *Let* $t>0$. $m(t\xi)\in\mathcal{M}(H^p)$ *if and only if* $m(\xi)\in\mathcal{M}(H^p)$ *and*

$$\|m(t\cdot)\|_{\mathcal{M}(H^p)}=\|m\|_{\mathcal{M}(H^p)} .$$

(ii) *If* m_1 *and* $m_2\in\mathcal{M}(H^p)$, *then* $m_1m_2\in\mathcal{M}(H^p)$ *and*

$$\|m_1m_2\|_{\mathcal{M}(H^p)}\leq\|m_1\|_{\mathcal{M}(H^p)}\|m_2\|_{\mathcal{M}(H^p)} .$$

(i) of Proposition A can be easily shown by using the equality

$$\|f(t\cdot)\|_{H^p(R^n)}=t^{-n/p}\|f\|_{H^p(R^n)} .$$

(ii) is clear by the definition of $\mathcal{M}(H^p)$.

PROOF OF LEMMA 1. We shall give proofs of (i) and (iii). Other results can be proved in a similar way.

Proof of (i). By the homogeneity of ϕ, it holds that

$$m_{1,k,t}(\xi)=m_{1,k,1}(t\xi) .$$

Hence $\|m_{1,k,t}\|_{\mathcal{M}(H^p)}$ does not depend on t by Proposition A, (i). Set

$$F(x)=k\int_0^1(1-s)^{k-1}e^{isx}ds$$

and

$$F(x)=C_k\phi(x)x^{-k}e^{ix}+\tilde{F}(x) ,$$

where

$$C_k=k\Gamma(k)e^{-\pi ik/2} .$$

Note that C_k is the constant that satisfies the equality

$$k\int_{-\infty}^1(1-s)^{k-1}e^{isx}ds=C_kx^{-k}e^{ix} ,\quad x>0 ,$$

where the left hand side shall be considered as the Fourier transform of a tempered distribution on $\boldsymbol{R}$ (see Gel'fand-Shilov [4], Chap. II). It can be shown that $\tilde{F}$ is smooth and

$$\frac{d^M}{dx^M}\tilde{F}(x)=O(x^{-M-1})\quad\text{as } x\to\infty$$

for every integer $M\geq0$ (see Sjöstrand [9]). Hence, using Theorem A, we have $\operatorname{Re}\tilde{F}(\phi(\xi))\in\mathcal{M}(H^p)$ for all $p>0$. Thus we have the desired result by observing that

$$m_{1,k,1}(\xi) = \mathrm{Re}\, F(\phi(\xi))$$
$$= k\Gamma(k)\psi(\phi(\xi))\phi(\xi)^{-k} \cos(\phi(\xi) - \pi k/2) + \mathrm{Re}\, \tilde{F}(\phi(\xi)) .$$

Proof of (iii). We rewrite $m_{3,s,t}(\xi)$ as follows:

$$m_{3,s,t}(\xi) = m_{s,t}(\xi) m_s(t\xi) ,$$

where

$$m_{s,t}(\xi) = (1+|\xi|^2)^{-s/2}(1+|t\xi|^2)^{s/2}$$

and

$$m_s(t\xi) = (1+|t\xi|^2)^{-s/2} \cos \phi(t\xi) .$$

Using Theorem A, we can show that $m_{s,t} \in \mathcal{M}(H^p)$, $m_{s,t}^{-1} \in \mathcal{M}(H^p)$, and

$$\|m_{s,t}\|_{\mathcal{M}(H^p)} \leq C(1+t)^s$$

for every $p>0$. Also by Theorem A,

$$(1-\psi(\phi(\xi)))m_s(\xi) \in \mathcal{M}(H^p) ,$$
$$\psi(\phi(\xi))(1+|\xi|^2)^{-s/2}|\xi|^s \in \mathcal{M}(H^p) ,$$

and

$$\psi(\phi(\xi))(1+|\xi|^2)^{s/2}|\xi|^{-s} \in \mathcal{M}(H^p)$$

for all $p>0$. Combining these results and using Proposition A, we can complete the proof.

Lemma 1 reduces the problems to the study of the Fourier multiplier of the following form:

$$m(\xi) = \psi(\phi(\xi))(a(\xi)e^{i\phi(\xi)} + b(\xi)e^{-i\phi(\xi)}) ,$$

where a and b are smooth (on $\boldsymbol{R}^n \setminus \{0\}$) homogeneous functions of degree $-k$, $k \geq 0$. For our argument given below, it is important to know the singularity of the distribution $K = \mathcal{F}^{-1}m$. In the next section we shall study the singularity of K in detail.

§ 3. Singularity of the kernel

We shall use the notations of Fréchet differential calculus. If X and Y are Banach spaces and $g: \mathcal{O} \to Y$ is a smooth map defined on an open subset $\mathcal{O}$ of X, then the Fréchet derivative $g'(a)$, $a \in \mathcal{O}$, is a linear operator $X \to Y$ and the second Fréchet derivative $g''(a)$, $a \in \mathcal{O}$, is a symmetric bilinear map $X \times X \to Y$. $g''(a)x^2$ denotes the value $(\in Y)$ of $g''(a)$ evaluated at $(x, x) \in X \times X$. In the following calculus, X and Y are finite dimensional Euclidean spaces $\boldsymbol{R}^n$ or linear subspaces of $\boldsymbol{R}^n$. If $X=\boldsymbol{R}^n$ and $Y=\boldsymbol{R}$, $g'(a)$ is given by $\operatorname{grad} g(a)$;

$$g'(a)x = x\cdot \operatorname{grad} g(a)\ .$$

If X is the Euclidean space $\boldsymbol{R}^n$ or a linear subspace of $\boldsymbol{R}^n$ and $Y=\boldsymbol{R}$, we define $\det g''(a)$, $\operatorname{rank} g''(a)$, and $\operatorname{sign} g''(a)$ as follows. $\det g''(a)$ is defined to be the determinant of the matrix

$$(g''(a)(x_i, x_j))\ ,$$

where (x_i) is an orthonormal basis of X. $\operatorname{rank} g''(a)$ and $\operatorname{sign} g''(a)$ are defined by

$$\operatorname{rank} g''(a) = \text{rank of the matrix } (g''(a)(y_i, y_j))$$

and

$$\operatorname{sign} g''(a) = p-q\ ,$$

$$p = \text{number of positive eigenvalues of the matrix } (g''(a)(y_i, y_j))\ ,$$
$$q = \text{number of negative eigenvalues of the matrix } (g''(a)(y_i, y_j))\ ,$$

where (y_i) is a linear space basis of X. It is to be noted that the above definitions of $\det g''(a)$, $\operatorname{rank} g''(a)$ and $\operatorname{sign} g''(a)$ do not depend on the choices of the orthonormal basis (x_i) and (not necessarily orthonormal) basis (y_i).

Throughout this section we assume that $n\geq 2$. Let ϕ be a positively homogeneous function of degree 1 which is smooth and positive on $\boldsymbol{R}^n\setminus\{0\}$. Consider the compact hypersurface

$$\Sigma = \{x\in \boldsymbol{R}^n;\ \phi(x)=1\}$$

and the *spherical map*

$$\nu : \Sigma \ni p \longmapsto \frac{\operatorname{grad}\phi(p)}{|\operatorname{grad}\phi(p)|} \in S^{n-1}\ . \tag{7}$$

Set $\phi_\lambda(x)=\phi(x)^\lambda$, $\lambda\in\boldsymbol{R}$, $\lambda\neq 0$. Then ϕ_λ is a positively homogeneous function of degree λ and the equation $\phi_\lambda(x)=1$ also defines the surface Σ. By the implicit function theorem, the equation

$$\phi(p+x+h\nu(p))=1\ ,\quad x\in T_p\Sigma,\ h\in \boldsymbol{R}$$

determines a smooth function $h=h_p(x)$ in a neighborhood of $x=0$ in $T_p\Sigma$, where $T_p\Sigma$ denotes the tangent space to Σ at the point p. We have

$$h_p'(0)x=0\ ,\quad x\in T_p\Sigma$$

and

$$(8) \qquad \begin{cases} h_p''(0)x^2 = -\dfrac{\phi''(p)x^2}{|\operatorname{grad}\phi(p)|} \\ \qquad = -(\operatorname{sign}\lambda)\, \dfrac{\phi_\lambda''(p)x^2}{|\operatorname{grad}\phi_\lambda(p)|}\,, \quad x \in T_p\Sigma\,. \end{cases}$$

The map

$$\mathcal{O} \ni x \longmapsto p + x + h_p(x)\nu(p) \in \Sigma$$

is an embedding onto a neighborhood of p in Σ if $\mathcal{O}$ is a sufficiently small neighborhood of 0 in $T_p\Sigma$. This parametrization of Σ will simplify some calculations in our arguments, which, however, we shall not give in detail. The *Gaussian curvature* $\kappa(p)$ of Σ at p with respect to the spherical map (7) is given by

$$(9) \qquad \kappa(p) = (-1)^{n-1} \det h_p''(0)$$

(see Matsumura [7], p. 340) or, by (8),

$$\begin{aligned} \kappa(p) &= |\operatorname{grad}\phi(p)|^{-n+1} \det(\phi''(p)\,|T_p\Sigma) \\ &= (\operatorname{sign}\lambda)^{n-1} |\operatorname{grad}\phi_\lambda(p)|^{-n+1} \det(\phi_\lambda''(p)\,|\,T_p\Sigma)\,, \end{aligned}$$

where $\phi_\lambda''(p)|T_p\Sigma$ denotes the restriction of $\phi_\lambda''(p)$ to $T_p\Sigma \times T_p\Sigma$. Differentiating Euler's equality

$$\phi_\lambda'(x)x = \lambda\phi_\lambda(x)$$

in the direction of y, we have

$$\phi_\lambda''(x)(y, x) + \phi_\lambda'(x)y = \lambda\phi_\lambda'(x)y\,.$$

Hence

$$\phi_\lambda''(x)(y, x) = 0 \quad \text{whenever } y \in T_x\Sigma$$

and

$$\phi_\lambda''(x)x^2 = \lambda(\lambda-1) \quad \text{for } x \in \Sigma\,.$$

This means that, if we take a basis $(y_1, \cdots, y_n)$ of $\boldsymbol{R}^n$ such that $(y_1, \cdots, y_{n-1})$ is an orthonormal basis of $T_p\Sigma$ and $y_n = p$, then the corresponding matrix of $\phi_\lambda''(p)$ is of the following form:

$$(\phi_\lambda''(p)(y_i, y_j))_{1 \le i,j \le n} = \begin{pmatrix} & & 0 \\ & A & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & \lambda(\lambda-1) \end{pmatrix},$$

$$A=((\phi''_\lambda(p)\,|\,T_p\Sigma)(y_i,\,y_j))_{1\le i,j\le n-1}$$
$$=-(\text{sign}\,\lambda)|\text{grad}\,\phi_\lambda(p)|(h''_p(0)(y_i,\,y_j))_{1\le i,j\le n-1}\,.$$

From this we see that

$$\text{sign}\,\phi''_\lambda(p)=-(\text{sign}\,\lambda)\ \text{sign}\,h''_p(0)+\text{sign}\,\lambda(\lambda-1)$$

and

$$\text{rank}\,\phi''_\lambda(p)=\begin{cases}\text{rank}\,h''_p(0)+1\,, & \lambda\neq 0,\ 1,\\ \text{rank}\,h''_p(0)\,, & \lambda=1.\end{cases}$$

To calculate $\det\phi''_\lambda(p)$, we use the orthonormal basis

$$(y_1,\,\cdots,\,y_{n-1},\,\nu(p))\,.$$

The relation of this basis to the basis $(y_1,\,\cdots,\,y_{n-1},\,y_n)$ is given by

$$(y_1,\,\cdots,\,y_{n-1},\,y_n)=(y_1,\,\cdots,\,y_{n-1},\,\nu(p))T\,,$$

$$T=\begin{pmatrix}1 & & & \xi_1\\ & \ddots & & \vdots\\ & & 1 & \xi_{n-1}\\ 0 & \cdots & 0 & \xi_n\end{pmatrix},$$

where

$$\xi_n=y_n\cdot\nu(p)=\frac{p\cdot\text{grad}\,\phi(p)}{|\text{grad}\,\phi(p)|}=\frac{\phi(p)}{|\text{grad}\,\phi(p)|}=\frac{1}{|\text{grad}\,\phi(p)|}\,.$$

Hence

$$\det\phi''_\lambda(p)=(\det T)^{-2}\det(\phi''_\lambda(p)(y_i,\,y_j))_{1\le i,j\le n}$$
$$=|\text{grad}\,\phi(p)|^2\lambda(\lambda-1)(-(\text{sign}\,\lambda)|\text{grad}\,\phi_\lambda(p)|)^{n-1}\det h''_p(0)\,,$$

and thus

(10) $$\det\phi''_\lambda(p)=(-1)^{n-1}\lambda^n(\lambda-1)|\text{grad}\,\phi(p)|^{n+1}\det h''_p(0).$$

Combining (9) and (10), we have

$$\kappa(p)=\frac{\det\phi''_\lambda(p)}{\lambda^n(\lambda-1)|\text{grad}\,\phi(p)|^{n+1}}\,,\quad \lambda\neq 0,\ 1.$$

We refer to the following

PROPOSITION 1. *The following conditions are mutually equivalent:*
(i) *the spherical map* $\nu:\Sigma\to S^{n-1}$ *is a smooth diffeomorphism;*
(ii) *the Gaussian curvature of* Σ *never vanishes on* Σ*;*

(iii) $\det \phi''_\lambda(x) \neq 0$ *for* $x \neq 0$ *and* $\lambda \neq 0, 1$;
(iv) $\operatorname{rank} \phi''(x) = n-1$ *for* $x \neq 0$;
(v) $\operatorname{sign} \phi''(x) = n-1$ *for* $x \neq 0$.

As for this proposition, see Matsumura [7], pp. 339-341, and the references given there. The equivalence of the conditions (ii)$\sim$(v) can be shown by the calculations given above.

Now we go to the study of the distribution K mentioned at the end of the previous section. Set

$$K^+ = \mathcal{F}^{-1}(\psi(\phi(\xi))a(\xi)e^{i\phi(\xi)})$$

and

$$K^- = \mathcal{F}^{-1}(\psi(\phi(\xi))a(\xi)e^{-i\phi(\xi)}) .$$

The behaviors of K^+ and K^- are described in the following

PROPOSITION 2. *Suppose that* $a(\xi)$ *is a positively homogeneous function of degree* λ, $\lambda \in \boldsymbol{R}$, *which is smooth on* $\boldsymbol{R}^n \backslash \{0\}$ *and that the Gaussian curvature of the surface* $\Sigma = \{\phi = 1\}$ *never vanishes on* Σ. *Then* K^+ *and* K^- *have the following behavior.*

(i) K^+ *is a smooth function in* $\boldsymbol{R}^n \backslash (-\Sigma^*)$, *where*

$$\begin{aligned} -\Sigma^* &= \{x \in \boldsymbol{R}^n ;\ |x| = |\operatorname{grad} \phi(\xi^-(x'))|\} \\ &= \{-\operatorname{grad} \phi(\xi) ;\ |\xi| = 1\} , \\ \xi^-(x') &= \nu^{-1}(-x/|x|) . \end{aligned}$$

For every differential monomial $(\partial/\partial x)^\alpha$ *and every* $M > 0$, *we have*

$$\left(\frac{\partial}{\partial x}\right)^\alpha K^+(x) = O(|x|^{-M}) \quad as\ |x| \to \infty .$$

In the neighborhood of $-\Sigma^*$, K^+ *has the following singularity (we shall abbreviate* $\xi^-(x')$ *to* ξ^-*): if* $-\lambda - |\alpha| - (n-1)/2 - 1 < 0$, *then*

$$\left(\frac{\partial}{\partial x}\right)^\alpha K^+(x) = O(||x| - |\operatorname{grad} \phi(\xi^-)||^{-\lambda - |\alpha| - (n-1)/2 - 1}) \quad as\ |x| \to |\operatorname{grad} \phi(\xi^-)| ; \tag{11}$$

if $-\lambda - (n-1)/2 - 1 < 0$, *then*

$$\begin{cases} K^+(rx') = A^- |\kappa(\xi^-)|^{-1/2} |\operatorname{grad} \phi(\xi^-)|^{-(n-1)/2-1} a(\xi^-) \left(1 - \dfrac{r}{|\operatorname{grad} \phi(\xi^-)|} + i0\right)^{-\lambda - (n-1)/2 - 1} \\ \qquad + o(|r - |\operatorname{grad} \phi(\xi^-)||^{-\lambda - (n-1)/2 - 1}) \quad as\ r \to |\operatorname{grad} \phi(\xi^-)| , \end{cases} \tag{12}$$

where A^- *is a constant depending only on* n *and* λ;

$$A^{-}=(2\pi)^{-1/2}\exp\left[\frac{\pi i}{2}(\lambda+n)\right]\Gamma\left(\lambda+\frac{n-1}{2}+1\right).$$

The O- and o-estimates in (11) *and* (12) *can be made uniform with respect to* $x'\in S^{n-1}$.

(ii) K^{-} *has the similar behavior as* K^{+}. *We have only to replace* $\xi^{-}(x')$ *by* $\xi^{+}(x')=\nu^{-1}(x/|x|)$, $-\Sigma^{*}$ *by*

$$\begin{aligned}\Sigma^{*}&=\{x\in \boldsymbol{R}^{n};\ |x|=|\operatorname{grad}\phi(\xi^{+}(x'))|\}\\&=\{\operatorname{grad}\phi(\xi);\ |\xi|=1\},\end{aligned}$$

$+i0$ *by* $-i0$ (*in* (12)) *and* A^{-} *by*

$$A^{+}=(2\pi)^{-1/2}\exp\left[-\frac{\pi i}{2}(\lambda+n)\right]\Gamma\left(\lambda+\frac{n-1}{2}+1\right).$$

Before going to the proof of this proposition, we prepare the following

LEMMA 2. *Let* $\lambda\in \boldsymbol{C}$. *For* $\varepsilon>0$, *set*

$$J_{\lambda,\varepsilon}(t)=(2\pi)^{-1/2}\int_0^\infty \phi(s)s^{\lambda}e^{ist-\varepsilon s}ds.$$

Then $J_{\lambda,\varepsilon}(t)$ *converges, as* ε *tends to zero, uniformly in* $|t|\geq\delta$ *for every* $\delta>0$, *and the resulting function* $J_{\lambda}(t)=\lim_{\varepsilon\downarrow 0}J_{\lambda,\varepsilon}(t)$ *has the following estimates*:

$$J_{\lambda}(t)=\begin{cases}A_{\lambda}(t+i0)^{-\lambda-1}+\tilde{J}_{\lambda}(t), & \lambda\neq-1,-2,\cdots,\\ A'_{\lambda}t^{-\lambda-1}+A''_{\lambda}t^{-\lambda-1}\log(t+i0)+\tilde{J}_{\lambda}(t), & \lambda=-1,-2,\cdots,\end{cases}$$

where $\tilde{J}_{\lambda}(t)$ *is smooth on* $\boldsymbol{R}$ *and* A_{λ}, A'_{λ}, *and* A''_{λ} *are constants depending only on* λ (*esp.*

$$A_{\lambda}=(2\pi)^{-1/2}\exp\left[\frac{\pi i}{2}(\lambda+1)\right]\Gamma(\lambda+1));$$

and

$$J_{\lambda}(t)=O(|t|^{-M})\quad \text{as } |t|\to\infty \text{ for every } M>0.$$

The existence of $\lim_{\varepsilon\downarrow 0}J_{\lambda,\varepsilon}(t)$ can be easily seen by rewriting $J_{\lambda,\varepsilon}(t)$ as

$$J_{\lambda,\varepsilon}(t)=\left(\frac{-1}{it-\varepsilon}\right)^{M}(2\pi)^{-1/2}\int_0^\infty\left(\frac{d}{ds}\right)^{M}(\phi(s)s^{\lambda})e^{ist-\varepsilon s}ds \tag{13}$$

(integration by parts). The first estimate for $J_{\lambda}(t)$ comes from the equality

$$\mathcal{F}^{-1}(s_{+}^{\lambda})(t)=\begin{cases}A_{\lambda}(t+i0)^{-\lambda-1}, & \lambda\neq-1,-2,\cdots,\\ A'_{\lambda}t^{-\lambda-1}+A''_{\lambda}t^{-\lambda-1}\log(t+i0), & \lambda=-1,-2,\cdots,\end{cases}$$

(see Gel'fand-Shilov [4], Chap. II) and the fact that $s_+^\lambda - \psi(s)s^\lambda$ is a distribution with compact support. The second estimate for $J_\lambda(t)$ is seen from (13).

PROOF OF PROPOSITION 2. We shall refine the calculus of Sjöstrand [9]. Since the proofs of (i) and (ii) are almost the same, we shall give the proof of (i). We give the estimates of $K^+(x)$ only; those of $(\partial/\partial x)^\alpha K^+(x)$, $|\alpha|>0$, can be obtained by replacing $a(\xi)$ by $\xi^\alpha a(\xi)$.

Writing $y \in \boldsymbol{R}^n \backslash \{0\}$ as

$$y = s\xi\ , \quad 0<s<\infty\ , \quad \xi \in \Sigma\ ,$$

we have

$$\begin{aligned} K^+(x) &= \lim_{\varepsilon \downarrow 0} K_\varepsilon^+(x) \\ &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-n/2} \int_{\boldsymbol{R}^n} \psi(\phi(y)) e^{-\varepsilon\phi(y) + i\phi(y) + iy\cdot x} a(y) dy \\ &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-n/2} \iint_{\substack{0<s<\infty \\ \xi \in \Sigma}} \psi(s) s^{\lambda+n-1} e^{-\varepsilon s + is + is\xi\cdot x} \frac{a(\xi)}{|\operatorname{grad}\phi(\xi)|} ds d\sigma(\xi)\ , \end{aligned}$$

where $d\sigma$ is the $(n-1)$-dimensional surface element of Σ. We shall show that the above limit exists uniformly on every compact subset of $\boldsymbol{R}^n \backslash (-\Sigma^*)$ and hence, in $\boldsymbol{R}^n \backslash (-\Sigma^*)$, the resulting function is equal to the inverse Fourier transform of $\psi(\phi(\xi)) a(\xi) e^{i\phi(\xi)}$ in the sense of tempered distribution.

First, integrate in the following way:

$$K_\varepsilon^+(x) = (2\pi)^{-1/2} \int_0^\infty \psi(s) s^{\lambda+n-1} e^{-\varepsilon s + is} \left((2\pi)^{-(n-1)/2} \int_\Sigma \frac{a(\xi) e^{isr\xi\cdot x'}}{|\operatorname{grad}\phi(\xi)|} d\sigma(\xi) \right) ds\ ,$$

where $r=|x|$ and $x'=x/|x|$. We can obtain the asymptotic behavior as $sr\to\infty$ of the inner integral by using the method of stationary phase. The phase function $\xi\cdot x'$ is stationary at $\xi^0 \in \Sigma$ if and only if the affine hyperplane $\{\xi \in \boldsymbol{R}^n;\ \xi\cdot x' = \xi^0\cdot x'\}$ is tangent to Σ, *i.e.*

$$\nu(\xi^0) = x' \quad \text{or} \quad -x'\ .$$

By Proposition 1, there are exactly two such points;

$$\xi^+ = \nu^{-1}(x') \quad \text{and} \quad \xi^- = \nu^{-1}(-x')\ .$$

The main contribution to the integral on Σ comes from the immediate neighborhood of these points. We have the following asymptotic expansions:

$$(2\pi)^{-(n-1)/2}\int_{\Sigma}\frac{a(\xi)e^{isr\xi\cdot x'}}{|\mathrm{grad}\,\phi(\xi)|}d\sigma(\xi)=I^{+}(sr,x')+I^{-}(sr,x')\ ,$$

$$I^{+}(sr,x')\sim e^{isr\xi^{+}\cdot x'}(sr)^{-(n-1)/2}\sum_{l=0}^{\infty}(sr)^{-l}p_l^{+}\quad \text{as } sr\to\infty,$$

$$I^{-}(sr,x')\sim e^{isr\xi^{-}\cdot x'}(sr)^{-(n-1)/2}\sum_{l=0}^{\infty}(sr)^{-l}p_l^{-}\quad \text{as } sr\to\infty,$$

uniformly with respect to $x'\in S^{n-1}$, where p_l^{+} and p_l^{-}, $l=0, 1, 2, \cdots$, are given by the values of $a(\xi)$, $\phi(\xi)$, and their derivatives at $\xi=\xi^{+}$ or ξ^{-} respectively, especially

$$\begin{aligned}p_0^{\pm}&=e^{\mp\pi i(n-1)/4}a(\xi^{\pm})|\kappa(\xi^{\pm})|^{-1/2}|\mathrm{grad}\,\phi(\xi^{\pm})|^{-1}\\&=e^{\mp\pi i(n-1)/4}2^{n/2}a(\xi^{\pm})|\det\phi_2''(\xi^{\pm})|^{-1/2}|\mathrm{grad}\,\phi(\xi^{\pm})|^{(n-1)/2}\ .\end{aligned}$$

Note that $p_l^{\pm}$, $l=0, 1, 2, \cdots$, are functions of x' only (they do not depend on s and r). (For more details, see Matsumura [7], pp. 330-346.) Executing the integration with respect to s, we have

$$\begin{aligned}K_\varepsilon^{+}(x)=&\sum_{l=0}^{M}r^{-(n-1)/2-l}p_l^{+}(x')(2\pi)^{-1/2}\int_0^\infty\phi(s)s^{\lambda+(n-1)/2-l}e^{-\varepsilon s+is(1+r\xi^{+}\cdot x')}ds\\&+\sum_{l=0}^{M}r^{-(n-1)/2-l}p_l^{-}(x')(2\pi)^{-1/2}\int_0^\infty\phi(s)s^{\lambda+(n-1)/2-l}e^{-\varepsilon s+is(1+r\xi^{-}\cdot x')}ds\\&+r^{-(n-1)/2}(2\pi)^{-1/2}\int_0^\infty\phi(s)s^{\lambda+(n-1)/2}e^{-\varepsilon s+is(1+r\xi^{+}\cdot x')}R_M^{+}(sr,x')ds\\&+r^{-(n-1)/2}(2\pi)^{-1/2}\int_0^\infty\phi(s)s^{\lambda+(n-1)/2}e^{-\varepsilon s+is(1+r\xi^{-}\cdot x')}R_M^{-}(sr,x')ds\ ,\end{aligned}$$

where

$$|R_M^{+}(sr,x')|\le C(sr)^{-M-1}\quad\text{and}\quad|R_M^{-}(sr,x')|\le C(sr)^{-M-1}$$

with some constant C independent of s, r and $x'\in S^{n-1}$. If we take M sufficiently large, then, by Lebesgue's convergence theorem, the integrals involving R_M^{+} and R_M^{-} converge, as $\varepsilon\downarrow 0$, uniformly in $r=|x|\ge\delta$, $\delta>0$, and the resulting functions are $O(|x|^{-(n-1)/2-M-1})$ as $|x|\to\infty$. The terms in the finite summations can be managed by Lemma 2. Using Lemma 2 and observing that

$$1+r\xi^{+}\cdot x'=1+r\frac{\xi^{+}\cdot\mathrm{grad}\,\phi(\xi^{+})}{|\mathrm{grad}\,\phi(\xi^{+})|}=1+\frac{r}{|\mathrm{grad}\,\phi(\xi^{+})|}\ge1$$

and

$$1+r\xi^{-}\cdot x'=1-r\frac{\xi^{-}\cdot\mathrm{grad}\,\phi(\xi^{-})}{|\mathrm{grad}\,\phi(\xi^{-})|}=1-\frac{r}{|\mathrm{grad}\,\phi(\xi^{-})|}\ ,$$

we can conclude that $\lim_{\varepsilon\downarrow0}K_\varepsilon^{+}(x)$ exists uniformly in

$$\left\{x\in \boldsymbol{R}^n;\ |x|\geq\delta,\ \left|1-\frac{r}{|\operatorname{grad}\phi(\xi^-(x'))|}\right|\geq\delta\right\},$$

for every $\delta>0$, and $K^+(x)=\lim K_\varepsilon^+(x)$ has the singularity in the neighborhood of $-\Sigma^*$ as described in the proposition.

Secondly, we shall show that $K_\varepsilon^+(x)$ converges uniformly in a neighborhood of $x=0$. But this can be easily seen by rewriting $K_\varepsilon^+(x)$ as

$$K_\varepsilon^+(x)=(2\pi)^{-(n-1)/2}\int_\Sigma \frac{a(\xi)}{|\operatorname{grad}\phi(\xi)|}\left((2\pi)^{-1/2}\int_0^\infty \phi(s)s^{\lambda+n-1}e^{-\varepsilon s+is(1+r\xi\cdot x')}ds\right)d\sigma(\xi)$$

and using Lemma 2. Thus we have completed the proof.

§ 4. Main results

The followings are our main results.

THEOREM 1. *Let $a(\xi)$ and $b(\xi)$ be positively homogeneous functions of degree $-k$, $k\geq 0$, and $\phi(\xi)$ be a positively homogeneous function of degree* 1. *Suppose that $a(\xi)$ and $b(\xi)$ are smooth on $\boldsymbol{R}^n\backslash\{0\}$ and at least one of them does not vanish identically. Also suppose that $\phi(\xi)$ is smooth and positive on $\boldsymbol{R}^n\backslash\{0\}$ and that the Gaussian curvature of the surface $\Sigma=\{\phi=1\}$ never vanishes on Σ. Then the function*

$$m(\xi)=\psi(\phi(\xi))(a(\xi)e^{i\phi(\xi)}+b(\xi)e^{-i\phi(\xi)})$$

is a Fourier multiplier for $H^p(\boldsymbol{R}^n)$ if and only if

$$(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|\leq k\,.$$

COROLLARY 1. (i) *Estimate* (3) *holds if and only if $(n-1)|1/p-1/2|\leq k$. In this case* (3) *holds with $C_p(t)=C_p$.*

(ii) *Estimate* (4) *holds if and only if $(n-1)|1/p-1/2|\leq k+1$. In this case* (4) *holds with $C_p(t)=C_pt$.*

(iii) *Estimate* (5) *holds if and only if $(n-1)|1/p-1/2|\leq s$. In this case* (5) *holds with $C_p(t)=C_p(1+t)^{(n-1)|1/p-1/2|}$.*

(iv) *Estimate* (6) *holds if and only if $(n-1)|1/p-1/2|\leq r+1$. In this case* (6) *holds with the following $C_p(t)$:*
when $r\geq 0$,

$$C_p(t)=C_pt(1+t)^{a(n,p)}\,,$$
$$a(n,\,p)=\max\{(n-1)|1/p-1/2|-1,\,0\};$$

when $-1\leq r<0$,

$$C_p(t)=\begin{cases} C_p t\,, & t\geq 1 \\ C_p t^{1+r}\,, & 0<t<1. \end{cases}$$

(*In* (3)~(6), *we replace the norm* $\|\ \|_p$ *by* $\|\ \|_{H^p}$ *if* $0<p\leq 1$.) *The same estimates hold if we replace* $U(t)$ *and* $V(t)$ *respectively by*

$$U_\phi(t)=T_m \quad with \quad m(\xi)=\cos t\phi(\xi)$$

and

$$V_\phi(t)=T_m \quad with \quad m(\xi)=\frac{\sin t\phi(\xi)}{\phi(\xi)}\,,$$

where $\phi(\xi)$ *is as mentioned in Theorem* 1.

REMARK. The constants $C_p(t)$'s given in Corollary 1 cannot be improved. This can be seen by using Theorem 2 below and the inequality

$$\|m\|_{L^\infty}\leq C_p\|m\|_{\mathcal{M}(H^p)}\,, \quad 0<p<\infty$$

(cf. Miyachi [8]).

COROLLARY 2. *Let* $\phi(\xi)$ *be as mentioned in Theorem* 1, $k>0$, $n\geq 2$, and $l=\max\{[nk/(n-1)]+1, [n/2]+1\}$. *Suppose that* $f\in C^l(\boldsymbol{R}^n\backslash\{0\})$ *and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha f(\xi)\right|\leq C|\xi|^{-k-|\alpha|}\,, \quad |\alpha|\leq l.$$

Then the functions

$$\psi(\phi(\xi))f(\xi)e^{\pm i\phi(\xi)}$$

are Fourier multipliers for $H^p(\boldsymbol{R}^n)$ *for* $(n-1)|1/p-1/2|\leq k$.

THEOREM 2. *Let* $a(\xi)$, $b(\xi)$ *and* $\phi(\xi)$ *be as mentioned in Theorem* 1 *and consider the multiplier*

$$m_t(\xi)=\psi(\phi(\xi))(a(\xi)e^{it\phi(\xi)}+b(\xi)e^{-it\phi(\xi)})\,.$$

Then there exist positive constants C *and* C' *such that*

$$Ct^{(n-1)|1/p-1/2|}\leq\|m_t\|_{\mathcal{M}(H^p(\boldsymbol{R}^n))}\leq C't^{(n-1)|1/p-1/2|}\,, \quad t\geq 1$$

for $(n-1)|1/p-1/2|\leq k$.

PROOF OF COROLLARY 1. Most parts of this corollary are direct consequences of Theorem 1 and Lemma 1. The nontrivial deductions are those of estimates of

$C_p(t)$ in (iii) and (iv). We shall prove the estimate of $C_p(t)$ in (iv); that in (iii) can be proved by a similar but simpler argument. We shall divide the proof into three cases.

Case (a): $r \geq 0$, $(n-1)|1/p-1/2| \geq 1$. Write

$$m_{4,r,t}(\xi)=(1+|\xi|^2)^{-r_1/2} m_{4,r_0,t}(\xi) ,$$

where $r_0=(n-1)|1/p-1/2|-1$ and $r_1=r-r_0 \geq 0$. $(1+|\xi|^2)^{-r_1/2} \in \mathcal{M}(H^p)$ by Theorem A and $m_{4,r_0,t} \in \mathcal{M}(H^p)$ by Theorem 1. Hence we obtain the desired estimate using Lemma 1.

Case (b): $r \geq 0$, $(n-1)|1/p-1/2|<1$. Theorem A and Theorem 1 show that $\phi(\xi)^{-1} \sin \phi(\xi) \in \mathcal{M}(H^p)$. Hence, by Proposition A,

$$\|m_{4,r,t}\|_{\mathcal{M}(H^p)} \leq t\|(1+|\xi|^2)^{-r/2}\|_{\mathcal{M}(H^p)} \left\| \frac{\sin \phi(t\xi)}{\phi(t\xi)} \right\|_{\mathcal{M}(H^p)} = Ct .$$

Case (c): $-1 \leq r < 0$, $(n-1)|1/p-1/2| \leq 1+r$. We decompose $m_{4,r,t}$ as follows:

$$m_{4,r,t}(\xi)=(1-\psi(|\xi|))m_{4,r,t}(\xi)+\psi(|\xi|)m_{4,r,t}(\xi) .$$

Theorem A and Theorem 1 show that $(1-\psi(|\xi|))(1+|\xi|^2)^{-r/2} \in \mathcal{M}(H^p)$ and $\phi(\xi)^{-1} \sin \phi(\xi) \in \mathcal{M}(H^p)$. Hence

$$\begin{aligned} &\|(1-\psi(|\xi|))m_{4,r,t}\|_{\mathcal{M}(H^p)} \\ &\quad \leq t\|(1-\psi(|\xi|))(1+|\xi|^2)^{-r/2}\|_{\mathcal{M}(H^p)} \left\| \frac{\sin \phi(t\xi)}{\phi(t\xi)} \right\|_{\mathcal{M}(H^p)} \\ &\quad = Ct. \end{aligned}$$

On the other hand

$$\begin{aligned} \|\psi(|\xi|)m_{4,r,t}\|_{\mathcal{M}(H^p)} &\leq C\|\psi(|\xi|)\phi(\xi)^{-r-1} \sin t\phi(\xi)\|_{\mathcal{M}(H^p)} \\ &= Ct^{1+r}\|\psi(|\xi|)\phi(t\xi)^{-r-1} \sin \phi(t\xi)\|_{\mathcal{M}(H^p)} \\ &\leq Ct^{1+r}\|\phi(\xi)^{-r-1} \sin \phi(\xi)\|_{\mathcal{M}(H^p)} \\ &= Ct^{1+r} \end{aligned}$$

since $\phi(\xi)^{-r-1} \sin \phi(\xi) \in \mathcal{M}(H^p)$ by Theorem A and Theorem 1. Thus we have

$$\|m_{4,r,t}\|_{\mathcal{M}(H^p)} \leq C \max\{t, t^{1+r}\} .$$

PROOF OF COROLLARY 2. Write

$$\psi(\phi(\xi))f(\xi)e^{\pm i\phi(\xi)}=m_1(\xi)m_2(\xi) ,$$

$$m_1(\xi)=\phi(\xi)^k f(\xi) ,$$

$$m_2(\xi)=\psi(\phi(\xi))\phi(\xi)^{-k}e^{\pm i\phi(\xi)} ,$$

and apply Theorem A to $m_1(\xi)$ and Theorem 1 to $m_2(\xi)$.

PROOF OF THEOREM 1. We shall prove Theorem 1 for $\phi(\xi)=|\xi|$ since the proof for general $\phi(\xi)$ can be performed with a little modification. If $n=1$, Theorem 1 can be proved by Theorem A and the fact that $e^{\pm i\xi}\in\mathcal{M}(H^p(\boldsymbol{R}))$, $0<p<\infty$. Hence we assume that $n\geq 2$.

Proof of the "if" part. We show that

$$m_\pm(\xi)=\psi(\phi(\xi))a(\xi)e^{\pm i\phi(\xi)}\in\mathcal{M}(H^p(\boldsymbol{R}^n))$$

under the assumption that

$$(n-1)(1/p-1/2)=k \quad \text{and} \quad 0<p\leq 1\ ,$$

from which, by using the interpolation theorem for the analytic family of operators (Calderón-Torchinsky [1], pp. 151-152; Coifman-Weiss [2], p. 597; Macías [6]) and the duality argument, the whole statement of the "if" part is derived. If we use the characterization of $H^p(\boldsymbol{R}^n)$ by the atom decomposition (Latter [5]) and that by the Riesz transforms, all we have to show is reduced to the following estimate:

$$\|T_{m_\pm}f\|_p\leq C\ ,\qquad f\in\mathcal{A}_r,\ 0<r<\infty, \tag{14}$$

where $\mathcal{A}_r$ is the set of functions f such that

$$\text{support} f\subset\{|x|\leq r\}\ ,\qquad \|f\|_{L^\infty}\leq r^{-n/p}$$

and

$$\int f(x)x^\alpha dx=0\ ,\qquad |\alpha|\leq[n/p-n].$$

See Miyachi [8] for details. We shall give the proof of (14) for $m_+(\xi)$; that for $m_-(\xi)$ is almost identical. We write $T=T_{m_+}$ and $K=\mathcal{F}^{-1}m_+$.

First suppose that $f\in\mathcal{A}_r$ with $r\geq 1/3$. By Hölder's inequality, we have

$$\begin{aligned}\left(\int_{|x|\leq 5r}|Tf(x)|^p dx\right)^{1/p}&\leq Cr^{n(1/p-1/2)}\|Tf\|_2\\ &\leq Cr^{n(1/p-1/2)}\|f\|_2\\ &\leq Cr^{n(1/p-1/2)}r^{-n/p+n/2}\\ &=C\ .\end{aligned}$$

If $|x|\geq 5r$, then, by Proposition 2,

$$\begin{aligned}|Tf(x)|=|K*f(x)|&\leq\int_{|y|\leq r}|K(x-y)||f(y)|dy\\ &\leq Cr^{-n/p+n}|x|^{-M}\end{aligned}$$

for any $M>0$. We take $M>n/p\geq n$. Then

$$\left(\int_{|x|\geq 5r} |Tf(x)|^p dx\right)^{1/p} \leq Cr^{-n/p+n}\left(\int_{|x|\geq 5r} |x|^{-Mp} dx\right)^{1/p}$$
$$= Cr^{-n/p+n} r^{-M+n/p}$$
$$= Cr^{n-M} \leq C .$$

Thus we have proved the estimate (14) for $r\geq 1/3$.

Next suppose that $f \in \mathcal{A}_r$ with $r\leq 1/3$. We shall write $N=[n/p-n]$. Since K is a smooth function on $\boldsymbol{R}^n \backslash S^{n-1}$, $Tf(x)$ is given by the integral

$$\int_{|y|\leq r} K(x-y)f(y)dy$$

in the region

$$\{x \in \boldsymbol{R}^n;\ \text{distance}\,(x, S^{n-1}) > r\} .$$

Since $f \in \mathcal{A}_r$ is orthogonal to all the polynomials of degree $\leq N$, the above integral can be rewritten as follows:

$$\int_{|y|\leq r} K(x-y)f(y)dy$$
$$= \int_{|y|\leq r} \left(K(x-y) - \sum_{|\alpha|\leq N} \frac{1}{\alpha !} D^\alpha K(x)(-y)^\alpha\right) f(y)dy$$
$$= (N+1) \sum_{|\alpha|=N+1} \iint_{\substack{0\leq t\leq 1 \\ |y|\leq r}} (1-t)^N \frac{1}{\alpha !} D^\alpha K(x-ty)(-y)^\alpha f(y)dtdy .$$

Hence

$$|Tf(x)| \leq Cr^{N+1-n/p+n} \sup_{\substack{|y|\leq r \\ |\alpha|=N+1}} |D^\alpha K(x-y)|$$

if distance $(x, S^{n-1}) > r$. From this and Proposition 2, it follows the following estimates:

$$(15) \qquad |Tf(x)| \leq \begin{cases} Cr^{N+1-n/p+n} |1-|x||^{k-N-1-(n-1)/2-1}, & ||x|-1| \geq 2r, \\ Cr^{N+1-n/p+n} |x|^{-M}, & |x| \geq 2, \end{cases}$$

where M can be taken arbitrarily large; we take $M > n/p$. Now decompose $\|Tf\|_p^p$ as follows

$$\|Tf\|_p^p = \int_{|x|\leq 1-2r} |Tf(x)|^p dx + \int_{1-2r\leq |x|\leq 1+2r} |Tf(x)|^p dx$$
$$+ \int_{1+2r\leq |x|\leq 2} |Tf(x)|^p dx + \int_{|x|\geq 2} |Tf(x)|^p dx$$
$$= I_1^p + I_2^p + I_3^p + I_4^p .$$

Using (15), we have

$$\begin{aligned}I_1&\leq Cr^{N+1-n/p+n}\left(\int_{|x|\leq 1-2r}(1-|x|)^{(k-N-1-(n-1)/2-1)p}dx\right)^{1/p}\\&\leq Cr^{N+1-n/p+n+k-N-1-(n-1)/2-1+1/p}\\&=C,\end{aligned}$$

where we used the relations

$$(k-N-1-(n-1)/2-1)p=(n/p-n-N-1-1/p)p<-1$$

and

$$N+1-n/p+n+k-N-1-(n-1)/2-1+1/p=0\,.$$

Similarly we have $I_3\leq C$ and also

$$\begin{aligned}I_4&\leq Cr^{N+1-n/p+n}\left(\int_{|x|\geq 2}|x|^{-Mp}\,dx\right)^{1/p}\\&\leq Cr^{N+1-n/p+n}\leq C\end{aligned}$$

since $N+1-n/p+n>0$ and $0<r\leq 1/3$. In order to estimate I_2, we use Hölder's inequality;

$$\text{(16)}\qquad\begin{cases}I_2\leq|\{1-2r\leq|x|\leq 1+2r\}|^{1/p-1/2}\|Tf\|_2\\ \quad\leq Cr^{1/p-1/2}\|Tf\|_2.\end{cases}$$

Now $f\in\mathcal{A}_r$ can be written as

$$f(x)=r^{-n/p}f_1(x/r)\,,\qquad f_1\in\mathcal{A}_1,$$

and hence

$$\hat{f}(\xi)=r^{-n/p+n}\hat{f}_1(r\xi)\,,\qquad f_1\in\mathcal{A}_1.$$

Here note that $f_1\in\mathcal{A}_1$ has the estimates

$$\|f_1\|_2\leq C\quad\text{and}\quad|\hat{f}_1(\xi)|\leq C|\xi|^{N+1}\,.$$

Using these estimates and Plancherel's theorem, we have

$$\begin{aligned}\|Tf\|_2^2&=\|m(\xi)\hat{f}(\xi)\|_2^2\\&\leq C\int_{|\xi|\geq 1}(|\xi|^{-k}r^{-n/p+n}|\hat{f}_1(r\xi)|)^2d\xi\\&\leq Cr^{2(-n/p+n)}\left\{\int_{1\leq|\xi|\leq 1/r}|\xi|^{-2k}r^{2(N+1)}|\xi|^{2(N+1)}d\xi+\int_{|\xi|\geq 1/r}|\xi|^{-2k}|\hat{f}_1(r\xi)|^2d\xi\right\}\\&\leq Cr^{2(-n/p+n)}\left\{r^{2(N+1)}(r^{-1})^{-2k+2(N+1)+n}+(r^{-1})^{-2k}\int_{R^n}|\hat{f}_1(r\xi)|^2d\xi\right\}\\&\leq Cr^{2(-n/p+n)+2k-n}\,,\end{aligned}$$

where we have used the relation

$$-2k+2(N+1)+n>-2(n-1)(1/p-1/2)+2(n/p-n)+n=2/p-1>0 .$$

Thus

$$\|Tf\|_2\leq Cr^{k-n(1/p-1/2)} .$$

Substituting this in (16), we have

$$I_2\leq Cr^{1/p-1/2+k-n(1/p-1/2)}=C .$$

Thus we have proved (14) for $0<r\leq 1/3$ and completed the proof of the "if" part.

Proof of the "only if" part. We shall refine the argument of Sjöstrand [9]. By the duality equation $\mathcal{M}(H^p)=\mathcal{M}(H^q)$, $1/p+1/q=1$ (with equality of the norms), it is sufficient to consider the case $0<p<2$. Suppose that $(n-1)(1/p-1/2)>k$ and $a(\xi)$ does not vanish identically. Take $\xi^0\in S^{n-1}$ such that $a(\xi^0)\neq 0$. Let $\varepsilon>0$ be a sufficiently small number and take a smooth (on $\boldsymbol{R}^n\backslash\{0\}$) positively homogeneous function $g(\xi)$ of degree $n/p-n-\varepsilon$ such that

$$g(\xi)=1 \quad \text{in a neighborhood of} \quad \xi^0$$

and

$$g(\xi)=0 \quad \text{in a neighborhood of} \quad -\xi^0.$$

Set $G=\mathcal{F}^{-1}(\psi(\phi(\xi))g(\xi))$. Then Proposition 2 shows that

$$T_mG(x)\sim A^-a(-x')(1-r+i0)^{k-n/p+n+\varepsilon-(n-1)/2-1} \quad \text{as } r\to 1$$

for x' in a neighborhood of $-\xi^0$. Hence, if we take $\varepsilon>0$ so small that

$$k-n/p+n+\varepsilon-(n-1)/2-1<-1/p ,$$

which is possible since

$$k-n/p+n-(n-1)/2-1<(n-1)(1/p-1/2)-n/p+n-(n-1)/2-1=-1/p ,$$

then $T_mG\notin H^p(\boldsymbol{R}^n)$. On the other hand $G\in H^p(\boldsymbol{R}^n)$ (see Lemma 3 below). Hence $m\notin\mathcal{M}(H^p(\boldsymbol{R}^n))$. Thus the proof is complete.

In the proof of Theorem 2, we shall use the following two lemmas.

LEMMA 3. *Let $g(\xi)$ be a positively homogeneous function of degree λ which is smooth on $\boldsymbol{R}^n\backslash\{0\}$. If $\lambda<n/p-n$, then $\mathcal{F}^{-1}(\psi(|\xi|)g(\xi))\in H^p(\boldsymbol{R}^n)$.*

LEMMA 4. *Let $m_t(\xi)$ be as described in Theorem 2. Suppose that $a(\xi^0)\neq 0$, $\xi^0=\xi^-(x_0')\in\Sigma$. Let $g(\xi)$ be a smooth (on $\boldsymbol{R}^n\backslash\{0\}$) positively homogeneous function of degree λ which is equal to 1 in a neighborhood of ξ^0 and to 0 in a neighborhood* *of* $\xi^+(x_0')$. *Suppose that* $k-\lambda-(n-1)/2-1<0$. *Then, there exist positive numbers* δ, c *and* M *and a neighborhood* $\mathcal{V}$ *of* x_0' *such that*

$$|\mathcal{F}^{-1}(m_t(\xi)\psi(\phi(\xi))g(\xi))(x)|\geq ct^{-(n-1)/2}\left|t-\frac{|x|}{|\operatorname{grad}\phi(\xi^-(x'))|}\right|^{k-\lambda-(n-1)/2-1}$$

for

$$x'\in\mathcal{V},\quad \left|t-\frac{|x|}{|\operatorname{grad}\phi(\xi^-(x'))|}\right|\leq\delta \text{ and } t\geq M\ .$$

We shall omit the proof of Lemma 4, which can be performed by a slight modification of that of Proposition 2. We give the

PROOF OF LEMMA 3. Write $m(\xi)=\phi(|\xi|)g(\xi)$. Take a function $\Psi\in C_0^\infty(\boldsymbol{R}^n)$ such that

$$\operatorname{support}\Psi\subset\{1/2\leq|\xi|\leq 2\},\quad \sum_{j=-\infty}^{\infty}\Psi(\xi/2^j)\equiv 1\quad(\xi\neq 0),$$

and decompose $m(\xi)$;

$$\begin{aligned} m(\xi)&=\sum_{j=0}^{\infty}\Psi(\xi/2^j)\phi(|\xi|)g(\xi)\\ &=m'(\xi)+\sum_{j=2}^{\infty}\Psi(\xi/2^j)g(\xi)\\ &=m'(\xi)+\sum_{j=2}^{\infty}2^{j\lambda}\Psi(\xi/2^j)g(\xi/2^j)\\ &=m'(\xi)+\sum_{j=2}^{\infty}2^{j\lambda}m''(\xi/2^j)\ . \end{aligned}$$

Then

$$\mathcal{F}^{-1}m=\mathcal{F}^{-1}m'+\sum_{j=2}^{\infty}2^{j(\lambda+n)}(\mathcal{F}^{-1}m'')(2^j\cdot)\ .$$

Certainly $\mathcal{F}^{-1}m'$ and $\mathcal{F}^{-1}m''\in H^p(\boldsymbol{R}^n)$ since m' and $m''\in C_0^\infty(\boldsymbol{R}^n\backslash\{0\})$. If $\lambda<n/p-n$, then, using the equality

$$\|f(2^j\cdot)\|_{H^p(\boldsymbol{R}^n)}=2^{-jn/p}\|f\|_{H^p(\boldsymbol{R}^n)}\ ,$$

we have

$$\sum_{j=2}^{\infty}\|2^{j(\lambda+n)}(\mathcal{F}^{-1}m'')(2^j\cdot)\|_{H^p(\boldsymbol{R}^n)}^p=\sum_{j=2}^{\infty}2^{j(\lambda+n-n/p)p}\|\mathcal{F}^{-1}m''\|_{H^p(\boldsymbol{R}^n)}^p<\infty$$

for $0<p<1$ and

$$\sum_{j=2}^{\infty}\|2^{j(\lambda+n)}(\mathcal{F}^{-1}m'')(2^j\cdot)\|_{H^p(\boldsymbol{R}^n)}=\sum_{j=2}^{\infty}2^{j(\lambda+n-n/p)}\|\mathcal{F}^{-1}m''\|_{H^p(\boldsymbol{R}^n)}<\infty$$

for $1\leq p<\infty$. Hence, by the completeness of $H^p(\boldsymbol{R}^n)$, $\mathcal{F}^{-1}m\in H^p(\boldsymbol{R}^n)$ if $\lambda<n/p-n$.

Now we go to the

PROOF OF THEOREM 2. The upper bound for $\|m_t\|_{\mathcal{M}(H^p(\boldsymbol{R}^n))}$ can be obtained by the following decomposition:

$$m_t(\xi)=t^{k_0}\{m_1^+(\xi)m_2^+(t\xi)+m_1^-(\xi)m_2^-(t\xi)\}\ ,$$

where

$$\begin{aligned}k_0&=(n-1)|1/p-1/2|\ ,\\ m_1^+(\xi)&=\psi(\phi(\xi))\phi(\xi)^{k_0}\,a(\xi)\ ,\\ m_1^-(\xi)&=\psi(\phi(\xi))\phi(\xi)^{k_0}\,b(\xi)\ ,\\ m_2^+(t\xi)&=\psi(\phi(t\xi))\phi(t\xi)^{-k_0}\,e^{i\phi(t\xi)}\end{aligned}$$

and

$$m_2^-(t\xi)=\psi(\phi(t\xi))\phi(t\xi)^{-k_0}\,e^{-i\phi(t\xi)}\ .$$

Applying Theorem A, Proposition A and Theorem 1, we obtain the upper bound. We go to show the lower bound.

It is sufficient to consider the case $0<p<2$; the case $p=2$ is clear and the case $2<p<\infty$ is reduced to the case $1<p<2$ by the duality equation $\mathcal{M}(H^p)=\mathcal{M}(H^q)$, $1/p+1/q=1$ (with equality of the norms). It is easy to show that the norm $\|m_t\|_{\mathcal{M}(H^p)}$ is bounded from below by a positive constant as t varies on any finite interval. Hence we have only to consider large t.

First assume that $(n-1)(1/p-1/2)\leq k$ and $k<-n/2+1/2+n/p$. Take a number λ such that

$$k-n/2-1/2<\lambda<n/p-n\ .$$

Let $g(\xi)$ be a positively homogeneous function of degree λ as mentioned in Lemma 4. Then, by Lemma 3, $G=\mathcal{F}^{-1}(\psi(\phi(\xi))g(\xi))\in H^p(\boldsymbol{R}^n)$ and, by Lemma 4,

$$\left\{\begin{aligned}\|T_{m_t}G\|_p&\geq Ct^{-(n-1)/2}\left(\int\left|t-\frac{|x|}{|\operatorname{grad}\phi(\xi^-(x'))|}\right|^{p(k-\lambda-n/2-1/2)}dx\right)^{1/p}\\&\geq Ct^{(n-1)(1/p-1/2)}\qquad\text{for}\quad t\geq M,\end{aligned}\right.\tag{17}$$

where the integral is taken over

$$\left\{x\in\boldsymbol{R}^n;\ x'\in\mathcal{V}\ \text{and}\ \left|t-\frac{|x|}{|\operatorname{grad}\phi(\xi^-(x'))|}\right|\leq\delta\right\}\ .$$

(17) gives the desired lower bound.

Next consider the general case; $(n-1)(1/p-1/2)\leq k$. Consider the following family of multipliers:

$$m_{t,z}(\xi)=m_t(\xi)\phi(\xi)^{kz}, \qquad z\in \boldsymbol{C},$$

which depend analytically on z. We have

$$\|m_{t,iy}\|_{\mathcal{M}(H^p)}\leq C(1+|y|)^M\|m_t\|_{\mathcal{M}(H^p)}, \qquad y\in \boldsymbol{R},$$

by Theorem A and also

$$\|m_{t,1+iy}\|_{\mathcal{M}(H^2)}\leq C, \qquad y\in \boldsymbol{R}.$$

Hence, by the interpolation theorem for the analytic family of operators, we have

$$\|m_{t,\theta}\|_{\mathcal{M}(H^q)}\leq C\|m_t\|_{\mathcal{M}(H^p)}^{1-\theta}, \tag{18}$$

where $0<\theta<1$ and $1/q-1/2=(1-\theta)(1/p-1/2)$. If θ is sufficiently near 1, then, as we have already shown,

$$\|m_{t,\theta}\|_{\mathcal{M}(H^q)}\geq Ct^{(n-1)(1/q-1/2)}. \tag{19}$$

Combining (18) and (19), we have the desired inequality. This completes the proof.

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