*Estimation of singular spectrum of boundary values for some semihyperbolic operators**

To the memory of the late professor Mikio ISE, with a paternal reverence.

By Akira KANEKO

(Communicated by H. Komatsu)

This article is a continuation of our earlier works [7], [8] on the estimation of the singular spectrum (S. S. for short) of boundary values of solutions of linear partial differential equations. In [7] we have examined the S. S. of boundary values of real analytic solutions for the equation with constant coefficients employing a Fourier transformation technique adapted to the non-convex analysis. In [8] we have extended the results of [7] to the case of equations with real analytic coefficients and also examined general hyperfunction solutions by a different method employing the Cauchy-Kowalevsky theorem, the method of sweeping out and Green's formula. Here we give a sharper result mainly in the case of constant coefficients. We introduce here two new tools: The first is a theorem on micro-local analyticity given in [10] which enables us to reduce the S. S. of the boundary data exactly to one directions. The second is the use of "a local version of Bochner's tube theorem" with some of the real and the imaginary coordinates interchanged. Our examples contain "the glancing region" for the partial Laplacian or for the Lewy-Mizohata operator. We remark that P. Schapira has already considered in [26] a fairly wide case containing also these two examples. We think that our method gives an elementary explication to these examples.

We also give some results on the propagation of singularities along the boundary bicharacteristics. It would be better to notice the relation of our standpoint with that of Melrose-Sjöstrand [23], Ivriĭ [4] or Wakabayashi [27] etc., to say nothing of the difference of the categories, C^∞ or analytic. They have considered the general solution with (elliptic) boundary condition, hence the S. S. (i.e. the wave front set) of the boundary values was first of all limited in the hyperbolic and the glancing region. On the other hand we have considered mainly the real

*) The main part of this work was done while the author was staying at the University of Grenoble under the scholarship of the French government.

analytic solution with no boundary condition, because we have started from the problem of continuation of regular solutions. Thus the S. S. of the boundary values was first of all limited in the elliptic and the glancing region. Nevertheless our method gives a result of a kind of reflection of S. S. for the hyperfunction solution in the hyperbolic region with no boundary condition. We suppose thus that we will finally be able to unify the two approaches apart from the "microglobal" boundary condition in a way purely microlocal along the boundary.

The preparation of elementary lemmas has made this article thicker than we expected. We think, however, that it will serve to reveal the matters usually hidden behind the cohomological expression.

Contents

§1. Reduction of the singular spectrum to one direction.

1. The following lemma is a faithful copy of Corollary 2.3 in [10] and is one of our main tool.

LEMMA 1.1. *Let $u(x)$ be a hyperfunction with compact support. Let $W(x, \omega)$, $\omega \in \boldsymbol{S}^{n-1}$ be the components of a curved wave decomposition of $\delta(x)$. Let $\nu \in \boldsymbol{S}^{n-1}$ be a direction. Consider the local operators $J(D_\omega)$ in the variables $\omega \in \boldsymbol{R}^{n-1}$ with constant coefficients, where ω are considered as some fixed local coordinates on $\boldsymbol{S}^{n-1}$ on a neighborhood of ν. Assume that for every such $J(D_\omega)$, $J(D_\omega)W(x, \omega)|_{\omega=\nu} * u(x)$ is real analytic on a neighborhood of the origin. Then* S. S. $u(x)$ *does not contain* $(0, \sqrt{-1}\nu dx\infty)$.

The local operator is a kind of differential operator of infinite order appearing in the theory of hyperfunctions. In the sequel we only need the fact that $J(D_\omega)W(z, \omega)|_{\omega=\nu}$ is holomorphic in z where $W(z, \nu)$ is. Therefore we omit the detailed definition and refer to the references given in [10].

As a curved wave decomposition we mainly employ the most familiar one:

$$\delta(x) = \int_{S^{n-1}} W(x, \omega) d\omega,$$

$$d\omega = \sum (-1)^{j-1}\xi_j d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n|_{|\xi|=1}, \quad \omega = \xi/|\xi|, \tag{1.1}$$
$$W(x,\omega) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \frac{(1-\sqrt{-1}x\omega)^{n-1} - (1-\sqrt{-1}x\omega)^{n-2}(x^2-(x\omega)^2)}{(x\omega+\sqrt{-1}(x^2-(x\omega)^2)+\sqrt{-1}0)^n}.$$

For the sake of simplicity assume that $\nu=(0,\cdots,0,1)$. Then $W(x,\nu)$ is a hyperfunction obtained as the boundary value of the function $W(z,\nu)$ which is holomorphic on

$$0 < \mathrm{Im}\,\{z_n + \sqrt{-1}(z_1^2 + \cdots + z_{n-1}^2)\} = y_n - (y_1^2 + \cdots + y_{n-1}^2) + (x_1^2 + \cdots + x_{n-1}^2). \tag{1.2}$$

Thus S. S. $W(x,\nu)$ contains only one direction $\sqrt{-1}\nu dx\infty$. Since we have on the other hand

$$\mathrm{Re}\,\{z_n + \sqrt{-1}(z_1^2 + \cdots + z_{n-1}^2)\} = x_n - 2(x_1y_1 + \cdots + x_{n-1}y_{n-1}),$$

$W(z,\nu)$ can be continued holomorphically up to the real axis outside the origin. Thus, as is well known we have S. S. $W(x,\nu) = \{(0,\sqrt{-1}\nu dx\infty)\}$. As remarked above these regularity properties are valid also for $J(D_\omega)W(z,\omega)|_{\omega=\nu}$ for every $J(D_\omega)$.

As a matter of fact the condition that the wave decomposition is really curved is not at all necessary for the validity of Lemma 1 1. We can employ any decomposition of the delta function, in particular the classical plane wave decomposition:

$$\delta(x) = \int_{S^{n-1}} W_0(x,\omega)d\omega, \qquad W_0(x,\omega) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \frac{1}{(x\omega+\sqrt{-1}0)^n}.$$

This component has the emission of singularity out of the origin along the hyperplane $x\omega=0$. Since in general we attempt to employ Lemma 1.1 to $u(x)$ whose support is made compact by cutting off, the ambiguity of the cutting near the boundary will cause the ambiguity of the singular spectrum of the result of convolution inside the domain, and one may think that there will be little use for such a decomposition. But there are some cases where one can avail a standard cutting off. (The most trivial one is the case where $u(x)$ is real analytic near the boundary. Then we can use the cutting off by the characteristic function of an approximating domain bounded by a real analytic hypersurface.) In such cases the fact that $W_0(z,\omega)$ can be continued with respect to z exactly to the half space $y\omega>0$ will play an important role.

2. Next we recall the definition of the boundary value given by Komatsu-Kawai [19]. To shorten the description we employ the following notation. For an open subset $U\subset \boldsymbol{R}^n$ we denote by $\mathcal{B}(U)$ the space of hyperfunctions on U. For an open subset U' of the coordinate subspace $\boldsymbol{R}^{n-1}$ with the coordinates $x'=(x_2, \cdots, x_n)$, we denote the space of hyperfunctions on U' by $'\mathcal{B}(U')$ to distinguish the number of the independent variables. For a compact subset $K\subset \boldsymbol{R}^n$ (resp. $K'\subset \boldsymbol{R}^{n-1}$), we denote by $\mathcal{B}[K]$ (resp. $'\mathcal{B}[K']$) the subspace of $\mathcal{B}(\boldsymbol{R}^n)$ (resp. $'\mathcal{B}(\boldsymbol{R}^{n-1})$) consisting of the elements whose support is contained in K (resp. K'). Now let $p(x, D)$ be a linear partial differential operator of order m with real analytic coefficients. Assume that $x_1=0$ is non-characteristic with respect to $p(x, D)$. Let u be a real analytic solution of $p(x, D)u=0$ defined on $\{0<x_1<\delta\}\times U'$. Then there exists a unique extension $[u]$ of u as a hyperfunction on $\{x_1<\delta\}\times U'$ satisfying $\operatorname{supp}[u]\subset \{x_1\geq 0\}$ and an identity of the form

$$p(x, D)[u]=\sum_{j=0}^{m-1} b_j^{\dagger}(u)\delta^{(m-1-j)}(x_1) \tag{1.3}$$

with some coefficients $b_j^{\dagger}(u)\in {}'\mathcal{B}(U')$. $[u]$ is called the canonical extension of u. The coefficients are also determined uniquely and correspond to the boundary values of u with respect to a normal system of boundary operators: $b_j^{\dagger}(u)=b_j(D)u|_{x_1\to+0}$. In the sequel we need not know the explicit form of these boundary conditions. On the other hand the results in this article are obviously valid for any normal system of boundary operators. Therefore we do not discuss them here in detail.

Now we give the microlocalization principle for our problem.

PROPOSITION 1.2. *Under the above situation, assume that the operator has constant coefficients and that U' is relatively compact. Let $\nu'\in \boldsymbol{S}^{n-2}$ be a direction and let $J(D_{\omega'})$ be a local operator with constant coefficients by a system of local coordinates ω' on $\boldsymbol{S}^{n-2}$ in a neighborhood of ν'. Then for every open subset $V'\Subset U'$ there exist $\delta'>0$ and a real analytic solution v of $p(D)v=0$ on $\{0<x_1<\delta'\}\times V'$ such that its boundary values agree with*

$$[b_j^{\dagger}(u)]*\{J(D_{\omega'})W(x', \omega')|_{\omega'=\nu'}\}|_{V'}, \qquad j=0, \cdots, m-1,$$

where $[b_j^{\dagger}(u)]$ is an (arbitrary) extension of $b_j^{\dagger}(u)$ as a hyperfunction with support in $\overline{U'}$.

PROOF. Choose a compact piece of non-characteristic real analytic hypersurface of the form $x_1=t(x')$, $x_1\leq\delta_1$, satisfying $|dt(x')|\neq 0$ on $U'\setminus\overline{V'}$ and

$$\{x'\in \boldsymbol{R}^{n-1};\ t(x')=\varepsilon\}\subset U'\setminus\overline{V'} \qquad \text{for} \qquad 0\leq\varepsilon\leq\delta_1.$$

This is possible if we choose $t(x')$ with a sufficiently small modulus of gradient

$|dt(x')|$ and accordingly a small δ_1. As a system of coordinates on this hypersurface we can employ x'. Put

$$T_\varepsilon=\{x_1=t(x')\}\cap\{0<x_1<\varepsilon\},$$
$$W'_\varepsilon=\{x'\in R^{n-1};\ t(x')<\varepsilon\},\qquad W'=W'_0,$$
$$W_\varepsilon=\{x_1>t(x')\}\cap\{x_1<\varepsilon\}.$$

Then for each ε we have $V'\Subset W'_\varepsilon\Subset U'$, for $\varepsilon_1<\varepsilon_2$ we have $W'_{\varepsilon_1}\Subset W'_{\varepsilon_2}$ and for each $\varepsilon>0$ W_ε is a neighborhood of $\{0\}\times W'$ in R^n (that is, W_ε contains $\{0\}\times W'$ as a closed subset). Consider the product $Y(x_1-t(x'))[u]$, where Y denotes the Heaviside function of one variable. This is a well-defined hyperfunction on $\{x_1<\delta_1\}\backslash(\{0\}\times\partial W')$. It is real analytic on $x_1>0$ except on T_{δ_1} where it contains the singular spectrum only at the conormal bundle of the hypersurface $x_1=t(x')$. On W_{δ_1} it agrees with the original canonical extension $[u]$. Thus on $W_{\delta_1}\cup\{x_1>0\}$ we have

$$p(D)\{Y(x_1-t(x'))[u]\}=\sum_{j=0}^{m-1}b_j^+(u)\delta^{(m-1-j)}(x_1)+\sum_{j=0}^{m-1}t_j^+(u)\delta^{(m-1-j)}(x_1-t(x')),$$

where the coefficients of the two terms in the right hand side represent the boundary values to $x_1=0$ and to $x_1=t(x')$ respectively. Now choose $f_j(x')\in{}'\mathscr{B}[\overline{W'}]$ such that $f_j(x')=b_j^+(u)$ in W'. Let $[[Y(x_1-t(x'))[u]]]$ be an extension of $Y(x_1-t(X'))[u]$ as a hyperfunction on $\{x_1<\delta_1\}$ such that its support is contained in $\overline{W}_{\delta_1}\cap\{x_1\geq 0\}$. Then the difference

$$(1.4)\qquad p(D)[[Y(x_1-t(x'))[u]]]-\sum_{j=0}^{m-1}f_j(x')\delta^{(m-1-j)}(x_1)$$

is a hyperfunction on $\{x_1<\delta_1\}$ such that its support is contained in the part $\{x_1\geq 0\}$ of the hypersurface $x_1=t(x')$ and that it agrees with

$$\sum_{j=0}^{m-1}t_j^+(u)\delta^{(m-1-j)}(x_1-t(x'))$$

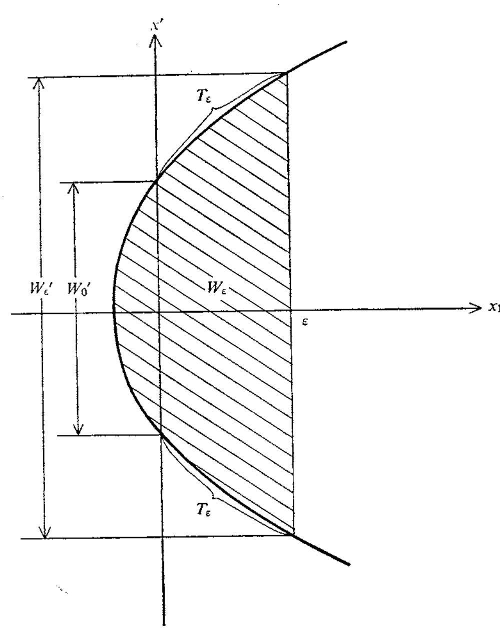

Fig. 1.

on $x_1>0$. Due to the theorem on division by Komatsu-Kawai (Theorem 2 in [19]) we can modify $[[Y(x_1-t(x'))[u]]]$ on the non-characteristic hypersurface $x_1=t(x')$ so that the result of (1.4) can be written in the form $\sum_{j=0}^{m-1}\alpha_j(x')\delta^{(m-1-j)}(x_1-t(x'))$. Since this modification is locally unique along this hypersurface, the coefficients α_j satisfy

$$\begin{aligned}&\operatorname{supp}\alpha_j\subset W'_{\delta_1}\backslash W'\\ &\alpha_j=t_j^{+}(u) \qquad \text{on} \qquad W'_{\delta_1}\backslash\overline{W'}.\end{aligned}$$

Especially $\alpha_j(x')$ are real analytic outside $\partial W'$. Thus we have obtained

$$p(D)[[Y(x_1-t(x'))[u]]]=\sum_{j=0}^{m-1}f_j(x')\delta^{(m-1-j)}(x_1)+\sum_{j=0}^{m-1}\alpha_j(x')\delta^{(m-1-j)}(x_1-t(x')), \tag{1.5}$$

where in the left hand side we have employed the same notation for the modified extension.

Now we make the convolution of both sides of (1.5) with $J(D_{\omega'})W(x',\omega')|_{\omega'=\nu'}$. It will cause no confusion if in the sequel we write simply $W(x',\nu')$ instead of this terrible factor. Then we obtain

$$p(D)v=\sum_{j=0}^{m-1}g_j(x')\delta^{(m-1-j)}(x_1)+\beta(x),$$

where we have put

$$\begin{aligned}&v(x)=[[Y(x_1-t(x'))[u]]]\underset{x'}{*}W(x',\nu'),\\ &g_j(x')=f_j(x')*W(x',\nu'), \qquad j=0,\cdots,m-1,\\ &\beta(x)=\sum_{j=0}^{m-1}\left(\frac{\partial}{\partial x_1}\right)^{m-1-j}\{\alpha_j(x')\delta(x_1-t(x'))\underset{x'}{*}W(x',\nu')\}.\end{aligned} \tag{1.6}$$

First of all, $v(x)$ is a hyperfunction on $\{x_1<\delta_1\}$ satisfying $\operatorname{supp}v\subset\{x_1\geq 0\}$. We claim that it is real analytic in $W_{\delta_1}\cap\{x_1>0\}$. In fact, recall the law that the S. S. of the product of two hyperfunctions can be estimated by taking the fiberwise convex combination by the great circle (or equivalently, the fiberwise vector sum of the conical representatives) of the S. S. of the factors (see Corollary 2.4.2 of Chapter I in [24]). We have then on $x_1>0$

$$\begin{aligned}&\text{S. S. } Y(x_1-t(y'))u(x_1,y')\subset\{(x_1,y';\ \pm\sqrt{-1}(dx_1-dt(y'))\infty);\ x_1=t(y')\},\\ &\text{S. S. } W(x'-y',\nu')\subset\{(x',y';\ \sqrt{-1}\nu'(dx'-dy')\infty);\ x'=y'\},\end{aligned} \tag{1.7}$$

hence

$$\begin{aligned}&\text{S. S. } Y(x_1-t(y'))u(x_1,y')W(x'-y',\nu')\\ &\quad\subset\{(x_1,x',y';\ \sqrt{-1}(\pm(1-\lambda)(dx_1-dt(y'))+\lambda\nu'(dx'-dy'))\infty);\\ &\qquad\qquad x_1=t(y'),x'=y',\ 0<\lambda<1\}\end{aligned} \tag{1.8}$$

$$\cup\{(x_1, x', y';\ \pm\sqrt{-1}(dx_1 - dt(y'))\infty);\ x_1 = t(y')\}$$
$$\cup\{(x_1, x', y';\ \sqrt{-1}\nu'(dx' - dy')\infty);\ x' = y'\}.$$

Recall next the law of estimation of the S. S. concerning the integration (see Theorem 2.3.1 of Chapter I in [24]; its expression being very abstract, Theorem 3.2.8 in [12] will help the understanding). It reads as follows: On the integration by y', it remains in the S. S. of the result only those points $(x, \sqrt{-1}\xi dx\infty)$ for which there exists a point y' verifying with the x, ξ all the enumerated conditions and reducing the coefficients of dy' to zero. Thus when we integrate with respect to y', the first component of the right hand side of (1.8) does not concern the S. S. of the result outside $x_1 = t(x')$. The second component does not cause the S. S., for we have assumed that $dt(y') \neq 0$, hence $dx_1 - dt(y')$ has no situation in which the coefficients of dy' vanish. Similarly the third component is also indifferent to the S. S. of the result of the integral.

The same argument applies to the term $\beta(x)$ on $x_1 > 0$. Let us examine it more in detail on $x_1 = 0$. We have

$$\begin{aligned}(1.9)\qquad \text{S. S. } &\alpha_j(y')\delta(x_1 - t(y'))W(x' - y', \nu')\\ &\subset\{(x_1, x', y';\ \sqrt{-1}((1-\lambda-\mu)\omega' dy' \pm \lambda(dx_1 - dt(y')) + \mu\nu'(dx' - dy'))\infty);\\ &\qquad\qquad y' \in \partial W', x_1 = t(y'), x' = y', \omega' \in R^{n-1}, \lambda > 0, \mu > 0, 0 < \lambda + \mu < 1\}\\ &\cup\{(x_1, x', y';\ \sqrt{-1}(\pm(1-\lambda)(dx_1 - dt(y')) + \lambda\nu'(dx' - dy'))\infty);\\ &\qquad\qquad x_1 = t(y'), x' = y', 0 < \lambda < 1\}\\ &\cup\{(x_1, x', y';\ \sqrt{-1}((1-\lambda)\omega' dy' \pm \lambda(dx_1 - dt(y')))\infty);\\ &\qquad\qquad y' \in \partial W', x_1 = t(y'), \omega' \in R^{n-1}, 0 < \lambda < 1\}\\ &\cup\cdots,\end{aligned}$$

where we have abbreviated the more tame terms. When we integrate with respect to y', the first and the second components do not affect the singular spectrum of the result on W_{δ_1}. The third component provides the direction $\pm\sqrt{-1}dx_1\infty$ on $x_1 = 0$, because the equation

$$(1-\lambda)\omega' dy' \mp \lambda dt(y') = 0$$

may be satisfied by some $\omega' \in \boldsymbol{R}^{n-1}$. Thus we have proved that

$$(1.10)\qquad \text{S. S. } \beta(x)|_{W_{\delta_1}} \subset \{x_1 = 0\} \times \{\pm\sqrt{-1}dx_1\infty\}.$$

Here we employ the following lemma. The following proof is due to K. Kataoka [16]. For self-containedness we will give in Appendix A a direct elementary proof with an accessory refinement which will be useful in § 4.

LEMMA 1.3. *Let $p(x, D)$ be a linear partial differential operator with real analytic coefficients. Assume that $x_1=0$ is non-characteristic with respect to p. Let $f(x)$ be a hyperfunction defined on $\{x_1<\delta\}\times U'$ such that*

$$\text{(1.11)}\qquad \begin{aligned}&\operatorname{supp} f\subset\{x_1\geq 0\},\\ &\mathrm{S.\,S.}\, f\subset\{x_1=0\}\times\{\pm\sqrt{-1}dx_1\infty\}\quad (\text{resp. } \mathrm{S.\,S.}\, f\subset\{x_1\geq 0\}\times\{\pm\sqrt{-1}dx_1\infty\}).\end{aligned}$$

Then, for any $V'\Subset U'$ there exist a positive constant $\delta'<\delta$ and a solution u of $p(x, D)u=f$ on $\{x_1<\delta'\}\times V'$ with the same property (1.11).

PROOF. On account of the Holmgren uniqueness theorem, it suffices to solve the equation locally on a neighborhood of each point on $x_1=0$. We employ the boundary value theory of Kataoka [14], [15] and for the details of the notation used below we refer to his original article. We put $M_+=\{x_1\geq 0\}$ and $X=\boldsymbol{C}^n$. $\mathcal{C}_{M_+/X}$ denotes the sheaf of relative microfunctions $H^n_{S^*_{M_+}X}(\pi^{-1}_{M_+/X}\mathcal{O}_X)$, where $S^*_{M_+}X$ is the conormal sphere bundle of M_+ in X and $\pi_{M_+/X}$: $S^*_{M_+}X\to X$ is the canonical projection. The pseudo-differential operator (or the micro-differential operator in new terminology) operates on $\mathcal{C}_{M_+/X}$ as a sheaf homomorphism and we have an isomorphism

$$\Gamma_{M_+}\mathcal{B}|_{x_1=0}=(\pi_{M_+/X})_*\mathcal{C}_{M_+/X}|_{x_1=0}$$

which is compatible with this operation. By the assumption the support of f, considered as a section of $\mathcal{C}_{M_+/X}$, is contained in

$$\{(x;\ (\zeta_1 dz_1+\sqrt{-1}\eta' dx')\infty)\in S^*_{M_+}X;\ x_1=0, \eta'=0\}.$$

(For the case in the parentheses $x_1=0$ is to be replaced by $x_1\geq 0$.) Since dz_1 is a non-characteristic direction of p along $x_1=0$, p^{-1} exists as a pseudo-differential operator on a neighborhood of this set. Thus we can define a section of $\mathcal{C}_{M_+/X}$ by $p^{-1}f$ there and by zero otherwise. Let u be the section of $\Gamma_{M_+}\mathcal{B}|_{x_1=0}$ corresponding to it by the above isomorphism. This is the desired solution. q.e.d.

END OF PROOF OF PROPOSITION 1.2. We apply the first case of this lemma to $\beta(x)$ and obtain a solution $w(x)$ of $p(D)w=\beta$ on $\{x_1<\delta'\}\times V'$ satisfying (1.11). Then we have on $\{x_1<\delta'\}\times V'$

$$p(D)(v-w)=\sum_{j=0}^{m-1} g_j(x')\delta^{(m-1-j)}(x_1).$$

In view of (1.6) this mostly proves our proposition. The remaining detail is the following: We have $g_j(x')=f_j(x')*W(x', \nu')$, where $f_j(x')\in{}'\mathcal{B}[\overline{W'}]$ is an extension of $b_j^+(u)|_{W'}$ but not an element of ${}'\mathcal{B}[\overline{U'}]$ extending $b_j^+(u)$ itself. But the difference is a real analytic function on a neighborhood of $\overline{V'}$, because we are now employing the curved wave decomposition (1.1). Thus by the Cauchy-Kowalevsky theorem we can once more adjust v perhaps for a smaller δ' in order to obtain the very assertion of our proposition. This argument shows in the same time that we can employ an arbitrarily fixed extension $[b_j^+(u)]\in{}'\mathcal{B}[\overline{U'}]$ of $b_j^+(u)$. q.e.d.

REMARK 1.4. In the statement of Proposition 1.2 we have assumed the analyticity of the solution u on $x_1>0$. But we can obtain the same conclusion assuming only that

$$\mathrm{S.\,S.}\ u\cap\{x_1>0\}\times\{\sqrt{-1}(\pm(1-\theta)dx_1+\theta\nu'dx')\infty;\ 0<\theta\leq 1\}=\varnothing.$$

In fact the proof is then modified as follows: First remark that on account of Sato's fundamental theorem we can then estimate S. S. u by

$$\{0<x_1<\delta\}\times U'\times\{\sqrt{-1}(\pm(1-\theta)dx_1+\theta\omega'dx')\infty;\ \omega'\neq\nu',\theta_0\leq\theta\leq 1\},\tag{1.12}$$

where θ_0 is a positive constant depending only on p and satisfying $|dt(x')|<\theta_0/(1-\theta_0)$. Thus the product $Y(x_1-t(x'))u(x_1,x')$ is well defined on $x_1>0$ and instead of (1.7) we have

$$\begin{aligned}&\mathrm{S.\,S.}\,Y(x_1-t(y'))u(x_1,y')\\&\quad\subset(1.12)_{x'\mapsto y'}\cup\{(x_1,y';\sqrt{-1}(\omega_1dx_1+\omega'dy')\infty);\ x_1=t(y'),(\omega_1,\omega')\in\boldsymbol{S}^{n-1}\},\end{aligned}$$

where $(1.12)_{x'\mapsto y'}$ denotes the set obtained from (1.12) replacing x' by y'. Thus we can reproduce the above calculation on S. S. with a little more delicacy and conclude this time that

$$p(D)v=\sum_{j=0}^{m-1}g_j(x')\delta^{(m-1-j)}(x_1)+\beta(x).$$

Here $g_j(x')$ are as above and v,β satisfy the estimates of the type

$$\operatorname{supp}v\subset\{x_1\geq 0\},\qquad \mathrm{S.\,S.}\ v|_{\{x_1<\delta'\}\times W'}\subset\{x_1\geq 0\}\times\{\pm\sqrt{-1}dx_1\infty\}.\tag{1.13}$$

In view of the second case of Lemma 1.3 we can find a solution w of $p(D)w=\beta$ satisfying itself (1.13). Thus we conclude finally

$$p(D)(v-w)=\sum_{j=0}^{m-1}g_j(x')\delta^{(m-1-j)}(x_1).$$

Since $v-w$ satisfies (1.13) also, it becomes in fact real analytic on $x_1>0$ because of Sato's fundamental theorem. Thus $v'=v-w$ is a required solution. This way of argument shows in the same time that we could have employed instead of $Y(x_1-t(x'))u(x)$ any modification of u to cut off the support. In fact we did not use here any regularity of the modification along the edge.

REMARK 1.5. In the proof of Proposition 1.2 we have employed the assumption that the coefficients of p be constant only in order to guarantee

$$(p(D)[[Y(x_1-t(x'))[u]]]) \underset{x'}{*} W(x',\nu')=p(D)([[Y(x_1-t(x'))[u]]] \underset{x'}{*} W(x',\nu')).$$

Thus the above proof goes without modification if only p has coefficients independent of x'. Thus Proposition 1.2 (and Remark 1.4) is valid e.g. for the generalized Lewy-Mizohata operator $D_1+\sqrt{-1}x_1^kD_2$ on $\boldsymbol{R}^n$. In the case where $p(x,D)$ has essentially variable coefficients, the above method does not work, nor is the assertion of Proposition 1.2 any longer valid in general. See Remark 4.3.

3. Now we consider a special case where the plane wave decomposition is available. For the sake of simplicity we fix ν' to be $(0,\cdots,0,1)$.

PROPOSITION 1.6. *Let $U'=U''\times\{|x_n|<a\}$, where $U''\Subset\boldsymbol{R}^{n-2}$ is open and convex. In the situation of Proposition* 1.2, *assume further that the boundary values $b_j^+(u)$ are real analytic near the part $\partial U''\times\{|x_n|<a\}$ of the boundary of U'. Then for every open subset $V''\Subset U''$ and $a'<a$ there exist $\delta'>0$ and a real analytic solution v of $p(D)v=0$ on $\{0<x_1<\delta'\}\times V'$ (where $V'=V''\times\{|x_n|<a'\}$) such that its boundary values agree with*

$$f_j(x')*\{J(D_{\omega'})W_0(x',\omega')|_{\omega'=\nu'}\}|_{V'}, \qquad j=0,\cdots,m-1$$

where $f_j(x')$ is a fixed element of $'\mathcal{B}[\overline{U'}]$ independent of $J(D_{\omega'})$ and satisfying

$$f_j(x')|_{V'}=b_j^+(u)|_{V'}.$$

PROOF. We repeat the proof of Proposition 1.2 adding the condition that the hypersurface $x_1=t(x')$ is strictly convex and that $b_j^+(u)$ is real analytic on a neighborhood of $\partial W'\cap\{|x_n|\le a'\}$. In view of the Cauchy-Kowalevsky theorem and the Holmgren uniqueness theorem, the solution u itself can be continued real analytically onto a neighborhood of $\{0\}\times(\partial W'\cap\{|x_n|\le a'\})$. Then by the local uniqueness of the canonical extension, we can assume that $[u]=Y(x_1)u$ there. Thus in this neighborhood the product $Y(x_1-t(x'))[u]=Y(x_1-t(x'))Y(x_1)u$ is meaningful in the usual sense, and this gives an explicit expression of $[[Y(x_1-t(x'))[u]]]$ there. Calculating $p(D)[[Y(x_1-t(x'))[u]]]$ by this expression, we find that we can assume the following regularity for the coefficients $\alpha_j(x')$ of (1.5):

$$\text{S. S. }\alpha_j(x')\cap\{|x_n|\le a'\}\subset\{(x';\ \pm\sqrt{-1}dt(x')\infty);\ t(x')=0\}. \tag{1.14}$$

Recall that $\partial W'=\{t(x')=0\}$. Now we convolute to both sides of (1.5) the component of the plane wave decompositions $W_0(x',\nu')$ (or, more precisely $J(D_{\omega'})W_0(x',\omega')|_{\omega'=\nu'}$). Then in the estimation of the S. S. starting from (1.7) we have to replace the condition $x'=y'$ everywhere by $(x'-y')\nu'=0$ because of the emission of singularity for $W_0(x',\nu')$. Nevertheless we claim first that

$$v(x)=\int Y(x_1-t(y'))u(x_1,y')W_0(x'-y',\nu')dy'$$

is real analytic in $W_{\delta_1}\cap\{x_1>0\}$. In fact, consider the estimate (1.8) with the above midification. Setting the coefficients of dy' to be equal to zero in the first component, we obtain

$$\pm(1-\lambda)dt(y')\infty=\lambda\nu' dy'\infty, \quad x_1=t(y'), \quad (x'-y')\nu'=0.$$

This implies that the hyperplane $(x'-y')\nu'=0$ in $\boldsymbol{R}^{n-1}$ passing through x' reaches the convex hypersurface $t(y')=x_1$ in $\boldsymbol{R}^{n-1}$ at the point y' where they are tangent. This is impossible if x' is in the region $x_1>t(x')$. Thus on the integration by y' the first component does not produce the S. S. inside $W_{\delta_1}\cap\{x_1>0\}$. There is no new problem for the second and the third components. The regularity of $\beta(x)$ on $W_{\delta_1}\cap\{x_1>0\}$ can be treated similarly.

We now examine the regularity of $\beta(x)$ on $x_1=0$. Consider therefore (1.9) with the same modification as above. In view of the regularity condition (1.14) the first component can be replaced by

$$\begin{aligned}(1.15)\quad &\{(x_1,x',y';\ \sqrt{-1}((1-\lambda-\mu)\omega' dy'\pm\lambda(dx_1-dt(y'))+\mu\nu'(dx'-dy'))\infty);\\ &\quad y'\in\partial W'\cap\{|x_n|>a'\},\ x_1=t(y'),\ (x'-y')\nu'=0,\ \omega'\in\boldsymbol{R}^{n-1},\ \lambda>0,\ \mu>0,\ 0<\lambda+\mu<1\}\\ &\cup\{(x_1,x',y';\ \sqrt{-1}(\pm(1-\lambda-\mu)dt(y')\pm\lambda(dx_1-dt(y'))+\mu\nu'(dx'-dy'))\infty);\\ &\quad y'\in\partial W'\cap\{|x_n|\le a'\},\ x_1=t(y'),\ (x'-y')\nu'=0,\ \lambda>0,\ \mu>0,\ 0<\lambda+\mu<1\}.\end{aligned}$$

Obviously the first component of (1.15) does not affect the S. S. of the integral by y' in $|x_n|\le a'$. As for the second component let the coefficients of dy' be equal to zero. Then we will again have the situation that the hyperplane $(x'-y')\nu'=0$ is tangent to $t(y')=x_1$. This is impossible for x' in $W'\cap\{|x_n|\le a'\}$. Thus this component also does not produce the S. S.

As for the second and the third components of (1.9) the situation is unchanged and thus we obtain the estimate (1.10). From now on the proof goes unchanged.

Remark that the ambiguity of the above extension $f_j(x')$ on $\partial W'\cap\{|x_n|>a'\}$ does not affect the regularity of $f_j(x')*W_0(x',\nu')$ on $W'\cap\{|x_n|<a'\}$. q.e.d.

4. Let us strengthen Proposition 1.6 as Remark 1.4 in replacing the analyticity by the micro-analyticity to the direction ν'. First we prepare

LEMMA 1.7 (cf. Schapira [25]). *Let $p(x, D)$ be as in Lemma 1.3. Let u be a hyperfunction solution of $p(x, D)u=0$ defined locally on $x_1>0$ on a neighborhood of the origin. Assume that the boundary values $b_j^+(u), j=0, \cdots, m-1$ do not contain $(0, \sqrt{-1}\nu' dx'\infty)$ in their singular spectrum. Then there exists a neighborhood Ω' of ν' in S^{n-2} and a neighborhood U of $0 \in R^n$ such that*

$$\text{S. S. } [u] \subset (U \cap \{x_1 \geq 0\}) \times \{\sqrt{-1}(\pm(1-\theta)dx_1 + \theta\omega' dx')\infty;\ \omega' \in S^{n-2} \backslash \Omega', 0 \leq \theta \leq 1\}, \tag{1.16}$$

where $[u]$ denotes the canonical extension of u.

PROOF. In fact we can prove the following equivalence:

$$\text{S. S. } b_j^+(u) \not\ni (0, \sqrt{-1}\nu' dx'\infty). \tag{1.17}$$

$$\Longleftrightarrow$$

$$\text{S. S. } [u] \cap \{(0, \sqrt{-1}(\pm(1-\theta)dx_1 + \theta\nu' dx')\infty);\ 0 < \theta \leq 1\} = \varnothing. \tag{1.18}$$

Let ρ be the projection from $\sqrt{-1}S_\infty^{*n-1} \backslash \{\pm\sqrt{-1}dx_1\infty\}$ to the equator $\omega_1=0$ along the longitudes, that is, $\rho(\sqrt{-1}\omega dx\infty) = \sqrt{-1}\omega' dx'\infty$ for $\omega = (\omega_1, \omega'), \omega' \neq 0$. Then the fiber in the braces of (1.18) is just equal to $\rho^{-1}(\sqrt{-1}\nu' dx'\infty)$. Now put $u_j(x') = b_j^+(u)$, $j=0, \cdots, m-1$ and recall the identity

$$p(x, D)[u] = \sum_{j=0}^{m-1} u_j(x')\delta^{(m-1-j)}(x_1). \tag{1.19}$$

The implication (1.18)⇒(1.17) then follows easily from a series of elementary calculus for S. S. Conversely assume (1.17). Then we will have

$$\text{S. S. } \sum_{j=0}^{m-1} u_j(x')\delta^{(m-1-j)}(x_1) \cap \{0\} \times \rho^{-1}(\sqrt{-1}\nu' dx'\infty) = \varnothing. \tag{1.20}$$

Because $p(x, D)$ is invertible as a pseudo-differential operator on a neighborhood $U \times \Omega$ of the non-characteristic points $U \times \{\pm\sqrt{-1}dx_1\infty\}$, we can obtain from (1.19)-(1.20) the estimate

$$\text{S. S. } [u] \cap \{0\} \times (\rho^{-1}(\sqrt{-1}\nu' dx'\infty) \cap \Omega) = \varnothing. \tag{1.21}$$

Owing to the theorem on watermelon slicing, the singular spectrum of the hyperfunction $[u]$ satisfying $\operatorname{supp}[u] \subset \{x_1 \geq 0\}$ has a fiber on each point of the boundary $x_1=0$ of the form $\rho^{-1}(G) \cup \{\pm\sqrt{-1}dx_1\infty\}$, with a suitable closed subset G of the equator. (See Corollary 4.2.3 in [15bis] (=Theorem 4.3.3 in [15]). See also Chapter

VIII, §2 of [12].) Thus (1.21) implies (1.18). In view of the closedness of the set S. S. $[u]$, this proves also (1.16). q.e.d.

For the sake of self-containedness we will give in Appendix A an alternative elementary proof of this lemma in the case of constant coefficients.

On account of this lemma we see that in the case of analytic wave front set it suffices to observe simply the WF_A of the boundary values in order to examine the regularity of a solution at the boundary. It seems to me that similar assertion is not obvious in the case of C^∞ wave front set.

Remark that because $[u]$ satisfies the equation $p(x, D)[u]=0$ on $x_1>0$, we can remove from (1.16) much more points by Sato's fundamental theorem, e.g. a neighborhood of the direction $\pm\sqrt{-1}dx_1\infty$ on $x_1>0$.

We now return to the situation of Proposition 1.6 and put once more $\nu'=(0, \cdots, 0, 1)$.

PROPOSITION 1.8. *Let U' be as in Proposition* 1.6. *We have the same conclusion as Proposition* 1.6 *only by assuming the following*:

1) *For every $V'\Subset U'$ there exist $\delta'>0$ and a neighborhood Ω' of ν' in S^{n-2} such that*

$$\text{S. S. } u|_{\{0<x_1<\delta'\}\times V'}\subset\{0<x_1<\delta'\}\times V'\times\{\sqrt{-1}(\pm(1-\theta)dx_1+\theta\omega'dx')\infty;\ \omega'\in S^{n-2}\backslash\Omega', 0\leq\theta\leq 1\}. \tag{1.22}$$

2) S. S. $b_j^+(u), j=0, \cdots, m-1$ *do not contain the direction $\sqrt{-1}\nu'dx'\infty$ near the part $\partial U''\times\{|x_n|<a\}$ of the boundary of U'.*

PROOF. We inherit the notation of the proof of Proposition 1.6. Assume that

$$\text{S. S. } b_j^+(u)\cap(\overline{W'\backslash V'}\cap\{|x_n|\leq a'\})\times\{\sqrt{-1}\omega'dx'\infty;\ \omega'\in\Omega'\}=\emptyset, \tag{1.23}$$

where Ω' is a neighborhood of ν' verifying in the same time (1.22) with V' there replaced by a neighborhood of $\overline{W}'$ here. On account of Lemma 1.7 we can assume without loss of generality that the canonical extension $[u]$ satisfies the estimate (1.16) on $\{0\leq x_1<\delta'\}\times\overline{W'\backslash V'}\cap\{|x_n|\leq a'\}$. Employing the cutting off by the hypersurface $x_1=t(x')$, we obtain the equality (1.5) as well. Let $\Delta'\Subset\Omega'$ be another neighborhood of $\nu'\in S^{n-2}$. Take the convolution of both sides of (1.5) with $W(x', \Delta')$, where

$$W(x', \Delta')=\int_{\Delta'}W(x', \omega')d\omega'.$$

Then we have

$$p(D)v(x)=\sum_{j=0}^{m-1}g_j(x')\delta^{(m-1-j)}(x_1)+\beta(x),$$

where v, g_j, β are defined by the obvious manner (cf. (1.6)). The same calculation as before shows that $\beta(x)$ satisfies (1.10) and that v is real analytic on $x_1>0$. Employing Lemma 1.3 we can modify v to remove the term $\beta(x)$. Thus we obtain

$$p(D)v=\sum_{j=0}^{m-1} g_j(x')\delta^{(m-1-j)}(x_1).$$

Here $g_j(x')=f_j(x')*W(x', \Delta')$ and $f_j(x')\in{}'\mathcal{B}[\overline{W'}]$ is an extension of $b_j^+(u)|_{W'}$. Thus we have obtained a real analytic solution v which, with the boundary values $\{g_j(x')\}$, satisfies the assumption of Proposition 1.6. Taking W' newly as U' we can thus manage this solution v. Consider next the convolution of (1.5) with $W(x', S^{n-2}\backslash\Delta')$. With the obvious notation we have

$$p(D)w=\sum_{j=0}^{m-1} h_j(x')\delta^{(m-1-j)}(x_1)+\gamma. \tag{1.24}$$

Here w and γ satisfy the following estimate

$$\text{S. S. } w(\text{resp. S. S. }\gamma)\subset\{x_1\geq 0\}\times\{\sqrt{-1}(\pm(1-\theta)dx_1+\theta\omega' dx')\infty;\ \omega'\in S^{n-2}\backslash\Delta', 0\leq\theta\leq 1\}.$$

Moreover, an elementary calculation of S. S. similar to (1.9) shows that their S. S. contains only the direction $\pm\sqrt{-1}dx_1\infty$ outside $\overline{W_{\delta'}}$. As for

$$h_j(x')=f_j(x')*W(x', S^{n-2}\backslash\Delta'),$$

they are real analytic outside $\overline{W'}$. Choose a convex open subset X' of R^{n-1} bounded by a real analytic hypersurface such that $W'\Subset X'\Subset U'$. Let $\chi(x')$ be its characteristic function. We can cut off the support of w, h_j, γ by this function without ambiguity (the product rule: see Corollary 2.4.2 of Chapter I in [24]). We obtain

$$p(D)(\chi w)=\sum_{j=0}^{m-1}\chi(x')h_j(x')\delta^{(m-1-j)}(x_1)+\chi\gamma+\gamma', \tag{1.25}$$

where χw and $\gamma'=p(D)(\chi w)-\chi p(D)w$ satisfy the following estimate (which we write only for γ' for the sake of simplicity):

$$\begin{aligned}\text{S. S. }\gamma'\subset\{&(x, \sqrt{-1}(\pm(1-\theta)dx_1+\theta\omega' dx')\infty);\\ &x_1\geq 0, (x'\in X', \omega'\in S^{n-2}\backslash\Delta')\ \text{ or }\ (x'\in\partial X', \omega'\perp\partial X' \text{ at } x'), 0\leq\theta\leq 1\}.\end{aligned}$$

If we make the convolution of $W_0(x', \nu')$ with (1.25), we obtain, again with the obvious notation,

$$p(D)w'=\sum_{j=0}^{m-1}\{\chi\cdot h_j*W_0(x', \nu')\}\delta^{(m-1-j)}(x_1)+\gamma'',$$

where w' and γ'' satisfy the estimates of the type

(1.26) $\quad \operatorname{supp} w' \subset \{x_1 \geq 0\}, \quad \text{S. S. } w'|_{\{x_1<\delta'\}\times X'} \subset \{x_1 \geq 0\} \times \{\pm\sqrt{-1}dx_1\infty\}.$

In fact, these can be verified by a calculation similar to (1.9) based on the above estimates for $\chi w, \gamma'$. We use in particular the fact that the S. S. of $\chi w, \gamma'$ does not contain the direction $\sqrt{-1}\nu' dx'\infty$ at the point of $\partial X'$ where the hyperplane $x'\nu' = \text{const.}$ is tangent to it.

Thus employing Lemma 1.3 we can replace w' by w'' in order to remove γ''. We have finally

$$p(D)w'' = \sum_{j=0}^{m-1} \{\chi h_j * W_0(x', \nu')\}\delta^{(m-1-j)}(x_1) \quad \text{on} \quad \{x_1 < \delta''\} \times W',$$

where w'' satisfies the estimates (1.26). In view of Sato's fundamental theorem, w'' becomes in fact real analytic on $\{0 < x_1 < \delta''\} \times W'$. Since we have clearly

$$\begin{aligned} g_j(x') + \chi h_j(x') &= f_j * W(x', \Delta') + f_j * W(x', S^{n-2} \setminus \Delta') = f_j(x') \\ &= b_j^\dagger(u) \quad \text{on} \quad W', \end{aligned}$$

the sum of $\chi h_j(x')$ and of the modification of $g_j(x')$ given by the first part of the proof (employing Proposition 1.6) is a required modification of $b_j^\dagger(u)$. q.e.d.

REMARK 1.9. The assumption 1) is stronger than what is in fact necessary: It suffices to assume that S. S. u does not contain the directions

$$\sqrt{-1}(\pm(1-\theta)dx_1 + \theta\nu' dx')\infty, \qquad 0 < \theta \leq 1.$$

For, the uniform estimate of the type (1.22) follows from the assumption 2) in view of Lemma 1.7 at least near $\{0\} \times \overline{W' \setminus V'} \cap \{|x_n| \leq a'\}$. Therefore we can practice the first part of the proof as well. The only difference is that v may not be real analytic inside $W_{\delta'}$, but S. S. v is still free from the directions mentioned above. Thus when we make the convolution with $W_0(x', \nu')$, the result becomes real analytic in $W_{\delta'}$. This is in fact all that is necessary.

5. In this paragraph we will sum up the results hitherto obtained to a theorem. We will employ the partition of variables $x' = (x^I, x^{II})$, where $x^I = (x_2, \cdots, x_k)$, $x^{II} = (x_{k+1}, \cdots, x_n)$. We will write correspondingly $\omega' = (\omega^I, \omega^{II})$. Consider

$$W_I(x', \omega') = \frac{(n-2)!}{(-2\pi\sqrt{-1})^{n-1}} \frac{(1-\sqrt{-1}x^I\omega^I)^{n-2} - (1-\sqrt{-1}x^I\omega^I)^{n-3}((x^I)^2 - (x^I\omega^I)^2)}{(x'\omega' + \sqrt{-1}((x^I)^2 - (x^I\omega^I)^2) + \sqrt{-1}0)^{n-1}}. \tag{1.27}$$

Since this is a special case of Example 1.2.5 in Chapter III of [24], we have always $\int_{S^{n-2}} W_I(x', \omega') d\omega' = \delta(x')$. For this component $W_I(x', \omega')$, the singularity flows out of the origin along the linear subvariety $x^I = 0, x^{II}\omega^{II} = 0$. That is, there is no propagation of singularity with respect to the variables x^I. Now choose $\nu'=(0,\cdots,0,1)$.

THEOREM 1.10. *Let $U'=U^I\times U^{II}$, where U^I resp. U^{II} is a convex open set in the x^I-space resp. x^{II}-space, and U^{II} is a cylinder with the generator $-a<x_n<a$ parallel to the x_n-axis. Let u be a hyperfunction solution of $p(D)u=0$ defined on $\{0<x_1<\delta\}\times U'$. Assume that* S. S. u *does not contain the directions*

$$\sqrt{-1}(\pm(1-\theta)dx_1+\theta\nu'dx')\infty, \qquad 0<\theta\leq 1,$$

and that S. S. $b_j^+(u)$ *do not contain the direction $\sqrt{-1}\nu'dx'\infty$ near the part $U^I\times\partial U^{II}\cap\{|x_n|<a\}$ of $\partial U'$. Then for every open subset $V'\Subset U'$ of the same type as U', there exist $\delta'>0$ and a real analytic solution v of $p(D)v=0$ on $\{0<x_1<\delta'\}\times V'$ such that its boundary values agree with*

$$f_j(x')*\{J(D_{\omega'})W_I(x',\omega')|_{\omega'=\nu'}\}|_{V'}, \qquad j=0,\cdots,m-1,$$

where $f_j(x')$ is a fixed element of $'\mathcal{B}[\overline{U'}]$ independent of $J(D_{\omega'})$ and satisfying

$$f_j(x')|_{V'}=b_j^+(u)|_{V'}.$$

The proof can be carried on imitating those of Proposition 1.2, Proposition 1.8 and Remark 1.9, so we omit it. Remark that in fact Proposition 1.2 resp. Proposition 1.8 corresponds to the case $II=\emptyset$ $(k=n)$ resp. $I=\emptyset$ $(k=1)$ of Theorem 1.10.

§ 2. Extension of solutions in the complex domain.

1. Let $p(D)$ be an m-th order linear partial differential operator with constant coefficients. Let $p_m(D)$ be its principal part. Assume that $x_1=0$ is non-characteristic with respect to p. We employ the following notation for the separation of the independent variables: $x=(x_1,x')=(x_1,x'',x_n)$ with $x''=(x_2,\cdots,x_{n-1})$ and similar one for the complexification $z=x+\sqrt{-1}y$ or for the dual variables $\zeta=\xi+\sqrt{-1}\eta$. In this paragraph we consider those $p(D)$ satisfying the following conditions: There exist positive constants b,c such that for $\mathrm{Re}\,\zeta_n<0$, $\zeta'\in C^{n-1}$ the roots $\zeta_1=\tau_j^0(\zeta')$, $j=1,\cdots,m$ of the homogeneous characteristic equation $p_m(\zeta_1,\zeta')=0$ satisfy

$$-\mathrm{Im}\,\tau_j^0(\zeta')\leq b|\mathrm{Im}\,\zeta_n|+c|\zeta''|. \tag{2.1}$$

REMARK 2.1. This is equivalent to the following inequality for the roots $\tau_j(\zeta')$ of the inhomogeneous characteristic equation $p(\zeta_1,\zeta')=0$:

$$-\mathrm{Im}\,\tau_j(\zeta')\leq a|\mathrm{Re}\,\zeta_n|^q+b|\mathrm{Im}\,\zeta_n|+c|\zeta''|+C, \tag{2.2}$$

where a,b,c,C and $q<1$ are some positive constants. In fact, by an elementary consideration for the roots of a polynomial (see e.g. [22], Chapter IV, Lemma 2.4), we see that the inequality (2.2) for $p(\zeta)$ imply a same type of inequality for any polynomial with the same principal part. Conversely, from the inequality (2.2) for a homogeneous polynomial $p_m(\zeta)$, we can drop the terms $a|\mathrm{Re}\,\zeta_n|^q$ and C by introducing a positive real factor $t\to+\infty$ and employing the homogeneity of the roots.

For the sake of simplicity we will call in the sequel $\tau_j^0(\zeta')$ resp. $\tau_j(\zeta')$ the *homogeneous* resp. the *inhomogeneous characteristic roots* of p.

PROPOSITION 2.2. *Under the above assumption consider the holomorphic Cauchy problem*

$$\begin{cases} p(D)F=0 \\ b_j(D)F|_{z_1=0}=F_j(z'), \qquad j=0,\cdots,m-1. \end{cases} \tag{2.3}$$

Here $\{b_j(D)\}$ *is a normal system of boundary operators with constant coefficients. The holomorphic data* $F_j(z')$ *are given on a domain of the form*

$$\{z'=x'+\sqrt{-1}y'\in \boldsymbol{C}^{n-1};\ |x'|<A,\ \varphi(|y''|)<y_n<B\}, \tag{2.4}$$

where $\varphi(t)$ *is a convex continuous function of* $t\geq 0$ *satisfying* $\varphi(t)\geq 0$, $\varphi(0)=0$ *and* $\varphi(t)/t\to 0$ *if* $t\to 0$. *Then the solution can be continued onto the domain*

$$\begin{aligned}\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ -\delta<x_1\leq 0,\ |x'|<A',\\ \lambda|y_1|+\varphi(c|x_1|)+\varphi(2|y''|)<y_n<B'\},\end{aligned} \tag{2.5}$$

where A', B', λ, δ *are suitably chosen positive constants.*

PROOF. First note that by a refined form of the Cauchy-Kowalevsky theorem the solution exists on a domain of the form

$$\tilde{\omega}=\{|z_1|<k(y_n-\varphi(|y''|)),\ |x'|<A/2,\ y_n<B/2\},$$

where k is a positive constant (see e.g. Leray [20]). Starting from this open set we try the sweeping out employing the device of Bony-Schapira [1]. Choose a point $(-t+\sqrt{-1}s, 0,\cdots,0,\sqrt{-1}\varepsilon)$, where $t>0$, $\varepsilon>0$. If every real characteristic hyperplane passing through this point intersects $\tilde{\omega}$, then the solution $u(z)$ can be continued up to the interior of $\mathrm{ch}[\{(-t+\sqrt{-1}s,0,\cdots,0,\sqrt{-1}\varepsilon)\}\cup\tilde{\omega}]$, where ch denotes the convex hull. We consider $\boldsymbol{C}^n$ as a real Euclidean space of dimension $2n$ by the inner product $-\mathrm{Re}\langle z,\sqrt{-1}\zeta\rangle=x\eta+y\xi$ between the two points $z=x+\sqrt{-1}y$ and $\zeta=\xi+\sqrt{-1}\eta$. This way of identification enjoys the compatibility with the notation of the Fourier transformation. Thus the calculation below becomes very similar to that of [7] although they have apparently very different origins. Now a characteristic hyperplane passing through $(-t+\sqrt{-1}s, 0, \cdots, 0, \sqrt{-1}\varepsilon)$ is expressed by the following equation

$$\begin{aligned}(2.6) \qquad & -\mathrm{Re}\langle z-(-t+\sqrt{-1}s, 0, \cdots, 0, \sqrt{-1}\varepsilon), \sqrt{-1}\zeta\rangle \\ & = x\eta+y\xi+t\eta_1-s\xi_1-\varepsilon\xi_n=0,\end{aligned}$$

where $\zeta=\xi+\sqrt{-1}\eta$ satisfies

$$(2.7) \qquad p_m(\zeta)=0.$$

The equation (2.7) and the non-characteristic assumption imply that there exists $M>0$ such that

$$|\zeta_1|\leq M|\zeta'|.$$

First consider the case $|\xi'|\leq|\eta'|$. This case can always be disposed by the assumption that $z_1=0$ is non-characteristic. In fact the point

$$x_1=0, \quad x'=\frac{-t\eta_1+s\xi_1}{|\eta'|^2}\eta', \quad y_1=y''=0, \quad y_n=\varepsilon$$

satisfies (2.6). Since we have

$$|\xi_1|\leq|\zeta_1|\leq M|\zeta'|\leq\sqrt{2}M|\eta'|,$$

and similarly $|\eta_1|\leq\sqrt{2}M|\eta'|$, this point is contained in $\tilde{\omega}$ provided that

$$(2.8) \qquad |x'|\leq\frac{t|\eta_1|+|s||\xi_1|}{|\eta'|}\leq\sqrt{2}M(t+|s|)<A/2.$$

Next consider the case $|\xi'|\geq|\eta'|$. For the sake of simplicity we choose for the moment $x=0$, $y_1=0$. Then the equation (2.6) becomes

$$(2.6)' \qquad y''\xi''+y_n\xi_n=-t\eta_1+s\xi_1+\varepsilon\xi_n.$$

If $|\xi_n|\leq|\xi''|$ we have

$$|\xi_1|\leq M|\zeta'|\leq\sqrt{2}M|\xi'|\leq 2M|\xi''|,$$

and similarly $|\eta_1|\leq 2M|\xi''|$. Then we consider the point

$$x=0, \quad y_1=0, \quad y''=\frac{-t\eta_1+s\xi_1-(t+|s|)\xi_n}{|\xi''|^2}\xi'', \quad y_n=\varepsilon+(t+|s|)$$

which satisfies (2.6)′ and lies in $\tilde{\omega}$ if

$$(2.9) \qquad \varphi(|y''|)\leq\varphi\left(\frac{t|\eta_1|+|s||\xi_1|+(t+|s|)|\xi_n|}{|\xi''|}\right)$$

$$\leq \varphi((2M+1)(t+|s|)) < \varepsilon + (t+|s|) < B/2.$$

It remains to check the most delicate case $|\xi_n| \geq |\xi''|$. This time we have $|\xi_1| \leq 2M|\xi_n|$ and $|\eta_1| \leq 2M|\xi_n|$. Without loss of generality we can assume that $-t\eta_1 + s\xi_1 \geq 0$. (If this is not the case we can replace ζ by $-\zeta$.) We further separate the cases. If $\xi_n > |\xi''|$ the point

$$x=0, \quad y_1 = y'' = 0, \quad y_n = \varepsilon + \frac{-t\eta_1 + s\xi_1}{\xi_n}$$

satisfies (2.6)′ and lies in $\tilde{\omega}$ provided that

$$y_n = \varepsilon + \frac{-t\eta_1 + s\xi_1}{\xi_n} \leq \varepsilon + 2M(t+|s|) < B/2. \tag{2.10}$$

Finally, if $\xi_n < -|\xi''|$ we can employ the assumption (2.1). We go back to the original equation (2.6) of the characteristic hyperplane. If $\eta_1 > 0$ we have $0 \leq -t\eta_1 + s\xi_1 \leq s\xi_1$, hence we put

$$x=0, \quad y_1 = \frac{-t\eta_1 + s\xi_1}{\xi_1}, \quad y''=0, \quad y_n = \varepsilon.$$

Because $|y_1| \leq |s|$ this solution of (2.6) is contained in $\tilde{\omega}$ if

$$|z_1| \leq |s| < k\varepsilon. \tag{2.11}$$

If $\eta_1 \leq 0$ we have, in view of (2.1), a decomposition of the form

$$\eta_1 = \alpha + \beta + \gamma,$$

where

$$|\alpha| \leq b|\eta_n|, \quad |\beta| \leq c|\xi''|, \quad |\gamma| \leq c|\eta''|.$$

Therefore we put

$$x_1 = 0, \quad x'' = \frac{-t\gamma}{|\eta''|^2}\eta'', \quad x_n = \frac{-t\alpha}{\eta_n}, \quad y_1 = s, \quad y'' = \frac{-t\beta}{|\xi''|^2}\xi'', \quad y_n = \varepsilon.$$

(Here we understand $x''=0$ if $\eta''=0$ etc.) This solution is contained in $\tilde{\omega}$ if

$$|x'| \leq t(b+c) < A/2, \qquad |z_1| \leq |s| \leq k(\varepsilon - \varphi(ct)). \tag{2.12}$$

Now (2.8)-(2.12) are satisfied if we choose

$$\varepsilon < B/4, \quad t+|s| < K, \quad |s| \leq k(\varepsilon - \varphi(ct)), \tag{2.13}$$

where

$$K = \min\left\{\frac{A}{2\sqrt{2}M}, \ \frac{B}{4}, \ \frac{1}{2M+1}\left[\frac{\varphi(t)}{t}\right]^{-1}\left(\frac{1}{2M+1}\right), \ \frac{B}{8M}, \ \frac{A}{2(b+c)}\right\},$$

$[\varphi(t)/t]^{-1}$ denoting the inverse function of $\varphi(t)/t$. Summing up we conclude that the solution $F(z)$ can be continued up to $\mathrm{ch}[\{(-t+\sqrt{-1}s, 0, \cdots, 0, \sqrt{-1}\varepsilon)\}\cup\tilde{\omega}]$ for every $t>0$, $\varepsilon>0$ and s satisfying (2.13). When we let them vary under these conditions and make the union of these convex domains, we clearly obtain a domain of the form (2.5). q.e.d.

We can give the following variant of the above proposition.

PROPOSITION 2.3. *Instead of* (2.1) *assume the following: There exist positive constants b, c and a constant q verifying $0<q<1$ such that for every positive $\varepsilon\le 1$ we have*

$$(2.14)\qquad -\mathrm{Im}\,\tau_j^0(\zeta')\le\varepsilon|\mathrm{Re}\,\zeta_n|+b\varepsilon^{-q}|\mathrm{Im}\,\zeta'|+c\varepsilon^{-q}|\mathrm{Re}\,\zeta''|,\quad \textit{if}\quad \zeta'\in \boldsymbol{C}^{n-1},\quad \mathrm{Re}\,\zeta_n<0.$$

Then given the Cauchy data $F_j(z')$ holomorphic on

$$\{z'=x'+\sqrt{-1}y'\in \boldsymbol{C}^{n-1};\ |x'|<A,\ |y''|^l<y_n<B\},$$

where $l>1$, we can find the solution $F(z)$ of (2.3) *defined on*

$$(2.5)'\qquad \{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ -\delta<x_1\le 0,\ |x'|<A',\ \lambda(|y_1|+|x_1|^{l/(ql+1)}+|y''|^l)<y_n<B'\}.$$

PROOF. In the proof of Proposition 2.2 we put $\varphi(t)=t^l$. It suffices to reexamine the case $|\xi'|\ge|\eta'|$, $\xi_n<-|\xi''|$ and $\eta_1\le 0$. In view of (2.14) we have the following decomposition

$$\eta_1=\alpha+\beta+\gamma,$$

where

$$|\alpha|\le\varepsilon|\xi_n|,\quad |\beta|\le b\varepsilon^{-q}|\eta'|,\quad |\gamma|\le c\varepsilon^{-q}|\xi''|.$$

Hence we choose the point

$$(2.15)\qquad x_1=0,\quad x'=\frac{-t\beta}{|\eta'|^2}\eta',\quad y_1=s,\quad y''=\frac{-t\gamma}{|\xi''|^2}\xi'',\quad y_n=\varepsilon-\frac{t\alpha}{\xi_n}$$

on (2.6). It is contained in $\tilde{\omega}$ if

$$(2.12)'\qquad |x'|\le tb\varepsilon^{-q}<A/2,\quad |z_1|\le|s|<k(\varepsilon(1-t)-(tc\varepsilon^{-q})^l).$$

If we pose the conditions

$$(2.16)\qquad |t|<1/2,\quad \varepsilon>t^{l/(ql+1)},$$

then the second inequality of (2.12)$'$ will be satisfied if

$$|s|<k\left(\frac{\varepsilon}{2}-c^l t^{l/(ql+1)}\right).$$

Now (2.8)-(2.11), (2.12)$'$ and (2.16) are satisfied if we choose

$$\text{(2.13)}' \qquad \varepsilon<\min\{1, B/4\}, \quad t+|s|<K, \quad |s|<\frac{k}{2}(\varepsilon-\mu t^{l/(ql+1)}),$$

where

$$\mu=\max\left\{\left(\frac{2b}{A}\right)^{1/q}, 2c^l, 1\right\},$$

$$K=\min\left\{\frac{A}{2\sqrt{2}M}, \frac{1}{2M+1}\left(\frac{1}{2M+1}\right)^{1/(l-1)}, \frac{B}{8M}, \frac{B}{4}, \frac{1}{2}\right\}.$$

The union of $\mathrm{ch}\,[\{(-t+\sqrt{-1}s, 0, \cdots, 0, \sqrt{-1}\varepsilon)\}\cup\tilde{\omega}]$ for every $t>0$, $\varepsilon>0$, s satisfying (2.13)$'$ obviously contains a domain of the form (2.5)$'$. q.e.d.

Remark that if we make l larger, then the exponent $l/(ql+1)$ increases and approaches $1/q$, hence finally exceeds 1. This fact will be used in the next section.

2. Now we generalize the condition (2.1) in an intrinsic form.

DEFINITION 2.4. Let $p(D)$ be an m-th order linear partial differential operator with constant coefficients. Let $p_m(D)$ be its principal part. Suppose that $x_1=0$ is non-characteristic with respect to p. Let $\nu'\in\boldsymbol{R}^{n-1}$ be a unit vector. We say that $p(D)$ is *$\sqrt{-1}\nu'dx'\infty$-semihyperbolic* to $x_1>0$ if there exist positive constants b, c such that the homogeneous characteristic roots $\tau_j^0(\zeta')$, $j=1, \cdots, m$ satisfy

$$\text{(2.17)} \qquad \mathrm{Im}\,\tau_j^0(\zeta')\leq b|\mathrm{Im}\,\zeta'|+c\sqrt{(\mathrm{Re}\,\zeta')^2-(\mathrm{Re}\,\zeta'\nu')^2} \quad \text{if} \quad \zeta'\in\boldsymbol{C}^{n-1},\ \mathrm{Re}\,\zeta'\nu'<0.$$

The $\sqrt{-1}\nu'dx'\infty$-semihyperbolicity to $x_1<0$ is defined by the inequality which is obtained from (2.17) by replacing $\mathrm{Im}\,\tau_j^0(\zeta')$ by $-\mathrm{Im}\,\tau_j^0(\zeta')$.

This definition is compatible with the coordinate transformation $x_1\mapsto -x_1$. Moreover, the substitution $\zeta\mapsto-\zeta$ in the equation shows that the following are equivalent.

1) $p(D)$ is $\sqrt{-1}\nu'dx'\infty$-semihyperbolic to $x_1>0$.

2) $p(D)$ is $-\sqrt{-1}\nu'dx'\infty$-semihyperbolic to $x_1<0$.

The significance of the semihyperbolicity is clarified by the following theorem.

THEOREM 2.5. *Assume that $p(D)$ is $\sqrt{-1}\nu'dx'\infty$-semihyperbolic to $x_1>0$. Assume that the hyperfunction data $u_j(x')$, $j=0, \cdots, m-1$ can be expressed as the boundary values of functions $F_j(z')$ holomorphic in $\{\nu'\,\mathrm{Im}\,z'>0\}\cap\{|z'|<\delta\}$. Then on a neighborhood of the origin we can solve the following boundary value problem:*

$$\begin{cases} p(D)u=0 \quad \text{on} \quad x_1>0, \\ b_j(D)u|_{x_1\to+0}=u_j(x'), \quad j=0, \cdots, m-1. \end{cases}$$

For the operator $\sqrt{-1}\nu'dx'\infty$-semihyperbolic to $x_1<0$, similar solvability holds for *the boundary value problem to* $x_1<0$.

PROOF. To utilize the calculus of Proposition 2.2 choose $\nu'=(0,\cdots,0,1)$ and consider the case $x_1<0$. With the initial data $F_j(z')$ we are going to solve the holomorphic Cauchy problem (2.3). Put $A=B=\delta/\sqrt{2}$. Then owing to a refined form of the Cauchy-Kowalevsky theorem the solution exists on a domain of the form

$$(2.18)\qquad \tilde{\omega}=\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ |z_1|<ky_n,\ \ y_n<B/2,\ \ |x'|<A/2,\ \ |y''|<B/2\}.$$

Choosing a point $(-t+\sqrt{-1}s,0,\cdots,0,\sqrt{-1}\varepsilon)$ we proceed as in the proof of Proposition 2.2. The calculation corresponding to (2.8) is valid without modification. For the case corresponding to (2.9) we can employ the same solution of the linear equation (2.6)′ and the new condition

$$(2.9)'\qquad (2M+1)(t+|s|)<B/2,\quad \varepsilon+(t+|s|)<B/2.$$

The cases corresponding to (2.10) and (2.11) are also valid without modification. Finally in the case corresponding to (2.12) we can also employ the same solution of (2.6) but the condition will be simply

$$(2.12)''\qquad t(2b+c)<A/2,\quad |s|<k\varepsilon,\quad (b+c)t<\frac{B}{2}.$$

Summing up the conditions (2.8), (2.9)′, (2.10), (2.11), (2.12)″ we obtain

$$(2.13)''\qquad \varepsilon<B/4,\quad t+|s|<K,\quad |s|<k\varepsilon,$$

where

$$K=\min\left\{\frac{B}{4(2M+1)},\ \frac{A}{2(2b+c)},\ \frac{B}{2(b+c)},\ \frac{A}{2\sqrt{2}M},\ \frac{B}{4}\right\}.$$

Since t varies now freely of ε, the union of $\mathrm{ch}\,[\{(-t+\sqrt{-1}s,0,\cdots,0,\sqrt{-1}\varepsilon)\}\cup\tilde{\omega}]$ now contains a domain of the form

$$(2.5)''\qquad \{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ -A'<x_1\leq 0,\ |x'|<A',\ |y_1|<ky_n<B',\ |y''|<B'\},$$

which is a wedge with its edge tangent to the real axis. Thus the holomorphic solution $F(z)$ continued there defines a hyperfunction solution $u(x)$ of $p(D)u=0$ on $x_1<0$ locally on a neighborhood of the origin. It remains to show that the boundary values of u agree with the given data u_j. Though this is intuitively clear we will give the proof in the following lemma.

LEMMA 2.6. *Let* Γ' *be a convex open cone in the coordinate subspace* $\boldsymbol{R}^{n-1}_{y'}$

containing the positive y_n-axis and contained properly in $y_n>0$. Let $F(z)$ be a holomorphic function defined on a neighborhood of the following set

$$\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ 0\le x_1<\delta,\ |x'|<A,\ y_1=0,\ y'\in\Gamma',\ y_n<B\}.$$

Assume that $F(z)$ satisfies there an m-th order linear partial differential equation $p(z, D)F(z)=0$ whose coefficients are holomorphic on a neighborhood of the closure of the above domain. Assume that $z_1=0$ is non-characteristic with respect to p on this neighborhood. Then $F(z)$ defines a hyperfunction solution $u(x)$ of $p(x, D)u=0$ on $\{0<x_1<\delta\}\times\{|x'|<A\}$ such that its boundary values $b_j^+(u)=b_j(x, D)u|_{x_1\to+0}$, $j=0,\cdots,m-1$ are just the hyperfunctions defined by the holomorphic functions $b_j(z, D)F(z)|_{z_1=0}$, $j=0,\cdots,m-1$ on $\{|x'|<A\}+\sqrt{-1}\{y'\in\Gamma', y_n<B\}$.

PROOF. First note that owing to a local version of Bochner's tube theorem we can extend $F(z)$ to a local wedge with its edge tangent to the real axis on a neighborhood of each point of $\{0<x_1<\delta\}\times\{|x'|<A\}$ so that we have a well defined hyperfunction $u(x)$ there. For the sake of simplicity we assume that the boundary system is the one which gives as the boundary values the coefficients of

$$p(x, D)[u]=\sum_{j=0}^{m-1} b_j^+(u)\delta^{(m-1-j)}(x_1). \tag{1.3bis}$$

Thus on account of the uniqueness of this expression it suffices to give a hyperfunction v which agrees with u on $\{0<x_1<\delta\}\times\{|x'|<A\}$ and satisfies $\operatorname{supp} v\subset\{x_1\ge 0\}$ and the above identity with $b_j^+(u)$ replaced by $F_j(x'+\sqrt{-1}\Gamma'0)$, where $F_j(z')=b_j(z, D)F(z)|_{z_1=0}$. Consider $F(x_1, x'+\sqrt{-1}y')Y(x_1)$, where Y is the Heaviside function. It is a well defined hyperfunction of x_1, x', y' on

$$\{x_1<\delta\}\times\{|x'|<A\}+\sqrt{-1}\{y'\in\Gamma', y_n<B\}.$$

It contains holomorphic parameters $x_j+\sqrt{-1}y_j$, $j=2,\cdots,n$ in the sense of [24], Chapter I, §3.2. Thus as is explained there it defines a hyperfunction $v(x_1, x')$ on $\{x_1<\delta\}\times\{|x'|<A\}$ as the boundary value. Since this correspondence is a sheaf homomorphism along $\{x_1<\delta\}\times\{|x'|<A\}$ we conclude that $\operatorname{supp} v\subset\{x_1\ge 0\}$ and $v=u$ on $x_1>0$. From the functorial definition of the boundary value the hyperfunction $p(x, D)v$ evidently agrees with the one defined as the boundary value of another hyperfunction with holomorphic parameters:

$$\begin{aligned}&p(x_1, x'+\sqrt{-1}y', D_x)\{F(x_1, x'+\sqrt{-1}y')Y(x_1)\}\\ &\quad=\sum_{j=0}^{m-1} b_j(x_1, x'+\sqrt{-1}y', D_x)F(x_1, x'+\sqrt{-1}y')|_{x_1=0}\delta^{(m-1-j)}(x_1).\end{aligned}$$

But the latter obviously agrees with $\sum F_j(x'+\sqrt{-1}\Gamma'0)\delta^{(m-1-j)}(x_1)$. q.e.d.

Now we combine the two semihyperbolic directions.

DEFINITION 2.7. We say that $p(D)$ is *$\sqrt{-1}\nu' dx'\infty$-hyperbolic* with respect to $x_1=0$ if $p(D)$ is $\sqrt{-1}\nu' dx'\infty$-semihyperbolic to both sides $\pm x_1>0$.

By what was remarked after Definition 2.4, $p(D)$ then becomes also $-\sqrt{-1}\nu' dx'\infty$-hyperbolic. In the case $\nu'=(0,\cdots,0,1)$ the $\sqrt{-1}dx_n\infty$-hyperbolicity is expressed by definition by the following inequality for the homogeneous characteristic roots $\tau_j^0(\zeta')$:

$$|\operatorname{Im}\tau_j^0(\zeta')|\leq b|\operatorname{Im}\zeta_n|+c|\zeta''|. \tag{2.19}$$

In view of Remark 2.1, it is also expressed by the following inequality for the inhomogeneous characteristic roots $\tau_j(\zeta')$:

$$|\operatorname{Im}\tau_j(\zeta')|\leq a|\operatorname{Re}\zeta_n|^q+b|\operatorname{Im}\zeta_n|+c|\zeta''|+C. \tag{2.20}$$

With this definition we have the following

COROLLARY 2.8. *Assume that $p(D)$ is $\sqrt{-1}\nu' dx'\infty$-hyperbolic with respect to $x_1=0$. Assume that the hyperfunction Cauchy data $f_j(x')$, $j=0,\cdots,m-1$ contain the coordinates supplementary to $\nu' x'$ in some linear coordinate system as complex holomorphic parameters (in the sense that we can complexify them to usual complex holomorphic parameters). Then the Cauchy problem*

$$\begin{cases} p(D)u=0, \\ b_j(D)u|_{x_1=0}=f_j(x'), \qquad j=0,\cdots,m-1 \end{cases}$$

admits a hyperfunction solution (which contains the same holomorphic parameters).

In fact, choose a system of coordinates such that $\nu'=(0,\cdots,0,1)$ and that $f_j(x')$ contain x'' as complex holomorphic parameters. By the definition of the complex holomorphic parameter, there exists locally a pair of functions $F_j^\pm(z')$ holomorphic, say, on $|x'|<A$, $|y''|<B$, $0<y_n<B$ (resp. $-B<y_n<0$) such that

$$f_j(x')=F_j^+(x',x_n+\sqrt{-1}0)-F_j^-(x',x_n-\sqrt{-1}0).$$

In view of the proof of Theorem 2.5, the holomorphic Cauchy problem for the data $F_j^\pm(z')$ admits a solution $F^\pm(z)$ holomorphic on a domain of the form

$$\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ |x_1|<\delta,\ |x'|<A',\ |y_1|<k|y_n|,\ |y''|<B', \\ 0<y_n<B' \quad (\text{resp. } -B'<y_n<0)\}.$$

Owing to the Holmgren uniqueness theorem, these local solutions may be glued together along the initial hyperplane $x_1=0$. Note that we cannot claim a fixed domain of existence in the real, because the constant δ in the above set depends on the constant B limiting $|y''|$.

3. For later use we give the following variant of the preceding paragraph. We continue to assume that $x_1=0$ is non-characteristic with respect to our operator.

DEFINITION 2.9. We say that $p(D)$ is *partially* $\sqrt{-1}\nu' dx'\infty$-*semihyperbolic* to $x_1>0$ (resp. to $x_1<0$) if the roots $\tau_j^0(\nu')$, $j=1,\cdots,m$ of the equation $p_m(\zeta_1,\nu')=0$ satisfy

$$\mathrm{Im}\,\tau_j^0(\nu')\geq 0 \tag{2.21}$$

(resp. $\mathrm{Im}\,\tau_j^0(\nu')\leq 0$). We say that $p(D)$ is *partially* $\sqrt{-1}\nu' dx'\infty$-*hyperbolic* with respect to $x_1=0$ if

$$\mathrm{Im}\,\tau_j^0(\nu')=0. \tag{2.22}$$

LEMMA 2.10. *Assume that* $\nu'=(0,\cdots,0,1)$. *Then* $p(D)$ *is partially* $\sqrt{-1}\nu' dx'\infty$-*semihyperbolic to* $x_1>0$ *if and only if the inhomogeneous characteristic roots* $\tau_j(\zeta')$, $j=1,\cdots,m$ *of* p *satisfy for any* $\varepsilon>0$,

$$\mathrm{Im}\,\tau_j(\zeta')\leq\varepsilon|\mathrm{Re}\,\zeta_n|+b|\mathrm{Im}\,\zeta_n|+C_{\zeta'',\varepsilon} \quad \textit{if} \quad \mathrm{Re}\,\zeta_n<0. \tag{2.23}$$

Similarly $p(D)$ *is partially* $\sqrt{-1}\nu' dx'\infty$-*hyperbolic with respect to* $x_1=0$ *if and only if*

$$|\mathrm{Im}\,\tau_j(\zeta')|\leq\varepsilon|\mathrm{Re}\,\zeta_n|+b|\mathrm{Im}\,\zeta_n|+C_{\zeta'',\varepsilon} \quad \textit{if} \quad \mathrm{Re}\,\zeta_n<0. \tag{2.24}$$

The same assertion holds if we replace $\tau_j(\zeta')$ *in these inequalities by the homogeneous characteristic roots* $\tau_j^0(\zeta')$.

PROOF. First note that assuming (2.23) for $\tau_j(\zeta')$ is equivalent to assuming the same inequality for the homogeneous characteristic roots $\tau_j^0(\zeta')$. In fact, this inequality is invariant by the lower order perturbation. Therefore assume (2.23) for $\tau_j^0(\zeta')$. Let $\lambda_j^0(\zeta_n)$, $j=1,\cdots,m$ be the homogeneous characteristic roots of the two dimensional operator $p(\zeta_1,0,\zeta_n)$ (that is, $\lambda_j^0(\zeta_n)=\tau_j^0(0,\zeta_n)$). Then (2.23) implies

$$\mathrm{Im}\,\lambda_j^0(\zeta_n)\leq\varepsilon|\mathrm{Re}\,\zeta_n|+b|\mathrm{Im}\,\zeta_n|+C_\varepsilon \quad \text{if} \quad \mathrm{Re}\,\zeta_n<0. \tag{2.25}$$

Employing the homogeneity of the roots as in the argument of Remark 2.1, we can deduce from this inequality the following new one

$$\mathrm{Im}\,\lambda_j^0(\zeta_n)\leq b|\mathrm{Im}\,\zeta_n| \quad \text{if} \quad \mathrm{Re}\,\zeta_n<0. \tag{2.26}$$

On the other hand the two dimensional homogeneous polynomial $p_m(\zeta_1,0,\zeta_n)$ admits the decomposition into linear factors of the form $\zeta_1-\alpha_j\zeta_n$, where $\alpha_j=\lambda_j^0(1)=\tau_j^0(\nu')$.

Then (2.26) obviously implies (2.21).

Conversely assume (2.21), that is, (2.26). For each fixed ζ'', $p(\zeta)$ has the same principal part as $p(\zeta_1, 0, \zeta_n)$ as a polynomial of ζ_1, ζ_n. The coefficients of ζ_1^{m-j} of these two polynomials are j-th order polynomials of ζ' and their difference is of order less than j with respect to ζ_n, hence majorated by

$$C_1|\zeta'|^j \sum_{k=0}^{j-1} \left(\frac{|\zeta''|}{|\zeta'|}\right)^{j-k} \left(\frac{|\zeta_n|}{|\zeta'|}\right)^k.$$

Then by an elementary consideration to the roots of a polynomial (see e.g. [22], Chapter IV, Lemma 2.4), we see that

$$\begin{aligned} |\tau_j^0(\zeta') - \lambda_j^0(\zeta')| &\leq C_2|\zeta'| \left\{ \sum_{k=0}^{m-1} \left(\frac{|\zeta''|}{|\zeta'|}\right)^{m-k} \left(\frac{|\zeta_n|}{|\zeta'|}\right)^k \right\}^{1/m} \\ &\leq C_3|\zeta'| \sum_{k=0}^{m-1} \left(\frac{|\zeta''|}{|\zeta'|}\right)^{(m-k)/m} \left(\frac{|\zeta_n|}{|\zeta'|}\right)^{k/m}, \end{aligned}$$

hence that

$$\operatorname{Im} \tau_j^0(\zeta') \leq a \sum_{k=0}^{m-1} |\zeta''|^{(m-k)/m} |\operatorname{Re} \zeta_n|^{k/m} + b|\operatorname{Im} \zeta_n|.$$

Thus noting the Hölder inequality

$$|\zeta''|^{k/m} |\operatorname{Re} \zeta_n|^{(m-k)/m} \leq \varepsilon |\operatorname{Re} \zeta_n| + C_\varepsilon |\zeta''|, \qquad C_\varepsilon = \frac{m-k}{m} \left(\frac{k}{m}\right)^{(k/m)\ ((m-k)/m)} \varepsilon^{-(k/m)\ ((m-k)/m)},$$

we obtain (2.23). q.e.d.

In correspondence with Theorem 2.5 we have the following

PROPOSITION 2.11. *Let $p(D)$ be an operator partially $\sqrt{-1}\nu' dx' \infty$-semihyperbolic to $x_1 > 0$ (resp. to $x_1 < 0$). Let $F_j(z')$ be functions holomorphic on $\{|x'| < A\} + \sqrt{-1}\{y'\nu' > 0\}$. Then the holomorphic Cauchy problem* (2.3) *admits a solution $F(z)$ holomorphic on a domain of the form*

$$\{z = x + \sqrt{-1}y \in \boldsymbol{C}^n;\ 0 < x_1 < \delta\ (\textit{resp.}\ -\delta < x_1 < 0),\ |x'| < A'\ \ |y'| < A',\ |y_1| < ky'\nu'\}.$$

The statement can be rewritten in an obvious way as a solvability theorem for the corresponding hyperfunction boundary value problem.

PROOF. Choose a linear change of coordinates such that $\nu' = (0, \cdots, 0, 1)$. To utilize the proof of Theorem 2.5 consider the case $x_1 < 0$. It suffices to reexamine the proof for the last case correspondingly with the replacement $B \mapsto +\infty$. Therefore assume $|\xi'| \geq |\eta'|$, $\xi_n < -|\xi''|$ and $\eta_1 \leq 0$. In view of Lemma 2.10, the assumption of partial $\sqrt{-1}\nu' dx' \infty$-semihyperbolicity implies the inequality

$$-\operatorname{Im}\tau_j^0(\zeta')\leq\varepsilon|\operatorname{Re}\zeta_n|+b|\operatorname{Im}\zeta_n|+C_{\zeta'',\varepsilon}.$$

Therefore we have a decomposition of the form $\eta_1=\alpha+\beta+\gamma$, where

$$|\alpha|\leq\varepsilon|\xi_n|,\quad |\beta|\leq b|\eta'|,\quad |\gamma|\leq C_{\zeta'',\varepsilon}.$$

We choose (2.15) as a solution of (2.6). As the condition that it be contained in $\tilde{\omega}$ of (2.18), we then obtain

$$tb<A/2,\quad |s|<k\varepsilon/2,\quad t<1/2.$$

Our assertion clearly follows from these estimates. q.e.d.

4. Here we discuss the relation between various notions of hyperbolicity which we have introduced until now. To make the comparison systematic, we first remember the notion of I-hyperbolicity: Let $I\subset \boldsymbol{S}^{n-2}$ be open. We will say that $p(D)$ is *I-semihyperbolic* to $x_1>0$ (resp. to $x_1<0$, resp. *I-hyperbolic* with respect to $x_1=0$) if $x_1=0$ is non-characteristic with respect to p and if the homogeneous characteristic roots $\tau_j^0(\zeta')$ all satisfy $\operatorname{Im}\tau_j^0(\zeta')\geq 0$ (resp. $\operatorname{Im}\tau_j^0(\zeta')\leq 0$, resp. $\operatorname{Im}\tau_j^0(\xi')=0$) for $\xi'\in I$. This corresponds to saying that $p(D)$ is $\boldsymbol{R}^n\times\sqrt{-1}Idx'\infty$-semihyperbolic by the terminology of [8]. Since we are treating the operators with constant coefficients, it is not necessary here to refer to the space variables.

Lemma 2.12. *Assume that $x_1=0$ is non-characteristic with respect to p. The following are equivalent.*

1) *There exists a neighborhood I of $\nu'=(0,\cdots,0,1)\in\boldsymbol{S}^{n-2}$ such that $p(D)$ is I-semihyperbolic to $x_1>0$.*

2) *There exists $c>0$ such that the homogeneous characteristic roots $\tau_j^0(\zeta')$ of p satisfy*

$$\operatorname{Im}\tau_j^0(\zeta')\leq b|\operatorname{Im}\zeta'|\quad \textit{if}\quad \zeta'\in\boldsymbol{C}^{n-1},\quad \operatorname{Re}\zeta_n\leq -c|\operatorname{Re}\zeta''|.$$

3) *There exist positive constants $q<1, a, b, c, C$ such that the inhomogeneous characteristic roots $\tau_j(\zeta')$ of p satisfy*

$$\operatorname{Im}\tau_j(\zeta')\leq a|\operatorname{Re}\zeta'|^q+b|\operatorname{Im}\zeta'|+C\quad \textit{if}\quad \zeta'\in\boldsymbol{C}^{n-1},\quad \operatorname{Re}\zeta_n\leq -c|\operatorname{Re}\zeta''|.$$

In fact, 1)$\Rightarrow$3) is proved in Corollary 2.4 of [7] employing the local Bochner theorem. The implication 3)$\Rightarrow$2) is similar to Remark 2.1. Finally 2)$\Rightarrow$1) is trivial.

Proposition 2.13. *Assume that there exists a neighborhood I of $\nu'\in\boldsymbol{S}^{n-2}$ such that $p(D)$ is I-semihyperbolic to $x_1>0$ (resp. to $x_1<0$, resp. I-hyperbolic with respect to $x_1=0$). Then given data $F_j(z')$ holomorphic on* (2.4), *we can find the solution $F(z)$ of the Cauchy problem* (2.3) *which can be continued up to a domain of* *the form*

$$\text{(2.27)} \quad \{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ 0<x_1<\delta \quad (\textit{resp.}\ -\delta<x_1<0,\ \textit{resp.}\ |x_1|<\delta),$$
$$|x'|<A',\ \lambda|y_1|+\varphi(c|y''|)<y_n<B'\}.$$

The proof is similar and is omitted. Note that this time we can even admit $\varphi(t)$ in (2.4) such that it only satisfies $\varphi(t)/t\leq\varepsilon$ with a suitable ε (whose smallness depends on the smallness of the neighborhood I). In fact, the case when $\varphi(t)$ is a linear function of t, that is, when (2.4) is a wedge domain, is already treated by Theorem 1.1 in [8].

Now we compare various hyperbolicity.

LEMMA 2.14. *Consider the three following propositions:*

$\mathrm{H}_1(\nu')$: *There exists a neighborhood I of $\nu'\in \boldsymbol{S}^{n-2}$ such that $p(D)$ is I-hyperbolic to $x_1>0$.*

$\mathrm{H}_2(\nu')$: *$p(D)$ is $\sqrt{-1}\nu' dx'\infty$-semihyperbolic to $x_1>0$.*

$\mathrm{H}_3(\nu')$: *$p(D)$ is partially $\sqrt{-1}\nu' dx'\infty$-semihyperbolic to $x_1>0$.*

We have the following implication relations:

1) $\mathrm{H}_1(\nu')\Rightarrow\mathrm{H}_2(\nu')\Rightarrow\mathrm{H}_3(\nu')$.

2) $\mathrm{H}_3(\omega')$ *for every ω' in a neighborhood of* $\nu'\Rightarrow\mathrm{H}_1(\nu')$.

3) $\mathrm{H}_3(\nu')$ *and $\tau_j^0(\nu')$ are distinct*$\Rightarrow\mathrm{H}_2(\nu')$.

PROOF. 1) follows easily employing Lemma 2.12. 2) is clear by definition. Assume finally that the roots $\alpha_j=\tau_j^0(\nu')$ of the equation $p_m(\zeta_1,\nu')=0$ are distinct and satisfy $\mathrm{Im}\,\alpha_j\geq 0$. Choose $\nu'=(0,\cdots,0,1)$ for the sake of simplicity. By the continuity of the roots, $\tau_j^0(\zeta')$ are also distinct if $|\zeta''|<k|\zeta_n|$ for a sufficiently small k. Since simple roots are holomorphic, hence Lipschitz continuous, we have

$$|\tau_j^0(\zeta')-\alpha_j\zeta_n|\leq c|\zeta''|,\quad \text{if}\quad |\zeta''|<k|\zeta_n|,$$

whence

$$\mathrm{Im}\,\tau_j^0(\zeta')\leq b|\mathrm{Im}\,\zeta_n|+c|\zeta''|,\quad \text{if}\quad |\zeta''|<k|\zeta_n|,\quad \mathrm{Re}\,\zeta_n<0.$$

The last inequality clearly holds for $|\zeta''|\geq k|\zeta_n|$. q.e.d.

Let $\nu'=(0,\cdots,0,1)$. A typical example of $p(D)$ which satisfies $\mathrm{H}_2(\nu')$ but not $\mathrm{H}_1(\nu')$ is $p(D)=D_1^2+\cdots+D_k^2$ $(k<n)$. A typical example of $p(D)$ which satisfies $\mathrm{H}_3(\nu')$ but not $\mathrm{H}_2(\nu')$ is $p(D)=D_1^2+D_2D_n$ $(n\geq 3)$. Note that for this operator the homogeneous characteristic equation $p_2(\zeta_1,\nu')=0$ has the same double root 0 as the preceding example. We will show that for this operator the assertion of Theorem 2.5 does not hold. Consider in fact the function

$$F(z_1, z_2, z_n)=(z_1^2+4(z_2+i)z_n)^{-1/2}.$$

This is a well defined holomorphic solution of $(D_1^2+D_2D_n)F=0$ near the part $|y_2|<1$, $y_n>0$ of the initial surface $z_1=0$. But F is singular e.g. along $y_1=0, z_2=0, x_n=0$, $x_1^2=4y_n$.

To conclude this paragraph we give some remarks about references. What we call I-semihyperbolicity is very close to the notion of *partial micro-hyperbolicity* beautifully defined in [13]. But the latter contains all the elliptic operators which is the main difference. (Indeed "micro" means treating S. S. instead of support.) We rather preferred the adjective "semi" instead of "partial" on account of the traditional use of the latter. That is, "partial" means discussing something in specializing some variables concerned. Thus our definition of "partial $\sqrt{-1}\nu'dx'\infty$-hyperbolicity" given in Definition 2.9 is coherent with the definition of "partial hyperbolicity modulo the hyperplane $\nu'x'=0$" given by Leray [21]. For such operators (even with analytic coefficients) Leray gives a theorem of the type Corollary 2.8 (with "hyperfunction" replaced by "Gevrey function") but under the additional assumption that the roots of $p_m(x, \nu')=0$ be distinct (see also [3]). Thus for the operator with constant coefficients, it reduces, in view of Lemma 2.14, to treating our $\sqrt{-1}\nu'dx'\infty$-hyperbolic operator. On account of the above example for $D_1^2+D_2D_n$, the partial $\sqrt{-1}\nu'dx'\infty$-hyperbolic operators do not form a good class from the viewpoint of seeking solvability theorem of the type Corollary 2.8.

5. Finally we list up miscellaneous intermediate cases. Put $x^I=(x_2, \cdots, x_k)$, $x^{II}=(x_{k+1}, \cdots, x_{n-1})$. That is, $x'=(x^I, x^{II}, x_n)$ and the notation is a little different of the one used in the final paragraph of §1. We employ the same notation for ζ.

PROPOSITION 2.15. *Assume that the homogeneous characteristic roots $\tau_j^0(\zeta')$ of $p(D)$ satisfy*

$$-\operatorname{Im}\tau_j^0(\zeta')\leq b(|\operatorname{Im}\zeta_n|+|\operatorname{Im}\zeta^{II}|)+c|\zeta^I|, \quad \text{if} \quad \operatorname{Re}\zeta_n<0. \tag{2.28}$$

Let $\varphi(t)$ be as in Proposition 2.2. Then given data $F_j(z')$ holomorphic on

$$\{z'=x'+\sqrt{-1}y'\in \boldsymbol{C}^{n-1};\ |x'|<A,\ |y^I|<B,\ \varphi(|y^{II}|)<y_n<B\},$$

we can find the solution of the Cauchy problem (2.3) *holomorphic on*

$$\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ -\delta<x_1\leq 0,\ |x'|<A',\ |y^I|<B',\ \lambda|y_1|+\varphi(2|y^{II}|)<y_n<B'\}.$$

In fact, it suffices to modify the calculation of the case $|\xi'|\geq|\eta'|$, $\xi_n<-|\xi''|$, $\eta_1\leq 0$. By the assumption we have a decomposition of the form $\eta_1=\alpha+\beta+\gamma$, where

$|\alpha|\le b|\eta_n|$, $|\beta|\le c|\xi^I|$, $|\gamma|\le(b+c)|\eta''|$. As a solution of (2.6) we choose

$$x_1=0,\quad x''=\frac{-t\gamma}{|\eta''|^2}\eta'',\quad x_n=\frac{-t\alpha}{\eta_n},\quad y_1=s,\quad y^I=\frac{-t\beta}{|\xi^I|^2}\xi^I,\quad y^{II}=0,\quad y_n=\varepsilon.$$

The remaining verification is similar.

PROPOSITION 2.16. *Assume that the homogeneous characteristic roots* $\tau_j^0(\zeta')$ *of* $p(D)$ *satisfy*

$$-\operatorname{Im}\tau_j^0(\zeta')\le\varepsilon|\operatorname{Re}\zeta_n|+b|\operatorname{Im}\zeta_n|+c|\zeta^{II}|+C_{\zeta^I,\varepsilon}. \tag{2.29}$$

Let $\varphi(t)$ *be as above. Then given data* $F_j(z')$ *holomorphic on*

$$\{z'=x'+\sqrt{-1}y'\in\boldsymbol{C}^{n-1};\ |x'|<A,\ \varphi(|y^{II}|)<y_n\}$$

we can find the solution of the Cauchy problem (2.3) *holomorphic on*

$$\{z=x+\sqrt{-1}y\in\boldsymbol{C}^n;\ -\delta<x_1\le 0,\ |x'|<A',\ |y_1|<A',\ \lambda|y_1|+\varphi(c|x_1|)+\varphi(2|y^{II}|)<y_n\}.$$

The proof is similar: We decompose $\eta_1=\alpha+\beta+\gamma^I+\gamma^{II}$, where $|\alpha|\le\varepsilon|\xi_n|$, $|\beta|\le(b+c)|\eta'|$, $|\gamma^{II}|\le c|\xi^{II}|$, $|\gamma^I|\le C_{\zeta^I,\varepsilon}$, and choose

$$\begin{aligned}&x_1=0,\quad x'=\frac{-t\beta}{|\eta'|^2}\eta',\\&y_1=s,\quad y^I=\frac{-t\gamma^I}{|\xi^I|^2}\xi^I,\quad y^{II}=\frac{-t\gamma^{II}}{|\xi^{II}|^2}\xi^{II},\quad y_n=\varepsilon-\frac{t\alpha}{\xi_n}.\end{aligned}$$

PROPOSITION 2.17. *Assume that the homogeneous characteristic roots* $\tau_j^0(\zeta')$ *of* $p(D)$ *satisfy*

$$-\operatorname{Im}\tau_j^0(\zeta')\le\varepsilon|\operatorname{Re}\zeta_n|+b|\operatorname{Im}\zeta_n|+c|\operatorname{Im}\zeta^{II}|+C_{\zeta^I,\varepsilon}. \tag{2.30}$$

Let $\varphi(t)$ *be as above. Then given data* $F_j(z')$ *holomorphic on*

$$\{z'=x'+\sqrt{-1}y'\in\boldsymbol{C}^{n-1};\ |x'|<A,\ \varphi(|y^{II}|)<y_n\},$$

we can find the solution of the Cauchy problem (2.3) *holomorphic on*

$$\{z=x+\sqrt{-1}y\in\boldsymbol{C}^n;\ -\delta<x_1\le 0,\ |x'|<A',\ |y_1|<A',\ \lambda|y_1|+\varphi(2|y^{II}|)<y_n\}.$$

In fact, this time we can choose $\gamma^{II}=0$ in the preceding case. This allows the term $\varphi(c|x_1|)$ to drop out.

In later section we apply Proposition 2.16 to $p(D)=D_1^2+\cdots+D_{n-2}^2-D_{n-1}D_n$ with $\zeta^{II}=(\zeta_2,\cdots,\zeta_{n-2})$, $\zeta^I=\zeta_{n-1}$ and Proposition 2.17 to $p(D)=D_1^2-D_2^2-\cdots-D_{n-2}^2$

$-D_{n-1}D_n$ with the same ζ^{II}, ζ^{I}. The latter example is legitimate on account of the following elementary calculation: Put

$$\zeta_1=\sqrt{\zeta_2^2+\cdots+\zeta_n^2+\zeta_{n-1}\zeta_n}.$$

Then for $|\eta^{II}|\geq|\xi^{II}|/2$ we have

$$\begin{aligned}|\mathrm{Im}\,\zeta_1|&\leq\sqrt{|\zeta_2^2+\cdots+\zeta_n^2|+|\zeta_{n-1}\zeta_n|}\leq\sqrt{5|\eta^{II}|^2+|\zeta_{n-1}\zeta_n|}\\&\leq\frac{5}{2}|\eta^{II}|+\varepsilon|\zeta_n|+C_\varepsilon|\zeta_{n-1}|.\end{aligned}$$

For $|\eta^{II}|\leq|\xi^{II}|/2\leq\sqrt{|\zeta_{n-1}\zeta_n|}$ we have

$$|\mathrm{Im}\,\zeta_1|\leq\sqrt{6|\zeta_{n-1}\zeta_n|}\leq\varepsilon|\zeta_n|+C'_\varepsilon|\zeta_{n-1}|.$$

Finally for $|\eta^{II}|\leq|\xi^{II}|/2$, $\sqrt{|\zeta_{n-1}\zeta_n|}\leq|\xi^{II}|/2$ we recall the elementary estimate

$$\mathrm{Im}\sqrt{a+\sqrt{-1}b}=\sqrt{\sqrt{a^2+b^2}-a}\leq|b|/\sqrt{2a}\qquad\text{for}\qquad a>0.$$

Then we have

$$\begin{aligned}|\mathrm{Im}\,\zeta_1|&\leq\frac{|\zeta_{n-1}\zeta_n|+2|\xi^{II}\eta^{II}|}{\sqrt{(\xi^{II})^2-(\eta^{II})^2-|\zeta_{n-1}\zeta_n|}}\leq\frac{\sqrt{|\zeta_{n-1}\zeta_n|}(1/2)|\xi^{II}|+2|\xi^{II}||\eta^{II}|}{|\xi^{II}|/\sqrt{2}}\\&\leq2\sqrt{2}|\eta^{II}|+\sqrt{|\zeta_{n-1}\zeta_n|}.\end{aligned}$$

§ 3. Singular spectrum of boundary values of real analytic solutions.

1. Now we are ready to state and prove our main results.

THEOREM 3.1. *Let $p(D)$ be an operator which is $\sqrt{-1}\nu' dx'\infty$-semihyperbolic to $x_1<0$ (resp. to $x_1>0$) in the sense of Definition 2.4. Assume further that $p(D_1/\sqrt{-1}, D')$ is I-hyperbolic with respect to $x_1=0$ for a neighborhood I of $\nu'\in S^{n-2}$. Then for every real analytic solution u of $p(D)u=0$ defined locally on $x_1>0$ (resp. on $x_1<0$), its boundary values become microanalytic to the direction $\sqrt{-1}\nu' dx'\infty$ (that is, this direction is not contained in* S. S. $b_j^{\pm}(u)$*).*

REMARK 3.2. To obtain the same conclusion it only suffices to assume, instead of the real analyticity of u, that S. S. u does not contain the directions

$$\sqrt{-1}((1-\theta)dx_1+\theta\nu' dx')\infty,\qquad 0<\theta\leq1.$$

This is obvious from the beginning of the proof below if the citation of Proposition 1.2 is replaced by Remark 1.4.

PROOF. Let U' be an open subset of $\boldsymbol{R}^{n-1}$ and assume that u is defined on $\{0<x_1<\delta\}\times U'$. Choose $f_j(x')\in{}'\mathscr{B}[\overline{U'}]$, $j=0,\cdots,m-1$ such that $f_j(x')=b_j^+(u)$ in U'. In view of Lemma 1.1 it suffices to show that for every local operator $J(D_{\omega'})$, $f_j(x')*J(D_{\omega'})W(x',\omega')|_{\omega'=\nu'}$ is real analytic in every subdomain $V'\Subset U'$. By virtue of Proposition 1.2 there exists another real analytic solution v of $p(D)v=0$ on $\{0<x_1<\delta'\}\times V'$ such that its boundary values agree with $f_j(x')*J(D_{\omega'})W(x',\omega')|_{\omega'=\nu'}$ on V'. Thus the theorem is reduced to the following lemma.

LEMMA 3.3. *Let $p(D)$ be an operator such that it is $\sqrt{-1}\nu'dx'\infty$-semihyperbolic to $x_1<0$ and that $p(D_1/\sqrt{-1},D')$ is I-hyperbolic with respect to $x_1=0$ for some neighborhood I of $\nu'\in\boldsymbol{S}^{n-2}$. Let v be a real analytic solution of $p(D)u=0$ defined on $\{0<x_1<\delta\}\times V'$. Assume that the singular spectrum of the boundary values $v_j(x')$ contains only the direction $\sqrt{-1}\nu'dx'\infty$. Then v can be continued as a real analytic function to a neighborhood of the origin of $\boldsymbol{R}^n$.*

We prepare a lemma which serves as a converse of Lemma 2.6.

LEMMA 3.4. *Let $p(x,D)$ be as in Lemma 2.6. Let u be a hyperfunction solution of $p(x,D)u=0$ defined on $\{0<x_1<\delta\}\times U'$. Let $\Gamma'\subset\boldsymbol{R}^{n-1}$ be a convex open cone with the vertex at the origin. Assume that the canonical extension $[u]$ of u satisfies*

(3.1) $\quad \mathrm{S.S.}\,[u]\subset\{x_1\geq 0\}\times\{\sqrt{-1}(\pm(1-\theta)dx_1+\theta\omega'dx')\infty;\ \omega'\in\Gamma'^{\circ}\cap\boldsymbol{S}^{n-2},\ 0\leq\theta\leq 1\},$

where Γ'° is the dual cone of Γ' defined by

$$\Gamma'^{\circ}=\{\omega'\in\boldsymbol{R}^{n-1};\ \omega'y'\geq 0\quad \textit{for any}\quad y'\in\Gamma'\}.$$

Then there exist a constant $k>0$ depending only on $p(x,D)$ and a holomorphic solution $F(z)$ of $p(z,D)F=0$ satisfying the following properties: For every $V'\Subset U'$ and for every open subcone $\Delta'\subset\Gamma'$ satisfying $\Delta'\cap\boldsymbol{S}^{n-2}\Subset\Gamma'\cap\boldsymbol{S}^{n-2}$ there exists $\varepsilon=\varepsilon(V',\Delta')$ such that $F(z)$ can be continued to

(3.2) $\quad \{z=x+\sqrt{-1}y\in\boldsymbol{C}^n;\ x_1<\varepsilon,\ x'\in V',\ |y_1|<\min\{0,x_1\}+k|y'|,\ y'\in\Delta'\cap\{|y'|<\varepsilon\}\}.$

Moreover $[u]$ agrees with the hyperfunction defined as the boundary value as $z'\mapsto x'+\sqrt{-1}\Gamma'0$ of the hyperfunction $Y(x_1)F(x_1,z')$ defined on $(3.2)\cap\{y_1=0\}$ and containing z' as holomorphic parameters.

PROOF. Choose W' such that $V'\Subset W'\Subset U'$. Let $[[u]]$ be a modification of $[u]$ satisfying $\mathrm{supp}\,[[u]]\subset\{x_1\geq 0\}\times\overline{W'}$ and $[[u]]=[u]$ in $\{x_1<\delta\}\times W'$. Put

$$v=[[u]]\underset{x'}{*}W(x',\Delta'^{\circ}),\qquad w=[[u]]\underset{x'}{*}W(x',\boldsymbol{S}^{n-2}\backslash\Delta'^{\circ}),$$

where $W(x',\Delta'^{\circ})=\int_{\Delta'^{\circ}\cap S^{n-2}}W(x',\omega')d\omega'$ etc. Note that

$$\operatorname{supp} v\subset\{x_1\geq 0\},\qquad \operatorname{supp} w\subset\{x_1\geq 0\}.$$

By the routine calculus of the S. S., we see from (3.1) that

$$\text{S. S. } w|_{\{x_1<\delta\}\times W'}\subset\{x_1<\delta\}\times W'\times\{\pm\sqrt{-1}dx_1\infty\}.$$

Hence there exist functions $W_{\pm}(z)$ holomorphic respectively on an infinitesimal wedge approximating $\{x_1<\delta\}\times W'+\sqrt{-1}\{\pm y_1>0\}$ from the inside such that

$$w=W_{+}(x_1+\sqrt{-1}0,x')-W_{-}(x_1-\sqrt{-1}0,x').$$

Here we have employed an abridged expression for the boundary value in the obvious way. Thus especially $W_{\pm}(z)$ agree with each other to a holomorphic function on $x_1<0$.

First of all we are going to prove that $W_{\pm}(z)$ are in fact holomorphic, respectively, on

$$\{|x|<\varepsilon\}+\sqrt{-1}(\{\pm y_1>0\}\times E')\cap\{|y|<\varepsilon\},$$

if we choose a proper subcone $E'\subset\Delta'$ and a sufficiently small $\varepsilon>0$. This is not an obvious conclusion from the estimate of S. S. w, because one asserts here that the domain of definition of $W_{\pm}(z)$ touches $y_1=0$ for small $y'\in E'$. Remark that $W(x',\Delta'^{\circ})$ is the boundary value of the holomorphic function $W(z',\Delta'^{\circ})=\int_{\Delta'^{\circ}\cap S^{n-2}}W(z',\omega')d\omega'$ from a *convex* wedge-like domain infinitesimally equal to $\boldsymbol{R}^{n-1}+\sqrt{-1}\Delta'$. We will denote this domain by $\boldsymbol{R}^{n-1}+\sqrt{-1}\Delta'0$ for the moment. Then v is the boundary value of the hyperfunction $v(x_1,z')=[[u]]\underset{x'}{*}W(z',\Delta'^{\circ})$ with holomorphic parameters z' from the same wedge. Remark that the latter has also support in $\{x_1\geq 0\}$. By the partial flabbiness of the sheaf $\mathcal{BO}$, we can extend $v(x_1,z')$ to $\boldsymbol{R}\times(\boldsymbol{R}^{n-1}+\sqrt{-1}\Delta'0)$ preserving the holomorphic parameters and with support in $0\leq x_1\leq\delta$. Denote this extended hyperfunction by the same letter v. Then put

$$V(z_1,z')=\frac{1}{2\pi\sqrt{-1}}\int_{-\infty}^{\infty}\frac{v(x_1,z')}{x_1-z_1}dx_1.$$

$V(z_1,z')$ is clearly holomorphic on $(\boldsymbol{C}\backslash[0,\delta])\times(\boldsymbol{R}^{n-1}+\sqrt{-1}\Delta'0)$ because it separately satisfies the Cauchy-Riemann equation in respective variables. We have obviously

$$v(x_1, z') = V(x_1 + \sqrt{-1}0, z') - V(x_1 - \sqrt{-1}0, z').$$

Put

$$\Delta_\pm = \{\pm y_1 > 0\} \times \Delta', \qquad \Delta = \boldsymbol{R} \times \Delta' = \Delta_+ + \Delta_-,$$

and similarly $E_\pm, E$ with E' instead of Δ'.

By the coherency of the boundary value operation we have

$$v(x) = V(x + \sqrt{-1}\Delta_+ 0) - V(x + \sqrt{-1}\Delta_- 0).$$

Now on $\{x_1 < \delta\} \times W'$ we have

$$p(x, D)(v + w) = p(x, D)[[u]] = p(x, D)[u] = \sum_{j=0}^{m-1} u_j(x') \delta^{(m-1-j)}(x_1).$$

By the estimate (3.3) we have an expression of the form $u_j(x') = F_j(x' + \sqrt{-1}\Gamma' 0)$. Applying the edge of the wedge theorem of the Epstein type, we thus conclude that the function

$$p(z, D_z)(V(z) + W_+(z)) - \sum_{j=0}^{m-1} F_j(z') \frac{1}{2\pi\sqrt{-1}} \frac{(-1)^{m-j}(m-1-j)!}{z_1^{m-j}}$$

agrees with $p(z, D_z)(V(z) + W_-(z))$ and hence can be continued holomorphically up to a domain of the form $\{|x| < \varepsilon\} \times \sqrt{-1}E \cap \{|y| < \varepsilon\}$, and similarly for the matter on the opposite side. Now let $F(z)$ be the solution of the holomorphic Cauchy problem $p(z, D_z)F(z) = 0$ with the data $F_j(z')$ by way of our boundary operators. Owing to the precise version of Cauchy-Kowalevsky's theorem, $F(z)$ is defined at least on a domain of the form

$$\{z \in \boldsymbol{C}^n;\ |z_1| < k|y'| < \varepsilon,\ x' \in V',\ y' \in \Delta'\}. \tag{3.4}$$

We have

$$p(z, D_z)\left\{F(z)\left(-\frac{1}{2\pi\sqrt{-1}}\right)\log(-z_1)\right\} = \sum_{j=0}^{m-1} F_j(z') \frac{1}{2\pi\sqrt{-1}} \frac{(-1)^{m-j}(m-1-j)!}{z_1^{m-j}}$$

whence we conclude that the function

$$p(z, D_z)\left\{V(z) + W_+(z) - F(z)\left(-\frac{1}{2\pi\sqrt{-1}}\right)\log(-z_1)\right\}$$

can be continued in a univalent holomorphic way onto a domain of the same form. Since the real hypersurfaces $y_1 = \mathrm{Im}\langle \zeta', z' \rangle$ are non-characteristic with respect to $p(z, D_z)$ for sufficiently small $|\zeta'|$, we can apply Bony-Schapira's method of sweeping out ([2]) to conclude that the function

$$V(z) + W_+(z) - F(z)\left(-\frac{1}{2\pi\sqrt{-1}}\right)\log(-z_1)$$

can be continued a little beyond the real hypersurface $y_1=0$. Hence especially $W_+(z)$ is also holomorphic on $\{|x|<\varepsilon\}+\sqrt{-1}E_+\cap\{|y|<\varepsilon\}$ for smaller $\varepsilon>0$. (See Fig. 2.)

In the above we have proved the following: There exist a neighborhood V' of the origin, constants $\delta'>0$, $\varepsilon>0$, and functions $F_\pm(z)$ holomorphic resp. on

$$\{|x_1|<\delta'\}\times V'+\sqrt{-1}(\Delta_\pm\cap\{|y|<\varepsilon\})$$

such that

$$[u]=F_+(x+\sqrt{-1}\Delta_+0)-F_-(x+\sqrt{-1}\Delta_-0).$$

In fact $F_\pm(z)=V(z)+W_\pm(z)$. Here we have rearranged the notation for the sake of simplicity. Since $\mathrm{supp}[u]\subset\{x_1\geq 0\}$, we see that $F_\pm$ agree and define a common holomorphic function on

$$(\{-\delta<x_1<0\}\times V'+\sqrt{-1}\Delta)\cap\{|y|<\varepsilon(|x_1|)\},$$

where $\varepsilon(|x_1|)$ may decrease to 0 with $|x_1|$.

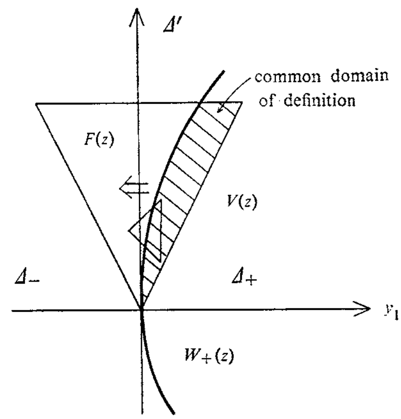

Fig. 2.

(This is a local version of the edge of the wedge theorem.) Now consider

$$f(x_1, z')=F_+(x_1+\sqrt{-1}0, z')-F_-(x_1-\sqrt{-1}0, z')$$

as a hyperfunction on $\{|x_1|<\delta'\}\times(V'+\sqrt{-1}\Delta'\cap\{|y'|<\varepsilon'\})$ with holomorphic parameters z'. By what is said above $f(x_1, z')$ has support contained in $x_1\geq 0$. In fact, this may not be clear on the very neighborhood of $x_1=0$ because $\varepsilon(|x_1|)$ may decrease to 0 with $|x_1|$. But we have the unique continuation property with respect to holomorphic parameters (see [12], Theorem 8.2.1; or apply the Holmgren uniqueness theorem to the partial Cauchy-Riemann system). Thus the zero propagates along $x_1=$const. up to where $f(x_1, z')$ is defined, and we can conclude that

$$\mathrm{supp} f(x_1, z')\subset\{x_1\geq 0\}.$$

We have

$$p(x_1, z', D_{x_1}, D_{z'})f(x_1, z')=\sum_{j=0}^{m-1}F_j(z')\delta^{(m-1-j)}(x_1). \tag{3.5}$$

In fact, if we let $z'\mapsto x'+\sqrt{-1}\Delta'0$ in both sides, we obtain, on account of the compatibility of the boundary value operation, the known equality

(1.3)bis $$p(x, D)[u]=\sum_{j=0}^{m-1} u_j(x')\delta^{(m-1-j)}(x_1).$$

On the other hand, the right hand side of (3.5) can also be written as

$$G_+(x_1+\sqrt{-1}0, z')-G_-(x_1-\sqrt{-1}0, z'),$$

where

$$G_\pm(z)=\sum_{j=0}^{m-1} F_j(z')\left(\frac{\partial}{\partial z_1}\right)^{m-1-j}\left(-\frac{1}{2\pi\sqrt{-1}}\frac{1}{z_1}\right)$$
$$\in \mathcal{O}(\{x_1<\delta'\}\times V'+\sqrt{-1}\Delta_\pm\cap\{|y|<\varepsilon\}).$$

Thus (1.3)bis means that the pair of holomorphic functions $(F_\pm - G_\pm)(z)$ defines 0 as hyperfunction on the real axis. Again by the edge of the wedge theorem we conclude that $F_\pm - G_\pm$ agree and define a holomorphic function on $\{|x_1|<\delta''\}\times V'+\sqrt{-1}(\Delta\cap\{|y|<\varepsilon''\})$ maybe with a smaller V'. Thus letting $y_1=0$ we have established (3.5).

Now compare (3.5) with the obvious relation

$$p(x_1, z', D_{x_1}, D_{z'})(Y(x_1)F(z))=\sum_{j=0}^{m-1} F_j(z')\delta^{(m-1-j)}(x_1)$$

which holds on $(3.4)\cap\{y_1=0\}$. By the Holmgren uniqueness theorem we conclude that there we have

$$Y(x_1)F(z)=f(x_1, z')=F_+(x_1+\sqrt{-1}0, z')-F_-(x_1-\sqrt{-1}0, z').$$

Next we try to continue $F(z)$ to a suitable wedge. On $x_1>0$, $[u]=u$ satisfies the equation $p(x, D)[u]=0$. Hence by Sato's fundamental theorem the estimate (3.1) can be strengthened to

(3.6) $$\text{S. S. }[u]|_{\{x_1>0\}}\subset\{x_1>0\}\times\sqrt{-1}\{(\pm(1-\theta)dx_1+\theta\omega' dx')\infty;$$
$$\omega'\in\Gamma'^{\circ}\cap S^{n-2},\ \theta_0<\theta\leq 1\}.$$

Put

$$\Delta_0=\{y\in \boldsymbol{R}^n;\ |y_1|<(\theta_0/(1-\theta_0))|y'|,\ y'\in\Delta'\}.$$

Then (3.6) implies that there exists $G(z)$ holomorphic on

$$(\{0\leq x_1<\delta'\}\times V'+\sqrt{-1}\Delta_0)\cap\{|y|<\varepsilon'(|x_1|)\}$$

such that $[u]=G(x+\sqrt{-1}\Delta_0 0)$. Thus

$$F_+(x+\sqrt{-1}\Delta_+0)-F_-(x+\sqrt{-1}\Delta_-0)=G(x+\sqrt{-1}\Delta_0 0),$$

and by the edge of the wedge theorem we conclude that $F_\pm(z)$ can be continued to

$$(\{0<x_1<\delta''\}\times V'+\sqrt{-1}\Delta_0)\cap\{|y|<\varepsilon''(|x_1|)\}$$

maybe for smaller V' and there satisfying $F_+(z)-F_-(z)=G(z)$. This especially implies that the hyperfunction with holomorphic parameters $f(x_1, z')=G(x_1, z')$ is real analytic on

$$\{0<x_1<\delta''\}\times V'+\sqrt{-1}\Delta'\cap\{|y'|<\varepsilon''(|x_1|)\}.$$

The real analyticity of a hyperfunction with holomorphic parameters z' also propagates along $x_1=\text{const.}$ (see [24], Chapter III, Theorem 2.2.8 or [12], Corollaire 8.2.3). Thus we conclude that $f(x_1, z')$ is real analytic on the set

$$(3.7) \qquad \{0<x_1<\delta''\}\times V'+\sqrt{-1}\Delta'\cap\{|y'|<\varepsilon''\}$$

with ε'' independent of x_1.

Summing up we have shown that the common real analytic function $F(x_1, z')=f(x_1, z')=G(x_1, z')$ defined on (3.4) $\cap$ (3.7) can be continued as a holomorphic function of (z_1, z') to a neighborhood of the set

$$\begin{aligned}&\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ -\delta''<x_1\leq 0,\ x'\in V',\ |x_1|+|y_1|<k|y'|,\ y'\in\Delta'\cap\{|y'|<\varepsilon''\}\}\\ &\cup\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ 0<x_1<\delta'',\ y_1=0,\ y'\in\Delta'\cap\{|y'|<\varepsilon''\}\}.\end{aligned}$$

Since this holomorphic function is a solution of $p(z, D)F=0$, it can be continued up to a wedge-like neighborhood of the second component employing the Cauchy-Kowalevsky theorem. Thus we have obtained a domain of the form (3.2) and a holomorphic function $F(z)$ on it having the required property. It is clear that the constant k has been prescribed by the characteristics of $p(x, D)$, hence independent of the other data.

Until now V' was a small neighborhood of the origin. Such a representation being unique, we can connect them along the boundary $x_1=0$ so that in the final expression V' may be arbitrarily close to U' with the sacrifice of the constant ε. q.e.d.

PROOF OF LEMMA 3.3. From now on we choose $\nu'=(0, \cdots, 0, 1)$. Owing to Lemma 1.7 the canonical extension $[v]$ of v satisfies (3.1) with $\Gamma'=\{y_n>0\}$. Hence in view of Lemma 3.4 we can assume that $v(x)$ is defined by a holomorphic solution $F(z)$ of $p(z, D)F=0$ defined on

$$\begin{aligned}(3.8)_\lambda \qquad &\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ |z_1|<k|y'|,\ |x'|<A,\ \lambda|y'|<y_n<\varepsilon(\lambda)\}\\ &\cup\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ 0<x_1<\delta,\ |x'|<A,\ |y_1|<k|y'|,\ \lambda|y'|<y_n<\varepsilon(\lambda)\}.\end{aligned}$$

Moreover we have a stronger assumption that $v(x)$ is real analytic on $x_1>0$.

Therefore by the edge of the wedge theorem $F(z)$ can be continued to a neighborhood of the edge $\{0<x_1<\delta\}\times\{|x'|<A\}$. Taking into account the fact that λ in (3.8) is arbitrary, we thus conclude that there exist in fact convex non-negative continuous functions $\varphi(t)$, $\phi(t)$ of $t\geq 0$ satisfying $\varphi(0)=\phi(0)=0$, $\varphi(t)/t\to 0$ if $t\to 0$ and $\phi(t)>0$ for $t>0$ such that $F(z)$ is holomorphic on

$$\text{(3.9)}\quad \{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ 0\leq x_1<\delta',\ |x'|<A',\ |y_1|<k|y'|,\ -\phi(x_1)+\varphi(|y''|)<y_n<B\}.$$

Hence especially $F_j(z')=b_j^{\dagger}(z, D)F(z)|_{z_1=0}$ are holomorphic on

$$\text{(3.10)}\qquad \{z'=x'+\sqrt{-1}y'\in \boldsymbol{C}^n;\ |x'|<A',\ \varphi(|y''|)<y_n<B\}.$$

(Recall that owing to Lemma 2.6 $F_j(z')$ define the hyperfunctions $v_j(x')$, that is, the boundary values of $v(x)$. Because $v_j(x')$ is now given by $f_j(x')*J(D_{\omega'})W(x', \omega')|_{\omega'=\nu'}$ and the latter is obviously defined by the holomorphic function

$$f_j(x')\underset{x'}{*}J(D_{\omega'})W(z', \omega')|_{\omega'=\nu'},$$

we see by the edge of the wedge theorem that $F_j(z')=f_j(x')\underset{x'}{*}J(D_{\omega'})W(z', \omega')|_{\omega'=\nu'}$ hence that we can employ $\varphi(t)=t^2$ in (3.10).)

Employing the observation made until now we try to continue $F(z)$ to a neighborhood of the origin. In view of the local character of the assertion this will prove our lemma. First we use the assumption of $\sqrt{-1}\nu'dx'\infty$-semihyperbolicity of p and employ Proposition 2.2 to solve the holomorphic Cauchy problem (2.3) with the data $F_j(z')$. Then the solution $F(z)$ can be continued to a domain of the form

$$\text{(3.11)}\quad \{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ -\delta<x_1\leq 0,\ |x'|<A'',\ \lambda|y_1|+\varphi(c|x_1|)+\varphi(2|y''|)<y_n<B'\}.$$

Thus we have continued $F(z)$ up to (3.9) $\cup$ (3.11). Next we recall that $p(D_1/\sqrt{-1}, D')$ is I-hyperbolic with respect to $x_1=0$ for some neighborhood I of $\nu'\in \boldsymbol{S}^{n-2}$. This permits us to apply Proposition 2.13 with the role of x_1 and y_1 interchanged. (That is, we solve the Cauchy problem starting from the each initial hyperplane $z_1=x_1$ (const.) to the direction of $\sqrt{-1}y_1$ with the origin of z'-space suitably translated.) Thus the solution $F(z)$ can be continued from (3.9) $\cup$ (3.11) to a domain of the form

$$\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ |x_1|<\delta'',\ |x'|<A''',\ |y_1|<C,\ \chi(x_1)+\varphi(4|y''|)<y_n<B''\},$$

where

$$\chi(x_1)=\begin{cases}-\phi(x_1) & \text{if } x_1\geq 0,\\ \varphi(c|x_1|) & \text{if } x_1\leq 0.\end{cases}$$

Put

$$\Omega=\{|x_1|<\delta'',\quad \chi(x_1)<y_n<B''\}.$$

Then the above domain contains the following set

$$\{(z_1, z_n)\in \boldsymbol{C}^2;\ y_1^2+x_n^2<C',\ (x_1, y_n)\in\Omega\}\times\{z''=0\}. \tag{3.12}$$

(Strictly speaking, we must let y'' vary near the origin. Because the necessary modification is easy, we put $y''=0$ in the sequel for the sake of simplicity.)

We assert that we can apply the local version of Bochner's tube theorem to the set (3.12) with respect to the variables $\sqrt{-1}z_1, z_2, \cdots, z_n$ in order to continue $F(z)$ to a neighborhood of the origin. (For the local version of Bochner's tube theorem which we use, see, e.g. [18]. Note that the role of x_1 and y_1 in the first coordinate is interchanged compared with the usual use.) Though the convex hull of Ω swallows the origin, we cannot affirm this directly because of the local character. The accurate proof will be carried in two steps: First we attach the wedge to the part $x_1\geq 0$ as in the figure 3-1. By this we can replace the curve $y_n=-\phi(x_1)$ by another which has a really negative angle of inclination at the origin. (See Corollary 2 in [18] and remark nearby. This is a variant of the local Bochner theorem and close to Kashiwara's lemma in idea.) Then we swallow the origin by the slope of the wedge employing the usual local Bochner theorem as in the figure 3-2. (This is possible because the wedge approaches a true prism as we make the "imaginary" part of it smaller by similarity compared with the "real" part.)

Thus we have continued $F(z)$ to a neighborhood of the origin, hence proved Lemma 3.3 and in the same time Theorem 3.1. q.e.d.

EXAMPLE 3.5. The partial Cauchy-Riemann operator $D_1+\sqrt{-1}D_2$ on $\boldsymbol{R}^n$ $(n\geq 3)$, the partial Laplace operator $D_1^2+\cdots+D_k^2$ on $\boldsymbol{R}^n$ $(k<n)$, or more generally an

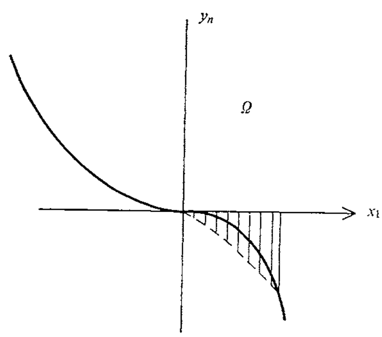

Fig. 3-1.

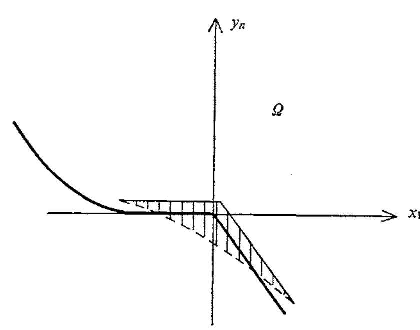

Fig. 3-2.

operator whose principal part is the product of some of them satisfy the hypothesis of Theorem 3.1 with $\nu'=(0,\cdots,0,1)$. Note that for these operators $p(D_1/\sqrt{-1}, D')$ is even hyperbolic with respect to $x_1=0$. For $p(D)=D_1+\sqrt{-1}D_2$ we can apply the theorem to every direction $\sqrt{-1}\omega' dx'\infty$ such that $\omega_2\geq 0$. Thus the S. S. of the boundary values of real analytic solutions on $x_1>0$ is contained in the open hemisphere

$$\boldsymbol{R}^{n-1}\times\{\sqrt{-1}\omega' dx'\infty;\ \omega_2<0\}. \tag{3.13}$$

This estimate improves our former work [7] on the equator $\omega_2=0$, and is the best one as well as we consider the whole real analytic solution at the same time. Similarly for $p(D)=D_1^2+\cdots+D_k^2$ the S. S. is accurately estimated by

$$\boldsymbol{R}^{n-1}\times\{\sqrt{-1}\omega' dx'\infty;\ \omega_2^2+\cdots+\omega_k^2\neq 0\}. \tag{3.14}$$

It is rather surprising that the sets (3.13), (3.14) are not closed though the S. S. of the boundary values for each solution is of course closed. As remarked in the introduction, these examples are already treated by Schapira [25], [26].

We add an artificial example of a polynomial which satisfies the assumptions of our theorem. Consider the operator

$$p(D)=D_1^4+4\sqrt{-1}D_1^3D_n+D_1^2(2D_2^2-4D_n^2)+4\sqrt{-1}D_1D_2^2D_n-2D_2^2D_n^2.$$

This is obtained expanding the doubly quadratic polynomial

$$(\zeta_1-\zeta_n)^4-2(\zeta_1-\zeta_n)^2(\zeta_2^2+\zeta_n^2)+\zeta_n^4,$$

and replacing ζ_1 by $\sqrt{-1}\zeta_1$. An elementary calculation shows that the latter polynomial has real roots for real ξ' satisfying $|\xi_n|>2|\xi_2|$. Moreover, for complex ζ' satisfying $|\zeta_n|>2|\zeta_2|$ the roots have the asymptotic form

$$2\zeta_n\pm O(|\zeta_2|),\qquad O(|\zeta_2|).$$

Because the roots of $p(\zeta)=0$ are $-\sqrt{-1}$ times these, we thus see that $p(D)$ is $\sqrt{-1}dx_n\infty$-semihyperbolic to $x_1<0$ and that $p(D_1/\sqrt{-1}, D')$ is I-hyperbolic with respect to $x_1=0$, where $I=\{\xi'\in\boldsymbol{S}^{n-2};\ |\xi_n|>2|\xi_2|\}$.

REMARK 3.6. The method of proof of Theorem 3.1 shows that it is applicable also to operators whose coefficients may depend on the variable x_1. In fact, the process by which we have reduced Theorem 3.1 to Lemma 3.3 is valid in view of Remark 1.5. Also, Proposition 2.2 or Proposition 2.13 concerning the solvability of the Cauchy problem for various semihyperbolic operators have respective correspondents in the case of variable coefficients (see [8]). Thus we can see e.g. that the generalized Lewy-Mizohata operator $D_1+\sqrt{-1}x_1^k D_2$ on $\boldsymbol{R}^n$ $(n\geq 3)$ comes in our example: Combining with our former result given in [8], we conclude that the S. S. of the boundary values of its real analytic solutions on $\pm x_1>0$ is accurately estimated by

$$\begin{aligned} &\boldsymbol{R}^{n-1}\times\{\sqrt{-1}\omega' dx'\infty;\ \omega_2<0\} &&\text{if } k \text{ is odd,}\\ &\boldsymbol{R}^{n-1}\times\{\sqrt{-1}\omega' dx'\infty;\ \pm\omega_2<0\} &&\text{if } k \text{ is even.}\end{aligned}$$

2. We suppose that the assumption of I-hyperbolicity of $p(D_1/\sqrt{-1}, D')$ in the hypothesis of Theorem 3.1 is too strong. That is, we expect that a holomorphic solution $F(z)$ of $p(D)F=0$ defined on a domain of the form (3.9) ∪ (3.11) can be continued to a neighborhood of the origin under a much weaker condition. But for the moment we do not know much better than the use of the local Bochner theorem based on the above hypothesis. In this respect we will give two miscellaneous results.

THEOREM 3.7. *Let $p(D)$ be an operator which is I-semihyperbolic to $x_1<0$ (resp. to $x_1>0$) for some neighborhood I of $\nu'\in\boldsymbol{S}^{n-2}$. Let u be a hyperfunction solution of $p(D)u=0$ defined locally on $x_1>0$ (resp. on $x_1<0$). If* S. S. u *does not contain the directions $\sqrt{-1}(\pm(1-\theta)dx_1+\theta\nu' dx')\infty$, $0<\theta\leq 1$, then the* S. S. *of the boundary values of u does not contain the direction $\sqrt{-1}\nu' dx'\infty$.*

PROOF. We can proceed just in the same way up to the estimate (3.9). Then this time we can directly apply Proposition 2.13 with the initial hyperplane $x_1=\varepsilon$ (and with the origin of $\boldsymbol{C}^{n-1}$ suitably translated). Considering the fact that the constant δ, the limit of continuation of the solution $F(z)$, is independent of ε, we can conclude directly that $F(z)$ can be continued to a neighborhood of the origin as $\varepsilon\to 0$. q.e.d.

This result is apparently a refinement of our former work [7] in the sense that we treated only the real analytic solutions formerly. (Our former method is, however, also capable of deducing Theorem 3.7 without essential amelioration.) Note that this theorem asserts a kind of reflection phenomena of the singularity at the boundary on the hyperbolic region. Consider for example the wave equation $\square=D_1^2+\cdots+D_{n-1}^2-D_n^2$ with the boundary $x_1=0$. Let u be a hyperfunction solution of $\square u=0$ defined locally on $x_1>0$ on a neighborhood of the origin. Theorem 3.7 asserts that if the S. S. of the boundary values of u contain the point $(0, \sqrt{-1}\xi' dx'\infty)$ in some component, where ξ' satisfy $\xi_2^2+\cdots+\xi_{n-1}^2<\xi_n^2$, then

S. S. u must contain a direction of the form $\sqrt{-1}(\pm(1-\theta)dx_1+\theta\xi'dx')\infty$, $0<\theta\leq 1$ on any neighborhood of the origin. On account of Sato's fundamental theorem there are but two possible directions, namely $\sqrt{-1}(\pm\xi_1 dx_1+\xi' dx')\infty$ with $\xi_1=(\xi_n^2-(\xi_2^2+\cdots+\xi_{n-1}^2))^{1/2}$. Thus considering the result on propagation of singularity along bicharacteristics we can conclude from this that S. S. u must contain either of the two half bicharacteristics hitting 0:

$$b^{\pm}=\{(\pm\xi_1 t,\ \xi' t;\ \sqrt{-1}(\pm\xi_1 dx_1+\xi' dx')\infty);\ t>0\}. \tag{3.15}$$

Since we do not pose any boundary condition, it is possible that another of these is outside S. S. u, that is, that the reflection does not occur and the "energy is absorbed in the boundary point".

In view of Lemma 1.7 the converse is also true. That is, if the S. S. of the boundary values does not contain $(0, \sqrt{-1}\xi' dx'\infty)$ with ξ' satisfying $\xi_2^2+\cdots+\xi_{n-1}^2<\xi_n^2$, then the two half bicharacteristics (3.15) are outside S. S. u.

Now we give another variant which is a little stronger than Theorem 3.1.

THEOREM 3.8. *Let $\nu'=(0,\cdots,0,1)$. Concerning the* S. S. *of the boundary values of a solution of $p(D)u=0$ on $x_1>0$, we can obtain the same conclusion as Theorem* 3.1 *or Remark* 3.2 *if only we assume, instead of the $\sqrt{-1}\nu' dx'\infty$-semi-hyperbolicity of $p(D)$ to $x_1<0$, the inequality* (2.14) *in Proposition* 2.3.

PROOF. We proceed in the same way as in the proof of Theorem 3.1. Then it suffices to prove an assertion corresponding to Lemma 3.3. Instead of (1.1) we employ a curved wave decomposition of the following form:

$$\begin{aligned} &\delta(x)=\int_{S^{n-1}} W^k(x,\omega)d\omega, \\ &W^k(x,\omega)=\frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\,\frac{J(x,\omega)}{(x\omega+\sqrt{-1}(x^2-(x\omega)^2)^k+\sqrt{-1}0)^n}, \end{aligned} \tag{3.16}$$

where k is a positive integer and $J(x,\omega)$ is a polynomial of x, (see [24], Chapter III, Example 1.2.5).

LEMMA 3.9. *The phase $W^k(z,\omega)$ is holomorphic on* $\mathrm{Im}\, z\omega>K(|\mathrm{Im}\, z|^2-(\mathrm{Im}\, z\omega)^2)^k$, *where $K>0$ is a constant.*

PROOF. Put

$$I=\mathrm{Im}\,\{z_1+\sqrt{-1}(z'^2)^k\}=y_1+\sum_{j=0}^{[(k-1)/2]}(-1)^j{}_kC_{2j}(x'^2-y'^2)^{k-2j}(2x'y')^{2j},$$

where we have employed the notation $z'^2=z_2^2+\cdots+z_n^2$ etc. If $x'^2-y'^2\geq N|2x'y'|$, we have

$$I \geq y_1 + (x'^2 - y'^2)^k \left\{1 - \sum_{j=1}^{[(k-1)/2]} {}_kC_{2j} N^{-2j}\right\}.$$

Thus if we choose N sufficiently large, we obtain $I \geq y_1$. If $x'^2 - y'^2 \leq N|2x'y'|$, then we have

$$x'^2 - y'^2 \leq N\left(\frac{x'^2}{2N} + 2Ny'^2\right),$$

hence

$$x'^2 \leq 2(2N^2+1)y'^2,$$
$$|x'^2 - y'^2| \leq x'^2 + y'^2 \leq (4N^2+3)y'^2, \qquad |2x'y'| \leq (4N^2+3)y'^2.$$

Thus we obtain

$$I \geq y_1 - \sum_{j=0}^{[(k-1)/2]} {}_kC_{2j} |x'^2 - y'^2|^{k-2j} (2x'y')^{2j} \geq y_1 - K(y'^2)^k,$$

where $K>0$ is a constant. This proves our lemma for $\omega = (1, 0, \cdots, 0)$.

End of Proof of Theorem 3.8. Now we apply Proposition 1.2 employing as $W(x', \omega')$ the components of the curved wave decomposition corresponding to (3.16) instead of (1.1). Then the real analytic solution $v(x)$ given there has the boundary values $v_j(x')$ of the form $f_j(x') * J(D_{\omega'}) W^k(x', \omega')|_{\omega'=\nu'}$ on V'. Therefore owing to Lemma 3.9 the hyperfunction $v_j(x')$ are given as the boundary values of some holomorphic functions $F_j(z')$ defined on a domain of the form (3.10), where $\varphi_{(t)} = Kt^{2k}$. Then Proposition 2.3 gives the solution of the holomorphic Cauchy problem (2.3) on a domain of the form

$$\{z = x + \sqrt{-1}y \in C^n;\ x_1 \leq 0,\ |x'| < A'',\ \lambda(|y_1| + |x_1|^{2k/(2kq+1)}) + K|y''|^{2k} < y_n < B''\}.$$

As remarked after the proof of Proposition 2.3 the exponent $2k/(2kq+1)$ becomes greater than 1 if we choose k sufficiently large. Thus the above domain enjoys the same property as (3.11) and from now on the proof goes in the same way.

q.e.d.

In [6bis] we gave a conjecture that we will have the conclusion of Theorem 3.1 assuming only that $p(D)$ is $\sqrt{-1}\nu' dx' \infty$-semihyperbolic to $x_1 < 0$, that is, assuming the inequality (2.2) for $\nu' = (0, \cdots, 0, 1)$. (There the two directions $\pm x_1 > 0$ were not yet separated.) For the present we have no confidence on this conjecture nor a counter-example. (If we are optimistic we will even be able to replace the inequality (2.2) by (2.14) of Theorem 3.8 in the above conjecture.)

§4. Propagation of singularity along the boundary.

1. We now enlarge the class of manageable operators by somewhat restricting the behaviour of the singular spectrum of boundary values.

THEOREM 4.1. *Let $p(D)$ be an m-th order operator with respect to which $x_1=0$ is non-characteristic. Assume that its homogeneous characteristic roots satisfy*

$$\mathrm{Re}\,\tau_j^0(\nu')=0, \quad \mathrm{Im}\,\tau_j^0(\nu')\leq 0, \qquad j=1,\cdots,m \tag{4.1}$$

for some direction ν'. Let $b_j^+(u)$, $j=0,\cdots,m-1$ be the boundary values of a hyperfunction solution u of $p(D)u=0$ defined locally on $x_1>0$. Assume that

1) *S. S. u does not contain the directions $\sqrt{-1}(\pm(1-\theta)dx_1+\theta\nu'dx')\infty$, $0<\theta\leq 1$;*

2) $\bigcup_{j=0}^{m-1}$ S. S. $b_j^+(u)$ *has compact cross section with a level plane $\{x'\nu'=\text{const.}\}\times\{\sqrt{-1}\nu'dx'\infty\}$ in the boundary cosphere bundle. Then* S. S. $b_j^+(u)$ *does not contain the direction $\sqrt{-1}\nu'dx'\infty$.*

Similar assertion holds for solutions on $x_1<0$ if we change the sign in (4.1).

PROOF. Choose a linear coordinate transformation such that $\nu'=(0,\cdots,0,1)$. Let U'' be a convex neighborhood of $0\in \boldsymbol{R}^{n-2}$ and put $U'=U''\times\{|x_n|<a\}$. We can assume that u is defined on $\{0<x_1<\delta\}\times U'$. Owing to Proposition 1.8-Remark 1.9, for every $V''\Subset U''$ and for every $a'<a$ there exist $\delta'>0$ and a real analytic solution v of $p(D)v=0$ on $\{0<x_1<\delta'\}\times V'$ (where $V'=V''\times\{|x_n|<a'\}$) such that its boundary values agree with

$$f_j(x')*\{J(D_{\omega'})W_0(x',\omega')|_{\omega'=\nu'}\}|_{V'}, \qquad j=0,\cdots,m-1. \tag{4.2}$$

Here $f_j(x')$ is a fixed element of $'\mathcal{B}[\overline{U'}]$ independent of $J(D_{\omega'})$ and satisfying

$$f_j(x')|_{V'}=b_j^+(u)|_{V'}.$$

Recall that Lemma 1.1 holds also for the plane wave decomposition $W_0(x',\omega')$ as remarked there. Thus it suffices to prove that (4.2) become real analytic on a neighborhood of the origin for every $J(D_{\omega'})$. Note that each hyperfunction in (4.2) is the boundary value of the function $F_j(z')=f_j(x')\underset{x'}{*}\{J(D_{\omega'})W_0(z',\omega')|_{\omega'=\nu'}\}$ which is holomorphic on $\boldsymbol{C}^{n-2}\times\{y_n>0\}$. Note also that our condition on $p(D)$ implies that $p(e^{\sqrt{-1}\theta}D_1, D')$ is partially $\sqrt{-1}\nu'dx'\infty$-semihyperbolic to $x_1>0$ in the sense of Definition 2.9 for every angle θ satisfying $\pi/2\leq\theta\leq 3\pi/2$. Thus we can apply Proposition 2.11 to every such direction z_1 and solve the holomorphic Cauchy problem (2.3) with the initial data $F_j(z')$ on $z_1=0$. The solution $F(z)$ is then holomorphic on a domain of the form

$$\{|z_1|<\delta,\quad \max(x_1, 0)<ky_n,\quad |x'|<A\}. \tag{4.3}$$

On the other hand, by Lemma 3.4 the hyperfunction $v(x)$ is represented as the boundary value of the hyperfunction with holomorphic parameters $Y(x_1)F(x_1, z')$ as $z'\mapsto x'+\sqrt{-1}\Delta'0$. Here Δ' is a cone arbitrarily close to the cone $\{y_n>0\}$ and $F(z)$ is a function holomorphic on a domain of the form (3.2). Here we have deliberately used the same symbol $F(z)$ as the above function by the very reason that they agree on account of the Holmgren uniqueness theorem. We claim that this common function $F(z)$ is in fact holomorphic on a neighborhood of the set

$$\{0<x_1<\delta',\quad y_1=0,\quad |x'|<A',\quad y_n>0\}. \tag{4.4}$$

In fact, recalling the proof of Lemma 3.4 we see that $F(x_1, z')$, as a hyperfunction with holomorphic parameters z', is given as the difference of the boundary values of $F_\pm(z)$ in the form

$$F(x_1, z')=F_+(x_1+\sqrt{-1}0, z')-F_-(x_1-\sqrt{-1}0, z'). \tag{4.5}$$

By the way, $F_\pm(z)$ is a pair of defining functions for the hyperfunction $v(x)$ which in turn is obtained via Lemma 1.3 by adjusting an identity e.g. of the form

$$p(D)\{[[Y(x_1-t(x'))[u]]]\underset{x'}{*}W_0(x', \nu')\}=\sum_{j=0}^{m-1} f_j(x')*W_0(x', \nu')\delta^{(m-1-j)}(x_1)+\gamma(x). \tag{4.6}$$

The hyperfunction under the operator $p(D)$ in the left hand side of (4.6) is the boundary value of the hyperfunction with holomorphic parameters

$$[[Y(x_1-t(x'))[u]]]\underset{x'}{*}W_0(z', \nu')$$

along the cone $\{y_n>0\}$. Hence it admits obviously a representation of the form (4.5) with $V_\pm(z)$ instead of $F_\pm(z)$ which are holomorphic resp. on

$$\{z\in \boldsymbol{C}^n;\ x_1<\delta',\quad |x'|<A',\quad \pm y_1>0,\quad y_n>0\}.$$

The same assertion holds also for the term $\gamma(x)$. Recall on the other hand that

$$\text{S. S. } \gamma(x)\subset\{0\leq x_1\leq\delta',\quad |x'|<A'\}\times\{\pm\sqrt{-1}dx_1\infty\}.$$

On account of the edge of the wedge theorem of the Bogoliubov type, this implies that the functions $W_\pm(z)$, defining $\gamma(x)$ as the boundary value from the above wedge, can also be continued resp. to a wedge infinitesimally equal to $\pm y_1>0$. This is even true if we translate the origin by a vector $\sqrt{-1}(y_2, \cdots, y_{n-1}, 0)$, because for our choice of ν' the singularity of the convolution factor $J(D_{\omega'})W_0(z', \omega')|_{\omega'=\nu'}$ is invariant under this translation. Moreover, the functions $W_\pm(z)$ agree on $x_1<0$

on account of the condition $\mathrm{supp}\,\gamma(x)\subset\{x_1\geq 0\}$. In fact, by the edge of the wedge theorem, $W_\pm(z)$ first agree as holomorphic functions on a neighborhood of the part $x_1<0$ of the real axis. Then, the hyperfunction $W_+(x_1+\sqrt{-1}0, z')-W_-(x_1-\sqrt{-1}0, z')$ with holomorphic parameters z', will vanish identically on the part $x_1<0$ of the boundary of our domain because of the unique continuation property. Therefore, again by the edge of the wedge theorem, $W_\pm(z)$ agree there. Thus in view of Corollary A.1 in the appendix we can deduce a little more careful conclusion that the modified hyperfunction $v(x)$ given by Lemma 1.3 admits also a same type of representation. Thus the real analyticity of the hyperfunction $F(x_1, z')$, which is already assured on a neighborhood of the edge $y=0$, propagates with respect to the holomorphic parameters z' along $x_1=\text{const.}$ up to the whole set (4.4).

Now we can apply Proposition 2.11 to extend $F(z)$ to the direction $\pm y_1$ starting from every initial plane $z_1=x_1^0=\text{const.}$ for $0<x_1^0<\delta'$. Summing up we have thus obtained a function $F(z)$ representing $v(x)$ and holomorphic on a domain of the form

$$\{|z_1|<\delta'',\quad |x'|<A'',\quad y_n>0\}.$$

Recalling the fact that $v(x)$ is real analytic on $x_1>0$ we see then that $F(z)$ is holomorphic on a domain of the form

$$\{z=x+\sqrt{-1}y\in \boldsymbol{C}^n;\ |x'|<A'',\quad |y_1|<\delta'',\quad (x_1, y_n)\in\Omega\},$$

where

$$\Omega=\{|x_1|\leq\delta''',\ y_n>0\}\cup\{\delta'''<x_1<2\delta''',\ y_n>-\varepsilon\}.$$

Applying the local Bochner theorem to this domain interchanging the role of x_1 and y_1, we can finally continue $F(z)$ to a new domain which rounds the corner of Ω to a smooth quadratic curve. The part of the boundary

$$\{y_1=0,\ x'=0\}\times[\{-\delta'''<x_1\leq\delta''',\ y_n=0\}\cup\{-\varepsilon<y_n<0,\ x_1=\delta'''\}]$$

is, hence especially the origin is, contained in this new domain. Thus $F(z)$, hence $v(x)$ and its boundary values (4.2) become analytic on a neighborhood of the origin.

q.e.d.

We can apply Theorem 4.1 e.g. to the operator $p(D)=D_1^2+D_2D_n$ $(n\geq 3)$ with $\nu'=(0,\cdots,0,1)$. Then the theorem asserts the following: If u is (for the sake of simplicity) a real analytic solution of $p(D)u=0$ on $x_1>0$ and if the S. S. of the boundary values contain the direction $\sqrt{-1}\nu' dx'\infty$ somewhere, then the intersection of S. S. $b_0^+(u)\cup$ S. S. $b_1^+(u)$ and $\boldsymbol{R}^{n-2}\times\{x_n=\text{const.}\}\times\{\sqrt{-1}\nu' dx'\infty\}$ cannot be compact. That is, the spectrum $\sqrt{-1}\nu' dx'\infty$ propagates along $x_n=\text{const.}$ in some manner. This is a kind of propagation of singularity phenomena in the boundary. In the next paragraph we will strengthen this consideration up to the level of bicharacteristic for the wave equation.

2. Thus we consider the wave equation

$$p(D)u=(D_1^2+\cdots+D_{n-1}^2-D_n^2)u=0, \tag{4.7}$$

where x_n is the time variable.

THEOREM 4.2. *Let u be hyperfunction solution of* (4.7) *defined locally on $x_1>0$. Let $b_j^+(u)$, $j=0,1$ be its boundary values. Let $\nu'\in\boldsymbol{R}^{n-1}$ be a direction satisfying $p_2(0,\nu')=0$, where p_2 is the principal part of* (4.7). *Assume that*

1) S. S. u *does not contain the directions* $\sqrt{-1}(\pm(1-\theta)dx_1+\theta\nu' dx'\infty$, $0<\theta\leq 1$.
2) S. S. $b_0^+(u)\cup$ S. S. $b_1^+(u)$ *has compact cross section with*

$$b_{\nu'}=\{x'^0+t(\text{grad }p_2)(0,\nu');\ t\in\boldsymbol{R}\}\times\{\sqrt{-1}\nu' dx'\infty\}.$$

Then S. S. $b_j^+(u)$ *has no intersection with* $b_{\nu'}$.

As the situation is symmetric the same assertion holds for a solution on $x_1<0$. Note that $b_{\nu'}$ is a bicharacteristic strip of (4.7) which lies entirely in the boundary $x_1=0$.

PROOF. By a suitable linear coordinate transformation which fixes the boundary $x_1=0$, we can always assume that $\nu'=(0,\cdots,0,1)$ $x'^0=0$ and that

$$p(D)=D_1^2+\cdots+D_{n-2}^2-D_{n-1}D_n. \tag{4.8}$$

Then we will have

$$b_{\nu'}=\{(0,\cdots,0,x_{n-1},0);\ x_{n-1}\in\boldsymbol{R}\}\times\{\sqrt{-1}dx_n\infty\}. \tag{4.9}$$

As the curved wave decomposition of $\delta(x')$ we employ the one introduced in the final paragraph of §1:

$$\delta(x')=\int_{S^{n-2}}W_I(x',\omega')d\omega',$$

where now

$$x^I=(x_2,\cdots,x_{n-2}),\qquad x^{II}=(x_{n-1},x_n),$$

that is, the wave component is really curved only for the first group of variables (see (1.27) for the detailed form of the component W_I).

Now assume that the solution u is defined on $\{0<x_1<\delta\}\times U'$, where $U'=U^I\times U^{II}$

and $U^I\subset\boldsymbol{R}^{n-3}$, $U^{II}\subset\boldsymbol{R}^2$ are convex open neighborhoods of the origin in the respective Euclidean spaces such that U^{II} is of the form $\{|x_{n-1}|<A\}\times\{|x_n|<a\}$. By the hypothesis of the theorem we can assume (if we choose a sufficiently small) that S. S. $b_0^+(u)\cup$ S. S. $b_1^+(u)$ does not contain the direction $\sqrt{-1}\nu' dx'\infty$ near the part $U^I\times\partial U^{II}\cap\{|x_n|<a\}$ of $\partial U'$. Thus we can apply Theorem 1.10 and obtain a real analytic solution v of $p(D)v=0$ on $\{0<x_1<\delta'\}\times V'$ such that its boundary values agree with

$$f_j(x')*\{J(D_{\omega'})W_I(x',\omega')|_{\omega'=\nu'}\}|_{V'},\qquad j=0,1, \tag{4.10}$$

where $V'\Subset U'$ and $f_j(x')\in{}'\mathcal{B}[\overline{U'}]$ is a fixed extension of $b_j^+(u)|_{V'}\in{}'\mathcal{B}(V')$. Thus by our repeatedly employed argument to prove the theorem it suffices to show that v can be continued analytically onto a neighborhood of the origin.

Note that the hyperfunction (4.10) is represented as the boundary value of the function $F_j(z')=f_j(x')\underset{x'}{*}\{J(D_{\omega'})W_I(z',\omega')|_{\omega'=\nu'}\}|_{V'}$ which is holomorphic on a domain of the form

$$\{y_n>y_2^2+\cdots+y_{n-2}^2\}.$$

To solve the holomorphic Cauchy problem (2.3) with the data $F_j(z')$ to the direction $x_1<0$, we can therefore apply Proposition 2.16 but this time with $\zeta^{II}=(\zeta_2,\cdots,\zeta_{n-2})$, $\zeta^I=\zeta_{n-1}$. The solution $F(z)$ exists on a domain of the form

$$\{z=x+\sqrt{-1}y\in\boldsymbol{C}^n;\ -\delta<x_1\leq 0,\ |x'|<A',\ |y_1|<A',\\ \lambda|y_1|+c^2x_1^2+4(y_2^2+\cdots+y_{n-2}^2)<y_n\}.$$

By the same argument as in the proof of Theorem 4.1, $F(z)$ is the defining function of the hyperfunction $v(x)$ on $x_1>0$. Moreover by the same argument employing Corollary A.1 we see also that $F(z)$ is holomorphic on a neighborhood of

$$\{0<x_1<\delta',\ y_1=0,\ |x'|<A',\ c'y_n>y_2^2+\cdots+y_{n-2}^2\}.$$

Thus we can on the other hand apply Proposition 2.17 to our operator $p(D)$ to continue $F(z)$ in the direction of y_1-axis starting from each initial plane $x_1=x_1^0=$ const. between $0\leq x_1^0<\delta'$. As a result we obtain a function $F(z)$ holomorphic on

$$\{z=x+\sqrt{-1}y\in\boldsymbol{C}^n;\ |y_1|<A'',\ |x'|<A'',\ (x_1,y')\in\Omega\}\\ \cup\ \text{a neighborhood of the edge } \{0<x_1<\delta;\ |x'|<A''\},$$

where

$$\Omega=\{-\delta''<x_1\leq 0,\ k(x_1^2+y_2^2+\cdots+y_{n-2}^2)<y_n\}\\ \cup\ \{0<x_1<\delta'',\ k(y_2^2+\cdots+y_{n-2}^2)<y_n\}.$$

Then we can apply the local Bochner theorem first to the corner

$$\left\{0<x_1\leq\frac{\delta''}{2},\ y'=0\right\}\cup\left\{x_1=\frac{\delta''}{2},\ y''=0,\ -\varepsilon<y_n<0\right\},$$

in order to make a positive angle with the x_1-axis along the y_1-axis, and next to swallow the y_1-axis making the profit of this angle. Thus $F(z)$, hence $v(x)$ has been continued to a neighborhood of the origin. q.e.d.

REMARK 4.3. We know that there exists a real analytic solution u on $x_1>0$ of the equation $p(D)u=0$ for the operator (4.8) such that the S. S. of its boundary values just agrees with $b_{\nu'}$. (Consider e.g. the global solution of $p(D)u=0$ whose S. S. agrees with $\{0\}\times b_{\nu'}$ constructed by Kawai [17] etc.) We do not know if we have a solution for which the S. S. of the boundary values agrees with the half of $b_{\nu'}$ nor if we can strengthen Theorem 4.2 up to the propagation of S. S. along the half boundary bicharacteristic. By the way, apply to this equation $p(D)u=0$ a coordinate transformation of the type $x_n=\phi(x_{n-1})$ in the boundary. Then we obtain an operator $p(x, D)$ such that for its real analytic solution on $x_1>0$ the S. S. of the boundary values propagate along the curve $x_n=\phi(x_{n-1})$, $x_2=\cdots=x_{n-2}=0$. Choose especially this curve in such a way that it intersects the plane $x_n=0$ only at the origin where they are tangent. Then we see that Proposition 1.2 never holds for this operator $p(x, D)$ with $\nu'=(0,\cdots,0,1)$. In fact consider the solution for this operator obtained from the above solution u of the original equation by this coordinate transformation. The modified solution v which is to be given by Proposition 1.2, after the inverse coordinate transformation, would then have to be a real analytic solution of the original equation whose boundary values have S. S. concentrated at the point $(0, \sqrt{-1}\nu' dx'\infty)$, hence void by Theorem 4.2. Since $J(D_{\omega'})$ is arbitrarily, this means that S. S. $b_j^+(u)$ do not contain $(0, \sqrt{-1}\nu' dx'\infty)$; a contradiction. It will be natural that one would have to use a decomposition into a kind of bicharacteristics in order to extend Proposition 1.2 to the operator with variable coefficients.

3. Applying the results of § 3 and § 4 we can deduce some results on continuation of real analytic solutions. See Theorem 3.1 in [7]. For example, a real analytic solution u of the partial Laplace equation $(D_1^2+\cdots+D_k^2)u=0$ $(k<n)$ defined outside the set $x_1=x_n=0$ can be continued as a hyperfunction solution to this set. The proof is just the same. Of course it suffices to assume that u is a hyperfunction solution such that S. S. u does not contain the directions

$$\sqrt{-1}(\pm(1-\theta)dx_1+\theta dx_n)\infty,\qquad 0<\theta\leq 1$$

to obtain the same conclusion. We neglect to formulate the precise theorem which follows by the same line from the results of § 3 and § 4, because it is a rather simple repetition. We can also add some results on the analyticity of minimal dimensional singularity for real analytic solutions. See [9] or [11]. For example, if we have a real analytic solution $u(x)$ of the partial Laplace equation $(D_1^2+\cdots+D_k^2)u=0$ defined outside a submanifold C of class C^1 in $x_1=0$ such that u cannot be extended as a hyperfunction solution to C, then each conormal $\nu'=(\nu_2,\cdots,\nu_n)$ of C in $\{x_1=0\}$ must satisfy $\nu_2^2+\cdots+\nu_k^2\neq 0$, hence in particular the dimension of C cannot be less than $n-k$. Further, if C is of dimension just equal to $n-k$ and if in addition $u(x)$ can be extended as a distribution to C, then C must be a real analytic submanifold. It seems difficult to give a direct proof to these simple facts for the harmonic function without employing the microlocal boundary value theory. Similarly we can improve the result on the "timelike" property of the singularity. In concluding we will formulate a result for the wave equation:

COROLLARY 4.4. *Let C be a weakly timelike curve of class C^1 contained in $x_1=0$. Assume that C is strongly timelike outside a compact set. If there exists a real analytic solution u of the wave equation* (4.7) *defined outside C such that it cannot be continued to C as a hyperfunction solution, then C must be in fact strongly timelike everywhere.*

It was already known in [9] that such a curve must be weakly timelike. What is improved is that we can now remove the isolated light-like directions employing Theorem 4.2.

Appendix A

In proving two fundamental lemmas in § 1, we cited the boundary value theory of Kataoka. Since the detail of his magnificent theory is not yet published at the time of preparation of this paper, we give here for the sake of self-containedness a direct elementary proof of them for a case a little restrictive but sufficient for our purpose.

ANOTHER PROOF OF LEMMA 1.3. Recall the condition

$$(1.11)\text{bis}\quad \begin{cases}\operatorname{supp} f\subset\{x_1\geq 0\},\\ \text{S. S.}\, f\subset\{x_1=0\}\times\{\pm\sqrt{-1}dx_1\infty\} \quad (\text{resp. S. S.}\, f\subset\{x_1\geq 0\}\times\{\pm\sqrt{-1}dx_1\infty\}).\end{cases}$$

Such a hyperfunction $f(x)$ can be represented as

$$f(x)=F_+(x+\sqrt{-1}\Gamma 0)-F_-(x-\sqrt{-1}\Gamma 0),$$

where $F_\pm(z)$ are holomorphic, say, on a domain of the form

(A.1) $$\{|x_1|<\delta,\ |x'|<A\}+\sqrt{-1}(\pm\Gamma\cap\{|y|<\varepsilon\})$$

respectively (and moreover in the case outside the parentheses they extend to a neighborhood of the part $x_1>0$ of the edge). Here Γ is a convex open cone containing the positive y_1 axis and it can be chosen as close to the cone $\{y_1>0\}$ as desired if only $\varepsilon>0$ is diminished accordingly. The condition $\operatorname{supp} f\subset\{x_1\geq 0\}$ implies that in either case $F_\pm(z)$ agree with each other to define a holomorphic function $F(z)$ on a neighborhood of the part $x_1<0$ of the edge. Now we are going to solve the equation

(A.2) $$p(z, D)U(z)=F(z)$$

for $U(z)$ which will enjoy the same properties as $F(z)$ enumerated above. For this purpose we make the coordinate transformation $z_1=Z_1^2$. (This idea is similar to one used in the elementary proof of the water-melon slicing theorem given in [12].) The equation (A.2) is transformed to

(A.3) $$p\left(Z_1^2,\ z';\ \frac{1}{2Z_1}D_{Z_1}, D'\right)U(Z_1^2, z')=F(Z_1^2, z').$$

Put $G(Z_1, z')=F(Z_1^2, z')$, $V(Z_1, z')=U(Z_1^2, z')$. The new domain where we consider the equation (A.3) is the image of (A.1) by this transformation, hence somewhat a neighborhood of a set of the form

$$\{Z_1=X_1+\sqrt{-1}Y_1\in\boldsymbol{C};\ |X_1|<\delta',\ 0<Y_1<\varepsilon'\}\times\{z'\in\boldsymbol{C}^{n-1};\ |x'|<A,\ y'=0\}.$$

Owing to Kashiwara's lemma a function holomorphic on such a domain can be continued up to a wedge:

(A.4) $$\{|X_1|<\delta'',\ |x'|<A',\ k|y'|<Y_1<\varepsilon''\}.$$

Moreover, if we remove any small neighborhood of $Z_1=0$, we can choose k arbitrarily small at the sacrifice of ε''. This follows from the fact that Γ in (A.1) was arbitrary and that our transformation is regular outside $Z_1=0$. Further, in the case outside the parentheses $G(Z_1, z')$ is also holomorphic on a neighborhood of the edge except for $X_1=0$. This follows from the corresponding property of $F(z)$. Now the equation (A.3), after cancelling the denominators, has the following form

(A.5) $$\{D_{Z_1}^m+Z_1 q(Z_1, z';\ D_{Z_1}, D')\}V(Z_1, z')=G(Z_1, z'),$$

where m is the order of p and q is of order less than $m-1$ in D_{Z_1}. (For the sake of simplicity we have re-employed the symbol G.) We try to solve the equation (A.5) for $V(Z_1, z')$ on the domain (A.4) giving the Cauchy data, say 0, on $Z_1=\sqrt{-1}s$ for a small $s>0$. Owing to the factor Z_1, our equation satisfies the condition of Theorems 4.1, 4.2 of Bony-Schapira [2] for any $k>0$ in (A.4) if only we choose δ'' and ε'' sufficiently small. Thus we can find the solution $V(Z_1, z')$ of (A.5) holomorphic on (A.4), which, in the case outside the parentheses, extends by the same Theorems also on a neighborhood of the edge except for $Z_1=0$.

Applying the inverse transformation $Z_1=\sqrt{z_1}$, we see easily that the function $U(z_1, z')=V(\sqrt{z_1}, z')$ is the desired holomorphic solution of (A.2), that is, that $U(z)$ becomes holomorphic on a domain of the form (A.1) for any Γ and that it defines a hyperfunction $u(x)$ satisfying (1.11) and the equation $p(x, D)u=f$. q.e.d.

Because of the elementary character of the above proof we can obtain the following a little more detailed variant which we needed in § 4:

COROLLARY A.1. *Let* (y^I, y^{II}) *be a grouping of the variables* $y_2, \cdots, y_{n-1}$. *Consider the case where the coefficients of* $p(x, D)$ *depend only on* x_1. *In the hypothesis of Lemma* 1.3 *assume further that the hyperfunction* $f(x)$ *also admits a representation of the form*

$$f(x)=F^1_+(x+\sqrt{-1}\Gamma^1_+0)-F^1_-(x+\sqrt{-1}\Gamma^1_-0),$$

where $\Gamma^1_\pm=\{\pm y_1>0, y_n>0\}$ *and* $F^1_\pm(z)$ *are holomorphic resp. on*

$$\{|x_1|<\delta,\ |x'|<A,\ 0<\pm y_1<\delta,\ (y^{II})^2<cy_n\}, \tag{A.6}$$

and also on a neighborhood of

$$\{|x_1|<\delta,\ |x'|<A,\ y^{II}=y_n=0,\ 0<\pm y_1<\delta\}.$$

Then the solution $u(x)$ *of* $p(x, D)u=f$ *given by Lemma* 1.3 *admits a representation of the form*

$$u(x)=U_+(x+\sqrt{-1}\Gamma 0)-U_-(x-\sqrt{-1}\Gamma 0)$$

such that $U_\pm(z)$ *extends also on a domain of the form* (A.6) *with smaller constants* δ, A, c.

Of course we include the case $y^{II}=\emptyset$, where the last inequality in (A.6) implies simply $y_n>0$.

PROOF. What Corollary A.1 asserts is the global character of the domain of existence in the parameters z'. Remark that $F^1_\pm(z)$ is nothing but the continuation of the function $F_{\pm}(z)$ holomorphic resp. on (A.1). After the transformation $z_1=Z_1^2$ used in the above proof they become therefore holomorphic on the union of

$$\{|X_1|<\delta',\ |x'|<A',\ 0<Y_1<\delta',\ cy_n>(y^{II})^2\}$$

and of

$$\{|X_1|<\delta',\ |x'|<A',\ k(|y^{II}|+|y_n|)<Y_1\}.$$

By the hypothesis that the coefficients of $p(z,D)$ are independent of z', the existence theorem of Bony-Schapira can be applied uniformly with respect to z' along the initial plane $Z_1=\sqrt{-1}s$. Thus we can assure a same type domain for the solution $V(Z_1,z')$, hence for its inverse transform $U(z)$. q.e.d.

PROOF OF LEMMA 1.7 in the case of constant coefficients. Assume that u is defined on $\{0<x_1<\delta\}\times U'$ and that

$$\text{S. S. } b_j^+(u)\cap U'\times\{\sqrt{-1}\omega' dx'\infty;\ \omega'\in\overline{\Omega'}\}=\emptyset, \tag{A.7}$$

where U' is a neighborhood of $0\in\boldsymbol{R}^{n-1}$ and Ω' is a neighborhood of $\nu'\in\boldsymbol{S}^{n-2}$. Choose a non-characteristic real analytic hypersurface $x_1=t(x')$ as in the proof of Proposition 1.2. By Sato's fundamental theorem the product $Y(x_1-t(x'))[u]$ still has a meaning on $\{x_1<\delta_1\}\backslash(\{0\}\times\partial W')$ even if u is a mear hyperfunction solution. Let $[[Y(x_1-t(x'))[u]]]$ be an extension with the smallest support. By the same reasoning as there, modifying this extension if necessary we can obtain the equality

$$p(D)[[Y(x_1-t(x'))[u]]]=\sum_{j=0}^{m-1}f_j(x')\delta^{(m-1-j)}(x_1)+\sum_{j=0}^{m-1}\alpha_j(x')\delta^{(m-1-j)}(x_1-t(x')), \tag{1.5)bis}$$

where $f_j(x')\in{}'\mathcal{B}[\overline{W'}]$ is an extension of $b_j^+(u)|_{W'}$ and $\alpha_j(x')\in{}'\mathcal{B}(W'_{\delta_1})$ satisfy $\operatorname{supp}\alpha_j(x')\subset W'_{\delta_1}\backslash W'$.

Now let $W(x',\omega')$ be the component of the curved wave decomposition (1.1) in dimension $n-1$. Put $W(x',\Omega')=\int_{\Omega'}W(x',\omega')d\omega'$. Then by the assumption (A.7) $g_j(x')=f_j(x')*W(x',\Omega')$ all become real analytic in W'. Take the convolution of $W(x',\Omega')$ with both sides of (1.5)bis. We obtain

$$p(D)v=\sum_{j=0}^{m-1}g_j(x')\delta^{(m-1-j)}(x_1)+\beta(x), \tag{A.8}$$

where v,β denote the hyperfunctions defined by the obvious manner (cf. (1.6)). We claim that

$$\text{(A.9)} \qquad \begin{cases} \operatorname{supp} \beta(x) \subset \{x_1 \geq 0\}, \\ \text{S. S. } \beta(x) \subset \{x_1 \geq 0\} \times \{\pm\sqrt{-1}dx_1\infty\}. \end{cases}$$

In fact this can be verified by the same calculation as (1.9) on $x_1=0$. Now let w be the solution of

$$p(D)w = \text{the right hand side of (A.8)}$$

given by Lemma 1.3, hence satisfying (A.9) also. Then we have

$$p(D)(v-w)=0, \qquad \operatorname{supp}(v-w) \subset \{x_1 \geq 0\}.$$

Thus by the Holmgren uniqueness theorem $v \equiv w$ in $\{x_1 < \delta'\} \times V'$, where $V' \Subset W'$ and $\delta' < \delta_1$ is chosen suitably small. This means that

$$\text{(A.10)} \qquad \text{S. S. } v \subset \{x_1 \geq 0\} \times \{\pm\sqrt{-1}dx_1\infty\} \quad \text{on} \quad \{x_1 < \delta'\} \times V'.$$

Recall that

$$[[Y(x_1-t(x'))[u]]] = v + [[Y(x_1-t(x'))[u]]] \underset{x'}{*} W(x', \boldsymbol{S}^{n-2} \backslash \Omega'),$$

where

$$W(x', \boldsymbol{S}^{n-2} \backslash \Omega') = \int_{S^{n-2} \backslash \Omega'} W(x', \omega') d\omega' = \delta(x') - W(x', \Omega').$$

Therefore we have

$$\text{(A.11)} \qquad \text{S. S. } [[Y(x_1-t(x'))[u]]] \underset{x'}{*} W(x', \boldsymbol{S}^{n-2} \backslash \Omega')$$
$$\subset \{x_1 \geq 0\} \times \{\sqrt{-1}(\pm(1-\theta)dx_1 + \theta\omega' dx')\infty;\ \omega' \in \boldsymbol{S}^{n-2} \backslash \Omega',\ 0 \leq \theta \leq 1\},$$

and we conclude that on the open set $\{x_1 < \delta'\} \times V'$, the S. S. of $[u] = [[Y(x_1-t(x'))[u]]]$ is contained in the union of (A.10) and (A.11). q.e.d

It is clear that the above proof goes also in the case where the operator contains the variable x_1 in its coefficients.

Appendix B

As a biproduct of an argument frequently used in this paper, we give here a detailed proof of the Green formula which was (or should have been) employed in §2 of [8][1].

[1] The original proof of [8] contained a mistake by using a device available only in the case of constant coefficients. The indication of Errata given in [9] was not suitable since it was available only when ${}^t p(x, D)$ is I^a-semihyperbolic to $x_1 > -\varepsilon$ for small $\varepsilon \geq 0$. The correct indication as it should have been was as follows: We must replace $w_k(x_1+\varepsilon, x', y', \omega')$ by $w_k^\varepsilon(x, y', \omega')$, the solution of

$$(1.8)' \qquad \begin{cases} {}^t p(x, D) w_k^\varepsilon = 0 \\ C_j(x, D) w_k^\varepsilon |_{x_1 \to +0} = \delta_{j, m-k-1} W(y'-x'+i\varepsilon\omega', \omega'), \qquad j=0, \cdots, m-1. \end{cases}$$

Recall that we are always assuming that $x_1=0$ is a non-characteristic boundary of the operator $p(x, D)$ of order m. To maintain the generality in accordance with the notation in [8], we consider here a general normal boundary system $\{B_j(x, D)\}_{j=0}^{m-1}$. Let $\{C_j(x, D)\}_{j=0}^{m-1}$ be its dual system. It is characterized by the classical Green formula: For a real analytic function $u(x)$ defined on a neighborhood of the origin we have

$$\text{(B.1)}\qquad \begin{aligned} &p(x, D)(u(x)Y(x_1)) \\ &\quad =(p(x, D)u(x))Y(x_1)+\sum_{j=0}^{m-1}{}^tC_{m-j-1}(x, D)(B_j(x, D)u|_{x_1=0}\delta(x_1)). \end{aligned}$$

This formula is also valid for a hyperfunction u which contains x_1 as real analytic parameter.

THEOREM B.1. *Let $u(x)$ be a local hyperfunction solution of $p(x, D)u=0$ defined on $x_1>0$ on a neighborhood of $\{0\}\times\{|x'|\le a\}$, and let $b_j^+(u)=B_j(x, D)u|_{x_1\to+0}$, $j=0, \cdots, m-1$ be the boundary values of $u(x)$. Let, on the other hand, $T\subset \boldsymbol{R}^\nu$ be an open set of the parameter space and let $w(x, t)$ be a hyperfunction solution of the adjoint equation ${}^tp(x, D)w(x, t)=0$ containing real analytic parameters $t\in T$ defined also on $x_1>0$ on a neighborhood of 0. Let $g_j(x', t)=C_j(x, D)w|_{x_1\to+0}$, $j=0, \cdots, m-1$ be the boundary values of $w(x, t)$. Assume that there exists an open convex cone Δ in $\boldsymbol{R}^{n-1+\nu}$ such that $\Delta\cap(\boldsymbol{R}^{n-1}\times\{0\})\neq\emptyset$ and that*

$$\text{(B.2)}\qquad \text{S. S. } g_j(x', t)\subset\{|x'|<a\}\times T\times\sqrt{-1}\Delta^\circ(dx', dt)\infty,$$

$$\text{(B.3)}\qquad (\text{S. S. } b_j^+(u))^a\cap\{|x'|<a\}\times\sqrt{-1}(\mathrm{pr}_{x'}\,\Delta^\circ)dx'\infty=\emptyset$$

for $j=0, \cdots, m-1$, where $\mathrm{pr}_{x'}$: $\boldsymbol{R}^{n+\nu}\to\boldsymbol{R}^n$ denotes the natural projection. Then, with possible modification on $a-\varepsilon<|x'|<a$, we can choose suitable cutting off $[[u]]$ of $[u]$ and $f_j(x')$ of $b_j^+(u)$ of support in $|x'|\le a$ such that for sufficiently small $\delta>0$ we have the following Green formula:

$$\text{(B.4)}\qquad \begin{aligned} &\sum_{j=0}^{m-1}\int_{\boldsymbol{R}^{n-1}}f_j(x')g_{m-j-1}(x', t)dx' \\ &\qquad -\sum_{j=0}^{m-1}\int_{\boldsymbol{R}^{n-1}}B_j(x, D)[[u]]|_{x_1=\delta}C_{m-j-1}(x, D)w(x, t)|_{x_1=\delta}dx' \\ &\quad =\int_{\boldsymbol{R}^n}v(x)w(x, t)dx. \end{aligned}$$

Here $v(x)$ is a hyperfunction with support in $\{0\le x_1\le\delta\}\times\{a-\varepsilon<|x'|<a\}$. The cutting off $[[u]]$ is so chosen that the product or the restriction appearing in the

formula have sense.

PROOF. Let $[[u]]$ be a suitable cutting off (with modification) of $[u]$ such that $[[u]]$ remains micro-analytic to the direction $\pm\sqrt{-1}dx_1\infty$ on $x_1>0$. Such a modification can be constructed as follows: First by the flabbiness of the sheaf $\mathcal{C}$ of microfunctions, we take a hyperfunction $u_1 \in \mathcal{B}(\boldsymbol{R}^n)$ such that

$$\mathrm{S.\,S.}\, u_1 \subset \overline{\mathrm{S.\,S.}\, ([u]|_{\{|x'|<a-\varepsilon/2\}})}$$

and that $u_2=[u]-u_1$ is real analytic in $\{|x'|<a-\varepsilon/2\}$. Then we put

$$[[u]]=\chi_{\{|x'|\leq a\}}(x')u_1+\chi_{\{|x'|\leq a-\varepsilon\}}(x')u_2.$$

Note that $\mathrm{supp}\,[[u]]\subset\{x_1\geq 0\}\times\{|x'|\leq a\}$ and that $[[u]]=[u]$ in $\{|x'|<a-\varepsilon\}$. If u is real analytic on $x_1>0$, we can of course simply take the product $\chi_{\{|x'|\leq a\}}(x')[u]$ outside $\{x_1=0,\ |x'|=a\}$ and then extend it to an element $[[u]]$ with the desired support. In this case we can take $\varepsilon=0$.

Now by the definition of the canonical extension and by the Leibniz rule we have

$$p(x, D)[[u]]=v+\sum_{j=0}^{m-1} {}^tC_{m-j-1}(x, D)(f_j(x')\delta(x_1)). \tag{B.5}$$

Here $f_j(x')$ is a suitable cutting off of $b_j^+(u)$ in $\{|x'|\leq a\}$ and v is a hyperfunction appearing by the effect of modification, hence with support contained in $\{x_1\geq 0,\ a-\varepsilon\leq|x'|\leq a\}$. Note that v is also micro-analytic to the direction $\pm\sqrt{-1}dx_1\infty$ on $x_1>0$.

Recall here that for the solution w whose boundary values satisfy (B.2) we can apply Lemma 1.7 and then Lemma 3.4 to conclude that $[w]$ can be expressed as the boundary value of the hyperfunction with holomorphic parameters $Y(x_1)F(x_1, z', \tau)$ as $(z', \tau)\mapsto(x', t)+\sqrt{-1}\Delta 0$. Here $F(z, \tau)$ is a holomorphic solution of ${}^tp(z, D)F(z, \tau)=0$ defined on a domain of the form (3.2). (We are considering here that the parameters τ are contained implicitly in the operator ${}^tp(x, D)$.) Moreover we know that the holomorphic function $g_j(z', \tau)=C_j(z, D)F(z, \tau)|_{z_1=0}$ gives the hyperfunction $g_j(x')$ as the boundary value.[2] Now choose a unit vector $\omega\in\Delta$. Let $\gamma>0$ be a small parameter and consider the following holomorphic Cauchy problem:

$$\begin{cases} {}^tp(z, D)F^\gamma(z, \tau)=0, \\ C_j(z, D)F^\gamma(z, \tau)|_{z_1=0}=g_j((z', \tau)+\sqrt{-1}\gamma\omega), \qquad j=0, \cdots, m-1. \end{cases}$$

[2] The situation in [8] was such that the solution w was given in this form from the beginning, hence Lemma 3.4 was unnecessary.

The solution $F^{\gamma}(z,\tau)$ exists on a conical neighborhood of the initial domain $\{z_1=0\}\times(\{|x'|\leq a\}\times T+\sqrt{-1}(\Delta 0-\gamma\omega))$. (Here $\Delta 0\subset \boldsymbol{R}^{n-1+\nu}$ denotes a fixed domain infinitesimally equal to the cone Δ at 0.) Hence $F^{\gamma}(z,\tau)$ is holomorphic on $\{0\leq x_1\leq\delta\}\times\{|x'|\leq a\}\times T$ if γ is large enough compared with δ. Unfortunately we do not know if $F^{\gamma}(z,\tau)$ continues to exist on a wedge-like domain touching the real axis when $\gamma\downarrow 0$. (This is true for the situation of [8] because ${}^{t}p(x,D)$ is I^{a}-semihyperbolic there, which simplifies the argument for the general case developed in the sequel.) We know however that $F^{\gamma}(z,\tau)$ continues to exist on a real neighborhood of $\{0\leq x_1\leq\delta\}\times\{|x'|=a\}\times T$ if δ is sufficiently small, since by the condition (B.2) the boundary data $g_j(x',t)$ are real analytic on a neighborhood of $\{|x'|=a\}\times T$.

Now multiply $Y(\delta-x_1)F^{\gamma}(x,t)$ to both sides of (B.5) and integrate with respect to x. Then we obtain

$$\begin{aligned}\text{(B.6)}\qquad &\int_{\boldsymbol{R}^n} Y(\delta-x_1)F^{\gamma}(x,t)p(x,D)[[u]]dx\\ &=\int_{\boldsymbol{R}^n} F^{\gamma}(x,t)Y(\delta-x_1)v(x)dx\\ &\quad+\sum_{j=0}^{m-1}\int_{\boldsymbol{R}^{n-1}} f_j(x')\{C_{m-j-1}(x,D)F^{\gamma}(x,t)\}|_{x_1=0}dx'.\end{aligned}$$

Here we have employed the integration by parts to deform the last term. Until now the calculus is obviously legitimate because of the condition of S. S. Note that we have the following formula which is adjoint to (B.1):

$${}^{t}p(x,D)(uY(x_1))=({}^{t}p(x,D)u)Y(x_1)-\sum_{j=0}^{m-1}{}^{t}B_{m-j-1}(x,D)(C_j(x,D)u|_{x_1=0}\delta(x_1)).$$

Thus applying the integration by parts to the left hand side of (B.6) we obtain

$$\begin{aligned}&\sum_{j=0}^{m-1}\int_{\boldsymbol{R}^{n-1}} C_{m-j-1}(x,D)F^{\gamma}(x,t)|_{x_1=\delta}B_j(x,D)[[u]]|_{x_1=\delta}dx'\\ &\quad+\int_{\boldsymbol{R}^n}{}^{t}p(x,D)F^{\gamma}(x,t)\cdot Y(\delta-x_1)[[u]]dx.\end{aligned}$$

Now rewrite v instead of $-Y(\delta-x_1)v(x)$ in the first term of the right hand side of (B.6). Then we obtain

$$\begin{aligned}\text{(B.7)}\qquad &\sum_{j=0}^{m-1}\int_{\boldsymbol{R}^{n-1}} f_j(x')\{C_{m-j-1}(x,D)F^{\gamma}(x,t)\}|_{x_1=0}dx'\\ &\quad-\sum_{j=0}^{m-1}\int_{\boldsymbol{R}^{n-1}} B_j(x,D)[[u]]|_{x_1=\delta}\{C_{m-j-1}(x,D)F^{\gamma}(x,t)\}|_{x_1=\delta}dx'\end{aligned}$$

$$=\int_{R^n} F^\gamma(x,t)v(x)dx.$$

Here we wish to let $\gamma\downarrow 0$. By what is said above $F^\gamma(z,\tau)$ converges to $F(z,\tau)$ when $\gamma\downarrow 0$ uniformly on a complex neighborhood of the real set $\{0\leq x_1\leq\delta\}\times\{a-\varepsilon\leq|x'|\leq a\}\times T$ if ε and δ are sufficiently small. Therefore owing to the topological duality between $\mathcal{A}(K)$ and $\mathcal{B}[K]$, where $K=\{0\leq x_1\leq\delta\}\times\{a-\varepsilon\leq|x'|\leq a\}$, the right hand side of (B.7) converges to that of (B.4) if we let $\gamma\downarrow 0$.

Next consider the first term of the left hand side of (B.7). It reads

$$\sum_{j=0}^{m-1}\int_{R^{n-1}} f_j(x')g_{m-j-1}((x',t)+\sqrt{-1}\gamma\omega)dx'.$$

Thus formally there is no difficulty of domain of existence when we let $\gamma\downarrow 0$. Since however the factors are not clearly distinguished as $\mathcal{A}(K)$ and $\mathcal{B}[K]$ as above, we cannot escape to the easy duality argument. This time we employ the most primitive argument based on the defining functions. On account of the condition (B.3) we can decompose $f_j(x')$ into the form

$$f_j(x')=f_{j0}(x')+\sum F_{jk}(x'+\sqrt{-1}\Gamma_k'0),$$

where $(-\Gamma_k'^{\circ})\cap \mathrm{pr}_{x'}(\Delta^\circ)=0$, $f_{j0}(x')$ is real analytic outside $\{a-\varepsilon\leq|x'|\leq a\}$, and $F_{jk}(z')$ are continued further holomorphically to the real axis on $|x'|>a$. The integral concerning $f_{j0}(x')$ can be decomposed as follows:

$$\begin{aligned}&\langle\chi_{\{|x'|\leq a+\varepsilon_2\}}(x')f_{j0}(x'),\ g_{m-j-1}((x',t)+\sqrt{-1}\gamma\omega)\rangle_{x'}\\ &\quad=\langle\chi_{\{a-\varepsilon_1\leq|x'|\leq a+\varepsilon_2\}}(x')f_{j0}(x'),\ g_{m-j-1}((x',t)+\sqrt{-1}\gamma\omega)\rangle_{x'}\\ &\qquad+\langle f_{j0}(x'),\ \chi_{\{|x'|\leq a-\varepsilon_1\}}(x')g_{m-j-1}((x',t)+\sqrt{-1}\gamma\omega)\rangle_{x'},\end{aligned}$$

where $\varepsilon_1>\varepsilon$ and $\varepsilon_2>0$ are so chosen that both f_{j0} and g_{m-j-1} are real analytic on $|x'|=a-\varepsilon_1$ and on $|x'|=a+\varepsilon_2$. The first term is similar to the one already treated, and we can see by the duality that it converges to

$$\chi_{\{a-\varepsilon_1\leq|x'|\leq a+\varepsilon_2\}}(x')f_{j0}(x'),\ g_{m-j-1}(x',t)\rangle_{x'}$$

when $\gamma\downarrow 0$. The second term can be rewritten as the complex integral for the defining functions:

$$\int_D f_{j0}(z')g_{m-j-1}((z',t)+\sqrt{-1}\gamma\omega)dz',$$

where the path of integral D is such that it deforms the real domain $\{|x'|\leq a-\varepsilon_1\}$

into the complex where both f_{j0} and g_{m-j-1} are holomorphic, fixing the boundary $\{|x'|=a-\varepsilon_1\}$. Then it is easy to see that it converges to the integral

$$\langle f_{j0}(x'), \chi_{\{|x'|\leq a-\varepsilon_1\}}(x')g_{m-j-1}(x', t)\rangle_{x'}$$

when $\gamma\downarrow 0$. The same argument is valid for the integral

$$\begin{aligned}&\int_{|x'|\leq a+\varepsilon_2} F_{jk}(x'+\sqrt{-1}\Gamma_k'0)g_{m-j-1}((x', t)+\sqrt{-1}\gamma\omega)dx'\\ &\quad=\int_D F_{jk}(z')g_{m-j-1}((z', t)+\sqrt{-1}\gamma\omega)dz',\end{aligned}$$

where the path of integral D is now such that it deforms the real domain $\{|x'|\leq a+\varepsilon_2\}$ into the complex, fixing the boundary $\{|x'|=a+\varepsilon_2\}$, in such a way that

$$x'\longmapsto x'+\sqrt{-1}\eta' \qquad \text{for} \qquad |x'|\leq a-\varepsilon,$$

with a fixed vector $\eta'\in\Gamma_k'\cap(\varDelta\cap\{\mathrm{Im}\,\tau=0\})$. The existence of such a vector η' is assured by the condition $(-\Gamma_k'^{\circ})\cap\mathrm{pr}_{x'}(\varDelta^{\circ})=0$. For

$$\begin{aligned}(-\Gamma_k'^{\circ})\cap\mathrm{pr}_{x'}(\varDelta^{\circ})=0 &\Longleftrightarrow (-\Gamma_k')+(\varDelta\cap\{\mathrm{Im}\,\tau=0\})=\boldsymbol{R}^n\\ &\Longleftrightarrow \Gamma_k'\cap(\varDelta\cap\{\mathrm{Im}\,\tau=0\})\neq\varnothing.\end{aligned}$$

Note that the path can be rather arbitrary on $a-\varepsilon\leq|x'|\leq a+\varepsilon_2$, because the integrand is analytic there. Thus the limit process $\gamma\downarrow 0$ can be executed as a simple substitution $\gamma=0$ in the holomorphic functions of the integrand. The result gives clearly a corresponding term in (the same type decomposition of) (B.4).

Finally we consider the second term of the left hand side of (B.7). This term is more delicate to treat since the second factor may loose the sense if we let γ small, because of the domain of definition of $F^{\gamma}(z, \tau)$. We proceed as follows: At first we consider γ sufficiently large so that this term has sense. Then choosing a vector η' as above we deform the integral into the complex domain on $|x'|\leq a-\varepsilon$. If η' is sufficiently large compared with δ (so that the points $x_1=\delta$, $z'=x'+\sqrt{-1}\eta'$ for $|x'|\leq a-\varepsilon$ are contained in the domain assured by the precise form of the Cauchy-Kowalevsky existence theorem), then the deformed integral will always have sense for any small $\gamma>0$. Thus we can let $\gamma\downarrow 0$. The limit integral, however, can be "re-deformed to the real axis" because the limit function $C_{m-j-1}(z, D)F(z, \tau)|_{z_1=\delta}$ is certainly holomorphic in a wedge domain so that it can define a hyperfunction. Thus in the limit we have obtained all the terms of (B.4).

q.e.d.

Finally we add the remark that K. Kataoka has extended the notion of this new product appearing formally in the proof of the Green formula to an elegant theory of mild hyperfunctions on a manifold with boundary. His article will appear in the same journal soon after [15] under the title "Micro local theory of boundary value problems I".

Notes added in proof 1. (Concerning Propositions 1.6–1.8) The convexity of U'' is not at all necessary. In fact, what we have used in the proof is the fact that for x' in $W'\cap\{|x_n|\leq a'\}$ the hyperplane $(x'-y')\nu'=x_n-y_n=0$ is never tangent to the hypersurface $t(y')=x_1$. By the same reason, the convexity of U^I, U^{II} in Theorem 1.10 is also unnecessary.

2. (Concerning Remark 4.3) We cannot strengthen Theorem 4.2 to the propagation along the half bicharacteristic as in the case of interior propagation phenomena. In fact, let $E(x)$ be the fundamental solution of the wave equation (4.7) with support in $x_1^2+\cdots+x_{n-1}^2\leq x_n^2$. Let $\nu'\in \boldsymbol{R}^{n-1}$ be a direction satisfying $p_2(0,\nu')=0$ and $\nu_n>0$. Then $E(x)$ is a solution of (4.7) in $x_1>0$ which is microanalytic to the directions $\sqrt{-1}(\pm(1-\theta)dx_1+\theta\nu'dx')\infty$, $0<\theta\leq 1$. But the boundary values have the singular spectrum of the direction $\sqrt{-1}\nu'dx'\infty$ just along the half bicharacteristic: $b_{\nu'}^{+}=\{t(\mathrm{grad}\, p_2)(0,\nu');\ t\geq 0\}\times\{\sqrt{-1}\nu'dx'\infty\}$. Applying Proposition 1.2–Remark 1.4 to this solution, we can obtain, if we wish it, an example in real analytic solution.

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