Asymptotic behaviour of singular solutions of linear partial differential equations in the complex domain

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Let $L(z, \partial)$ be a linear differential operator of order m and $K=\{z\in C^{n+1}; \varphi(z)=0\}$ be a characteristic surface of $L(z, \partial)$. If K is simply characteristic, then we can construct singular homogeneous solutions of $L(z, \partial)$ in the form

$$u(z)=\sum_{j\geq j_0} a_j(z)\Phi_j(\varphi(z)), \tag{0.1}$$

where $a_j(z)$ $(j\geq j_0)$ are holomorphic and $\{\Phi_j(t)\}$ is a sequence of functions of one variable, singular at $t=0$, and satisfying

$$\frac{d}{dt}\Phi_j(t)=\Phi_{j-1}(t). \tag{0.2}$$

The method has been generalized by Hamada [3], Hamada, Leray and Wagschal [9] and Persson [7] to the case where K is a characteristic surface with constant multiplicity, so that we can in general construct singular homogeneous solutions in the form:

$$v(z)=\sum_{j=-\infty}^{\infty} a_j(z)\Phi_j(\varphi(z)). \tag{0.3}$$

This method of construction of $u(z)$ and $v(z)$ is traced back to Hadamard [10], Lax [11], Ludwig [12] and others. It is useful in order to construct various solutions. Hamada [2], [3], Wagschal [8], Hamada, Leray and Wagschal [9], De Paris [1], Persson [7] and Komatsu [5] used it to solve singular Cauchy problems. Mizohata [13], Persson [14] and Komatsu [5] used it to construct null solutions in the real domain. And many mathematicians have employed this idea to construct fundamental solutions.

If we set

$$\begin{cases} \Phi_j(t)=\dfrac{t^j}{\Gamma(j+1)}(\log t-\gamma(j+1)), & j\geq 0, \\ \Phi_j(t)=-\Gamma(-j)(-t)^j, & j<0, \end{cases} \tag{0.4}$$

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where $\gamma(s)$ is digamma function and $\Gamma(s)$ is gamma function, then $u(z)$ has a pole and a logarithmic singularity on K and $v(z)$ has an essential singularity on K. Therefore, the behaviour of $u(z)$ near K is simple, but that of $v(z)$ is much complicated.

In this paper we shall investigate the behaviour of homogeneous solutions of $L(z, \partial)$, which are singular along a characteristic surface K of constant double multiplicity under the condition of non-vanishing subprincipal symbol and show the *Stokes phenomenon* of solutions. And we shall show that solutions constructed in the following sections are unique modulo holomorphic functions under some conditions.

Let us give a simple example:

$$\text{(0.5)} \qquad \begin{cases} \left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial}{\partial y}\right)u(x, y)=0, \\ u(0, y)=\dfrac{1}{y}, \\ \dfrac{\partial u(0, y)}{\partial x}=0. \end{cases}$$

According to (0.3) the solution $u(x, y)$ is expressed in the form

$$\text{(0.6)} \qquad u(x, y)=\frac{1}{y}\sum_{n=0}^{\infty}\frac{n!}{(2n)!}\left(\frac{x^2}{y}\right)^n.$$

However, this does not give sufficient information near the characteristic surface $\{y=0\}$. Our method shows that there exist two functions $u_1(x, y)$ and $u_2(x, y)$ such that

$$\text{(0.7)} \qquad u(x, y)=u_1(x, y)+u_2(x, y)$$

and for every $\varepsilon>0$

$$\text{(0.8)} \qquad u_1(x, y)=\frac{\sqrt{\pi}\,x}{2y\sqrt{y}}\exp\left(\frac{x^2}{4y}\right)\left(1+O\left(\left(\frac{\sqrt{y}}{x}\right)^4\exp\left(-\frac{x^2}{4y}\right)\right)\right),$$

$$\left(\left|\frac{x}{\sqrt{y}}\right|\to\infty,\ \left|\arg\frac{x}{\sqrt{y}}\right|\le\frac{\pi}{4}-\varepsilon\right),$$

$$\text{(0.9)} \qquad u_2(x, y)=-\frac{\sqrt{\pi}\,x}{2y\sqrt{y}}\exp\left(\frac{x^2}{4y}\right)\left(1+O\left(\left(\frac{\sqrt{y}}{x}\right)^4\exp\left(-\frac{x^2}{4y}\right)\right)\right),$$

$$\left(\left|\frac{x}{\sqrt{y}}\right|\to\infty,\ \left|\arg\frac{x}{\sqrt{y}}-\pi\right|\le\frac{\pi}{4}-\varepsilon\right),$$

$$\text{(0.10)} \qquad u_1(x, y)=\frac{1}{x^2}\left(1+O\left(\frac{y}{x^2}\right)\right) \quad \left(\left|\frac{x}{\sqrt{y}}\right|\to\infty,\ \pi\ge\left|\arg\frac{x}{\sqrt{y}}\right|\ge\frac{\pi}{4}+\varepsilon\right),$$

$$u_2(x, y)=\frac{1}{x^2}\Big(1+O\Big(\frac{y}{x^2}\Big)\Big) \quad \Big(\Big|\frac{x}{\sqrt{y}}\Big|\to\infty,\ \pi\ge\Big|\arg\frac{x}{\sqrt{y}}-\pi\Big|\ge\frac{\pi}{4}+\varepsilon\Big).$$

The case of operators with multiplicity more than 2 and applications will be treated in forthcoming papers.

In § 1 notations and summary are given. In § 2 we construct a formal solution with a parameter λ and in § 3 we give its estimates. In § 4 another solution is constructed by the method due to Hamada, Wagschal and others. In § 5 the relation of two solutions constructed in § 2 and § 4 is stated. In § 6 and § 7 we give a homogeneous solution represented by an integral and we analyze it to obtain its asymptotic behaviour near the characteristic surface. In § 8 we prove a uniqueness theorem stated in § 1.

§ 1. Notations and summary.

Let $z=(z_0, z_1, \cdots, z_n)$ denote the point in the complex $(n+1)$ dimensional space C^{n+1} and we write $z=(z_0, z')$, $z'=(z_1, z_2, \cdots, z_n)$ and $z''=(z_2, z_3, \cdots, z_n)$. Dual variable of z is denoted by $\xi=(\xi_0, \xi_1, \cdots, \xi_n)$. Let

(1.1) $$L(z, \partial)=\sum_{|\alpha|\le m} a_\alpha(z)\partial^\alpha$$

be a linear partial differential operator of order m with coefficients $a_\alpha(z)$ holomorphic in a neighbourhood Ω of $z=0$. We employ notations

(1.2) $$\partial^\alpha=(\partial_0)^{\alpha_0}(\partial_1)^{\alpha_1}\cdots(\partial_n)^{\alpha_n}=\Big(\frac{\partial}{\partial z_0}\Big)^{\alpha_0}\Big(\frac{\partial}{\partial z_1}\Big)^{\alpha_1}\cdots\Big(\frac{\partial}{\partial z_n}\Big)^{\alpha_n},$$

(1.3) $$\xi^\alpha=(\xi_0)^{\alpha_0}(\xi_1)^{\alpha_1}\cdots(\xi_n)^{\alpha_n}$$

and

(1.4) $$|\alpha|=\alpha_0+\alpha_1+\cdots+\alpha_n.$$

We set

(1.5) $$L_k(z, \xi)=\sum_{|\alpha|=k} a_\alpha(z)\xi^\alpha.$$

In particular the principal symbol of $L(z, \partial)$ is denoted by $l(z, \xi)$.

Let $K=\{\varphi(z)=0\}$ be a nonsingular characteristic surface of $L(z, \partial)$ through the origin, that is, $\mathrm{grad}_z\varphi(z)\neq0$ on K and $\varphi(0)=0$. In this paper we assume that K is a characteristic surface with constant double multiplicity. We impose

Condition A. The differential operator $L(z, \partial)$ is expressed in the form

(1.6) $$L(z, \partial)=P(z, \partial)^2Q(z, \partial)+R(z, \partial),$$

where $P(z, \partial)$ and $Q(z, \partial)$ are homogeneous differential operators of order m_1

and m_2 $(2m_1+m_2=m)$ respectively and $R(z, \partial)$ is an operator of order $(m-1)$, whose coefficients are holomorphic in Ω. Moreover

$$p\left(z, \frac{\partial\varphi}{\partial z}\right)=0 \quad \text{and} \quad \mathrm{grad}_\xi p\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0 \qquad \text{on } K \tag{1.7}$$

and

$$q\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0 \qquad \text{on } K. \tag{1.8}$$

We note that from (1.7) there is a holomorphic function $a(z)$ with $a(0)=1$ such that $p\left(z, \frac{\partial(a(z)\varphi(z))}{\partial z}\right)=0$ identically. This was remarked in Komatsu [5]. We shall give a proof of it different from in [5] in Appendix. For operators with constant multiple characteristics, we refer to Komatsu [5] and Matsuura [6]. From now on we assume that $p\left(z, \frac{\partial\varphi}{\partial z}\right)=0$ identically.

Next we impose a condition on lower order terms:

Condition B. The operator $R(z, \partial)$ in (1.6) satisfies

$$r\left(z, \frac{\partial\varphi}{\partial z}\right)=R_{m-1}\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0 \qquad \text{on } K. \tag{1.9}$$

Condition B is equivalent to

Condition B′. The subprincipal symbol of $L(z, \partial)$ does not vanish on K,

$$\sum_{i=0}^{n}\frac{1}{2}\frac{\partial^2}{\partial z_i\partial\xi_i}L_m\left(z, \frac{\partial\varphi}{\partial z}\right)-L_{m-1}\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0 \qquad \text{on } K. \tag{1.10}$$

Now let S be an n dimentional complex surface through $z=0$ to which the bicharacteristic curve issuring from $\left(0, \frac{\partial\varphi}{\partial z}(0)\right)$ is transversal. We can choose coordinates so as $S=\{z_0=0\}$, $T=S\cap K=\{z_0=z_1=0\}$, $(\varphi(0, z')=z_1)$, and

$$\frac{\partial p\left(z, \frac{\partial\varphi}{\partial z}\right)}{\partial\xi_0}\neq 0 \qquad \text{on } S \tag{1.11}$$

by (1.7).

Our purpose is to investigate solutions with singularity on K of

$$\begin{cases} L(z, \partial)u(z)=0 \\ \left(\dfrac{\partial}{\partial z_0}\right)^k u(0, z')=v_k(z') \qquad (k=0, 1), \end{cases} \tag{1.12}$$

where $v_k(z')$ is holomorphic in a neighbourhood of $z'=0$ except on T. We have

THEOREM 1. *Under above conditions, if $v_0(z')$ and $v_1(z')$ are meromorphic, then there is a solution $u(z)$ of* (1.12) *in a neighbourhood of $z=0$ with singularity only on K which behaves as follows:*

Let $\phi(z)$ be a solution of

$$\left\{\begin{aligned} & q\left(z, \frac{\partial\varphi}{\partial z}\right)\left(\sum_{i=0}^{n} \frac{\partial p\left(z, \frac{\partial\varphi}{\partial z}\right)}{\partial\xi_i} \frac{\partial}{\partial z_i}\phi(z)\right)^2 \\ & \qquad + \frac{1}{2}\sum_{i=0} \frac{\partial^2}{\partial z_i \partial\xi_i} L_m\left(z, \frac{\partial\varphi}{\partial z}\right) - L_{m-1}\left(z, \frac{\partial\varphi}{\partial z}\right) = 0\,, \\ & \phi(0, z') = 0 \end{aligned}\right. \tag{1.13}$$

and set

$$\omega(z) = \phi(z)\varphi(z)^{-1/2}\,. \tag{1.14}$$

(i) *If* $|\arg \pm\omega(z)| < \frac{\pi}{4} - \varepsilon$,

$$u(z) \sim \left(\frac{\phi(z)}{\varphi(z)}\right)^{p_\pm} \exp\left(\frac{1}{4}\frac{\phi(z)^2}{\varphi(z)}\right)\omega(z)^{-1}(\nu^\pm(z) + O(\omega(z)^{-1})) \tag{1.15}$$

as $|\omega(z)| \to \infty$ in $\alpha < \arg\varphi(z) < \beta$ for any α, β and $\varepsilon > 0$, where $\nu^\pm(z)$ are holomorphic functions and $p_\pm$ are integers or $-\infty$. If $p_+(p_-) = -\infty$, $u(z)$ is bounded as $|\omega(z)| \to \infty$ in $|\arg\omega(z)| < \frac{\pi}{4} - \varepsilon$ $\left(|\arg -\omega(z)| < \frac{\pi}{4} - \varepsilon\right)$.

(ii) *If* $\frac{\pi}{4} + \varepsilon < \arg\omega(z) < \frac{3}{4}\pi - \varepsilon$ *and* $\alpha < \arg\varphi(z) < \beta$,

$$|u(z)| \le C_{\alpha,\beta,\varepsilon}|\phi(z)|^{-p'}\,, \tag{1.16}$$

where $C_{\alpha,\beta,\varepsilon} > 0$ and $p' \ge 0$.

For another solution $\tilde{u}(z)$ of (1.12) we have

THEOREM 2. *Let $\tilde{u}(z)$ be a solution of* (1.12) *with singularity only on K. Then $(u(z) - \tilde{u}(z))$, where $u(z)$ is the one in Theorem* 1, *is holomorphic in a neighbourhood of $z=0$.*

Thus we conclude that every solution of (1.12) with singularity only on K behaves as stated in Theorem 1, which shows the *Stokes phenomenon* of solutions of (1.12). These will be stated more precisely with proofs in the following sections.

§ 2. Construction of a formal solution.

In this section we shall construct a formal solution in a sense specified later with a parameter λ of the following problem

$$(2.1)\qquad \begin{cases} L(z, \partial)U(z, \lambda)=0 \\ U(0, z', \lambda)=\exp(-\lambda^2 z_1)u_0(z') \\ \dfrac{\partial}{\partial z_0}U(0, z, \lambda)=\exp(-\lambda^2 z_1)u_1(z'), \end{cases}$$

which has the form

$$(2.2)\qquad U(z, \lambda)=\exp(-\lambda^2\varphi(z))W(z, \lambda),$$

where $u_0(z')$ and $u_1(z')$ are holomorphic in z'. Later we shall find a homogeneous solution $u(z)$ of $L(z, \partial)$ with singularity on K by integrating in λ after multiplying some function $h(\lambda)$.

To determine $W(z, \lambda)$ in (2.2), we give an elementary lemma.

LEMMA 2.1. *Let $M(z, \partial)$ be a linear partial differential operator of order m and let $H(z)$ and $u(z)$ be holomorphic functions. Then we have*

$$(2.3)\qquad M(z, \partial)\exp(\mu H(z))u(z)=\sum_{j=0}^{m}\mu^{m-j}\exp(\mu H(z))\mathcal{M}_j(z, \partial)u(z),$$

where $\mathcal{M}_j(z, \partial)$ is a linear partial differential operator of order j which is independent of $u(z)$ and

$$(2.4)\qquad \mathcal{M}_0(z, \partial)=m\left(z, \frac{\partial H}{\partial z}\right),$$

$$(2.5)\qquad \mathcal{M}_1(z, \partial)=\sum_{i=0}^{n}\frac{\partial m\left(z, \dfrac{\partial H}{\partial z}\right)}{\partial\xi_i}\frac{\partial}{\partial z_i}+a\ function.$$

This lemma follows from the Leibniz formula of differential operators. In the following calculations we shall often use Lemma 2.1.

Now, substituting (2.2) into (2.1), we have

$$(2.6)\qquad L(z, \partial)U(z, \lambda)=\exp(-\lambda^2\varphi(z))\sum_{j=0}^{m}\lambda^{2(m-j)}\mathcal{L}_j(z, \partial)W(z, \lambda).$$

Since $p\left(z, \dfrac{\partial\varphi}{\partial z}\right)=0$, we have

$$(2.7)\qquad \mathcal{L}_0(z, \partial)=(-1)^m p\left(z, \frac{\partial\varphi}{\partial z}\right)^2 q\left(z, \frac{\partial\varphi}{\partial z}\right)=0,$$

$$(2.8)\qquad \mathcal{L}_1(z, \partial)=(-1)^{m-1}r\left(z, \frac{\partial\varphi}{\partial z}\right)$$

and

(2.9) $$\mathcal{L}_2(z, \partial)=(-1)^{m-2}q\Big(z, \frac{\partial\varphi}{\partial z}\Big)\Big(\sum_{i=0}^{n}\frac{\partial p\big(z, \frac{\partial\varphi}{\partial z}\big)}{\partial\xi_i}\frac{\partial}{\partial z_i}\Big)^2$$

$$+\text{lower order terms.}$$

$\mathcal{L}_1(z, \partial)$ becomes an operator of order 0. Hence, if we set

(2.10) $$\mathcal{R}_0(z)=(-1)^{m-1}r\Big(z, \frac{\partial\varphi}{\partial z}\Big),$$

we have

(2.11) $$L(z, \partial)U(z, \lambda)=\exp(-\lambda^2\varphi(z))\{\lambda^{2(m-2)}\mathcal{L}_2+\lambda^{2(m-1)}\mathcal{R}_0+\sum_{j=3}^{m}\lambda^{2(m-j)}\mathcal{L}_j\}\,W(z, \lambda).$$

We seek for $W(z, \lambda)$ in the form

(2.12) $$W(z, \lambda)=\sum_{n=0}^{\infty}\lambda^3\int_C\exp(-\lambda\zeta)w_n(z, \zeta)d\zeta,$$

where C will be determined later so as to justify the arguments concerning integrations by parts. Set

(2.13) $$W_n(z, \lambda)=\lambda^3\int_C\exp(-\lambda\zeta)w_n(z, \zeta)d\zeta.$$

From (2.11) we have

(2.14) $$L(z, \partial)U(z, \lambda)=\exp(-\lambda^2\varphi(z))\lambda^3$$

$$\times\sum_{n=0}^{\infty}\int_C\Big\{\Big(-\frac{\partial}{\partial\zeta}\Big)^{2(m-2)}\exp(-\lambda\zeta)\mathcal{L}_2w_n+\Big(-\frac{\partial}{\partial\zeta}\Big)^{2(m-1)}\exp(-\lambda\zeta)\mathcal{R}_0w_n$$

$$+\sum_{j=3}^{m}\Big(-\frac{\partial}{\partial\zeta}\Big)^{2(m-j)}\exp(-\lambda\zeta)\mathcal{L}_jw_n\Big\}d\zeta$$

$$=\exp(-\lambda^2\varphi(z))\lambda^3\sum_{n=0}^{\infty}\int_C\exp(-\lambda\zeta)\mathcal{L}(z, \partial_z, \partial_\zeta)w_nd\zeta,$$

where

(2.15) $$\mathcal{L}(z, \partial_z, \partial_\zeta)=\Big(\frac{\partial}{\partial\zeta}\Big)^{2(m-2)}\mathcal{L}_2+\Big(\frac{\partial}{\partial\zeta}\Big)^{2(m-1)}\mathcal{R}_0+\sum_{j=3}^{m}\Big(\frac{\partial}{\partial\zeta}\Big)^{2(m-j)}\mathcal{L}_j.$$

In order to construct formal solutions $w_n(z, \zeta)$, we introduce auxilliary functions:

(2.16) $$f_j(\eta)=\frac{(-1)^{j+1}}{2\pi i}\Gamma(-j)\eta^j\qquad(j<0),$$

$$f_j(\eta)=\frac{1}{2\pi i}\frac{\eta^j}{\Gamma(j+1)}(\log\eta-\gamma(j+1)) \qquad (j\geq 0)\,, \tag{2.17}$$

where $\Gamma(s)$ is gamma function and $\gamma(s)$ is digamma function and $\gamma(j+1)=-\gamma_0+\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{j}\right)$ (γ_0; Euler's constant). We remark that

$$\frac{df_j(\eta)}{d\eta}=f_{j-1}(\eta)\,. \tag{2.18}$$

$U(z,\lambda)=\exp(-\lambda^2\varphi(z))W(z,\lambda)$ is said to be a formal solution of (2.1), if $w_n(z,\zeta)$ satisfy the following equations and initial conditions:

$$(2.19)_0 \qquad \begin{cases} \mathcal{L}_2 w_0+\left(\dfrac{\partial}{\partial\zeta}\right)^2\mathcal{R}_0 w_0=0\,, \\ w_0(0,z',\zeta)=u_0(z')f_2(\zeta)\,, \\ \dfrac{\partial w_0}{\partial z_0}=u_1(z')f_2(\zeta)+\dfrac{\partial\varphi}{\partial z_0}(0,z')u_0(z')f_0(\zeta)\,, \end{cases}$$

$$(2.19)_{n+1} \qquad \begin{cases} \left(\dfrac{\partial}{\partial\zeta}\right)^{2(m-2)}\left(\mathcal{L}_2+\mathcal{R}_0\left(\dfrac{\partial}{\partial\zeta}\right)^2\right)w_{n+1}=-\displaystyle\sum_{j=3}^{m}\left(\dfrac{\partial}{\partial\zeta}\right)^{2(m-j)}\mathcal{L}_j w_{n+3-j}\,, \\ w_{n+1}(0,z',\zeta)=\dfrac{\partial w_{n+1}(0,z',\zeta)}{\partial z_0}=0 \qquad \text{for } n\geq 0\,. \end{cases}$$

For initial conditions, we note that

$$\begin{aligned} U(0,z',\lambda)&=\exp(-\lambda^2 z_1)\lambda^3\sum_{n=0}^{\infty}\int_C \exp(-\lambda\zeta)w_n(0,z',\zeta)d\zeta \\ &=\exp(-\lambda^2 z_1)\lambda^3\int_C \exp(-\lambda\zeta)u_0(z')f_2(\zeta)d\zeta \\ &=\exp(-\lambda^2 z_1)\lambda^3\left(\int_0^\infty \exp(-\lambda\zeta)\frac{\zeta^2}{2}d\zeta\right)u_0(z') \\ &=\exp(-\lambda^2 z_1)u_0(z') \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \frac{\partial U(0,z',\lambda)}{\partial z_0}&=\exp(-\lambda^2 z_1)\lambda^3\sum_{n=0}^{\infty}\left(\int_C(-\lambda^2)\exp(-\lambda\zeta)\frac{\partial\varphi(0,z')}{\partial z_0}w_n(0,z',\zeta)d\zeta\right. \\ &\qquad\left.+\int_C\exp(-\lambda\zeta)\frac{\partial w_n(0,z',\zeta)}{\partial z_0}d\zeta\right) \\ &=\exp(-\lambda^2 z_1)\lambda^3\int_C\exp(-\lambda\zeta)\left\{\frac{\partial w_0(0,z',\zeta)}{\partial z_0}-\frac{\partial\varphi(0,z')}{\partial z_0}\frac{\partial^2 w_0(0,z,\zeta)}{\partial\zeta^2}\right\}d\zeta \end{aligned} \tag{2.21}$$

$$=\exp(-\lambda^2 z_1)\lambda^3\int_0^\infty \exp(-\lambda\zeta)u_1(z')\frac{\zeta^2}{2}d\zeta$$

$$=\exp(-\lambda^2 z_1)u_1(z),$$

if the curve C can be deformed to the one enclosing the positive real axis.

We want to find $w_n(z, \zeta)$ of the form

$$w_n(z, \zeta)=\sum_{-\infty<j\leq 2n+2} w_{n,j}(z)f_j(\zeta). \tag{2.22}$$

Substituting $w_0(z, \zeta)$ into $(2.19)_0$ and $w_{n+1}(z, \zeta)$ into $(2.19)_{n+1}$, we have

$$\sum_{-\infty<j\leq 2}(\mathcal{L}_2 w_{0,j}+\mathcal{R}_0 w_{0,j+2})f_j(\zeta)=0 \tag{2.23}$$

and

$$\sum_{-\infty<j\leq 2n+2}\{\mathcal{L}_2 w_{n+1,j}+\mathcal{R}_0 w_{n+1,j+2}+\sum_{s=3}^{m}\mathcal{L}_s w_{n+3-s,j-2(s-2)}\}f_{j-2(m-2)}(\zeta)=0. \tag{2.24}$$

Hence $w_0(z, \zeta)=\sum_{-\infty<j\leq 2n+2} w_{0,j}(z)f_j(\zeta)$ is a formal solution of $(2.19)_0$, if $w_{0,j}(z)$ satisfy

$$(2.25)_2 \quad \begin{cases} \mathcal{L}_2 w_{0,2}=0, \\ w_{0,2}(0, z')=u_0(z'), \quad \dfrac{\partial w_{0,2}(0, z')}{\partial z_0}=u_1(z'), \end{cases}$$

$$(2.25)_1 \quad \begin{cases} \mathcal{L}_2 w_{0,1}=0, \\ w_{0,1}(0, z')=\dfrac{\partial w_{0,1}(0, z')}{\partial z_0}=0, \end{cases}$$

$$(2.25)_0 \quad \begin{cases} \mathcal{L}_2 w_{0,0}+\mathcal{R}_0 w_{0,2}=0, \\ w_{0,0}(0, z')=0, \quad \dfrac{\partial w_{0,0}(0, z')}{\partial z_0}=\dfrac{\partial\varphi(0, z')}{\partial z_0}u_0(z'), \end{cases}$$

$$(2.25)_j \quad \begin{cases} \mathcal{L}_2 w_{0,j}+\mathcal{R}_0 w_{0,j+2}=0, \\ w_{0,j}(0, z')=\dfrac{\partial w_{0,j}(0, z')}{\partial z_0}=0 \qquad (j\leq -1). \end{cases}$$

And also, if $w_{n+1,j}$ $(n\geq 0)$ satisfy the equations

$$(2.26) \quad \begin{cases} \mathcal{L}_2 w_{n+1,j}+\mathcal{R}_0 w_{n+1,j+2}+\sum_{s=3}^{m}\mathcal{L}_s w_{n+3-s,j-2(s-2)}=0, \\ w_{n+1}(0, z')=\dfrac{\partial w_{n+1}(0, z')}{\partial z_0}=0, \end{cases}$$

then $w_{n+1}(z, \zeta)=\sum_{-\infty<j\leq 2n+2} w_{n+1,j}(z)f_j(\zeta)$ is a formal solution of $(2.19)_{n+1}$.

We note that from (2.9), the principal symbol $l_2(z, \xi)$ of $\mathcal{L}_2=\mathcal{L}_2(z, \partial)$ is

$$(2.27)\qquad l_2(z,\xi)=(-1)^{m-2}q\Big(z,\frac{\partial\varphi}{\partial z}\Big)\Big(\sum_{i=0}^{n}\frac{\partial p\big(z,\frac{\partial\varphi}{\partial z}\big)}{\partial\xi_i}\xi_i\Big)^2.$$

This implies that $\mathcal{L}_2$ is a partial differential operator of order 2 whose second order term is an ordinary differential operator along bicharacteristic curves on the characteristic surface $\varphi(z)=\text{const}$. Therefore, equations $(2.25)_j$ and (2.26) have holomorphic solutions $w_{n,j}(z)$ in a certain neighbourhood V of $z=0$ which does not depend on n and j. Thus, summing up, we have

THEOREM 2.2. *Suppose that $L(z,\partial)$ and $K=\{z\in\Omega\,;\,\varphi(z)=0\}$ satisfy Condition A. Then there exists a formal solution $U(z,\lambda)$ of problem* (2.1) *in the form*

$$(2.28)\qquad U(z,\lambda)=\exp(-\lambda^2\varphi(z))\sum_{n=0}^{\infty}W_n(z,\lambda),$$

where

$$(2.29)\qquad W_n(z,\lambda)=\lambda^3\int_C\exp(-\lambda\zeta)\sum_{-\infty<j\le 2n+2}w_{n,j}(z)f_j(\zeta)d\zeta$$

and $w_{n,j}(z)$ are holomorphic in a certain neighbourhood V of $z=0$, $V\subset\Omega$, independent of n and j.

In the next section we shall show that the formal sum $\sum_{-\infty<j\le 2n+2}w_{n,j}(z)f_j(\zeta)$ is convergent by obtaining the estimates of $w_{n,j}(z)$.

§ 3. Estimates of formal solutions.

In this section we shall obtain estimates of $w_{n,j}(z)$. To get them we shall make use of the method due to Wagschal [8] and Hamada [3]. The method is also explained in Komatsu [5].

When $a(z)$ and $b(z)$ are formal power series, $a(z)\ll b(z)$ means that each Taylor coefficient of $b(z)$ bounds the absolute value of the corresponding coefficient of $a(z)$. In the following we assume that $0<r<R'<R$. Let $\Theta(t)$ be a formal power series in one variable t such that

$$(3.1)\qquad \Theta(t)\gg 0,$$

$$(3.2)\qquad (R'-t)\Theta(t)\gg 0.$$

From now on, we put $t=\rho z_0+z_1+\cdots+z_n$ with a constant $\rho\ge 1$ to be determined later. Put $\Theta^{(j)}(t)=\Big(\frac{d}{dt}\Big)^j\Theta(t)$.

LEMMA 3.1. (Wagschal) *Let*

(3.3) $$A(z, \partial)=\sum_{\substack{\alpha_0\leq m_0\\|\alpha|\leq m}} a_\alpha(z)\partial^\alpha$$

be a linear partial differential operator with coefficients $a_\alpha(z)$ *holomorphic on a neighbourhood of the polydisk* $C_R=\{z\in C^{n+1};\ |z_i|\leq R\}$. *Then there is a constant* C *which is independent of* $\Theta(t)$ *and* $\rho\geq 1$ *such that if*

(3.4) $$u(z)\ll\Theta^{(j)}(t),$$

then

(3.5) $$A(z, \partial)u(z)\ll C\rho^{m_0}\Theta^{(j+m)}(t).$$

LEMMA 3.2. (De Paris) *Let*

(3.6) $$B(z, \partial)=\sum_{\substack{\alpha_0<k\\|\alpha|\leq k}} b_\alpha(z)\partial^\alpha$$

be a linear partial differential operator with coefficients $b_\alpha(z)$ *holomorphic on a neighbourhood of the polydisk* C_R. *Then there are constants* $\rho\geq 1$ *and* C_1 *such that if*

(3.7) $$v(z)\ll\Theta^{(j+k)}(t)$$

and

(3.8) $$u_l(0, z')\ll\Theta^{(j+l)}(t)|_{z_0=0}\qquad (l=0, 1, \cdots, (k-1)),$$

then the solution $u(z)$ *of the initial value problem*

(3.9) $$\begin{cases}\left(\dfrac{\partial}{\partial z_0}\right)^k u(z)=B(z, \partial)u(z)+v(z)\\[2ex] \left(\dfrac{\partial}{\partial z_0}\right)^l u(0, z')=u_l(z')\qquad (l=0, 1, \cdots, (k-1))\end{cases}$$

satisfies

(3.10) $$u(z)\ll C_1\Theta^{(j)}(t).$$

Now we introduce majorant functions due to Wagschal and Hamada:

(3.11) $$\theta^{(k)}(t)=\frac{k!}{(r-t)^{k+1}}\qquad (k\geq 0),$$

(3.12) $$\theta^{(k)}(t)=\sum_{s=0}^{\infty}\frac{s!}{(|k|+s)!}\frac{t^{|k|+s}}{r^{s+1}}\qquad (k\leq 0).$$

We note that

(3.13) $$\left(\frac{d}{dt}\right)^l\theta^{(k)}(t)=\theta^{(k+l)}(t).$$

For $\theta^{(k)}(t)$, we have

LEMMA 3.3. *Let $R>2r$. For $k<0$ we have*

$$\frac{1}{(R-t)}\theta^{(k)}(t)\ll\frac{2^{|k|}}{(R-2r)}\theta^{(k)}(t). \tag{3.14}$$

Now we set

$$H_k(t)=\frac{R'}{(R'-t)}\theta^{(k)}(t) \qquad (k\in Z). \tag{3.15}$$

$H_k(t)$ satisfies conditions (3.1) and (3.2). We note the following facts concerning derivatives $H_k^{(j)}(t)=\left(\frac{d}{dt}\right)^j H_k(t)$:

LEMMA 3.4.
(a)

$$H_k^{(j)}(t)\ll R'H_k^{(j+1)}(t). \tag{3.16}$$

(b) *If $k<l$*

$$H_l^{(j)}(t)\ll H_k^{(j-k+l)}(t). \tag{3.17}$$

(c) *If $k\geq 0$*

$$H_k^{(j)}(t)\ll\frac{R'}{(R'-r)}\theta^{(j+k)}(t). \tag{3.18}$$

(d) *If $k<0$ and $R'>2r$*

$$H_k^{(j)}(t)\ll\frac{2^{|k|}R'}{(R'-2r)}\theta^{(j+k)}(t). \tag{3.19}$$

For proofs of Lemmas 3.1~3.4 we refer to Komatsu [5].

Now let us estimate $w_{n,j}(z)$. We choose R so that coefficients of the operators $\mathcal{L}_s(z,\partial)$ and $\mathcal{R}_0(z)$ in (2.11) and $u_0(z')$, $u_1(z')$ in (2.1) are holomorphic on the polydisk C_R. First we estimate $w_{0,j}(z)$.

LEMMA 3.5. *There are constants $\rho\geq 1$, $M_1>0$ and $A_1>0$ which are independent of j such that*

$$w_{0,j}(z)\ll M_1A_1^{-j}H_j(t) \qquad (j\leq 2). \tag{3.20}$$

PROOF. First we note that we can choose $C>0$ such that

$$w_{0,2}(z)\ll CH_2(t), \qquad w_{0,0}(z)\ll CH_0(t). \tag{3.21}$$

Obviously, if j is odd, $w_{0,j}(z)=0$. Suppose that (3.20) holds for $j\geq J+2$. Then by Lemmas 3.1 and 3.4 we have

$$\mathcal{R}_0(z)w_{0,J+2}(z) \ll M_1C_1A_1^{-(J+2)}H_{J+2}(t) \ll M_1C_1A_1^{-(J+2)}H_J^{(2)}(t) \tag{3.22}$$

for a constant $C_1>0$.
Hence by Lemma 3.2 we have for some constants $C_2>0$ and $\rho\geq 1$

$$w_{0,J}(z) \ll M_1C_1C_2A_1^{-(J+2)}H_J(t). \tag{3.23}$$

Hence for some A_1 and M_1 we have (3.20).

For $w_{n,j}(z)$ we have

PROPOSITION 3.6. *There are constants $\rho\geq 1$, A_2, B_2 and M which are independent of $H_j(t)$ such that*

$$w_{n,j}(z) \ll MA_2^{2n-j}B_2^nH_{j-2n}^{(n)}(t) \qquad (j\leq 2n+2). \tag{3.24}$$

PROOF. We prove this lemma by induction. Assume that (3.24) is valid, when $0\leq n\leq N$, or $n=N+1$ and $j\geq J+2$. Since by Lemmas 3.1 and 3.4

$$\mathcal{R}_0(z)w_{N+1,J+2} \ll C_1MA_2^{2(N+1)-(J+2)}B_2^{N+1}H_{J+2-2(N+1)}^{(N+1)}(t) \ll C_1MA_2^{2N-J}B_2^{N+1}H_{J-2(N+1)}^{(N+3)}(t) \tag{3.25}$$

and

$$\mathcal{L}_s(z,\partial)w_{N+3-s,J-2(s-2)} \ll C_sMA_2^{2N-J+2}B_2^{N+3-s}H_{J-2N-2}^{(N+3)}(t), \tag{3.26}$$

we have by (2.26) and Lemma 3.2

$$w_{N+1,J} \ll C_2C_1MA_2^{2N-J}B_2^{N+1}H_{J-2(N+1)}^{(N+1)}(t) + C_2\sum_{3\leq s\leq m} C_sMA_2^{2N-J+2}B_2^{N+3-s}H_{J-2N-2}^{(N+1)}(t). \tag{3.27}$$

Therefore for some M, A_2 and B_2 we have (3.24).

PROPOSITION 3.7. *There are positive constants M, A, B and C which are independent of n and j, and a neighbourhood V of $z=0$ such that for $z\in V$*

$$|w_{n,j}(z)| \leq MA^nB^{n-j}\frac{|z|^{n-j}}{(n-j)!} \qquad (-\infty<j\leq n), \tag{3.28}$$

$$|w_{n,j}(z)| \leq MA^nC^{j-n}(j-n)! \qquad (2n+2\geq j\geq n). \tag{3.29}$$

PROOF. By Lemma 3.4, we have, when $j\geq 2n$,

$$H_{j-2n}^{(n)}(t) \ll \frac{R'}{(R'-r)}\theta^{(j-n)}(t) \tag{3.30}$$

and, when $j<2n$,

$$H_{j-2n}^{(n)}(t) \ll \frac{2^{2n-j}R'}{R'-2r}\theta^{(j-n)}(t). \tag{3.31}$$

If $0\leq t\leq r/2$, we have

$$\theta^{(k)}(t)\leq\left(\frac{2}{r}\right)^{k+1}k! \qquad (k\geq 0) \tag{3.32}$$

and

$$\theta^{(k)}(t)\leq 2r^{-1}t^{|k|}/|k|! \qquad (k<0). \tag{3.33}$$

From (3.30) and (3.32), we have for $j\geq 2n$

$$|H_{j-2n}^{(n)}(t)|\leq D_1\left(\frac{2}{r}\right)^{j-n}(j-n)! \tag{3.34}$$

for some $D_1>0$. By (3.31) and (3.32), there is a constant $D_2>0$ such that for $n\leq j\leq 2n$

$$|H_{j-2n}^{(n)}(t)|\leq D_2 2^{2n-j}\left(\frac{2}{r}\right)^{j-n}(j-n)!. \tag{3.35}$$

Finally, for $j<n$ there is a constant D_3 such that

$$|H_{j-2n}^{(n)}(t)|\leq D_3 2^{2n-j}|t|^{n-j}/(n-j)! \tag{3.36}$$

Hence by Proposition 3.6 we have, when $j\geq n$,

$$|w_{n,j}(z)|\leq MA_2^{2n-j}B_2^n D_4 2^{2n-j}(2/r)^{j-n}(j-n)!, \tag{3.37}$$

and when $j\leq n$,

$$|w_{n,j}(z)|\leq MA_2^{2n-j}B_2^n D_5 2^{2n-j}|z|^{n-j}/(n-j)! \tag{3.38}$$

for some D_4 and D_5. From (3.37) and (3.38) Proposition 3.7 follows.

Set

$$w_n^+(z,\zeta)=\sum_{0\leq j\leq 2n+2} w_{n,j}(z)f_j(\zeta) \tag{3.39}$$

and

$$w_n^-(z,\zeta)=\sum_{-\infty<j<0} w_{n,j}(z)f_j(\zeta). \tag{3.40}$$

Proposition 3.8. *There is a constant c_0 independent of n such that $w_n^-(z,\zeta)$ converges in $z\in V$ and $|\zeta|>c_0|z|$.*

Proof. By Proposition 3.7 we have

$$\sum_{j=-1}^{-\infty}|w_{n,j}(z)||f_j(\zeta)|\leq\sum_{j=1}^{\infty}MA^nB^{n+j}\frac{|z|^{n+j}}{(n+j)!}\frac{(j-1)!}{|\zeta|^j}. \tag{3.41}$$

Hence, if $|\zeta|>2B|z|$, we have

$$\sum_{j=1}^{\infty}\left(\frac{zB}{|\zeta|}\right)^j\frac{(j-1)!}{(n+j)!}\leq\frac{1}{n!}. \tag{3.42}$$

This completes the proof.

Set

$$w^{+}(z, \zeta)=\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{2n+2} w_{n,j}(z)f_j(\zeta)\Big)=\sum_{j=0}^{\infty}\bar{w}_j^{+}(z)f_j(\zeta), \tag{3.43}$$

where

$$\bar{w}_j^{+}(z)=\sum_{n\geq(j-2)/2}^{\infty} w_{n,j}(z) \qquad (j\geq 0). \tag{3.44}$$

PROPOSITION 3.9. *The series in* (3.43) *converges uniformly on any compact set in* $\{\zeta;\ \alpha<\arg\zeta<\beta,\ \zeta\neq 0\}$ *and* $z\in V$.

PROOF. By Proposition 3.7, we have

$$\begin{aligned} |\bar{w}_j^{+}(z)| &\leq \sum_{n=j}^{\infty}|w_{n,j}(z)|+\sum_{j>n\geq(j-2)/2}|w_{n,j}(z)| \\ &\leq \sum_{n=j}^{\infty} MA^n B^{n-j}\frac{|z|^{n-j}}{(n-j)!}+\sum_{j>n>(j-2)/2} MA^n C^{j-n}(j-n)! \\ &\leq MA^j(\exp(AB|z|)+D\Gamma(j/2+1)) \\ &\leq M'A^j\Gamma(j/2+1). \end{aligned} \tag{3.45}$$

Hence we have

$$\begin{aligned} |w^{+}(z, \zeta)| &\leq \sum_{j=0}^{\infty}|\bar{w}_j^{+}(z)|\,|f_j(\zeta)| \\ &\leq \sum_{j=0}^{\infty} M'A^j\Gamma(j/2+1)\frac{|\zeta|^j(|\log\zeta|+|\gamma_{(j+1)}|)}{\Gamma(j+1)}\leq M\exp(c_1|\zeta|^2). \end{aligned} \tag{3.46}$$

This completes the proof.

§4. Construction of another formal solution.

In this section we shall construct another solution $\tilde{w}_n$ of $(2.19)_n$:

$$(4.1)_0 \qquad \begin{cases} \mathcal{L}_2\tilde{w}_0+\Big(\dfrac{\partial}{\partial\zeta}\Big)^2\mathcal{R}_0\tilde{w}_0=0, \\ \tilde{w}_0(0, z', \zeta)=u_0(z')f_2(\zeta), \\ \dfrac{\partial\tilde{w}_0(0, z', \zeta)}{\partial z_0}=u_1(z)f_2(\zeta)+\dfrac{\partial\varphi}{\partial z_0}(0, z')u_0(z')f_0(\zeta), \end{cases}$$

$$(4.1)_{n+1} \qquad \begin{cases} \Big(\dfrac{\partial}{\partial\zeta}\Big)^{2(m-2)}\Big\{\mathcal{L}_2+\mathcal{R}_0\Big(\dfrac{\partial}{\partial\zeta}\Big)^2\Big\}\tilde{w}_{n+1}=-\sum_{s=3}^{m}\Big(\dfrac{\partial}{\partial\zeta}\Big)^{2(m-s)}\mathcal{L}_s\tilde{w}_{n+3-s} \\ \tilde{w}_{n+1}(0, z', \zeta)=\dfrac{\partial w_{n+1}(0, z', \zeta)}{\partial z_0}=0. \end{cases}$$

The initial values of $\tilde{w}_0(z, \zeta)$ have a singularity at $\zeta=0$. We construct $\tilde{w}_n$ by the method due to Hamada, Wagschal and others.

Now we shall find a characteristic function of the operator $\mathcal{L}_2(z, \partial)+\mathcal{R}_0(z, \partial)\left(\frac{\partial}{\partial\zeta}\right)^2$ through $\zeta=0$ at $z_0=0$. Let $\phi(z)$ satisfy the equation

$$(4.2)\qquad \begin{cases} q\left(z, \frac{\partial\varphi}{\partial z}\right)\left(\sum_{i=0}^{n} \frac{\partial p\left(z, \frac{\partial\varphi}{\partial z}\right)}{\partial\xi_i}\frac{\partial\phi}{\partial z_i}\right)^2 - r\left(z, \frac{\partial\varphi}{\partial z}\right)=0, \\ \phi(0, z')=0. \end{cases}$$

Then $(\zeta+\phi(z))$ and $(\zeta-\phi(z))$ are simple characteristic functions. It follows from a result in Hamada [2] and Wagschal [8] that there exist $\tilde{w}_n$ $(n=0, 1, \cdots)$ such that

$$(4.3)\qquad \tilde{w}_n(z, \zeta)=\sum_{j=1}^{\infty}(\tilde{w}_{n,j}^{+}(z)f_j(\zeta+\phi(z))+\tilde{w}_{n,j}^{-}(z)f_j(\zeta-\phi(z))),$$

where $\tilde{w}_{n,j}^{+}(z)$ and $\tilde{w}_{n,j}^{-}(z)$ are holomorphic in a neighbourhood V_2 of $z=0$ which is independent of n and j.

We shall give lemmas concerning estimates of $\tilde{w}_{n,j}^{\pm}(z)$ without proofs. We also make use of $H_k(t)$ in (3.15) and assume that the polydisk $C_R=\{|z_i|\leq R\}$ is contained in V_2 and $0<r<R'<R$.

LEMMA 4.1. *There exists $\rho\geq 1$, A and E_1 such that*

$$(4.4)\qquad \tilde{w}_{0,j}^{\pm}(z)\ll AE_1^j H_0^{(j')}(t) \qquad (j\geq 1).$$

Let us consider the equation

$$(4.5)\qquad \begin{cases} \left(\mathcal{L}_2+\mathcal{R}_0\left(\frac{\partial}{\partial\zeta}\right)^2\right)u=\sum_{j=J}^{\infty}(a_j(z)f_j(\zeta+\phi(z))+b_j(z)f_j(\zeta-\phi(z))), \\ u(0, z', \zeta)=\frac{\partial u(0, z', \zeta)}{\partial z_0}=0. \end{cases}$$

We have

LEMMA 4.2. *There is a neighbourhood V_3 of $z=0$ such that for $a_j(z)$ and $b_j(z)$ holomorphic in V_3 there exists a formal solution $u(z)$ of* (4.5) *of the form*

$$(4.6)\qquad u(z)=\sum_{j=J+1}^{\infty}(u_j^{+}(z)f_j(\zeta+\phi(z))+u_j^{-}(z)f_j(\zeta-\phi(z))),$$

where $u_j^{\pm}(z)$ are holomorphic in V_3. Moreover, if

$$(4.7)\qquad a_j(z),\ b_j(z)\ll A\Theta^{(j-J+s)}(t) \qquad (s\geq 2),$$

there are $\rho\geq 1$ and F_1 such that

(4.8) $$u_j^+(z),\ u_j^-(z) \ll AF_1\Theta^{(j-J+s-2)}(t),$$

where $\Theta(t)$ *satisfies conditions* (3.1) *and* (3.2).

By making use of these lemmas we get estimates of $\tilde{w}_{n,j}^{\pm}(z)$:

LEMMA 4.3. *There are constants* G, L *and* K *such that*

(4.9) $$\tilde{w}_{n,j}^{\pm}(z) \ll GL^n K^j H_{-n}^{(j)}(t).$$

We can prove Lemma 4.3 by the method similar to that used in § 3. Lemma 4.3 means that the power series

(4.10) $$\sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \tilde{w}_{n,j}^{\pm}(z)(\zeta \pm \phi(z))^j / j!$$

are uniformly convergent, if $|(\zeta \pm \phi(z))| \leq c_1$ for some c_1. Thus we have

PROPOSITION 4.4. *There is a constant* c_2 *such that* $\tilde{w}(z, \zeta) = \sum_{n=0}^{\infty} \tilde{w}_n(z, \zeta)$ *is holomorphic in* $\{(z, \zeta);\ |\zeta| + |z| < c_2,\ (\zeta \pm \phi(z)) \neq 0\}$.

We note that $w_n(z, \zeta)$ constructed in § 2 is holomorphic in $\{(z, \zeta);\ z \in V, |\zeta| > c_0|z|\}$ but $\tilde{w}_n(z, \zeta)$ in this section is holomorphic in a neighbourhood of $z = \zeta = 0$ except on $\{\zeta \pm \phi(z) = 0\}$.

§ 5. The difference $w_n(z, \zeta) - \tilde{w}_n(z, \zeta)$.

In this section we shall show that $w_n(z, \zeta) - \tilde{w}_n(z, \zeta)$ is holomorphic in some neighbourhood of $z = \zeta = 0$. We note that, though their constructions are different, they satisfy the same equations (2.19) (or (4.1)). First we give a lemma.

Set $x = (x_0, x_1, \cdots, x_n) = (x_0, x') \in C^{n+1}$ and

(5.1) $$M(x, \zeta, \partial_x, \partial_\zeta) = \left(\frac{\partial}{\partial x_0}\right)^2 + \left(\frac{\partial}{\partial \zeta}\right)^2 + M_1(x, \zeta, \partial_x, \partial_\zeta),$$

where $M_1(x, \zeta, \partial_x, \partial_\zeta)$ is an operator of order 1 whose coefficients are holomorphic in a neighbourhood of $x = \zeta = 0$.

Let us consider solutions of the equation

(5.2) $$\begin{cases} \left(\dfrac{\partial}{\partial \zeta}\right)^k M(x, \zeta, \partial_x, \partial_\zeta) u(x, \zeta) = f(x, \zeta) \\ u(0, x', \zeta) = \dfrac{\partial u(0, x', \zeta)}{\partial x_0} = 0. \end{cases}$$

LEMMA 5.1. *Suppose that for small* ε *and* γ *with* $|\gamma| < r$ $u(x, \zeta)$ *is holomorphic on* $\omega_1 = \{(x, \zeta);\ |x_0| \leq r,\ |x'| \leq r',\ |\zeta - \gamma| \leq \varepsilon\}$. *If* $f(x, \zeta)$ *is holomorphic on*

$\omega=\{(x, \zeta);\ |x_0|\leq r,\ |x_0|+|\zeta|\leq 2r,\ |x'|\leq r'\}$, *then* $u(x, \zeta)$ *is also holomorphic on* ω.

PROOF. Set $v(x, \zeta)=M(x, \zeta, \partial_x, \partial_\zeta)u(x, \zeta)$. $v(x, \zeta)$ satisfies

$$\left(\frac{\partial}{\partial\zeta}\right)^k v(x, \zeta)=f(x, \zeta).$$

From the assumption $v(x, \zeta)$ is also holomorphic on ω.

Now consider the Cauchy problem

$$\text{(5.3)}\qquad \begin{cases} M(x, \zeta, \partial_x, \partial_\zeta)\bar{u}(x, \zeta)=v(x, \zeta) \\ \bar{u}(0, x', \zeta)=\dfrac{\partial\bar{u}(0, x', \zeta)}{\partial x_0}=0. \end{cases}$$

Since the principal part of $M(x, \zeta, \partial_x, \partial_\zeta)$ is $\left(\frac{\partial}{\partial x_0}\right)^2+\left(\frac{\partial}{\partial\zeta}\right)^2$, Cauchy problem (5.3) has a unique solution $\bar{u}(x, \zeta)$ on ω. From uniqueness we have $\bar{u}(x, \zeta)=u(x, \zeta)$. Hence $u(x, \zeta)$ is holomorphic on ω. This completes the proof.

Let us apply Lemma 5.1 to the functions

$$\text{(5.4)}\qquad \beta_n(z, \zeta)=w_n(z, \zeta)-\tilde{w}_n(z, \zeta).$$

They satisfy

$$\text{(5.5)}_0\qquad \begin{cases} \left\{\mathcal{L}_2+\mathcal{R}_0\left(\dfrac{\partial}{\partial\zeta}\right)^2\right\}\beta_0(z, \zeta)=0, \\ \beta_0(0, z', \zeta)=\dfrac{\partial\beta_0}{\partial z_0}(0, z', \zeta)=0, \end{cases}$$

$$\text{(5.5)}_{n+1}\qquad \begin{cases} \left(\dfrac{\partial}{\partial\zeta}\right)^{2(m-2)}\left(\mathcal{L}_2+\mathcal{R}_0\left(\dfrac{\partial}{\partial\zeta}\right)^2\right)\beta_{n+1}(z, \zeta) \\ \qquad =-\displaystyle\sum_{s=3}^{m}\left(\frac{\partial}{\partial\zeta}\right)^{2(m-s)}\mathcal{L}_s\beta_{n+3-s}(z, \zeta), \\ \beta_n(0, z', \zeta)=\dfrac{\partial\beta_n(0, z', \zeta)}{\partial z_0}=0. \end{cases}$$

PROPOSITION 5.2. *There is a neighbourhood* V *of* $z=\zeta=0$ *independent of* n *such that* $\beta_n(z, \zeta)$ *is holomorphic on* V.

PROOF. We note that $\beta_n(z, \zeta)$ is holomorphic in

$$\text{(5.6)}\qquad A=\{(z, \zeta);\ |\zeta|>c_0|z|,\ 0<|\zeta\pm\phi(z)|<c_1,\ |z_0|<a_0,\ |z'|<a'\}$$

by Propositions 3.8 and 4.4 for some constant c_0, c_1, a_0, $a'>0$. Hence for small $\gamma\neq 0$, $\beta_n(z, \zeta)$ is holomorphic in a neighbourhood of $(0, \gamma)$. We can find a transformation $z=\chi(x)$ which maps $x_0=0$ to $z_0=0$ and $\chi(0)=0$ and is biholomorphic near $z=0$ such that the operator $\left(\frac{\partial}{\partial\zeta}\right)^{2(m-2)}\left(\mathcal{L}_2+\mathcal{R}_0\left(\frac{\partial}{\partial\zeta}\right)^2\right)$ is transformed into $\left(\frac{\partial}{\partial\zeta}\right)^{2(m-2)}a(x)\left(\left(\frac{\partial}{\partial x_0}\right)^2+\left(\frac{\partial}{\partial\zeta}\right)^2+\text{lower order terms}\right)$ $(a(0)\neq 0)$.

Thus we can apply Lemma 5.1. Obviously $\beta_0(z,\zeta)=0$. Since $\beta_n(\chi(x),\zeta)$ is holomorphic near $(x,\zeta)=(0,\gamma)$, we can find r, r' and ε so that $\beta_n(\chi(x),\zeta)$ is holomorphic on

$$\omega_1=\{(x,\zeta)\,;\,|x_0|\leq r,\ |x'|\leq r',\ |\zeta-\gamma|\leq\varepsilon\}. \tag{5.7}$$

Set, as in Lemma 5.1,

$$\omega=\{(x,\zeta)\,;\,|x_0|\leq r,\ |x_0|+|\zeta|\leq 2r,\ |x'|\leq r'\}. \tag{5.8}$$

Let us prove by induction on n that $\beta_n(\chi(x),\zeta)$ is holomorphic on ω. Suppose that $\beta_n(\chi(x),\zeta)$, $0\leq n\leq N$, are holomorphic on ω. Then the right hand side of $(5.5)_{N+1}$ is holomorphic on ω. It follows from Lemma 5.1 that $\beta_{N+1}(\chi(x),\zeta)$ is holomorphic on ω. Set $V=\chi(\omega)$. This completes the proof.

§ 6. Integral representation of solutions.

In this section we shall give a singular homogeneous solution $u(z)$ of the operator $L(z,\partial)$ such that

$$\begin{cases} L(z,\partial)u(z)=0\,, \\ u(0,z')=\Gamma((\beta+1)/2)\dfrac{u_0(z')}{z_1^{(\beta+1)/2}}\,, \\ \dfrac{\partial u(0,z')}{\partial z_0}=\Gamma((\beta+1)/2)\dfrac{u_1(z')}{z_1^{(\beta+1)/2}}\quad(\beta\in N)\,, \end{cases} \tag{6.1}$$

whose singularity lies on the surface $K=\{\varphi(z)=0\}$. We shall construct $u(z)$ by making use of $U(z,\lambda)$ in the previous sections.

Now recall that $U(z,\lambda)=\sum_{n=0}^{\infty}\exp(-\lambda\varphi(z))W_n(z,\lambda)$, where

$$\begin{aligned} W_n(z,\lambda)&=\lambda^3\int_C \exp(-\lambda\zeta)w_n(z,\zeta)d\zeta \\ &=\lambda^3\int_C \exp(-\lambda\zeta)\sum_{-\infty<j\leq 2n+2} w_{n,j}(z)f_j(\zeta)d\zeta\,. \end{aligned} \tag{6.2}$$

Let us define the path C_θ as follows:

$$C_\theta=C_\theta^+\cup C_\theta^d\cup C_\theta^-\qquad(d>0)\,,$$

$$(6.3)\qquad \begin{cases} C_\theta^+ = \{\zeta = r\exp(i\theta),\ d \le r < \infty\}, \\ C_\theta^d = \{\zeta = d\exp(i\rho),\ \theta \le \rho \le \theta + 2\pi\}, \\ C_\theta^- = \{\zeta = r\exp(i(\theta+2\pi)),\ d \le r < \infty\}, \end{cases}$$

where the path C_θ starts at $\infty \exp(i\theta)$, goes around the origin on C_θ^d and ends at $\infty \exp(i(\theta+2\pi))$ on C_θ^-. Set $C=C_0$ in (6.2). $W_n(z, \lambda)$ is holomorphic in $|\arg \lambda| < \pi/2$. By varying C in (6.2) we can extend $W_n(z, \lambda)$ holomorphically in λ. We have

PROPOSITION 6.1. *$W_n(z, \lambda)$ is expanded as follows:*

$$(6.4)\qquad W_n(z, \lambda) = \sum_{-\infty < j \le 2n+2} w_{n,j}(z)\lambda^{-j+2}.$$

Hence $W_n(z, \lambda)$ has a most a pole at $\lambda = 0$.

PROOF. If $|\arg \lambda| < \pi/2$, we have

$$(6.5)\qquad W_n(z, \lambda) = \lambda^3 \sum_{-\infty < j \le 2n+2} w_{n,j}(z) \int_{C_0} \exp(-\lambda\zeta) f_j(\zeta) d\zeta.$$

Since

$$(6.6)\qquad \int_{C_0} \exp(-\lambda\zeta) f_j(\zeta) d\zeta = \lambda^{-(j+1)},$$

we have (6.4).

Set

$$(6.7)\qquad u_n(z) = \frac{1}{2\pi i}\int_{\Gamma_\theta} \exp(-\lambda^2\varphi(z)) W_n(z, \lambda) h(\lambda) d\lambda$$

and

$$(6.8)\qquad g_k^\beta(s) = \frac{1}{2\pi i}\int_{\Gamma_\theta} \exp(-\lambda^2 s)\lambda^{-(k+1)} h(\lambda) d\lambda \qquad (k \in Z),$$

where $h(\lambda) = 2\lambda^\beta(\log \lambda)$ and Γ_θ is a path which is defined by the same way as C_θ. We will show that

$$(6.9)\qquad u(z) = \sum_{n=0}^{\infty} u_n(z)$$

is a solution of (6.1) and from (6.9) we will derive an integral representation of $u(z)$. To do so we estimate $g_k^\beta(s)$.

LEMMA 6.2. *$g_k^\beta(s)$ is a holomorphic function of s except at $s=0$. More precisely*

$$(6.10)\qquad g_k^\beta(s) = \Gamma((\beta-k)/2) s^{(k-\beta)/2} \qquad (\beta \ge k+1),$$

$$(6.11)\qquad g_k^\beta(s) = \delta_{k-\beta} s^{(k-\beta)/2} - \frac{(-1)^{(\beta-k)/2}}{\Gamma((k-\beta)/2+1)} s^{(k-\beta)/2}(\log s)$$

$$(k+1>\beta \textit{ and } k-\beta \textit{ is even}),$$

(6.12) $$g_k^\beta(s)=\delta_{k-\beta}s^{(k-\beta)/2}$$

$$(k+1>\beta \textit{ and } k-\beta \textit{ is odd}),$$

where $\delta_{k-\beta}$ *is a constant with*

(6.13) $$|\delta_{k-\beta}|\leq Nh^{k-\beta}/\Gamma((k-\beta)/2+1)$$

for some N *and* h.

PROOF. It follows easily from varying the path $\Gamma=\Gamma_\theta$ that $g_k^\beta(s)$ is a holomorphic function of s $(s\neq 0)$. If $\beta\geq k+1$, we have for $s>0$

(6.14) $$\begin{aligned} g_k^\beta(s)&=\frac{1}{2\pi i}\int_{\Gamma_0}\exp(-\lambda^2 s)2\lambda^{\beta-(k+1)}(\log\lambda)d\lambda \\ &=2\int_0^\infty \exp(-\lambda^2 s)\lambda^{\beta-(k+1)}d\lambda \\ &=\Gamma((\beta-k)/2)s^{(k-\beta)/2}. \end{aligned}$$

If $\beta<k+1$, then we have

(6.15) $$\begin{aligned} g_k^\beta(s)&=\frac{1}{\pi i}\int_{\Gamma_0}\exp(-\lambda^2 s)\lambda^{\beta-(k+1)}(\log\lambda)d\lambda \\ &=\frac{1}{\pi i}\int_{\Gamma_0}\exp(-\mu^2)\mu^{\beta-k-1}s^{(k-\beta)/2}(\log\mu-\log\sqrt{s})d\mu. \end{aligned}$$

Since

(6.16) $$\frac{1}{2\pi i}\int_{\Gamma_0}\exp(-\mu^2)\mu^l d\mu=\begin{cases} 0 & l\geq 0 \text{ or even} \\ \dfrac{(-1)^{(l+1)/2}}{\Gamma((-l+1)/2)} & l<0 \text{ and odd}, \end{cases}$$

we have (6.12) by setting

(6.17) $$\delta_l=\frac{1}{\pi i}\int_{\Gamma_0}\exp(-\mu^2)\mu^l(\log\mu)d\mu \qquad (l<0).$$

Now we estimate δ_l. Substituting $\mu=|l|^{1/2}\rho$, we have

(6.18) $$\delta_l=\frac{1}{\pi i}|l|^{(-|l|+1)/2}\int_{\Gamma_0}\exp(-|l|\rho^2)\rho^{-|l|}(\log|l|^{1/2}\rho)d\rho.$$

By deformating the path Γ_0 to the one that starts at ∞ on the real axis, goes around the origin once on the circle $|\rho|=1$ and goes to $\infty\exp(i(2\pi))$ on the real axis, we can show that there are N and h such that (6.13) is valid by making use of Stirling's formula.

Form Lemma 6.2 we have the following proposition which is obtained in

Hamada [3]. Our construction of a solution of (6.1) is different from that in Hamada [3].

PROPOSITION 6.3. *$u(z)$ ((6.9)) is a solution of (6.1) which is holomorphic in a neighbourhood of $z=0$ except on $K=\{\varphi(z)=0\}$. Moreover $u(z)$ is expressed in the form*

$$u(z)=\begin{cases}\sum_{k=1}^{\infty} a_k(z)\varphi(z)^{-k}+b(z)(\log\varphi(z))+c(z) & \beta \text{ odd}.\\ \sum_{k=0}^{\infty} a_k(z)\varphi(z)^{-k+1/2} & \beta \text{ even}.\end{cases} \tag{6.19}$$

where $a_k(z)$, $b(z)$ and $c(z)$ are holomorphic functions in a neighbourhood of $z=0$.

In general $u(z)$ has a term of infinite negative powers of $\varphi(z)$. So it is not easy to know the behaviour of $u(z)$ when z tends to K, $\varphi(z)\to 0$. Our integral representation given later will enable us to know it.

PROOF OF PROPOSITION 6.3. First we note that

$$u_n(z)=\sum_{-\infty<j\leq 2n+2} w_{n,j}(z)g^{\beta}_{j-3}(\varphi(z)), \tag{6.20}$$

and $w_{n,j}(z,\lambda)=0$, if j is odd.
Hence

$$u(z)=\sum_{n=0}^{\infty} u_n(z)=\sum_{n=0}^{\infty}\sum_{-\infty<j\leq 2n+2} w_{n,j}(z)g^{\beta}_{j-3}(\varphi(z)). \tag{6.21}$$

By Proposition 3.7 for $j\leq 0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}|w_{n,j}(z)| &\leq M\sum_{n=0}^{\infty}A^nB^{n-j}|z|^{(n-j)}/(n-j)!\\ &\leq M\exp(|z|AB)(B|z|)^{|j|}/|j|!.\end{aligned} \tag{6.22}$$

For $j\geq 1$, we have

$$\begin{aligned}\sum_{n\geq(j-2)/2}|w_{n,j}(z)| &= \sum_{(j-2)/2\leq n<j}|w_{n,j}(z)|+\sum_{n\geq j}|w_{n,j}(z)|\\ &\leq \sum_{(j-2)/2\leq n<j} MA^nC^{j-n}(j-n)!+\sum_{n\geq j} MA^nB^{n-j}|z|^{n-j}/(n-j)!\\ &\leq MA'^j\Gamma(j/2+1).\end{aligned} \tag{6.23}$$

We return to (6.21):

$$\begin{aligned}|u(z)| \leq &\sum_{j\leq 0} M'((B|z|)^{|j|}/|j|!)|g^{\beta}_{j-3}(\varphi(z))|\\ &+\sum_{j\geq 1} M'A'^j|\Gamma(j/2+1)|g^{\beta}_{j-3}(\varphi(z))|.\end{aligned} \tag{6.24}$$

We have

$$\sum_{j\geq\beta+3} A'^j|\Gamma(j/2+1)||g^{\beta}_{j-3}(\varphi(z))| \tag{6.25}$$

$$\leq \sum_{j\geq\beta+3} A'^{j}\Gamma(j/2+1)|\varphi(z)|^{(j-\beta-3)/2}(|\delta_{j-\beta-3}|+\Gamma((j-\beta-1)/2)^{-1}|\log\varphi(z)|)$$

$$\leq \sum_{j\geq\beta+3} \tilde{M}\tilde{A}^{j}\frac{\Gamma(j/2+1)}{\Gamma((j-\beta-1)/2)}|\varphi(z)|^{(j-\beta-3)/2}(1+|\log\varphi(z)|).$$

Hence (6.25) converges, if $|\varphi(z)|^{1/2}\tilde{A}\leq 1/2$.

On the other hand

$$\sum_{-\infty<j\leq 0}((B|z|)^{|j|}/|j|!)|g^{\beta}_{j-3}(\varphi(z))| \tag{6.26}$$

$$\leq \sum_{-\infty<j\leq 0}(B|z|)^{|j|}(\Gamma((\beta-j-3)/2)/|j|!)|\varphi(z)|^{-(\beta-j-3)/2}.$$

Obviously (6.26) converges uniformly on any compact set in $\{r>|\varphi(z)|>0\}$ for some r. Expression (6.19) is clear. From the method of construction of $U(z,\lambda)=\sum_{n=0}^{\infty}\exp(-\lambda^2\varphi(z))W_n(z,\lambda)$ it follows that

$$u(0,z')=\frac{1}{2\pi i}\int_{\Gamma}\exp(-\lambda^2 z_1)u_0(z')2\lambda^{\beta}(\log\lambda)d\lambda \tag{6.27}$$

$$=\Gamma((\beta+1)/2)\frac{u_0(z')}{z_1^{(\beta+1)/2}}$$

and

$$\frac{\partial u(0,z')}{\partial z_0}=\frac{1}{2\pi i}\int_{\Gamma}\exp(-\lambda^2 z_1)u_1(z')2\lambda^{\beta}(\log\lambda)d\lambda \tag{6.28}$$

$$=\Gamma((\beta+1)/2)\frac{u_1(z')}{z_1^{(\beta+1)/2}}.$$

$u(z)$ is a solution of (6.1).

We have, by making use of $W_n(z,\lambda)$ and $w_n(z,\zeta)$,

THEOREM 6.4. *There is a solution $u(z)$ of* (6.1) *in a neighbourhood of $z=0$ which is expressed in the form*

$$u(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{\Gamma}\exp(-\lambda^2\varphi(z))W_n(z,\lambda)2\lambda^{\beta}(\log\lambda)d\lambda, \tag{6.29}$$

where

$$W_n(z,\lambda)=\lambda^3\int_C\exp(-\lambda\zeta)w_n(z,\zeta)d\zeta \tag{6.30}$$

and

$$w_n(z,\zeta)=\sum_{-\infty<j\leq 2n+2}w_{n,j}(z)f_j(\zeta). \tag{6.31}$$

Now let us represent $u(z)$ in a form different from (6.29) with the aid of

$\tilde{w}_n(z, \zeta)$ constructed in §4, which is convenient for analysis of singularity. To do so let us recall

$$w_n^+(z, \zeta)=\sum_{j=0}^{2n+2} w_{n,j}(z)f_j(\zeta) \tag{6.32}$$

and

$$w_n^-(z, \zeta)=\sum_{-\infty<j\leq -1} w_{n,j}(z)f_j(\zeta). \tag{6.33}$$

Hence we have

$$\begin{aligned} W_n(z, \lambda)&=\lambda^3\int_C \exp(-\lambda\zeta)w_n(z, \zeta)d\zeta \\ &=\lambda^3\int_C \exp(-\lambda\zeta)(w_n^+(z, \zeta)+w_n^-(z, \zeta))d\zeta. \end{aligned} \tag{6.34}$$

Since $w_n^-(z, \zeta)$ is single valued, we have

$$\begin{aligned} W_n(z, \lambda)=&\lambda^3\int_C \exp(-\lambda\zeta)w_n^+(z, \zeta)d\zeta \\ &+\lambda^3\int_{|\zeta|=d} \exp(-\lambda\zeta)w_n^-(z, \zeta)d\zeta. \end{aligned} \tag{6.35}$$

From Proposition 5.2 there are holomorphic functions $\beta_n(z, \zeta)$ $(n=0, 1, \cdots)$ such that

$$w_n(z, \zeta)=\tilde{w}_n(z, \zeta)+\beta_n(z, \zeta). \tag{6.36}$$

Hence for small d we have

$$\begin{aligned} W_n(z, \lambda)=&\lambda^3\int_{|\zeta|=d} \exp(-\lambda\zeta)(\tilde{w}_n(z, \zeta)+\beta_n(z, \zeta)-w_n^+(z, \zeta))d\zeta \\ &+\lambda^3\int_C \exp(-\lambda\zeta)w_n^+(z, \zeta)d\zeta \\ =&\lambda^3\int_{|\zeta|=d} \exp(-\lambda\zeta)(\tilde{w}_n(z, \zeta)-w_n^+(z, \zeta))d\zeta \\ &+\lambda^3\int_C \exp(-\lambda\zeta)w_n^+(z, \zeta)d\zeta. \end{aligned} \tag{6.37}$$

Thus we have by Proposition 3.9

THEOREM 6.5. *In a neighbourhood of $z=0$, the solution $u(z)$ of the problem* (6.1) *defined by* (6.29) *is represented as follows*:

$$u(z)=\frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2\varphi(z))2\lambda^{\beta+3}(\log\lambda)d\lambda \tag{6.39}$$

$$\times\int_{|\zeta|=d}\exp(-\lambda\zeta)(\tilde{w}(z,\zeta)-w^{+}(z,\zeta))d\zeta$$

$$+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{\Gamma}\exp(-\lambda^{2}\varphi(z))2\lambda^{\beta}W_{n}^{+}(z,\lambda)(\log\lambda)d\lambda,$$

where

(6.40) $$w^{+}(z,\zeta)=\sum_{n=0}^{\infty}w_{n}^{+}(z,\zeta),$$

(6.41) $$\tilde{w}(z,\zeta)=\sum_{n=0}^{\infty}\tilde{w}_{n}(z,\zeta)$$

and

(6.42) $$W_{n}^{+}(z,\lambda)=\lambda^{3}\int_{C}\exp(-\lambda\zeta)w_{n}^{+}(z,\zeta)d\zeta.$$

In §7 we will show that in integral representation (6.39) the essential part of $u(z)$ when z tends to $K=\{\varphi(z)=0\}$ is

(6.43) $$u_{I}(z)=\frac{1}{2\pi i}\int_{\Gamma}\exp(-\lambda^{2}\varphi(z))2\lambda^{\beta+3}(\log\lambda)d\lambda\int_{|\zeta|=d}\exp(-\lambda\zeta)\tilde{w}(z,\zeta)d\zeta.$$

§7. Asymptotic behaviour of singular solutions.

In this section we shall give informations of asymptotic behaviour of $u(z)$. In order to do so we shall investigate functions defined by integrals. Let us recall that the path C_{θ}^{d} starts at $d\exp(i\theta)$ and ends to $d\exp(i(\theta+2\pi))$ on the circle $|\zeta|=d$.

Now set

(7.1) $$I_{1,\theta}(z)=\frac{1}{2\pi i}\int_{\Gamma}\exp(-\lambda^{2}\varphi(z))2\lambda^{\beta+3}(\log\lambda)d\lambda\int_{C_{\theta}^{d}}\exp(-\lambda\zeta)w^{+}(z,\zeta)d\zeta,$$

(7.2) $$I_{2,\theta}(z)=\frac{1}{2\pi i}\int_{\Gamma}\exp(-\lambda^{2}\varphi(z))2\lambda^{\beta+3}(\log\lambda)d\lambda\int_{C_{\theta}^{d}}\exp(-\lambda\zeta)\tilde{w}(z,\zeta)d\zeta$$

and

(7.3) $$I_{3}(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{\Gamma}\exp(-\lambda^{2}\varphi(z))W_{n}^{+}(z,\lambda)2\lambda^{\beta}(\log\lambda)d\lambda.$$

For $I_{3}(z)$ we have

LEMMA 7.1. *$I_{3}(z)$ is expressed as follows:*

(7.4) $$I_{3}(z)=\sum_{j=0}^{\infty}\overline{w}_{j}^{+}(z)g_{j-3}^{\beta}(\varphi(z)),$$

where $g^{\beta}_{j-3}(s)$ *and* $\overline{w}^{+}_{j}(z)$ *are defined by* (6.8) *and* (3.44) *respectively.*

The proof is the same as that of Proposition 6.3.

Now to investigate $I_{1,\theta}(z)$ and $I_{2,\theta}(z)$ we need asymptotic behaviour of the functions

$$W^{+}_{\theta}(z, \lambda)=\int_{C^{d}_{\theta}} \exp(-\lambda\zeta)w^{+}(z, \zeta)d\zeta \tag{7.5}$$

and

$$\widetilde{W}_{\theta}(z, \lambda)=\int_{C^{d}_{\theta}} \exp(-\lambda\zeta)\tilde{w}(z, \zeta)d\zeta \tag{7.6}$$

as $|\lambda|\to\infty$.

LEMMA 7.2. *For any* $\varepsilon>0$,

$$W^{+}_{\theta}(z, \lambda)\sim \sum_{j=0}^{\infty} \overline{w}^{+}_{j}(z)\lambda^{-(j+1)} \tag{7.7}$$

as $|\lambda|\to\infty$ *in the sector* $|\arg\lambda+\theta|\leq\pi/2-\varepsilon$.

PROOF. We have

$$W^{+}_{\theta}(z, \lambda)=\int_{0}^{d\exp(i\theta)} \exp(-\lambda\zeta)\sum_{j=0}^{\infty}\overline{w}^{+}_{j}(z)\frac{\zeta^{j}}{\Gamma(j+1)}d\zeta. \tag{7.8}$$

Substituting $\lambda=\mu\exp(-i\theta)$ and $\zeta=\eta\exp(i\theta)$, we have

$$W^{+}_{\theta}(z, \mu\exp(-i\theta))=\int_{0}^{d}\exp(-\mu\eta)\sum_{j=0}^{\infty}\overline{w}^{+}_{j}(z)\eta^{j}\frac{\exp(i(j+1)\theta)}{\Gamma(j+1)}d\eta. \tag{7.9}$$

By Watson's lemma for asymptotic expansions, we have

$$W^{+}_{\theta}(z, \lambda)\sim \sum_{j=0}^{\infty} \overline{w}^{+}_{j}(z)\lambda^{-(j+1)} \tag{7.10}$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda+\theta|\leq\pi/2-\varepsilon$.

To investigate $\widetilde{W}_{\theta}(z, \lambda)$ let us introduce functions

$$f(\zeta)=\sum_{j=0}^{\infty} a_{j}f_{j}(\zeta+b), \tag{7.11}$$

where $|a_{j}|\leq A\Gamma(j+1)/(d')^{j}$, $d'/2>d>|b|/\kappa$ $(0<\kappa<1/2)$, and

$$F(\lambda, b, \theta)=\int_{C^{d}_{\theta}} \exp(-\lambda\zeta)f(\zeta)d\zeta. \tag{7.12}$$

LEMMA 7.3. *We have*

$$F(\lambda, b, \theta)\sim\exp(\lambda b)\sum_{j=0}^{\infty} a_{j}\lambda^{-(j+1)} \tag{7.13}$$

as $|\lambda|\to\infty$ *in the sector* $|\arg\lambda+\theta|<\cos^{-1}(\kappa)$.

PROOF. We have

$$F(\lambda, b, \theta)=\exp(\lambda b)\sum_{j=0}^{\infty} a_j\int_0^{b+d\exp(i\theta)} \exp(-\lambda\zeta)\frac{\zeta^j}{\Gamma(j+1)}d\zeta. \tag{7.14}$$

Since $-\mathrm{Re}\,\lambda(b+d\exp(i\theta))\leq-\delta|\lambda|$ for some δ in the sector $|\arg\lambda+\theta|<\cos^{-1}(\kappa)$ we have (7.13) from Watson's lemma.

Set

$$\tilde{W}_\theta(z, \lambda)=\tilde{W}_\theta^+(z, \lambda)+\tilde{W}_\theta^-(z, \lambda), \tag{7.15}$$

where

$$\tilde{W}_\theta^\pm(z, \lambda)=\int_{C_\theta^d}\exp(-\lambda\zeta)\sum_{j=1}^{\infty}\tilde{w}_j^\pm(z)f_j(\zeta\pm\phi(z))d\zeta. \tag{7.16}$$

From Lemma 7.3 we have

LEMMA 7.4. *In a small neighbourhood of* $z=0$ *which depends on* κ *we have*

$$\tilde{W}_\theta^\pm(z, \lambda)\sim\exp(\pm\lambda\phi(z))\sum_{j=1}^{\infty}\tilde{w}_j^\pm(z)\lambda^{-(j+1)} \tag{7.17}$$

as $|\lambda|\to\infty$ *in the sector* $|\arg\lambda+\theta|<\cos^{-1}(\kappa)$.

PROOF. Since $\phi(0)=0$, we can take a neighbourhood of $z=0$ such that $|\phi(z)|<d\kappa$ there. Hence we can apply Lemma 7.3.

Next we consider integrals in λ. To do so let us investigate functions

$$K^1(a, \theta)=\frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2 a^2)k^1(\lambda, \theta)(\log\lambda)d\lambda \tag{7.18}$$

and

$$K^2(a, b, \theta)=\frac{1}{2\pi i}\int_\Gamma \exp(-\lambda^2 a^2)k^2(\lambda, b, \theta)(\log\lambda)d\lambda, \tag{7.19}$$

where $a, b\in C^1$ and $|b|<r$. $k^1(\lambda, \theta)$ and $k^2(\lambda, b, \theta)$ have properties:

(i) $k^1(\lambda, \theta)$ and $k^2(\lambda, b, \theta)$ are entire functions of λ and satisfy

$$|k^1(\lambda, \theta)|,\ |k^2(\lambda, b, \theta)|\leq C\exp(c|\lambda|), \tag{7.20}$$

(ii)

$$k^1(\lambda, \theta)\sim\sum_{j=-J_1}^{\infty}k_j^1\lambda^{-(j+1)} \tag{7.21}$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda+\theta|<\pi/2-\varepsilon$,

(iii)

$$k^2(\lambda, b, \theta)\sim\exp(\lambda b)\sum_{j=-J_2}^{\infty}k_j^2(b)\lambda^{-(j+1)} \tag{7.22}$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda+\theta|<\cos^{-1}(\kappa)$ $(0<\kappa<1/2)$, where $k_j^2=k_j^2(b)$ is bounded.

For $K^1(a, \theta)$ we have

LEMMA 7.5.
(i) *$K^1(a, \theta)$ is a single valued holomorphic function of a except at $a=0$.*
(ii) *For any $\varepsilon>0$ we have in the sector $|\arg a-\theta|<(3/4)\pi-\varepsilon$*

$$K^1(a, \theta)=\sum_{i=0}^{J_1} k_{i-J_1}^1 g_{i-J_1}^0(a)+a \text{ bounded function,} \tag{7.23}$$

where $g_k^0(s)$ is defined by (6.8).

PROOF. Since

$$K^1(a, \theta)=\frac{1}{a}\int_0^\infty \exp(-\mu^2)k^1(\mu/a, \theta)d\mu, \tag{7.24}$$

(i) follows. To prove (ii) we may assume that $\theta=0$ and set $K^1(a)=K^1(a, 0)$ and $k^1(\lambda)=k^1(\lambda, 0)$. For any a with $|\arg a|<3\pi/4-\varepsilon$ we can find an $\arg\lambda=\tilde{\theta}$ $(|\tilde{\theta}|<\pi/2)$ such that $\operatorname{Re}(\lambda a)^2>0$. Hence from the assumption of $k^1(\lambda)$ we have

$$\begin{aligned} K^1(a)&=\frac{1}{2\pi i}\int_{\Gamma_{\tilde{\theta}}}\exp(-\lambda^2a^2)k^1(\lambda)(\log\lambda)d\lambda \\ &=\frac{1}{2\pi i}\int_{\Gamma_{\tilde{\theta}}}\exp(-\lambda^2a^2)\Big(\sum_{j=-J_1}^{0} k_j^1\lambda^{-(j+1)}+O(\lambda^{-2})\Big)(\log\lambda)d\lambda \end{aligned} \tag{7.25}$$

in $|\arg a|<(3/4)\pi-\varepsilon$. Thus we have (7.23).

For $K^2(a, b, \theta)$ we have

LEMMA 7.6.
(i) *$K^2(a, b, \theta)$ is a single valued holomorphic function of a except at $a=0$.*
(ii) *Let $|\arg a-\theta|<\cos^{-1}\kappa-\pi/4$ and set $\omega=b/a$. Then for any $\varepsilon>0$ we have*

$$K^2(a, b, \theta)\sim(\omega/a)^{J_2}\exp(\omega^2/4)\sum_{j=1}^{\infty}\nu_j\omega^{-j} \tag{7.26}$$

as $|\omega|\to\infty$ in the sector $|\arg\omega|<\pi/4-\varepsilon$, where $\nu_1=2^{1-J_2}\sqrt{\pi}\,k_{-J_2}^2$ and in the sector $|\arg\omega-\pi|<(3/4)\pi-\varepsilon$

$$\begin{cases} |K^2(a, b, \theta)|\le C_\varepsilon|b|^{-J_2} & (J_2\ge 1) \\ |K^2(a, b, \theta)|\le C_\varepsilon|\log b| & (J_2=0) \\ |K^2(a, b, \theta)|\le C_\varepsilon & (J_2\le -1). \end{cases} \tag{7.27}$$

PROOF. Since

$$K^2(a, b, \theta)=\frac{1}{a}\int_0^\infty \exp(-\mu^2)k^2(\mu/a, b, \theta)d\mu, \tag{7.28}$$

(i) is clear. To prove (ii) we may assume that $\theta=0$. We set $K^2(a, b)=K^2(a, b, 0)$, $k^2(\lambda, b)=k^2(\lambda, b, 0)$ and $k^{2*}(\lambda)=\exp(-\lambda b)k^2(\lambda, b)$. We have

$$K^2(a, b)=\frac{1}{a}\int_0^\infty \exp(-\mu^2+\omega\mu)k^{2*}(\mu/a)d\mu = \frac{1}{a}\exp(\omega^2/4)\int_{-\omega/2}^\infty \exp(-\mu^2)k^{2*}((\mu+\omega/2)/a)d\mu. \tag{7.29}$$

Let $|\arg\omega|<\pi/4-\varepsilon$. Then we have $\left|\arg\frac{\omega}{a}(\rho+1/2)\right|<\cos^{-1}\kappa-\varepsilon$, provided $\rho>-1/2$. Hence we have

$$K^2(a, b)=\frac{\omega}{a}\exp(\omega^2/4)\int_{-1/2}^\infty \exp(-(\omega\rho)^2)k^{2*}\Big(\frac{\omega}{a}(\rho+1/2)\Big)d\rho \tag{7.30}$$

and

$$k^{2*}\Big(\frac{\omega}{a}(\rho+1/2)\Big)\sim\sum_{j=-J_2}^\infty \frac{k_j^2}{\Big(\frac{\omega}{a}(\rho+1/2)\Big)^{j+1}} \tag{7.31}$$

as $\left|\frac{\omega}{a}(\rho+1/2)\right|\to\infty$ in the sector $\left|\arg\frac{\omega}{a}(\rho+1/2)\right|<\cos^{-1}\kappa$. Let us set

$$K_1^2(a, b)=\frac{\omega}{a}\int_{-1/2}^{-1/4}\exp(-(\omega\rho)^2)k^{2*}\Big(\frac{\omega}{a}(\rho+1/2)\Big)d\rho \tag{7.32}$$

and

$$K_2^2(a, b)=\frac{\omega}{a}\int_{-1/4}^{\infty}\exp(-(\omega\rho)^2)k^{2*}\Big(\frac{\omega}{a}(\rho+1/2)\Big)d\rho. \tag{7.33}$$

Then we have from (7.31)

$$|K_1^2(a, b)|\le\left|\frac{\omega}{a}\right|\int_{-1/2}^{-1/4}\exp(-\mathrm{Re}\,(\omega\rho)^2)C\Big(1+\frac{\omega}{a}(\rho+1/2)\Big)^{J_2-1}d\rho \le\left|\Big(\frac{\omega}{a}\Big)\right|^{J_2}C'\exp(-(\mathrm{Re}\,\omega)^2/16) \tag{7.34}$$

and

$$K_2^2(a, b)=\Big(\frac{\omega}{a}\Big)\int_{-1/4}^\infty \exp(-(\omega\rho)^2)\Big\{\sum_{-J_2\le j\le N}k_j^2\Big(\frac{\omega}{a}(\rho+1/2)\Big)^{-j-1} +O\Big(\frac{\omega}{a}(\rho+1/2)\Big)^{-(N+2)}\Big\}d\rho. \tag{7.35}$$

We note that

$$A_j(\omega)=\int_{-1/4}^\infty \exp(-(\omega\rho)^2)(\rho+1/2)^{-j-1}d\rho\sim\sum_{k=0}^\infty \alpha_{j,k}\omega^{-2k-1},\ \alpha_{j,0}=2^{j+1}\sqrt{\pi}, \tag{7.36}$$

as $|\omega|\to\infty$ in $|\arg\omega|<\pi/4-\varepsilon$ for any $\varepsilon>0$. Therefore

$$K^2(a, b)=\left(\frac{\omega}{a}\right)\exp(\omega^2/4)\left(\sum_{-J_2\leq j\leq N} k_j^2\left(\frac{\omega}{a}\right)^{-j-1}A_j(\omega)\right.$$
$$\left.+O\left(\left(\frac{\omega}{a}\right)^{-N-2}\right)+K_1^2(a, b)\right) \tag{7.37}$$

as $|\omega|\to\infty$. Thus we have

$$K^2(a, b)\sim\left(\frac{\omega}{a}\right)^{J_2}\exp(\omega^2/4)\left(\sum_{i=1}^{\infty}\nu_i\omega^{-i}\right) \tag{7.38}$$

as $|\omega|\to\infty$ in the sector $|\arg\omega|<\pi/4-\varepsilon$, where $\nu_1=2^{1-J_2}\sqrt{\pi}\,k_{-J_2}^2$.

Next suppose that $|\arg\omega-\pi|<(3/4)\pi-\varepsilon$. Let us return to (7.19). For (a, b) with $|\arg a|<\cos^{-1}\kappa-\pi/4$ and $|\arg\omega-\pi|<3\pi/4-\varepsilon$, we can find an $\arg\lambda=\theta_1$ with $|\theta_1|<\cos^{-1}\kappa$ such that $|\exp(-(\lambda a)^2+\lambda b)|\leq 1$. Since $|\arg\lambda|<\cos^{-1}\kappa$,

$$k^{2*}(\lambda)\sim\sum_{j=-J_2}^{\infty}k_j^2\lambda^{-(j+1)} \tag{7.39}$$

as $|\lambda|\to\infty$. Thus we have

$$K^2(a, b)=\frac{1}{2\pi i}\int_{\Gamma_{\theta_1}}\exp(-(\lambda a)^2+\lambda b)\left(\sum_{j=-J_2}^{-1}k_j^2\lambda^{-(j+1)}\right.$$
$$\left.+\frac{k_0^2}{\lambda}+O(\lambda^{-2})\right)(\log\lambda)d\lambda\,. \tag{7.40}$$

Since

$$\int_0^{\infty\exp(i\theta_1)}\exp(-(\lambda a)^2+\lambda b)\lambda^{\alpha}d\lambda=(-\omega a)^{-(1+\alpha)}(\Gamma(\alpha+1)+O(|\omega|^{-2}))\quad(\alpha\geq 0) \tag{7.41}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_{\theta_1}}\exp(-(\lambda a)^2+\lambda b)\frac{(\log\lambda)}{\lambda}d\lambda=-\log(-b)+O(1)\quad(|\omega|\to\infty)\,. \tag{7.42}$$

Therefore we have

$$K^2(a, b)=\sum_{j=J_2}^{1}a^{-j}(k_{-j}^2(-\omega)^{-j}\Gamma(j)+O(|\omega|^{-j-1}))-k_0^2\log(-b)+O(1)$$
$$(|\omega|\to\infty)\,. \tag{7.43}$$

Consequently the statement of (ii) of this lemma follows.

REMARK 7.7. We do not make use of the higher order terms of asymptotic developments (7.21) and (7.22). They are concerned with smoothness up to the boundaries. (7.21) is connected with smoothness of $K^1(a, \theta)$ on the set $\{a\,;\,|\arg a-\theta|<(3/4)\pi-\varepsilon\}$, (7.22) is connected with smoothness of $K^2(a, b, \theta)$ when (a, b) tends to $(0, 0)$.

Now let us return to the functions $I_{1,\theta}(z)$ and $I_{2,\theta}(z)$.

PROPOSITION 7.8. *For any* $\varepsilon>0$, *we have*

$$I_{1,\theta}(z)=\sum_{j=0}^{\beta+3}\bar{w}_j^+(z)g_{j-(\beta+3)}^0(\sqrt{\varphi(z)})+O(1) \tag{7.44}$$

as $|\varphi(z)|\to 0$ *in* $|\arg\sqrt{\varphi(z)}-\theta|<(3/4)\pi-\varepsilon$.

PROOF. From Lemma 7.2

$$\lambda^{\beta+3}W_\theta^+(z,\lambda)\sim\sum_{j=0}^{\infty}\bar{w}_j^+(z)\lambda^{\beta+2-j} \tag{7.45}$$

$$\sim\sum_{j=0}^{\beta+3}\bar{w}_j^+(z)\lambda^{\beta+2-j}+O(\lambda^{-2})$$

as $|\lambda|\to\infty$ in $|\arg\lambda+\theta|\leq(\pi/2)-\varepsilon$. A simple application of Lemma 7.5 leads to (7.45).

REMARK 7.9. We note that the asymptotic development of $W_\theta^+(z,\lambda)$ and the formal sum $\sum_{n=0}^{\infty}W_n^+(z,\lambda)$ are equal. This implies that $(I_{1,\theta}(z)-I_3(z))$ is an infinitely differentiable function on the set $\overline{\{z;\ |\arg\sqrt{\varphi(z)}-\theta|<(3/4)\pi-\varepsilon\}}$.

Set

$$\omega(z)=\psi(z)\varphi(z)^{-1/2} \tag{7.46}$$

and

$$I_{2,\theta}^{\pm}(z)=\frac{1}{2\pi i}\int_{\Gamma_\theta}\exp(-\lambda^2\varphi(z))\lambda^{\beta+3}\widetilde{W}_\theta^{\pm}(z,\lambda)(\log\lambda)d\lambda. \tag{7.47}$$

PROPOSITION 7.10. *Let* $|\arg\sqrt{\varphi(z)}-\theta|<\cos^{-1}\kappa-\pi/4$ $(0<\kappa<1/2)$. *Then there are* $\nu_j^{\pm}(z)$ $(j=1,2,\cdots)$ *holomorphic in a neighbourhood of* $z=0$, $p_\pm$ *integers or* $-\infty$ *and constants* $C_\varepsilon>0$, $p'_\pm\geq 0$ *such that*

(i) *if* $|\arg\omega(z)|<(\pi/4)-\varepsilon$,

$$I_{2,\theta}^{+}(z)\sim\left(\frac{\psi(z)}{\varphi(z)}\right)^{p_+}\exp\left(\frac{1}{4}\frac{\psi(z)^2}{\varphi(z)}\right)\left(\sum_{j=1}^{\infty}\nu_j^+(z)\omega(z)^{-j}\right) \tag{7.48}$$

as $|\omega(z)|\to\infty$, *and*

if $|\arg\omega(z)-\pi|<(3/4)\pi-\varepsilon$,

$$|I_{2,\theta}^{+}(z)|\leq C_\varepsilon|\psi(z)|^{-p'_+} \tag{7.49}$$

and

(ii) *if* $|\arg(-\omega(z))|<(\pi/4)-\varepsilon$

$$I_{2,\theta}^{-}(z)\sim\left(\frac{-\psi(z)}{\varphi(z)}\right)^{p_-}\exp\left(\frac{1}{4}\frac{\psi(z)^2}{\varphi(z)}\right)\left(\sum_{j=1}^{\infty}\nu_j^-(z)\omega(z)^{-j}\right) \tag{7.50}$$

as $|\omega(z)|\to\infty$, *and*

if $|\arg(-\omega(z)-\pi)|<(3/4)\pi-\varepsilon$,

$$|I^-_{2,\theta}(z)|\leq C_\varepsilon|\phi(z)|^{-p'_-}, \tag{7.51}$$

where if p_+ (*or* p_-)$=-\infty$, $I^+_{2,\theta}(z)$ (*or* $I^-_{2,\theta}(z)$)$=0$.

Proposition 7.10 follows from Lemma 7.4 and Lemma 7.6. We note that the (7.48)$\sim$(7.51) are valid, if we replace $\arg\omega(z)$ by $\arg\omega(z)+2m\pi$ ($m\in Z$). This follows from the proof of Lemma 7.6.

Now suppose that $|\arg\sqrt{\varphi(z)}-\theta|<\cos^{-1}\kappa-(\pi/4)$. Then $(I_{1,\theta}(z)-I_3(z))$ is bounded by Lemma 7.1 and Lemma 7.5. Set $I_\theta(z)=I^+_{2,\theta}(z)+I^-_{2,\theta}(z)-I_{1,\theta}(z)+I_3(z)$. The asymptotic behaviour of $I_\theta(z)$ is essentially dependent on $I^+_{2,\theta}(z)$ and $I^-_{2,\theta}(z)$.

Let us return to the integral representation (6.29) of $u(z)$. Put $\Gamma=\Gamma_{-\theta}$ and $C=C_\theta$. Then we have $u(z)=I_\theta(z)$. By applying above results we have the main theorem

THEOREM 7.11. *$u(z)$ behaves asymptotically near the characteristic surface $K=\{\varphi(z)=0\}$ as follows:*

Let $\alpha<\arg\varphi(z)<\beta$.

(i)

$$u(z)\sim\left(\frac{\phi(z)}{\varphi(z)}\right)^{p_+}\exp\left(\frac{1}{4}\frac{\phi(z)^2}{\varphi(z)}\right)\left(\sum_{j=1}^{\infty}\nu^+_j(z)\omega(z)^{-j}\right), \tag{7.52}$$

as $|\omega(z)|\to\infty$ *in the sector* $|\arg\omega(z)|<\pi/4-\varepsilon$ *and*

$$u(z)\sim\left(\frac{-\phi(z)}{\varphi(z)}\right)^{p_-}\exp\left(\frac{1}{4}\frac{\phi(z)^2}{\varphi(z)}\right)\left(\sum_{j=1}^{\infty}\nu^-_j(z)\omega(z)^{-j}\right), \tag{7.53}$$

as $|\omega(z)|\to\infty$ *in the sector* $|\arg(-\omega(z))|<\pi/4-\varepsilon$ *for any* $\varepsilon>0$, *where if* $p_+(p_-)=-\infty$, *$u(z)$ is bounded in* $|\arg\omega(z)|<\pi/4-\varepsilon$ ($|\arg-\omega(z)|<\pi/4-\varepsilon$).

(ii) *For any* $\varepsilon>0$ *there are* $C_{\alpha\beta\varepsilon}>0$ *and* $p'\geq 0$ *such that*

$$|u(z)|\leq C_{\alpha\beta\varepsilon}|\phi(z)|^{-p'} \tag{7.54}$$

in the domain $(\pi/4)+\varepsilon<|\arg\omega(z)|<(3/4)\pi-\varepsilon$.

PROOF. (i) and (ii) follow from Proposition 7.10 and the fact that $\tilde{w}^\pm_j(z)$ ($j=1, 2, \cdots$) are independent of θ.

REMARK 7.12. If $u_0(z')$ or $u_1(z')$ in (6.1) is not identically 0, at least one of p_+ or p_- is positive. This follows from the method of determination of $w_{n,j}(0, z')$.

By exchanging $h(\lambda)$ in the integral λ, we can get homogeneous solutions of $L(z, \partial)$ with singularities on K in various form. And by investigating integral representations, we obtain their informations near the characteristic surface K.

Thus Theorem 1 is proved. For the solution $u(z)$ of (0.5), $p_+=p_-=2$, $\tilde{w}_1^+(x, y) =\tilde{w}_1^-(x, y)=0$ and $\tilde{w}_2^+(x, y)=\tilde{w}_2^-(x, y)=1/2$. $u(x, y)$ is represented in the form

$$u(x, y)=2\int_0^\infty \exp(-\lambda^2 y)\cosh(\lambda x)\lambda\, d\lambda. \tag{7.55}$$

§8. Proof of Theorem 2.

In this section we show Theorem 2. To do so we need existence and uniqueness of Goursat's problem.

Consider the differential equation

$$(\partial_x)^\beta u(x)=\sum_{|\alpha|\le m} a_\alpha(x)(\partial_x)^\alpha u(x)+f(x) \qquad (|\beta|=m), \tag{8.1}$$

where $a_\alpha(x)$ and $f(x)$ are holomorphic on the polydisk $C_R=\{x\in C^{n+1};\ |x_i|\le R\}$. Pose the boundary conditions

$$(\partial_{x_i})^k(u(x)-v(x))=0 \quad \text{when } x_i=0 \quad \text{for } 0\le k\le \beta_i-1 \tag{8.2}$$

$(i=0, 1, \cdots, n)$, where $v(x)$ is also holomorphic on C_R.

The following lemma follows from Theorems 5.1.1 and 5.1.1′ in Hörmander [4].

LEMMA 8.1. *Let A be the set of multi-indices of the right hand side of* (8.1) *such that $a_\alpha(x)\not\equiv 0$, and assume that β does not belong to the convex hull of A considered as a subset in R^{n+1}. Then the boundary value problem* (8.1) *and* (8.2) *has one and only one holomorphic solution $u(x)$ in C_r for some r with $0<r<R$ which depends only on β, A and $M=\max_{x\in C_R}\sum_{|\alpha|=m}|a_\alpha(x)|$ and is independent of $f(x)$ and $v(x)$.*

We apply Lemma 8.1 to solutions of

$$(\partial_{x_0})^{\beta_0}(\partial_{x_1})^{\beta_1}u(x)=\sum_{|\alpha|\le m} a_\alpha(x)(\partial_x)^\alpha u(x)+f(x). \tag{8.3}$$

where $|\beta|=m$.

LEMMA 8.2. *Suppose that* (8.3) *satisfies the assumption of Lemma* 8.1. *Let $u(x)$ be holomorphic on $C_R-\{x_1=0\}$ and satisfy* (8.3) *and*

$$(\partial_{x_0})^k u(0, x')=0, \qquad 0\le k\le \beta_0-1. \tag{8.4}$$

If $f(x)$ is holomorphic on C_R, then $u(x)$ is holomorphic in a neighbourhood of $x=0$.

PROOF. Let $0<|\varepsilon|<R/2$. Consider the problem

$$(\partial_{x_0})^{\beta_0}(\partial_{x_1})^{\beta_1}u_\varepsilon(x)=\sum_{|\alpha|\le m} a_\alpha(x)(\partial_x)^\alpha u_\varepsilon(x)+f(x), \tag{8.5}$$

$$(\partial_{x_0})^k(u_\varepsilon(x)-v_\varepsilon(x))=0 \quad \text{when } x_0=0 \quad \text{for } 0\le k\le \beta_0-1, \tag{8.6}$$

and

(8.7) $$(\partial_{x_1})^k(u_\varepsilon(x)-v_\varepsilon(x))=0 \quad \text{when } x_1=\varepsilon \quad \text{for } 0\le k\le\beta_1-1\,,$$

where

(8.8) $$v_\varepsilon(x)=\sum_{p=0}^{\beta_1-1}\frac{(x_1-\varepsilon)^p}{p\,!}(\partial_{x_1})^p u(x_0,\,\varepsilon,\,x'')\,.$$

By the assumption $v_\varepsilon(x)$ is holomorphic on C_R and $(\partial_{x_0})^k v_\varepsilon(x)=0$ on $x_0=0$ if $0\le k\le\beta_0-1$. By Lemma 8.1 $u_\varepsilon(x)$ exists in $C_{r'}^\varepsilon=\{x\in C^{n+1};\,|x_0|<r',\,|x_1-\varepsilon|<r',\,|x''|<r'\}$ for some r' with $0<r'<R/2$ and independent of ε. Uniqueness implies that $u_\varepsilon(x)=u(x)$. Thus $u(x)$ is holomorphic in $\bigcup_{0<|\varepsilon|<(R/2)}\{x\in C^{n+1};\,|x_0|<r',\,|x_1-\varepsilon|<r',\,|x''|<r'\}$. This set contains $x=0$ as an interior point.

Let us return to the operator $L(z,\partial_z)=P(z,\partial_z)^2Q(z,\partial_z)+\cdots$. We shall apply Lemma 8.2 to $L(z,\partial_z)$. Since $\varphi(z)$ is a simple characteristic function of $P(z,\partial)$ with $\frac{\partial}{\partial\xi_0}p\Big(z,\frac{\partial\varphi(z)}{\partial z}\Big)\neq 0$ and $\varphi(0,z')=z_1$, it follows that by means of the coordinate transformation

(8.9) $$x_0=z_0,\qquad x_1=\varphi(z),\qquad x_j=z_j\quad (j\ge 2)$$

$P(z,\partial)u$ is written

(8.10) $$p(x,\partial_x)u=(a_0\partial_{x_0}+\sum_{j=2}^n a_j\partial_{x_j}+a)(\partial_{x_1})^{m_1-1}u+\cdots\,,$$

where the dots indicate terms of order (m_1-2) with respect to ∂_{x_1} and $a_0\neq 0$. Thus $L(z,\partial)u$ has the form

(8.11) $$L(x,\partial_x)u=c(x)(\partial_{x_0})^2(\partial_{x_1})^{m-2}u+\cdots\,,$$

where $c(0)\neq 0$ and the dots indicate terms of order $(m-3)$ with respect to ∂_{x_1}. Thus we have

THEOREM 8.3. *Suppose that $u(z)$ satisfy*

(8.12) $$L(z,\partial_z)u(z)=f(z)\,,$$

where $f(z)$ is holomorphic in a neighbourhood of $z=0$, and $u(0,z')$ and $\frac{\partial}{\partial z_0}u(0,z')$ are holomorphic in a neighbourhood of $z=0$. If $u(z)$ is holomorphic except on $K=\{\varphi(z)=0\}$, then $u(z)$ is holomorphic in a neighbourhood of $z=0$.

Theorem 2 in §1 is an easy consequence of Theorem 8.3.

Appendix

Let K be a holomorphic non-singular surface through $z=0$, that is, $K=\{z;\,\varphi(z)=0\}$, $\varphi(0)=0$ and $\mathrm{grad}_z\varphi(z)\neq 0$ on K. Suppose that K is a characteristic surface of $P(z, \partial)$, $p\left(z, \frac{\partial\varphi}{\partial z}\right)=0$ on K. We prove the following lemma which is shown in Komatsu [5] by a method different from ours.

LEMMA. *If* $\operatorname{grad}_\xi p\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0$ *on* K, *there is a function* $a(z)$ *holomorphic in a neighbourhood* U *of* $z=0$ *such that* $a(0)=1$ *and* $p\left(z, \frac{\partial(a(z)\varphi(z))}{\partial z}\right)\equiv 0$ *identically on* U.

PROOF. We seek for $a(z)$ to satisfy

$$p\left(z, \frac{\partial a}{\partial z}\varphi+a\frac{\partial\varphi}{\partial z}\right)\equiv 0. \tag{9.1}$$

By the Taylor expansion of $p(z, \xi)$, we have

$$p\left(z, a\frac{\partial\varphi}{\partial z}\right)+\varphi(z)\sum_{i=0}^{n}\frac{\partial a}{\partial z_i}\frac{\partial}{\partial\xi_i}p\left(z, a\frac{\partial\varphi}{\partial z}\right) + \sum_{2\le|\alpha|\le m}|\varphi|^{|\alpha|}\left(\frac{\partial a}{\partial z}\right)^{\alpha}a^{m-|\alpha|}g_\alpha(z)\equiv 0, \tag{9.2}$$

where m is the order of $P(z, \partial)$. By dividing (9.2) by $\varphi(z)a(z)^{m-1}$, we have

$$\sum_{i=0}^{n}\frac{\partial}{\partial\xi_i}p\left(z, \frac{\partial\varphi}{\partial z}\right)\frac{\partial a}{\partial z_i}+\varphi(z)\sum_{2\le|\alpha|\le m}\left(\frac{\partial a}{\partial z}\right)^{\alpha}a^{1-|\alpha|}h_\alpha(z)+h(z)a=0, \tag{9.3}$$

where $h(z)=p\left(z, \frac{\partial\varphi}{\partial z}\right)\Big/\varphi(z)$ which is holomorphic from the assumption.

Now we may assume that $\frac{\partial}{\partial\xi_0}p\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0$. We shall show that (9.3) has a solution $a(z)$ with $a(0, z')=1$. Since on $z_0=0$ (9.3) becomes

$$\frac{\partial p\left(z, \frac{\partial\varphi}{\partial z}\right)}{\partial\xi_0}\frac{\partial a}{\partial z_0}\bigg|_{z_0=0}+\varphi(0, z')\sum_{2\le\alpha_0\le m}\left(\frac{\partial a}{\partial z_0}\right)^{\alpha_0}h_{\alpha_0}(0, z')+h(0, z')\equiv 0 \tag{9.4}$$

and $\varphi(0)=0$, we have for small z

$$\frac{\partial a}{\partial z_0}=H\left(z, \frac{\partial a}{\partial z'}, a\right), \tag{9.5}$$

where $H(z, \xi', a)$ is holomorphic in a neighbourhood of $z=0$, $\xi'=0$ and $a=1$. (9.5) has a solution $a(z)$ with $a(0, z')=1$.

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