An integral representation of singular solutions of linear partial differential equations in the complex domain

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(Communicated by H. Komatsu)

Let $L(z, \partial)$ be a linear partial differential operator with coefficients holomorphic in Ω ($C^{n+1} \supset \Omega \ni 0$). Let $K = \{z; \varphi(z) = 0\}$ be a characteristic surface with constant multiplicity k. The purpose of this paper is to give an integral representation of homogeneous solutions of $L(z, \partial)$ with singularity on K and to show the Stokes phenomenon in the theory of partial differential equations.

According to the results due to De Paris [2], Hamada [3], [4], Persson [8], Wagschal [9] and Hamada, Leray and Wagschal [10], we can construct homogeneous solutions of $L(z, \partial)$ in a neighbourhood of $z=0$ which are holomorphic except on K.

Let us give an outline of their results. Suppose that the bicharacteristic curve of the characteristic surface K through $z=0$ is transversal to the surface $S = \{z_0 = 0\}$ and set $T = S \cap K$. Consider the problem

$$(0.1) \qquad \begin{cases} L(z, \partial)u(z) = 0 \\ \left(\dfrac{\partial}{\partial z_0}\right)^s u(0, z') = u_s(z') \qquad (0 \leqq s \leqq k-1), \end{cases}$$

where $u_s(z')$ is holomorphic in a neighbourhood ω of $z=0$ except on T. Then we can construct a solution $u(z)$ of (0.1) holomorphic in a neighbourhood of $z=0$ except on K as an infinite series. In particular if $u_s(z')$ has a pole or logarithmic singularity on T, we can find $u(z)$ in the form

$$(0.2) \qquad u(z) = \sum_{j=1}^{\infty} \frac{a_j(z)}{(\varphi(z))^j} + b(z)(\log \varphi(z)) + c(z),$$

where $a_j(z)$, $b(z)$ and $c(z)$ are holomorphic in a neighbourhood of $z=0$. If the characteristic surface K does not satisfy the Levi condition, in general $u(z)$ has an essential singularity on K. When z tends to K, expression (0.2) does not give sufficient informations of the asymptotic behaviour of $u(z)$.

We shall consider characteristic surfaces which do not satisfy the Levi condition. Our integral representation enables us to get informations of asymptotic behaviour of $u(z)$ near K.

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*) Partly supported by Fūjukai Foundation.

We shall find a homogeneous solution $u(z)$ of (0.1) in the form

(0.3) $$u(z)=\frac{1}{(2\pi qi)}\sum_{n=0}^{\infty}\int_{\Gamma}\exp(-\lambda\varphi(z))W_n(z, \lambda)(\log\lambda)d\lambda,$$

where

(0.4) $$W_n(z, \lambda)=\int_C \exp(-\lambda^\alpha\zeta)w_n(z, \lambda, \zeta)d\zeta,$$

(0.5) $$w_n(z, \lambda, \zeta)=\sum_{-\infty<j\leq p_n} w_{n,j}(z, \lambda)f_j(\zeta).$$

Here the functions $f_j(\zeta)$ satisfy $\frac{df_j(\zeta)}{d\zeta}=f_{j-1}(\zeta)$, $\alpha=1-\sigma^{-1}=l/q$ $(l, q\in \boldsymbol{N})$, where σ is the irregularity of K, and Γ and C are suitable paths. We shall also give an integral representation of $u(z)$ derived from (0.3) which is convenient for the analysis of asymptotic behaviour. We note that if there is another solution $\tilde{u}(z)$ of (0.1) holomorphic except on K, then $u(z)-\tilde{u}(z)$ is holomorphic at $z=0$ (see Ōuchi [7]).

In §1 we shall give notations, definitions and assumptions on $L(z, \partial)$. We shall define the subcharacteristic polynomial of $L(z, \partial)$ which plays a very important role in our investigation.

In §2 we shall determine $W(z, \lambda)=\sum_{n=0}^{\infty} W_n(z, \lambda)$ so as to satisfy

(0.6) $$L(z, \partial)\exp(-\lambda\varphi(z))W(z, \lambda)=0$$

formally. By the integral transformation (0.4), the multiplication by λ^α is transformed to the differential operator $\frac{\partial}{\partial\zeta}$. Hence for some operator $\mathcal{L}\left(z, \lambda, \frac{\partial}{\partial z}, \frac{\partial}{\partial\zeta}\right)$, the function $w(z, \lambda, \zeta)=\sum_{n=0}^{\infty} w_n(z, \lambda, \zeta)$ satisfies

(0.7) $$\mathcal{L}\left(z, \lambda, \frac{\partial}{\partial z}, \frac{\partial}{\partial\zeta}\right)w(z, \lambda, \zeta)=0.$$

$\mathcal{L}\left(z, \lambda, \frac{\partial}{\partial z}, \frac{\partial}{\partial\zeta}\right)$ has relations with the subcharacteristic polynomial of $L(z, \partial)$. We shall give a kind of formal solution $w(z, \lambda, \zeta)$ of (0.7) of the form (0.5). Therefore our method of construction is different from that of others.

In §3 we shall estimate $w_{n,j}(z, \lambda)$.

In §4 by making use of the estimates of $w_{n,j}(z, \lambda)$, we shall construct a solution $u(z)$ of the form (0.3).

In §5 we shall set an assumption on the subcharacteristic polynomial. Under it we shall construct another solution $\tilde{w}(z, \lambda, \zeta)$ of (0.7) in a neighbourhood of $\zeta=0$ by the method of Hamada, Wagschal and others.

In §6 we shall show that the difference $w(z, \lambda, \zeta)-\tilde{w}(z, \lambda, \zeta)$ is holomorphic in some neighbourhood of $z=\zeta=0$.

In §7 replacing $w(z, \lambda, \zeta)$ by $\tilde{w}(z, \lambda, \zeta)$ near $\zeta=0$, we shall give an integral representation which consists of three parts $u_{\mathrm{I}}(z)$, $u_{\mathrm{II}}(z)$ and $u_{\mathrm{III}}(z)$.

In §8 we shall investigate the asymptotic behaviour of $u(z)$, when z tends to the characteristic surface K under the condition that the trace of $u(z)$ on S, $\left(\frac{\partial}{\partial z_0}\right)^s u(0, z')$ $(0\leqq s\leqq k-1)$, has a pole on T. We shall clarify that $u_{\mathrm{I}}(z)$ is the most essential part of the three.

We show our method by a simple example:

Consider

$$(0.8)\qquad \begin{cases} L(z, \partial)u(z)=\left\{\left(\frac{\partial}{\partial z_0}\right)^3+a\left(\frac{\partial}{\partial z_1}\right)\right\}u(z)=0 \\ u(0, z_1)=\left(\frac{\partial}{\partial z_0}\right)u(0, z_1)=0\,, \quad \left(\frac{\partial}{\partial z_0}\right)^2 u(0, z_1)=\frac{1}{z_1}\,, \end{cases}$$

where $z=(z_0, z_1)$ and a is a non-zero constant. $K=\{\varphi(z)=z_1=0\}$ and $\sigma=3$, $\alpha=1/3$. $u(z)$ is expressed in the form

$$(0.9)\qquad u(z)=\sum_{n=0}^{\infty}\frac{n!\,a^n}{(3n+2)!}\frac{(z_0)^{3n+2}}{(z_1)^{n+1}}.$$

This expression, however, does not give sufficient informations of $u(z)$ near K.

Following our method, we have

$$(0.10)\qquad \mathcal{L}\left(z, \lambda, \frac{\partial}{\partial z}, \frac{\partial}{\partial \zeta}\right)=\left(\frac{\partial}{\partial z_0}\right)^3-a\left(\frac{\partial}{\partial \zeta}\right)^3+a\left(\frac{\partial}{\partial z_1}\right).$$

The solution of

$$(0.11)\qquad \begin{cases} \mathcal{L}\left(z, \lambda, \frac{\partial}{\partial z}, \frac{\partial}{\partial \zeta}\right)w(z, \lambda, \zeta)=0\,, \\ w(0, z_1, \lambda, \zeta)=\left(\frac{\partial}{\partial z_0}\right)w(0, z_1, \lambda, \zeta)=0\,, \quad \left(\frac{\partial}{\partial z_0}\right)^2 w(0, z_1, \lambda, \zeta)=\frac{\lambda^{1/3}}{(2\pi i)}\log\zeta\,, \end{cases}$$

has the following representation:

$$(0.12)\qquad w(z, \lambda, \zeta)=\sum_{k=0}^{2} w^k(z, \lambda, \zeta)\,,$$

where

$$(0.13)\qquad w^k(z, \lambda, \zeta)=\frac{\lambda^{1/3}\omega^k}{12\pi i\; a^{2/3}}(\zeta+a^{1/3}\omega^k z_0)^2(\log(\zeta+a^{1/3}\omega^k z_0)-3/2)\,,$$

$$\omega=\exp(i(2/3)\pi)\,.$$

Hence the solution $u(z)$ of (0.8) is also represented for $\mathrm{Re}\, z_1>0$ in the form

(0.14)
$$u(z)=u_{\mathrm{I}}(z)=\sum_{k=0}^{2} u_{\mathrm{I},k}(z),$$

where

(0.15)
$$u_{\mathrm{I},k}(z)=\int_0^\infty \exp(-\lambda z_1)d\lambda\int_C \exp(-\lambda^{1/3}\zeta)w^k(z,\lambda,\zeta)d\zeta .$$

C is an infinite contour enclosing the singularities of $w(z, \lambda, \zeta)$. In this case $u_{\mathrm{II}}(z)=u_{\mathrm{III}}(z)=0$.

To get informations of $u(z)$ near K, we first study the behaviour of the function

(0.16)
$$F_k(z,\lambda)=\int_C \exp(-\lambda^{1/3}\zeta)w^k(z,\lambda,\zeta)d\zeta$$

as $|\lambda|\to\infty$ and next investigate the integral in λ. For this simple equation we have

(0.17)
$$F_k(z,\lambda)=\frac{\omega^k}{3(\lambda a)^{2/3}}\exp((\lambda a)^{1/3}\omega^k z_0).$$

Hence we have

(0.18)
$$u_{\mathrm{I},0}(z)=\frac{1}{3a^{2/3}}\int_0^\infty \exp(-\lambda z_1+(\lambda a)^{1/3}z_0)\lambda^{-2/3}d\lambda$$
$$=a^{-2/3}z_1^{-1/3}\int_0^\infty \exp(-\mu^3+(a^{1/3}z_0z_1^{-1/3})\mu)d\mu .$$

Set $\tau(z)=a^{1/3}z_0z_1^{-1/3}$. Then we have

(0.19)
$$u_{\mathrm{I},0}(z)=a^{-2/3}z_1^{-1/3}\int_0^\infty \exp(-\mu^3+\tau\mu)d\mu .$$

Thus we have for any $\varepsilon>0$ the following results:
(i) if $|\arg\tau(z)|<(1/3)\pi-\varepsilon$,

(0.20)
$$u_{\mathrm{I},0}(z)=a^{-2/3}z_1^{-1/3}\tau(z)^{-1/4}\exp\left(\frac{2}{3\sqrt{3}}\tau(z)^{3/2}\right)(A+O(\tau(z)^{-3/2})),$$

as $|\tau(z)|\to\infty$, where A is a positive constant, and
(ii) if $|\arg(-\tau(z))|<(2/3)\pi-\varepsilon$, there is a constant C_ε such that

(0.21)
$$|z_1^{1/3}u_{\mathrm{I},0}(z)|\leqq C_\varepsilon|\tau(z)|^{-1}$$

as $|\tau(z)|\to\infty$.

For general equations, solutions of $\mathcal{L}\left(z,\lambda,\frac{\partial}{\partial z},\frac{\partial}{\partial\zeta}\right)w(z,\lambda,\zeta)=0$ have not always a global expression in ζ such as (0.12) and (0.13). To get around this difficulty we shall use $w(z,\lambda,\zeta)$ and $\tilde{w}(z,\lambda,\zeta)$.

When K is of double characteristic, similar results have been obtained in Ōuchi [7].

§1. Let $z=(z_0, z_1, \cdots, z_n)$ denote a point in the complex $(n+1)$-dimensional space C^{n+1}. We write $z=(z_0, z')$, $z'=(z_1, z_2, \cdots, z_n)$. The dual variable of z is denoted by $\xi=(\xi_0, \xi_1, \cdots, \xi_n)$. Let

$$L(z, \partial)=\sum_{|\alpha|\leq m} a_\alpha(z)\partial^\alpha \tag{1.1}$$

be a linear partial differential operator of order m with coefficients $a_\alpha(z)$ holomorphic in a neighbourhood Ω of $z=0$. Recall that a multi-index α is an $(n+1)$-tuple $(\alpha_0, \alpha_1, \cdots, \alpha_n)$ of non-negative integers and

$$|\alpha|=\alpha_0+\alpha_1+\cdots+\alpha_n\,. \tag{1.2}$$

We employ usual notations

$$\partial^\alpha=\partial_0^{\alpha_0}\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}=\left(\frac{\partial}{\partial z_0}\right)^{\alpha_0}\left(\frac{\partial}{\partial z_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial z_n}\right)^{\alpha_n} \tag{1.3}$$

and

$$\xi^\alpha=\xi_0^{\alpha_0}\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}\,. \tag{1.4}$$

We shall denote the principal symbol of the operator $L(z, \partial)$ by

$$l(z, \xi)=\sum_{|\alpha|=m} a_\alpha(z)\xi^\alpha\,. \tag{1.5}$$

Let $K=\{z\in\Omega\,;\ \varphi(z)=0\}$ be a non-singular surface, where $\varphi(z)$ is a holomorphic function defined on Ω such that $\varphi(0, z')=z_1$ and set $S=\{z_0=0\}$. From now on we assume that K is a characteristic surface with constant multiplicity k. More precisely we impose the following conditions:

The principal symbol $l(z, \xi)$ of $L(z, \partial)$ is expressed in the form

$$l(z, \xi)=p(z, \xi)^k q(z, \xi)\,, \tag{1.6}$$

where $p(z, \xi)$ and $q(z, \xi)$ are homogeneous polynomials of degree m_1 and m_2 $(km_1+m_2=m)$ respectively with coefficients holomorphic on Ω. Moreover

$$q\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0 \quad \text{on } K \tag{1.7}$$

and

$$p\left(z, \frac{\partial\varphi}{\partial z}\right)=0 \quad \text{and } \operatorname{grad}_\xi p\left(z, \frac{\partial\varphi}{\partial z}\right)\neq 0 \quad \text{on } K. \tag{1.8}$$

Now we note that it follows from (1.8) that there exists a holomorphic function $a(z)$ such that $a(0)=1$ and the function $\tilde{\varphi}(z)=a(z)\varphi(z)$ satisfies $p\left(z, \frac{\partial\tilde{\varphi}}{\partial z}\right)=0$ identically. (See Komatsu [6] or Ōuchi [7].) Hence hereafter we may assume that $p\left(z, \frac{\partial\varphi}{\partial z}\right)=0$ identically.

Secondly we assume that the bicharacteristic curve of the characteristic surface K through $z=0$ is transversal to the surface S. This implies

$$\frac{\partial p}{\partial \xi_0}\left(0, \frac{\partial\varphi}{\partial z}(0)\right)\neq 0. \tag{1.9}$$

For operators with constant multiple characteristics we have

LEMMA 1.1. *There exist non-negative integers or* $+\infty$, $k_0, k_1, \cdots, k_m=k$ *and linear homogeneous partial differential operators* $Q_i(z, \partial)$ *such that*

$$L(z, \partial)=\sum_{i=0}^{m} Q_i(z, \partial)P(z, \partial)^{k_i} \tag{1.10}$$

and that if $k_i<\infty$, $Q_i(z, \partial)\not\equiv 0$, $p(z, \xi)$ *does not divide* $q_i(z, \xi)$ *(the symbol fo* $Q_i(z, \partial)$*) and* ord. $Q_i(z, \partial)P(z, \partial)^{k_i}=i$ *and if* $k_i=+\infty$, $Q_i(z, \partial)\equiv 0$.

For the details we refer to Komatsu [6].

Set

$$\Sigma_p=\{(z, \xi);\ p(z, \xi)=0,\quad z\in\Omega,\quad \xi\neq 0\}. \tag{1.11}$$

DEFINITION 1.2. The irregularity σ of the characteristic elements Σ_p is defined by

$$\sigma=\max\{1, (k-k_i)/(m-i);\ m-k+1\leqq i\leqq m-1\}. \tag{1.12}$$

The irregularity of characteristic elements is introduced in Komatsu [6] and it is independent of the representation (1.10) of $L(z, \partial)$. We note that if $k_i>k-m+i$ $(m-k+1\leqq i\leqq m-1)$, then from (1.10) we have $Q_i(z, \partial)P(z, \partial)^{k_i}= Q_i(z, \partial)P(z, \partial)^{k_i+m-i-k}P(z, \partial)^{k+i-m}=\tilde{Q}_i(z, \partial)P(z, \partial)^{k_i'}$, $k_i'=k+i-m$. So we may assume that

$$k_i\leqq\max(0, k-m+i)\qquad (i=0, 1, \cdots, m), \tag{1.13}$$

because this assumption does not change the irregularity σ.

From now on we assume that $\sigma>1$. We note this implies $k\geqq 2$.

DEFINITION 1.3. The subcharacteristic polynomial $l_{\mathrm{sub}}(p;z, \xi, \eta)$ of $L(z, \partial)$ is defined by

$$l_{\mathrm{sub}}(p;z, \xi, \eta)=\sum_{i\in\Delta_0} q_i(z, \xi)\eta^{k_i}, \tag{1.14}$$

where

$$\Delta_0=\{i;\ \sigma=(k-k_i)/(m-i),\quad m-k+1\leqq i\leqq m-1\}\cup\{m\} \tag{1.15}$$

and $(z, \xi)\in\Sigma_p$.

The subcharacteristic polynomial will play an important role in §5~§8.

§2. In this section we shall give a kind of homogeneous solution $U(z, \lambda)$ of $L(z, \partial)$ with a parameter λ. Later we shall construct a homogeneous solution by integrating it with respect to λ.

Now let us consider the problem

$$\begin{cases} L(z, \partial)U(z, \lambda)=0\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s U(0, z', \lambda)=\exp(-\lambda z_1)\delta_{s,t}\hat{v}(z', \lambda)\,, \qquad 0\leqq s\leqq k-1\,, \end{cases} \tag{2.1}$$

where $\delta_{s,t}$ is Kronecker's delta, $0\leqq t\leqq k-1$ and $\hat{v}(z', \lambda)$ is a holomorphic function in λ and z' $(z'\in S\cap\Omega,\ \lambda\in C^1)$.

We seek for $U(z, \lambda)$ of the form

$$U(z, \lambda)=\exp(-\lambda\varphi(z))W(z, \lambda)\,, \tag{2.2}$$

where

$$W(z, \lambda)=\sum_{n=0}^{\infty} W_n(z, \lambda)=\sum_{n=0}^{\infty}\int_C \exp(-\lambda^\alpha\zeta)w_n(z, \lambda, \zeta)d\zeta\,. \tag{2.3}$$

The sum is formal and the path C will be determined later.

We give a lemma which will be often used and is a consequence of the Leibniz formula of differential operators.

LEMMA 2.1. *Let $M(z, \partial)$ be a linear partial differential operator of order m with the principal symbol $m(z, \xi)$. Then for holomorphic functions $\Phi(z)$ and $v(z)$ we have*

$$M(z, \partial)\{\exp(\mu\Phi(z))v(z)\}=\exp(\mu\Phi(z))\sum_{i=0}^{m}\mu^{m-i}\mathcal{M}_i(z, \partial)v(z)\,, \tag{2.4}$$

where $\mathcal{M}_i(z, \partial)$ is a linear partial differential operator of order i and is independent of $v(z)$. In particular

$$\mathcal{M}_0(z, \partial)=m\left(z, \frac{\partial\Phi}{\partial z}\right) \tag{2.5}$$

and

$$\mathcal{M}_1(z, \partial)=\sum_{i=0}^{n}\frac{\partial m\left(z, \dfrac{\partial\Phi}{\partial z}\right)}{\partial\xi_i}\frac{\partial}{\partial z_i}+a\ \text{function}. \tag{2.6}$$

The proof is omitted.

Let us derive an equation which $W(z, \lambda)$ satisfies. Substituting (2.2) into

(2.1), we have, by Lemma 1.1 and Lemma 2.1,

$$L(z, \partial)U(z, \lambda)=\sum_{i=0}^{m} Q_i(z, \partial)P(z, \partial)^{k_i}U(z, \lambda) \tag{2.7}$$

$$=\exp(-\lambda\varphi(z))\mathcal{L}(\lambda)W(z, \lambda),$$

where

$$\mathcal{L}(\lambda)=\sum_{i=0}^{m} \lambda^{i-k_i}\left\{\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)(\mathcal{P}_1)^{k_i}+\sum_{j=1}^{i-k_i}\lambda^{-j}\mathcal{L}_{i,k_i+j}\right\}, \tag{2.8}$$

$\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)=(-1)^{i-k_i}q_i\left(z, \frac{\partial\varphi}{\partial z}\right)$, $\mathcal{L}_{i,s}$ is an operator of order s and

$$\mathcal{P}_1=\sum_{i=0}^{n}\frac{\partial p\left(z, \frac{\partial\varphi}{\partial z}\right)}{\partial\xi_i}\frac{\partial}{\partial z_i}+\text{a function}. \tag{2.9}$$

Set

$$\alpha=1-\sigma^{-1}=l/q, \quad l, q\in \boldsymbol{N}, \quad (l, q)=1. \tag{2.10}$$

For $i\in\Delta_0$, we have

$$\lambda^{i-k_i}\left\{\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)(\mathcal{P}_1)^{k_i}+\sum_{j=1}^{i-k_i}\lambda^{-j}\mathcal{L}_{i,k_i+j}\right\} \tag{2.11}$$

$$=\lambda^{i-k_i-\alpha(m-k_i)}\left\{\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)\lambda^{\alpha(m-k_i)}(\mathcal{P}_1)^{k_i}+\sum_{j=1}^{i-k_i}\lambda^{\alpha(m-k_i)-j}\mathcal{L}_{i,k_i+j}\right\}$$

$$=\lambda^{(m-k)(1-\alpha)}\left\{\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)\lambda^{\alpha(m-k_i)}(\mathcal{P}_1)^{k_i}+\sum_{j=1}^{i-k_i}\lambda^{\alpha(m-k_i-j)-(1-\alpha)j}\mathcal{L}_{i,k_i+j}\right\}.$$

For $i\notin\Delta_0$, there exist a $\beta_i>0$ such that

$$i-k_i=\alpha(k-k_i)+(m-k)-\beta_i \tag{2.12}$$

and $q\beta_i\in\boldsymbol{N}$. Hence we have

$$\lambda^{i-k_i}\left\{\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)(\mathcal{P}_1)^{k_i}+\sum_{j=1}^{i-k_i}\lambda^{-j}\mathcal{L}_{i,k_i+j}\right\} \tag{2.13}$$

$$=\lambda^{(m-k)(1-\alpha)}\lambda^{-\beta_i}\left\{\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)\lambda^{\alpha(m-k_i)}(\mathcal{P}_1)^{k_i}+\sum_{j=1}^{i-k_i}\lambda^{\alpha(m-k_i-j)}\lambda^{-(1-\alpha)j}\mathcal{L}_{i,k_i+j}\right\}.$$

Thus we have

$$\mathcal{L}(\lambda)=\lambda^{(m-k)(1-\alpha)}\left[\sum_{i\in\Delta_0}\left\{\tilde{q}_i\left(z, \frac{\partial\varphi}{\partial z}\right)\lambda^{\alpha(m-k_i)}(\mathcal{P}_1)^{k_i}\right.\right. \tag{2.14}$$

$$\left.+\sum_{j=1}^{i-k_i}\lambda^{\alpha(m-k_i-j)}\lambda^{-(1-\alpha)j}\mathcal{L}_{i,k_i+j}\right\}$$

$$+\sum_{i\in\Delta_0^c}\left\{\tilde{q}_i\left(z,\frac{\partial\varphi}{\partial z}\right)\lambda^{\alpha(m-k_i)}\lambda^{-\beta_i}(\mathcal{P}_1)^{k_i}+\sum_{j=1}^{i-k_i}\lambda^{\alpha(m-k_i-j)}\lambda^{-(1-\alpha)j-\beta_i}\mathcal{L}_{i,k_i+j}\right\}\Bigg]$$

$$=\lambda^{(m-k)(1-\alpha)}\tilde{\mathcal{L}}(\lambda).$$

Recall that we find $W(z,\lambda)$ in the form (2.3). The path C will be chosen so as to justify the following calculations of integration by parts. From (2.14) we get

$$\text{(2.15)}\qquad \mathcal{L}(\lambda)W_n(z,\lambda)=\lambda^{(m-k)(1-\alpha)}\tilde{\mathcal{L}}(\lambda)\int_C\exp(-\lambda^{\alpha}\zeta)w_n(z,\lambda,\zeta)d\zeta$$

$$=\lambda^{(m-k)(1-\alpha)}\sum_{i\in\Delta_0}\left\{\int_C\exp(-\lambda^{\alpha}\zeta)\left[\tilde{q}_i\left(z,\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial}{\partial\zeta}\right)^{m-k_i}(\mathcal{P}_1)^{k_i}\right.\right.$$

$$\left.\left.+\sum_{j=1}^{i-k_i}\lambda^{-(1-\alpha)j}\left(\frac{\partial}{\partial\zeta}\right)^{(m-k_i-j)}\mathcal{L}_{i,k_i+j}w_n(z,\lambda,\zeta)\right]d\zeta\right\}$$

$$+\sum_{i\in\Delta_0^c}\left\{\int_C\exp(-\lambda^{\alpha}\zeta)\left[\tilde{q}_i\left(z,\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial}{\partial\zeta}\right)^{m-k_i}\lambda^{-\beta_i}(\mathcal{P}_1)^{k_i}\right.\right.$$

$$\left.\left.+\sum_{j=1}^{i-k_i}\left(\frac{\partial}{\partial\zeta}\right)^{(m-k_i-j)}\lambda^{-\beta_i-(1-\alpha)j}\mathcal{L}_{i,k_i+j}w_n(z,\lambda,\zeta)\right]d\zeta\right\}.$$

Therefore we obtain

$$\text{(2.16)}\qquad \mathcal{L}(\lambda)W_n(z,\lambda)=\lambda^{(m-k)(1-\alpha)}\int_C\exp(-\lambda^{\alpha}\zeta)\mathcal{L}(z,\lambda,\partial_z,\partial_\zeta)w_n(z,\lambda,\zeta)d\zeta,$$

where

$$\text{(2.17)}\qquad \mathcal{L}(z,\lambda,\partial_z,\partial_\zeta)=\left(\frac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))+\sum_{l=k+1}^{m}\left(\frac{\partial}{\partial\zeta}\right)^{m-l}\lambda^{-(1-\alpha)(l-k)}\mathcal{L}_l(\lambda),$$

$$\text{(2.18)}\qquad \mathcal{L}_k^0=\sum_{i\in\Delta_0}\tilde{q}_i\left(z,\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial}{\partial\zeta}\right)^{k-k_i}(\mathcal{P}_1)^{k_i},$$

$$\text{(2.19)}\qquad \mathcal{L}_k^1(\lambda)=\sum_{i\in\Delta_0^c}\tilde{q}_i\left(z,\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial}{\partial\zeta}\right)^{k-k_i}\lambda^{-\beta_i}(\mathcal{P}_1)^{k_i}$$

and $\mathcal{L}_l(\lambda)$ $(l=k+1,\cdots,m)$ are operators of ∂_z order at most l with a parameter λ and their coefficients are polynomials of $\lambda^{-1/q}$ with coefficients holomorphic in z.

We shall determine $w_n(z,\lambda,\zeta)$ successively as follows:

$$\text{(2.20)}\qquad \left(\frac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))w_0=0,$$

$$(2.21)\quad \left(\frac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))w_{n+1}+\sum_{l=k+1}^{m}\left(\frac{\partial}{\partial\zeta}\right)^{m-l}\lambda^{-(1-\alpha)(l-k)}\mathcal{L}_l(\lambda)w_{n+1+k-l}=0\,.$$

Formally $w(z, \lambda, \zeta)=\sum_{n=0}^{\infty} w_n(z, \lambda, \zeta)$ satisfies $\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)w(z, \lambda, \zeta)=0$. To construct $w_n(z, \lambda, \zeta)$ we shall introduce auxiliary functions $f_j(\zeta)$ used in Hamada [3] and others:

$$(2.22)\quad f_j(\zeta)=\frac{1}{2\pi i}\frac{\zeta^j}{\Gamma(j+1)}(\log\zeta-\gamma_{j+1})\qquad (j\geqq 0)\,,$$

$$\gamma_{j+1}=-\gamma_0+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{j},\ \gamma_0\text{: Euler constant,}$$

and

$$(2.23)\quad f_j(\zeta)=\frac{(-1)^{j+1}}{2\pi i}\Gamma(-j)\zeta^j\qquad (j<0)\,.$$

We note that

$$(2.24)\quad \frac{d}{d\zeta}f_j(\zeta)=f_{j-1}(\zeta)\,.$$

By making use of $f_j(\zeta)$, we shall find $w_n(z, \lambda, \zeta)$ in the form

$$(2.25)\quad w_n(z, \lambda, \zeta)=\sum_{j=-\infty}^{p_n} w_{n,j}(z, \lambda)f_j(\zeta)\,,$$

where p_n will be determined later.

Now let us determine initial conditions for $w_n(z, \lambda, \zeta)$ so as to satisfy conditions (2.1). We have

$$(2.26)\quad \left(\frac{\partial}{\partial z_0}\right)^s W_n(z, \lambda)=\int_C \exp(-\lambda^\alpha\zeta)\left(\frac{\partial}{\partial z_0}\right)^s w_n(z, \lambda, \zeta)d\zeta\,.$$

For $n\geqq 1$, we impose the conditions

$$(2.27)\quad \left(\frac{\partial}{\partial z_0}\right)^s w_n(0, z', \lambda, \zeta)=0\,,\qquad 0\leqq s\leqq k-1\,,$$

so that we have

$$(2.28)\quad \left(\frac{\partial}{\partial z_0}\right)^s W_n(0, z', \lambda)=0\,,\qquad 0\leqq s\leqq k-1\,.$$

Then the initial conditions $\left(\frac{\partial}{\partial z_0}\right)^s U(0, z', \lambda)$ for $0\leqq s\leqq k-1$ are written

$$(2.29)\quad \left(\frac{\partial}{\partial z_0}\right)^s U(0, z', \lambda)$$

$$=\exp(-\lambda z_1)\left\{\left(\frac{\partial}{\partial z_0}\right)^s W_0(0, z', \lambda)+\sum_{l=0}^{s-1}\lambda^{s-l}\mathcal{A}_l^s\left(z', \frac{\partial}{\partial z_0}\right)W_0(0, z', \lambda)\right\}$$

with operators $\mathcal{A}_l^s\left(z', \dfrac{\partial}{\partial z_0}\right)$ of order l.

Set $p_0=0$. So

$$W_0(z, \lambda)=\int_C \sum_{j=-\infty}^{0} \exp(-\lambda^\alpha\zeta) w_{0,j}(z, \lambda) f_j(\zeta) d\zeta . \tag{2.30}$$

Let us impose for $j\leqq -1$

$$\left(\frac{\partial}{\partial z_0}\right)^s w_{0,j}(0, z', \lambda)=0, \qquad 0\leqq s\leqq k-1 . \tag{2.31}$$

Then we have

$$\left(\frac{\partial}{\partial z_0}\right)^s U(0, z', \lambda)=\exp(-\lambda z_1)\int_C \exp(-\lambda^\alpha\zeta)\Big\{\left(\frac{\partial}{\partial z_0}\right)^s w_{0,0}(z, \lambda) + \sum_{l=0}^{s-1} \lambda^{s-l}\mathcal{A}_l^s\left(z', \frac{\partial}{\partial z_0}\right) w_{0,0}(z, \lambda)\Big\}\Big|_{z_0=0} f_0(\zeta) d\zeta . \tag{2.32}$$

If we can choose the path C so that it starts at ∞, encloses the origin $\zeta=0$ in the positive direction and ends to ∞, it follows from the equality

$$\int_C \exp(-\lambda^\alpha\zeta) f_0(\zeta) d\zeta=\frac{1}{2\pi i}\int_C \exp(-\lambda^\alpha\zeta)(\log\zeta) d\zeta=\lambda^{-\alpha} \tag{2.33}$$

that

$$\left(\frac{\partial}{\partial z_0}\right)^s U(0, z', \lambda) =\exp(-\lambda z_1)\Big\{\left(\frac{\partial}{\partial z_0}\right)^s w_{0,0}+ \sum_{l=0}^{s-1} \lambda^{s-l}\mathcal{A}_l^s\left(z', \frac{\partial}{\partial z_0}\right) w_{0,0}\Big\}\Big|_{z_0=0}\lambda^{-\alpha} . \tag{2.34}$$

Let us define the initial conditions for $w_{0,0}(z, \lambda)$ as follows:

$$\left(\frac{\partial}{\partial z_0}\right)^s w_{0,0}+ \sum_{l=0}^{s-1} \lambda^{s-l}\mathcal{A}_l^s\left(z', \frac{\partial}{\partial z_0}\right) w_{0,0}\Big|_{z_0=0}\lambda^{-\alpha}=\delta_{s,t}\hat{v}(z', \lambda), \qquad 0\leqq s\leqq k-1 . \tag{2.35}$$

From (2.35) we have for $0\leqq s\leqq t-1$

$$\left(\frac{\partial}{\partial z_0}\right)^s w_{0,0}|_{z_0=0}=0 . \tag{2.36}$$

If $t\leqq s\leqq k-1$, there exists $\tilde{v}_s(z', \lambda)$, which depends only on the derivatives of $\hat{v}(z', \lambda)$, and $\tilde{v}_t(z', \lambda)=\hat{v}(z, \lambda)$ such that

$$\left(\frac{\partial}{\partial z_0}\right)^s w_{0,0}(0, z', \lambda)=\lambda^{s-t+\alpha}\tilde{v}_s(z', \lambda) . \tag{2.37}$$

Thus, let us determine $w_n(z, \lambda, \zeta)=\sum_{j=-\infty}^{p_n} w_{n,j}(z, \lambda) f_j(\zeta)$ so as to satisfy the equations with initial conditions

$$
(2.38)\qquad \begin{cases} \left(\dfrac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))w_0(z,\lambda,\zeta)=0\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_0(0,z',\lambda,\zeta)=0\,, \qquad 0\leqq s\leqq t-1\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_0(0,z',\lambda,\zeta)=\lambda^{s-t+\alpha}\tilde{v}_s(z',\lambda)f_0(\zeta)\,, \qquad t\leqq s\leqq k-1 \end{cases}
$$

and

$$
(2.39)\qquad \begin{cases} \left(\dfrac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))w_{n+1}(z,\lambda,\zeta) \\ \quad +\displaystyle\sum_{l=k+1}^{m}\left(\dfrac{\partial}{\partial\zeta}\right)^{m-l}\lambda^{-(1-\alpha)(l-k)}\mathcal{L}_l(\lambda)w_{n+k+1-l}(z,\lambda,\zeta)=0 \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_{n+1}(0,z',\lambda,\zeta)=0\,, \qquad 0\leqq s\leqq k-1\,. \end{cases}
$$

We shall show the equations (2.38) and (2.39) have a formal solution. Substituting $w_n(z,\lambda,\zeta)=\sum_{j=-\infty}^{p_n} w_{n,j}(z,\lambda)f_j(\zeta)$ into the equations, we have

$$
(2.40)\qquad \begin{cases} L_k w_{0,j}+\displaystyle\sum_{0\leq i\leq m-1} L_{k_i}(\lambda)w_{0,j+k-k_i}=0\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_{0,0}\Big|_{z_0=0}=0 \qquad (0\leqq s\leqq t-1)\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_{0,0}\Big|_{z_0=0}=\lambda^{s-t+\alpha}\tilde{v}_s(z',\lambda) \qquad (t\leqq s\leqq k-1)\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)w_{0,j}\Big|_{z_0=0}=0 \qquad (j\leqq -1,\ 0\leqq s\leqq k-1) \end{cases}
$$

and

$$
(2.41)\qquad \begin{cases} L_k w_{n+1,j}+\displaystyle\sum_{0\leq i\leq m-1} L_{k_i}(\lambda)w_{n+1,j+k-k_i} \\ \quad +\displaystyle\sum_{l=k+1}^{m}\lambda^{-(1-\alpha)(l-k)}\mathcal{L}_l(\lambda)w_{n+k+1-l,j+k-l}=0\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_{n+1,j}\Big|_{z_0=0}=0 \qquad (0\leqq s\leqq k-1)\,. \end{cases}
$$

Here

$$
(2.42)\qquad L_k=\tilde{q}\left(z,\frac{\partial\varphi}{\partial z}\right)\left(\sum_{i=0}^{n}\frac{\partial p\left(z,\dfrac{\partial\varphi}{\partial z}\right)}{\partial\xi_i}\frac{\partial}{\partial z_i}+\text{ a function}\right)^k
$$

and

$$(2.43)\qquad L_{k_i}(\lambda)=\tilde{q}_i\Big(z, \frac{\partial\varphi}{\partial z}\Big)\lambda^{-\beta_i}\Big(\sum_{i=0}^{n}\frac{\partial p\Big(z, \dfrac{\partial\varphi}{\partial z}\Big)}{\partial\xi_i}\frac{\partial}{\partial z_i}+\text{ a function}\Big)^{k_i}.$$

L_{k_i} is an ordinary differential operator of order k_i along the bicharacteristic curves on $\varphi(z)=$constant and $\beta_i=0$ if $i\in\varDelta_0$.

Thus, by putting $p_n=n$, we can determine $w_{n,j}(z, \lambda)$ which are holomorphic in z in a neighbourhood V of $z=0$ and also holomorphic in $\lambda^{1/q}$. V is independent of n, j and $|\lambda|\geqq 1$. Summing up we have

PROPOSITION 2.2. *There exist formal solutions $w_n(z, \lambda, \zeta)$ of* (2.38) *and* (2.39) *which are expressed in the form*

$$(2.44)\qquad w_n(z, \lambda, \zeta)=\sum_{j=-\infty}^{n} w_{n,j}(z, \lambda)f_j(\zeta).$$

§ **3.** In this section we shall give estimates of the functions $w_{n,j}(z, \lambda)$ constructed in § 2. In order to do so we shall employ the majorant method due to Wagschal [9], Hamada [4] and Hamada, Leray and Wagschal [10]. Its details are explained in Komatsu [6].

Let $f(z)$, $g(z)$ be formal power series of z. $f(z)\ll g(z)$ means that every coefficient of $g(z)$ bounds the absolute value of the corresponding coefficient of $f(z)$.

Let us recall the functions $\theta^{(k)}(t)$ and $\Theta_k(t)$ introduced by Wagschal [9] and Hamada [4]:

$$(3.1)\qquad \theta^{(k)}(t)=\frac{k\,!}{(r-t)^{k+1}}\qquad (k\geqq 0),$$

$$(3.2)\qquad \theta^{(-k)}(t)=\sum_{s=0}^{\infty}\frac{s\,!}{(k+s)\,!}\frac{t^{k+s}}{r^{s+1}}\qquad (k>0),$$

$$(3.3)\qquad \Theta_k(t)=\frac{R'}{R'-t}\theta^{(k)}(t),\qquad \text{for } 0<r<R'.$$

They satisfy

$$(3.4)\qquad \frac{d}{dt}\theta^{(k)}(t)=\theta^{(k+1)}(t)$$

and

$$(3.5)\qquad (R'-t)\Theta_k(t)\gg 0.$$

Set

$$(3.6)\qquad \Theta_k^{(j)}(t)=\Big(\frac{d}{dt}\Big)^j\Theta_k(t).$$

Then the functions $\Theta_k^{(j)}(t)$ have the following properties:

LEMMA 3.1. *We have*

(i)

$$\frac{d}{dt}\Theta_k^{(j)}(t)=\Theta_k^{(j+1)}(t)\,, \tag{3.7}$$

(ii)

$$\Theta_k^{(j)}(t)\ll R'\Theta_k^{(j+1)}(t)\,, \tag{3.8}$$

(iii) *for* $R'<R$

$$\frac{1}{(R-t)}\Theta_k^{(j)}(t)\ll\frac{1}{R-R'}\Theta_k^{(j)}(t) \tag{3.9}$$

and

(iv) *if* $k<l$,

$$\Theta_k^{(j)}(t)\ll\Theta_k^{(j-k+l)}(t) \qquad j\geqq 0\,. \tag{3.10}$$

For the proof we refer to Komatsu [6]. Hereafter we assume that $0<r<R'<R$ and set $t=\rho z_0+z_1+\cdots+z_n$ $(\rho\geqq 1)$. By Lemma 3.1 we obtain the following lemmas. We shall omit proofs.

LEMMA 3.2. (Wagschal) *Let*

$$B(z,\partial)=\sum_{\substack{\alpha_0\leqq m_0\\ |\alpha|\leqq m}} b_\alpha(z)\partial^\alpha \tag{3.11}$$

be a linear partial differential operator with coefficients $b_\alpha(z)$ holomorphic on $\{z\in C^{n+1};\ |z_i|\leqq R\}$. Then there exists a constant B_1 independent of $\Theta_k^{(j)}(t)$ such that if

$$u(z)\ll\Theta_k^{(j)}(t)\,, \tag{3.12}$$

then

$$B(z,\partial)u(z)\ll B_1\rho^{m_0}\Theta_k^{(j+m)}(t)\,. \tag{3.13}$$

LEMMA 3.3. (De Paris) *Let*

$$C(z,\partial)=\sum_{\substack{\alpha_0<p\\ |\alpha|\leqq p}} c_\alpha(z)\partial^\alpha \tag{3.14}$$

be a linear partial differential operator with coefficients holomorphic on the poly-disk $\{z\in C^{n+1};\ |z_i|\leqq R\}$. We consider the Cauchy problem

$$\begin{cases}(\partial_0)^p u(z)=C(z,\partial)u(z)+f(z)\,,\\ (\partial_0)^s u(z)|_{z_0=0}=u_s(z'),\ 0\leqq s\leqq p-1\,,\end{cases} \tag{3.15}$$

where $u_s(z')$ *and* $f(z)$ *are holomorphic in a neighbourhood of* $z=0$. *If*

$$f(z) \ll \Theta_k^{(j+p)}(t) \tag{3.16}$$

and

$$u_s(z') \ll \Theta_k^{(j+s)}(t)|_{z_0=0} \qquad (0 \leqq s \leqq p-1), \tag{3.17}$$

then there are constants $\rho \geqq 1$ *and* C_1 *which do not depend on* j *and* k *such that the solution* $u(z)$ *fulfills*

$$u(z) \ll C_1 \Theta_k^{(j)}(t). \tag{3.18}$$

Now let us get estimates of $w_{n,j}(z, \lambda)$. We choose $R>0$ so that the coefficients of $L(z, \partial)$ and $f(z)$ are holomorphic and $\dfrac{\partial p\left(z, \dfrac{\partial \varphi}{\partial z}\right)}{\partial \xi_0} q\left(z, \dfrac{\partial \varphi}{\partial z}\right) \neq 0$ on the polydisk $\{z \in C^{n+1};\ |z_i| \leqq R\}$. The functions $w_{n,j}(z, \lambda)$ depend on the function $\hat{v}(z', \lambda)$ in (2.1). Set

$$h(\lambda) = \sup_{|z_i| \leq R} |\hat{v}(z', \lambda)|. \tag{3.19}$$

An assumption on $h(\lambda)$ will be set in §4. Under it we will integrate $U(z, \lambda)$ with respect to λ.

PROPOSITION 3.4. *There exist constants* M, A, B *and* $\rho \geqq 1$ *independent of* n *and* j *such that for* $|\lambda| \geqq 1$

$$w_{n,j}(z, \lambda) \ll M A^n B^{n-j} |\lambda|^{-(1-\alpha)n} h_1(\lambda) \Theta_{j-n}^{(n)}(t), \tag{3.20}$$

where $h_1(\lambda) = |\lambda|^{k-1+t+\alpha} h(\lambda)$ *and* $j \leqq n$.

PROOF. First of all we note that the coefficients of the operator $\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)$ are polynomials of $\lambda^{-1/q}$. Hence they are bounded if $|\lambda| \geqq 1$ and $|z_i| \leqq R$ $(i=0, 1, \cdots, n)$. We shall prove the proposition by induction on n and j. First we investigate the case $n=0$. In view of (2.40) there exists an M such that

$$w_{0,0}(z, \lambda) \ll M h_1(\lambda) \Theta_0(t) \qquad \text{for } |\lambda| \geqq 1. \tag{3.21}$$

Assume that (3.20) is valid when $n=0$ and $j \geqq J+1$. Recall that $w_{0,J}(z, \lambda)$ satisfies

$$\begin{cases} L_k w_{0,J} + \sum_{1 \leq i \leq m-1} L_{k_i}(\lambda) w_{0,J+k-k_i} = 0 \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_{0,J}\Big|_{z_0=0} = 0 \qquad (0 \leqq s \leqq k-1). \end{cases} \tag{3.22}$$

Hence we have by Lemma 3.1 and Lemma 3.2

$$(3.23)\qquad \begin{aligned} L_{k_i}(\lambda)w_{0,J+k-k_i} &\ll Mh_1(\lambda)B^{-J-k+k_i}C'\Theta^{(k_i)}_{J+k-k_i}(t) \\ &\ll Mh_1(\lambda)B^{-J-k+k_i}C'\Theta^{(k)}_{J}(t). \end{aligned}$$

By Lemma 3.3 there are $\rho \geqq 1$ and B such that

$$(3.24)\qquad w_{0,J} \ll Mh_1(\lambda)B^{-J}\Theta^{(\rho)}_{J}(t).$$

Secondly we assume that (3.20) is valid when $n \leqq N$ and when $n=N+1$ and $j \geqq J+1$. Let us also recall that $w_{N+1,J}$ satisfies

$$(3.25)\qquad \begin{cases} L_k w_{N+1,J} + \sum_{1\leqq i\leqq m-1} L_{k_i}(\lambda)w_{N+1,J+k-k_i} \\ \quad + \sum_{l=k+1}^{m} \lambda^{-(1-\alpha)(l-k)}\mathcal{L}_l(\lambda)w_{N+k+1-l,J+k-l}=0, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s w_{N+1,J}\Big|_{z_0=0}=0 \qquad (0\leqq s\leqq k-1). \end{cases}$$

From Lemma 3.1 and Lemma 3.2, it follows that for some C''

$$(3.26)\qquad \begin{aligned} \mathcal{L}_l(\lambda)w_{N+k+1-l,J+k-l} &\ll MC''A^{N+k+1-l}B^{N+1-J} \\ &\quad \times|\lambda|^{-(1-\alpha)(N+k+1-l)}h_1(\lambda)\Theta^{(N+k+1)}_{J-(N+1)}(t) \\ &\ll MC''A^N B^{N+1-J}|\lambda|^{-(1-\alpha)(N+k+1-l)}h_1(\lambda)\Theta^{(N+k+1)}_{J-(N+1)}(t). \end{aligned}$$

Hence

$$(3.27)\qquad \begin{aligned} &\lambda^{-(1-\alpha)(l-k)}\mathcal{L}_l(\lambda)w_{N+k+1-l,J+K-l} \\ &\ll MC''A^N B^{N+1-J}|\lambda|^{-(1-\alpha)(N+1)}h_1(\lambda)\Theta^{(N+k+1)}_{J-(N+1)}(t). \end{aligned}$$

And we have

$$(3.28)\qquad \begin{aligned} &L_{k_i}(\lambda)w_{N+1,J+k-k_i} \\ &\ll M\tilde{C}A^{N+1}B^{N+1-(J+k-k_i)}|\lambda|^{-(1-\alpha)(N+1)}h_1(\lambda)\Theta^{(N+1+k_i)}_{J+k-k_i-(N+1)}(t) \\ &\ll M\tilde{C}A^{N+1}B^{N+1-(J+k-k_i)}|\lambda|^{-(1-\alpha)(N+1)}h_1(\lambda)\Theta^{(N+k+1)}_{J-(N+1)}(t) \\ &\ll M\tilde{C}A^{N+1}B^{N-J}|\lambda|^{-(1-\alpha)(N+1)}h_1(\lambda)\Theta^{(N+k+1)}_{J-(N+1)}(t). \end{aligned}$$

Consequently by taking account of (3.27) and (3.28), it follows from Lemma 3.3 that there exist large M, A and B and $\rho\geqq 1$ such that for $|\lambda|\geqq 1$

$$(3.29)\qquad w_{N+1,J} \ll MA^{N+1}B^{N+1-J}|\lambda|^{-(1-\alpha)(N+1)}h_1(\lambda)\Theta^{(N+1)}_{J-(N+1)}(t).$$

This completes the proof.

We give a lemma in order to show convergence of $\sum_{n=0}^{\infty}\sum_{-\infty<j\leqq n}|w_{n,j}(z,\lambda)|\,|f_j(\zeta)|$.

LEMMA 3.5.
(i) *If* $k \geqq 0$

$$\Theta_k^{(j)}(t) \ll \frac{R'}{(R'-r)} \theta^{(j+k)}(t) . \tag{3.30}$$

(ii) *If* $k<0$ *and* $R'>2r$

$$\Theta_k^{(j)}(t) \ll \frac{2^{|k|} R'}{R'-2r} \theta^{(j+k)}(t) . \tag{3.31}$$

For the proof we refer to Komatsu [6].
Now we can show

PROPOSITION 3.6. *There exist constants* $\tilde{M}$, $\tilde{A}$ *and* $\tilde{B}$ *and a neighbourhood* Ω' *of* $z=0$ *such that for* $z \in \Omega'$ *and* $|\lambda| \geqq 1$,
(i) *if* $0 \leqq j \leqq n$

$$|w_{n,j}(z, \lambda)| \leqq \tilde{M} \tilde{A}^n \tilde{B}^{n-j} |\lambda|^{-(1-\alpha)n} h_1(\lambda) j! \tag{3.32}$$

and
(ii) *if* $j \leqq -1$

$$|w_{n,j}(z, \lambda)| \leqq \tilde{M} \tilde{A}^n \tilde{B}^{n-j} |\lambda|^{-(1-\alpha)n} h_1(\lambda) \frac{|t|^{|j|}}{|j|!} . \tag{3.33}$$

PROOF. In view of Proposition 3.4 and Lemma 3.5 it follows that there is C such that

$$\begin{aligned} w_{n,j}(z, \lambda) &\ll M A^n B^{n-j} |\lambda|^{-(1-\alpha)n} h_1(\lambda) \Theta_{j-n}^{(n)}(t) \\ &\ll M A^n B^{n-j} |\lambda|^{-(1-\alpha)n} h_1(\lambda) C^{n-j+1} \theta^{(j)}(t) . \end{aligned} \tag{3.34}$$

Let $0 \leqq t \leqq r/2$. Then we have, if $j \geqq 0$

$$\theta^{(j)}(t) \leqq (2/r)^{j+1} j! \tag{3.35}$$

and if $j<0$

$$\theta^{(j)}(t) \leqq 2r^{-1} t^{|j|} / |j|! . \tag{3.36}$$

Hence, if $0 \leqq j \leqq n$

$$|w_{n,j}(z, \lambda)| \leqq \tilde{M} \tilde{A}^n \tilde{B}^{n-j} |\lambda|^{-(1-\alpha)n} j! \, h_1(\lambda) \tag{3.37}$$

and if $j \leqq -1$

$$|w_{n,j}(z, \lambda)| \leqq \tilde{M} \tilde{A}^n \tilde{B}^{n-j} |\lambda|^{-(1-\alpha)n} (|t|^{|j|} / |j|!) h_1(\lambda) . \tag{3.38}$$

This completes the proof.
Set

(3.39) $$w_n^-(z,\lambda,\zeta)=\sum_{j=-\infty}^{-1} w_{n,j}(z,\lambda)f_j(\zeta),$$

(3.40) $$w_n^+(z,\lambda,\zeta)=\sum_{0\leq j\leq n} w_{n,j}(z,\lambda)f_j(\zeta)$$

and

(3.41) $$K(r_1,r_2,\alpha,\beta)=\{\zeta=|\zeta|e^{i\theta};\ 0<r_1\leqq|\zeta|\leqq r_2,\ \alpha\leqq\theta\leqq\beta\}.$$

PROPOSITION 3.7. *There are constants Λ, Λ_K and c_0 and a neighbourhood Ω' of $z=0$ such that*
(i) *the series*

(3.42) $$w^-(z,\lambda,\zeta)=\sum_{n=0}^{\infty} w_n^-(z,\lambda,\zeta)$$

converges when $|\lambda|\geqq\Lambda$, $|\zeta|\geqq c_0|t|$ and $z\in\Omega'$, and is holomorphic there, and
(ii) *the series*

(3.43) $$w^+(z,\lambda,\zeta)=\sum_{n=0}^{\infty} w_n^+(z,\lambda,\zeta)$$

converges when $|\lambda|\geqq\Lambda_K$, $z\in\Omega'$, uniformly on $K=K(r_1,r_2,\alpha,\beta)$ for any r_1, r_2, α and β, and is holomorphic there.

PROOF. In view of Proposition 3.6 we have

(3.44) $$|w_n^-(z,\lambda,\zeta)|\leqq\sum_{j=1}^{\infty}|w_{n,-j}(z,\lambda)||f_{-j}(\zeta)|$$
$$\leqq M(\tilde{A}\tilde{B})^n|\lambda|^{-(1-\alpha)n}h_1(\lambda)\sum_{j=1}^{\infty}\left(\frac{\tilde{B}|t|}{|\zeta|}\right)^j.$$

Hence if $(B|t|/|\zeta|)\leqq1/2$,

(3.45) $$|w_n^-(z,\lambda,\zeta)|\leqq M(AB)^n|\lambda|^{-(1-\alpha)n}h_1(\lambda).$$

Thus if $|\lambda|^{-(1-\alpha)}(\tilde{A}\tilde{B})\leqq1/2$, the series (3.42) converges.

Let us show (ii). By Proposition 3.6 we have

(3.46) $$|w_n^+(z,\lambda,\zeta)|\leqq\sum_{j=0}^{n}|w_{n,j}(z,\lambda)||f_j(\zeta)|$$
$$\leqq M\sum_{j=0}^{n}\tilde{A}^n\tilde{B}^{n-j}|\lambda|^{-(1-\alpha)n}h_1(\lambda)j!|f_j(\zeta)|.$$

So

(3.47) $$\sum_{n=0}^{\infty}|w_n^+(z,\lambda,\zeta)|\leqq Mh_1(\lambda)\sum_{n=0}^{\infty}\sum_{0\leq j\leq n}\tilde{A}^n\tilde{B}^{n-j}|\lambda|^{-(1-\alpha)n}j!|f_j(\zeta)|$$
$$=M\,h_1(\lambda)\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}(\tilde{A}^n\tilde{B}^{n-j}|\lambda|^{-(1-\alpha)n})j!|f_j(\zeta)|$$

$$=M\,h_1(\lambda)\sum_{j=0}^{\infty}\Big(\sum_{p=0}^{\infty}\tilde{A}^{p+j}|\lambda|^{-(1-\alpha)(p+j)}\tilde{B}^{p}\Big)j\,!|f_j(\zeta)|$$

$$=M\,h_1(\lambda)\Big(\sum_{j=0}^{\infty}\tilde{A}^{j}|\lambda|^{-(1-\alpha)j}j\,!|f_j(\zeta)|\Big)\Big(\sum_{p=0}^{\infty}\tilde{A}^{p}|\lambda|^{-(1-\alpha)p}\tilde{B}^{p}\Big).$$

Hence if $|\tilde{A}\tilde{B}|\lambda|^{-(1-\alpha)}|\leqq 1/2$,

$$\sum_{n=0}^{\infty}|w_n^+(z,\lambda,\zeta)|\leqq M\sum_{j=0}^{\infty}\tilde{A}^{j}|\lambda|^{-(1-\alpha)j}j\,!|f_j(\zeta)|\,h_1(\lambda). \tag{3.48}$$

Since $f_j(\zeta)=\dfrac{1}{2\pi i}\dfrac{\zeta^j}{\Gamma(j+1)}(\log\zeta-\gamma_j)$, $\gamma_j=-\gamma_0+1+1/2+\cdots+1/j$, we can find a Λ_K such that the series (3.43) converges uniformly on $K(r_1, r_2, \alpha, \beta)$. This completes the proof.

We remark that $w^-(z,\lambda,\zeta)$ is single valued with respect to ζ and has singularities in $|\zeta|\leqq c_0|t|$ and $w^+(z,\lambda,\zeta)$ has logarithmic singularity at $\zeta=0$.

§ **4.** In this section we construct a solution $u(z)$ of the problem

$$\begin{cases} L(z,\partial)u(z)=0\,, \\ \Big(\dfrac{\partial}{\partial z_0}\Big)^s u(0,z')=\delta_{s,t}u_t(z') \qquad (0\leqq s\leqq k-1)\,, \end{cases} \tag{4.1}$$

where

$$u_t(z')=\frac{1}{2\pi qi}\int_\Gamma \exp(-\lambda z_1)\hat{v}(z',\lambda)\log\lambda\, d\lambda\,. \tag{4.2}$$

$\hat{v}(z',\lambda)$ is a holomorphic function on $\{|z_i'|\leqq R,\ 1\leqq i\leqq n\}\times C$ and we assume $|\hat{v}(z',\lambda)|\leqq h(\lambda)$, where $h(\lambda)$ satisfies the condition:

For any $\varepsilon>0$ there is a constant C_ε such that

$$h(\lambda)\leqq C_\varepsilon\exp(\varepsilon|\lambda|)\quad\text{in}\quad |\arg\lambda-\theta_0|\leqq\pi/2\,. \tag{4.3}$$

The path Γ will be determined later. The path $\Gamma(\theta)$ is defined as follows: Set

$$\begin{cases} \Gamma^+(d,\theta)=\{\lambda=r\exp(i\theta)\,;\ d\leqq r<\infty\}\,, \\ \Gamma^0(d,\theta)=\{\lambda=d\exp(i\rho)\,;\ \theta\leqq\rho\leqq\theta+2q\pi\}\,, \\ \Gamma^-(d,\theta)=\{\lambda=r\exp(i(\theta+2q\pi))\,;\ d\leqq r<\infty\}\,. \end{cases} \tag{4.4}$$

$\Gamma(\theta)$ is the path which starts at $\infty\exp(i\theta)$ and goes to $d\exp(i\theta)$ on $\Gamma^+(d,\theta)$ and goes around the origin on $\Gamma^0(d,\theta)$ and ends to $\infty\exp(i(\theta+2q\pi))$ on $\Gamma^-(d,\theta)$. We note that q is the same as in (2.10).

In (4.2) set $\Gamma=\Gamma(\theta)$ with $|\theta-\theta_0|\leqq\pi/2$. Then $u_t(z')$ is holomorphic on

$$\omega_{\theta_0}=\{z'\,;\ |z_i|\leqq R,\ |\arg z_1+\theta_0|<\pi\}\,. \tag{4.5}$$

In order to construct a solution of (4.1) we make use of $U(z, \lambda)=\exp(-\lambda\varphi(z))W(z, \lambda)$ given in the previous sections.

Now let us study the functions

$$W_n(z, \lambda)=\sum_{j=-\infty}^{n}\int_C \exp(-\lambda^\alpha\zeta)w_{n,j}(z, \lambda)f_j(\zeta)d\zeta. \tag{4.6}$$

We define the path $C(\theta)$ slightly different from the path $\Gamma(\theta)$.

Set

$$\begin{cases} C^+(d, \theta)=\{\zeta=r\exp(i\theta)\,;\ d\leqq r<\infty\}, \\ C^0(d, \theta)=\{\zeta=d\exp(i\rho)\,;\ \theta\leqq\rho\leqq\theta+2\pi\}, \\ C^-(d, \theta)=\{\zeta=r\exp(i(\theta+2\pi)) :\ d\leqq r<\infty\}. \end{cases} \tag{4.7}$$

$C(\theta)=C^+(d, \theta)\cup C^0(d, \theta)\cup C^-(d, \theta)$ and the direction of the curve $C(\theta)$ is the same as $\Gamma(\theta)$ but the path $C(\theta)$ goes around the origin once on $C^0(d, \theta)$.

Set $C=C(\theta)$ in (4.6). Then in view of the proof of Proposition 3.7 $W_n(z, \lambda)$ is holomorphic on $\{\lambda;\ |\lambda|\geqq\Lambda,\ |\arg\lambda^\alpha+\theta|<\pi/2\}$. By varying θ, we now claim that $W_n(z, \lambda)$ is holomorphic on $\{|\lambda|\geqq\Lambda\}$, which may be multi-valued. More precisely

LEMMA 4.1. *$W_n(z, \lambda)$ has a convergent expansion*

$$W_n(z, \lambda)=\sum_{j=-\infty}^{n} w_{n,j}(z, \lambda)\lambda^{-\alpha(j+1)}. \tag{4.8}$$

Therefore $W_n(z, \lambda)$ is a function of $\lambda^{1/q}$.

PROOF. Let $|\arg\lambda^\alpha|<\pi/2$ and $C=C(0)$. From the equality

$$\int_{C(0)}\exp(-\lambda^\alpha\zeta)f_j(\zeta)d\zeta=\lambda^{-\alpha(j+1)} \tag{4.9}$$

we have (4.8). $w_{n,j}(z, \lambda)$ is a holomorphic function of $\lambda^{1/q}$.

Set

$$u_n(z)=\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)}\exp(-\lambda\varphi(z))W_n(z, \lambda)(\log\lambda)d\lambda \tag{4.10}$$

and

$$u_{n,j}(z)=\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)}\exp(-\lambda\varphi(z))w_{n,j}(z, \lambda)\lambda^{-\alpha(j+1)}(\log\lambda)d\lambda. \tag{4.11}$$

To investigate $u_n(z)$ and $u_{n,j}(z)$, we shall give lemmas concerning the function $V(a, \theta_0)$ defined by

$$V(a, \theta_0)=\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)}\exp(-\lambda a)\hat{v}(\lambda)\log\lambda\, d\lambda. \tag{4.12}$$

Here $\hat{v}(\lambda)$ is a holomorphic function of λ, which may be multi-valued and is defined on $\{|\lambda|\geqq\Lambda\}$ and has the property:

For any $\varepsilon>0$, there is a constant C_ε such that if $|\arg\lambda-\theta_0+2m\pi|\leqq\pi/2$ $(m\in Z)$, then

$$|\hat{v}(\lambda)|\leqq C_\varepsilon\exp(\varepsilon|\lambda|)M|\lambda|^p, \tag{4.13}$$

where M and p are independent of ε.

LEMMA 4.2. *$V(a, \theta_0)$ is a holomorphic function of a in $\{a\,;\ |\arg a+\theta_0|<\pi\}$.*

PROOF. By varying the path of integration in (4.12) and by (4.13), we can show that $V(a, \theta_0)$ is holomorphic in $\{a\,;\ |\arg a+\theta_0|<\pi\}$.

Let us note that $V(a, \theta_0)$ is represented in the form

$$V(a, \theta_0)=\frac{1}{(2\pi qi)a}\int_{\Gamma(\theta')}\exp(-\mu)\hat{v}(\mu/a)(\log\mu-\log a)d\mu, \tag{4.14}$$

where $|\theta'|<\pi/2$ and $-\pi/2+\theta'<\arg a+\theta_0<\pi/2+\theta'$.

Put

$$V_1(a, \theta_0)=\frac{1}{(2\pi qi)a}\int_{\Gamma(\theta')}\exp(-\mu)\hat{v}(\mu/a)(\log\mu)d\mu \tag{4.15}$$

and

$$V_2(a, \theta_0)=\frac{-\log a}{(2\pi qi)a}\int_{\Gamma(\theta')}\exp(-\mu)\hat{v}(\mu/a)d\mu. \tag{4.16}$$

Hence

$$V(a, \theta_0)=V_1(a, \theta_0)+V_2(a, \theta_0). \tag{4.17}$$

LEMMA 4.3. *Assume that condition* (4.13) *holds. Then for any $\varepsilon>0$ there exist a G independent of ε and C'_ε such that if $\varepsilon<|a|<1/\Lambda$ and $|\arg a+\theta_0|<\pi/2$, then*

$$|V_1(a, \theta_0)|\leqq C'_\varepsilon MG^{p+1}\Gamma(p+1)/|a|^{p+1}, \qquad p\geqq 0, \tag{4.18}$$

and

$$|V_1(a, \theta_0)|\leqq C'_\varepsilon MG^{|p|+1}/|a|^{p+1}\Gamma(|p|+1), \qquad p<0. \tag{4.19}$$

PROOF. First we consider the case $p\geqq 0$. We have

$$\begin{aligned} V_1(a, \theta_0)=\frac{1}{(2\pi qi)a}\Big\{&\int_{\Gamma^0(p+1,0)}\exp(-\mu)v(\mu/a)\log\mu\,d\mu \\ &+\int_{\Gamma^+(p+1,0)\cup\Gamma^-(p+1,0)}\exp(-\mu)v(\mu/a)\log\mu\,d\mu\Big\} \\ =V_{1,1}&(a, \theta_0)+V_{1,2}(a, \theta_0). \end{aligned} \tag{4.20}$$

By (4.13), we have, for $|\arg a+\theta_0|<\pi/2$ and $\varepsilon<|a|<1/\Lambda$,

$$|V_{1,1}(a,\theta_0)|\leqq M\exp(p+1)C_\varepsilon\exp(\varepsilon(p+1)/|a|)\frac{(p+1)^{p+2}}{|a|^{p+1}} \leqq C'_\varepsilon MG_1^{p+1}\Gamma(p+1)/|a|^{p+1}. \tag{4.21}$$

For $V_{1,2}(a,\theta_0)$, we have

$$\begin{aligned}|V_{1,2}(a,\theta_0)| &\leqq \frac{2MC_\varepsilon}{|a|}\int_{p+1}^{\infty}\exp(-\mu(1-\varepsilon/2|a|))|\mu/a|^p(|\log\mu|+q\pi)d\mu\\ &\leqq \frac{2MC'_\varepsilon}{|a|^{p+1}}\int_{p+1}^{\infty}\exp(-\mu/2)\mu^p(|\log\mu|+q\pi)d\mu\\ &\leqq C'_\varepsilon MG_2^{p+1}\Gamma(p+1)/|a|^{p+1}.\end{aligned} \tag{4.22}$$

Consequently if $p\geqq 0$, we can choose C'_ε and G such that (4.18) holds.

Secondly suppose that $p<0$. We set $p=-p'$. We have

$$\begin{aligned}V_1(a,\theta_0) &= \frac{1}{(2\pi qi)a}\int_{\Gamma^0(p'+1,0)}\exp(-\mu)\hat{v}(\mu/a)(\log\mu)d\mu\\ &\quad+\frac{1}{(2\pi qi)a}\int_{\Gamma^+(p'+1,0)\cup\Gamma^-(p'+1,0)}\exp(-\mu)\hat{v}(\mu/a)(\log\mu)d\mu\\ &= V_{1,1}(a,\theta_0)+V_{1,2}(a,\theta_0).\end{aligned} \tag{4.23}$$

For $V_{1,1}(a,\theta_0)$ we have

$$\begin{aligned}|V_{1,1}(a,\theta_0)| &\leqq C_\varepsilon M\exp\Big((p'+1)+\varepsilon\frac{(p'+1)}{|a|}\Big)(p'+1)^{-p'}|a|^{p'-1}\\ &\leqq C_\varepsilon MG_1^{p'+1}|a|^{p'-1}\Gamma(p'+1)^{-1}.\end{aligned} \tag{4.24}$$

For $V_{1,2}(a,\theta_0)$, we have

$$\begin{aligned}|V_{1,2}(a,\theta_0)| &\leqq \frac{2M}{|a|}\int_{p'+1}^{\infty}C_\varepsilon\exp\Big(-\mu+\frac{\varepsilon\mu}{2|a|}\Big)|a/\mu|^{p'}(|\log\mu|+q\pi)d\mu\\ &\leqq 2C_\varepsilon M|a|^{p'-1}\int_{p'+1}^{\infty}\exp(-\mu/2)\mu^{-p'}(|\log\mu|+q\pi)d\mu\\ &\leqq C'_\varepsilon MG_2^{p'+1}|a|^{p'-1}\Gamma(p'+1)^{-1}.\end{aligned} \tag{4.25}$$

Thus, if $p<0$

$$|V_{1,2}(a,\theta_0)|\leqq\frac{C_\varepsilon MG^{|p|+1}}{|a|^{p+1}}\Gamma(|p|+1)^{-1}. \tag{4.26}$$

A slight modification of the proof of Lemma 4.3 gives

LEMMA 4.4. *Under condition* (4.13) *there exist an H independent of ε and C''_ε such that*

$$|V_2(a,\theta_0)|\leqq C''_\varepsilon MH^{p+1}\frac{|\log a|}{|a|^{p+1}}\Gamma(p+1)\qquad(p\geqq 0) \tag{4.27}$$

and

$$|V_2(a, \theta_0)| \leqq C''_\varepsilon M H^{|p|+1} \frac{|\log a|}{|a|^{p+1}\Gamma(|p|+1)} \qquad (p<0), \tag{4.28}$$

if $\varepsilon<|a|<1/A$ *and* $|\arg a+\theta_0|<\pi/2$.

Now we apply Lemmas 4.1~4.4 to $u_{n,j}(z)$. By Lemma 4.1 $u_{n,j}(z)$ is holomorphic in $\{z;\ |\arg \varphi(z)+\theta_0|<\pi\}$. We have

$$u_{n,j}(z)=u^1_{n,j}(z)+u^2_{n,j}(z) \qquad (|\arg \varphi(z)+\theta_0|<\pi/2), \tag{4.29}$$

where

$$u^1_{n,j}(z)=\frac{1}{(2\pi qi)\varphi(z)}\int_{\Gamma(0)} \exp(-\mu) w_{n,j}\Big(z, \frac{\mu}{\varphi(z)}\Big)\Big(\frac{\mu}{\varphi(z)}\Big)^{-\alpha(j+1)}(\log \mu)d\mu \tag{4.30}$$

and

$$u^2_{n,j}(z)=\frac{-\log \varphi(z)}{(2\pi qi)\varphi(z)}\int_{\Gamma(0)} \exp(-\mu) w_{n,j}\Big(z, \frac{\mu}{\varphi(z)}\Big)\Big(\frac{\mu}{\varphi(z)}\Big)^{-\alpha(j+1)} d\mu \tag{4.31}$$

By Proposition 3.6 we have

$$|w_{n,j}(z, \lambda)| \leqq M_{n,j} h_1(\lambda)|\lambda|^{-(1-\alpha)n}, \tag{4.32}$$

where

$$M_{n,j}=\begin{cases} MA^n B^{n-j}\Gamma(j+1) & (0\leqq j\leqq n) \\ MA^n B^{n-j}|t|^{|j|}/\Gamma(|j|+1) & (j<0) \quad (t=\rho z_0+z_1+\cdots+z_n) \end{cases} \tag{4.33}$$

and $h_1(\lambda)$ satisfies the condition:

For any $\varepsilon>0$ there is a constant C_ε such that

$$|h_1(\lambda)| \leqq C_\varepsilon \exp(\varepsilon|\lambda|) \qquad \text{for } |\arg \lambda-\theta_0+2m\pi| \leqq \pi/2. \tag{4.34}$$

Set $p=\alpha(n-j)-n-\alpha$. By Lemma 4.3 and Lemma 4.4 for any $\varepsilon>0$ there are constants C_ε and K such that

$$\begin{cases} |u^1_{n,j}(z)| \leqq C_\varepsilon K^{p+1}|\varphi(z)|^{-(p+1)} M_{n,j}\Gamma(p+1) & (p\geqq 0), \\ |u^1_{n,j}(z)| \leqq C_\varepsilon K^{p+1}|\varphi(z)|^{-(p+1)} M_{n,j}\Gamma(|p|+1)^{-1} & (p<0) \end{cases} \tag{4.35}$$

and

$$\begin{cases} |u^2_{n,j}(z)| \leqq |\log \varphi(z)| C_\varepsilon K^{p+1}|\varphi(z)|^{-(p+1)} M_{n,j}\Gamma(p+1) & (p\geqq 0), \\ |u^2_{n,j}(z)| \leqq |\log \varphi(z)| C_\varepsilon K^{p+1}|\varphi(z)|^{-(p+1)} M_{n,j}\Gamma(|p|+1)^{-1} & (p<0), \end{cases} \tag{4.36}$$

if $|\arg \varphi(z)+\theta_0|<\pi/2$ and $\varepsilon<|\varphi(z)|<1/A$.

In order to study convergence of $\sum_{n=0}^{\infty}\Big(\sum_{j=-\infty}^{n} u_{n,j}(z)\Big)$, we need some inequalities concerning Γ function:

LEMMA 4.5. (i) *If* $p \geqq 1$, $q \geqq 1$, *then*

$$\Gamma(p)\Gamma(q) \leqq \Gamma(p+q) \leqq 3^{p+q}\Gamma(p)\Gamma(q). \tag{4.37}$$

(ii) *If* $\alpha \geqq 1$, $\beta \geqq 0$ *and* $\alpha-\beta \geqq 1$, *then*

$$3^{-(\alpha+1)}\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)} \leqq \Gamma(\alpha-\beta) \leqq \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}. \tag{4.38}$$

PROOF. Let us recall the relation of Γ function and the beta function $B(t, s)$

$$\Gamma(t+s)B(t, s)=\Gamma(t)\Gamma(s). \tag{4.39}$$

When $p \geqq 1$, $q \geqq 1$,

$$3^{-(p+q)} \leqq B(p, q) \leqq 1. \tag{4.40}$$

From (4.39) and (4.40), the inequality (4.37) follows. Set $p=\alpha-\beta$ and $q=\beta+1$ in (4.37). We obtain

$$\Gamma(\alpha-\beta)\Gamma(\beta+1) \leqq \Gamma(\alpha+1) \leqq 3^{\alpha+1}\Gamma(\alpha-\beta)\Gamma(\beta+1). \tag{4.41}$$

This implies inequality (4.38).

We apply Lemma 4.5 to $\Gamma(p+1)$ $(p \geqq 0)$ and $\Gamma(|p|+1)$ $(p<0)$. Recall that $p=\alpha(n-j)-n-\alpha$. If $p \geqq 0$, then

$$\begin{aligned}\Gamma(p+1)=\Gamma(\alpha(n-j)-n-\alpha+1) &\leqq \frac{\Gamma(\alpha(n-j)+2-\alpha)}{\Gamma(n+1)} \\ &\leqq 3^{\alpha(n-j)+2-\alpha}\frac{\Gamma(1-\alpha)\Gamma(\alpha(n-j)+1)}{\Gamma(n+1)}.\end{aligned} \tag{4.42}$$

If $p<0$, then

$$\begin{aligned}\Gamma(|p|+1)=\Gamma(n-\alpha(n-j)+\alpha+1) &= \Gamma((n+2)-(1-\alpha+\alpha(n-j)) \\ &\geqq \frac{\Gamma(n+3)}{\Gamma(2-\alpha+\alpha(n-j))}3^{-(n+3)} \\ &\geqq \frac{\Gamma(n+3)}{\Gamma(1-\alpha)\Gamma(\alpha(n-j)+1)}3^{-(\alpha(n-j)+n+5-\alpha)}.\end{aligned} \tag{4.43}$$

Hence

$$\Gamma(|p|+1)^{-1} \leqq 3^{\alpha(n-j)+n+5-\alpha}\frac{\Gamma(1-\alpha)\Gamma(\alpha(n-j)+1)}{\Gamma(n+1)}. \tag{4.44}$$

PROPOSITION 4.6. *For any* $\varepsilon>0$, *there exist* K, A *and* B *independent of* ε *and* C_ε *such that the following estimates hold in* $\Omega'_\varepsilon=\{z \in \Omega'; |\varphi(z)|>\varepsilon, |\arg\varphi(z)+\theta_0|<\pi/2\}$:

If $0 \leqq j \leqq n$,

$$(4.45)\qquad |u^1_{n,j}(z)|\leqq\frac{C_\varepsilon KA^nB^{n-j}\Gamma(\alpha(n-j)+1)\Gamma(j+1)}{|\varphi(z)|^{\alpha(n-j)-n}\Gamma(n+1)}$$

and

$$(4.46)\qquad |u^2_{n,j}(z)|\leqq\frac{C_\varepsilon KA^nB^{n-j}|\log\varphi(z)|}{|\varphi(z)|^{\alpha(n-j)-n}}\;\frac{\Gamma(\alpha(n-j)+1)\Gamma(j+1)}{\Gamma(n+1)}.$$

If $j<0$,

$$(4.47)\qquad |u^1_{n,j}(z)|\leqq\frac{CKA^nB^{n-j}|z|^{|j|}}{\varphi(z)^{\alpha(n-j)-n}}\;\frac{\Gamma(\alpha(n-j)+1)}{\Gamma(n+1)\Gamma(|j|+1)}$$

and

$$(4.48)\qquad |u^2_{n,j}(z)|\leqq\frac{C_\varepsilon KA^nB^{n-j}|z|^{|j|}|\log\varphi(z)|}{|\varphi(z)|^{\alpha(n-j)-n}}\;\frac{\Gamma(\alpha(n-j)+1)}{\Gamma(n+1)\Gamma(|j|+1)}.$$

PROOF. We shall estimate $M_{n,j}\Gamma(p+1)$ $(p\geqq0)$ and $M_{n,j}\Gamma(|p|+1)^{-1}$ $(p<0)$ in (4.35) and (4.36). If $0\leqq j\leqq n$, it follows from (4.42) and (4.44) that there are A_1, B_1 and M_1 such that

$$(4.49)\qquad M_{n,j}\Gamma(p+1)\leqq M_1A_1^nB_1^{n-j}\frac{\Gamma(\alpha(n-j)+1)\Gamma(j+1)}{\Gamma(n+1)}\qquad(p\geqq0)$$

and

$$(4.50)\qquad M_{n,j}\Gamma(|p|+1)^{-1}\leqq M_1A_1^nB_1^{n-j}\frac{\Gamma(\alpha(n-j)+1)\Gamma(j+1)}{\Gamma(n+1)}\qquad(p<0).$$

If $j<0$, there exist A_2, B_2 and M_2 such that

$$(4.51)\qquad M_{n,j}\Gamma(p+1)\leqq M_2A_2^nB_2^{n-j}|z|^{|j|}\frac{\Gamma(\alpha(n-j)+1)}{\Gamma(n+1)\Gamma(|j|+1)}\qquad(p\geqq0)$$

and

$$(4.52)\qquad M_{n,j}\Gamma(|p|+1)^{-1}\leqq M_2A_2^nB_2^{n-j}|z|^{|j|}\frac{\Gamma(\alpha(n-j)+1)}{\Gamma(n+1)\Gamma(|j|+1)}\qquad(p<0).$$

Taking account of (4.35) and (4.36) we can find A, B and K independent of ε and C_ε such that estimates (4.47) and (4.48) hold.

Thus concerning the convergence of $\sum_{n=0}^{\infty}\sum_{-\infty<j\leqq n}u_{n,j}(z)$ we get

THEOREM 4.7. *There is an* $\varepsilon_0>0$ *such that the series* $u(z)=\sum_{n=0}^{\infty}\sum_{-\infty<j\leqq n}u_{n,j}(z)$ *converges uniformly on the set*

$$(4.53)\qquad \Omega^{\theta_0}_{\varepsilon,\varepsilon_0}=\{z\in\Omega\,;\ \varepsilon<\varphi(z)<\varepsilon_0,\ |\arg\varphi(z)+\theta_0|<\pi/2\}$$

for any $\varepsilon>0$ $(0<\varepsilon<\varepsilon_0)$. *Consequently* $u(z)$ *is holomorphic in* $\Omega^{\theta_0}_{0,\varepsilon_0}$ *and is a solution of* (4.1).

PROOF. Set

$$I_1 = \sum_{-\infty<j<0} \sum_{n=0}^{\infty} |u^1_{n,j}(z)| \tag{4.54}$$

and

$$I_2 = \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} |u^1_{n,j}(z)| . \tag{4.55}$$

By Proposition 4.6, if $\varepsilon<|\varphi(z)|<1/A$ and $|\arg\varphi(z)+\theta_0|<\pi/2$, then

$$\begin{aligned} I_1 &\leqq C_\varepsilon K\Big\{\sum_{j=1}^{\infty} \frac{B^j}{\Gamma(j+1)}|\varphi(z)|^{-\alpha j}|z|^j\Big(\sum_{n=0}^{\infty}\frac{(AB)^n\Gamma(\alpha(n+j)+1)}{\Gamma(n+1)|\varphi(z)|^{n(1-\alpha)}}\Big)\Big\} \\ &\leqq C'_\varepsilon K \sum_{j=1}^{\infty}\frac{(|z|B)^j}{\Gamma(j+1)}|\varphi(z)|^{-\alpha j}\Gamma(\alpha j+1). \end{aligned} \tag{4.56}$$

Hence I_1 converges when $\varepsilon<|\varphi(z)|<1/A$.

For I_2, also by Proposition 4.6, we have

$$\begin{aligned} I_2 &\leqq C_\varepsilon K \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} A^n B^{n-j}|\varphi(z)|^{n-\alpha(n-j)}\frac{\Gamma(\alpha(n-j)+1)\Gamma(j+1)}{\Gamma(n+1)} \\ &\leqq C_\varepsilon K \sum_{j=0}^{\infty}\Gamma(j+1)|\varphi(z)|^{\alpha j}\Big(\sum_{p=0}^{\infty}\frac{\Gamma(\alpha p+1)A^{p+j}B^p|\varphi(z)|^{(1-\alpha)(p+j)}}{\Gamma(p+j+1)}\Big) \\ &\leqq C'_\varepsilon K \sum_{j=0}^{\infty}(A|\varphi(z)|)^j . \end{aligned} \tag{4.57}$$

Therefore, if $\varepsilon<|\varphi(z)|<1/2A$, then I_2 converges. The convergence of $\sum_{n=0}^{\infty}\sum_{-\infty<j\leqq n} u^2_{n,j}(z)$ follows from the same reason. Thus $u(z)$ converges in $\Omega^{\theta_0}_{0,\varepsilon_0}$, for some ε_0.

REMARK 4.8. If the trace $u(z)$ on S, $u_t(z')$ in (4.1), is holomorphically extensible around $z_1=0$, we can show by the same method that there is a function $u(z)$ which is holomorphic except on K and holomorphically extensible around K. We refer this fact to Persson [8] and Hamada, Leray and Wagschal [10].

§5. In §4 we constructed a solution $u(z)$ of problem (4.1) by making use of $w(z, \lambda, \zeta)=\sum_{n=0}^{\infty} w_n(z, \lambda, \zeta)$ obtained in §2. From the construction of $w(z, \lambda, \zeta)$, it fulfills the equation

$$\begin{cases} \mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)w(z, \lambda, \zeta)=0, \\ \Big(\dfrac{\partial}{\partial z_0}\Big)^s w(z, \lambda, \zeta)\Big|_{z_0=0}=\lambda^{s-t+\alpha}\tilde{v}_s(z, \lambda)\dfrac{(\log\zeta)}{2\pi i}, \quad (0\leqq s\leqq k-1) \end{cases} \tag{5.1}$$

and from Proposition 3.7, it is holomorphic in the set $\{\zeta\in C^1;\ 2\tilde{\gamma}>|\zeta|>c_0|z|\}$ and $\{|\lambda|>\Lambda\}$. In this section we shall give another solution of problem (5.1). To do so let us set an assumption on the subcharacteristic polynomial $l_{\text{sub}}(p;\ z,\ \xi,\ \eta)$ introduced in §1:

ASSUMPTION. The subcharacteristic equation $l_{\text{sub}}(p;\ z,\ \xi,\ \eta)=0$ of $L(z,\partial)$ with respect to $\Sigma_p=\{(z,\xi);\ p(z,\xi)=0,\ \xi\neq 0\}$ has distinct k roots at $z=0$, $\xi=\frac{\partial\varphi}{\partial z}(0)$, that is,

$$l_{\text{sub}}\Big(p;\ 0,\ \frac{\partial\varphi}{\partial z}(0),\ \eta\Big)=q_m\Big(0,\ \frac{\partial\varphi}{\partial z}(0)\Big)\eta^k+\sum_{i\in\Delta_0-\{m\}}q_i\Big(0,\ \frac{\partial\varphi}{\partial z}(0)\Big)\eta^{k_i}=0 \tag{5.2}$$

has distinct k roots.

Since the operator $\mathcal{L}(z,\lambda,\partial_z,\partial_\zeta)$ has the form

$$\Big(\frac{\partial}{\partial\zeta}\Big)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))+\sum_{l=k+1}^{m}\Big(\frac{\partial}{\partial\zeta}\Big)^{m-l}\lambda^{-(1-\alpha)(l-k)}\mathcal{L}_l(\lambda), \tag{5.3}$$

for large λ its principal part becomes

$$\Big(\frac{\partial}{\partial\zeta}\Big)^{m-k}\Big(\sum_{i\in\Delta_0}\tilde{q}_i\Big(z,\ \frac{\partial\varphi}{\partial z}\Big)\Big(\frac{\partial}{\partial\zeta}\Big)^{k-k_i}(H_p)^{k_i}+O(\lambda^{-1/q})\Big), \tag{5.4}$$

where

$$H_p=\sum_{i=0}^{n}\frac{\partial p\Big(z,\ \frac{\partial\varphi}{\partial z}\Big)}{\partial\xi_i}\frac{\partial}{\partial z_i} \tag{5.5}$$

and $O(\lambda^{-1/q})$ is an operator of order m whose coefficients $c(z,\lambda)$ satisfy

$$\overline{\lim_{|\lambda|\to\infty}}|\lambda|^{1/q}\sup_{z\in\Omega}|c(z,\lambda)|<\infty. \tag{5.6}$$

We seek for characteristic functions $\tilde{\phi}(z,\lambda,\zeta)$ of the operator (5.3) in the form $\tilde{\phi}(z,\lambda,\zeta)=\zeta+\phi(z,\lambda)$. By taking account of the fact that $\left.\frac{\partial p\left(z,\ \frac{\partial\varphi}{\partial z}\right)}{\partial\xi_0}\right|_{z=0}\neq 0$, we now assume that the coordinate system is chosen so that $H_p\Big(z,\ \frac{\partial\varphi}{\partial z}\Big)=\frac{\partial}{\partial z_0}$. The characteristic equation of $\mathcal{L}\Big(z,\lambda,\frac{\partial}{\partial z},\frac{\partial}{\partial\zeta}\Big)$ is

$$l_{\text{sub}}\Big(p;\ z,\ \frac{\partial\varphi}{\partial z}(z),\ (-1)^\alpha H_p\phi\Big)+l_1\Big(z,\ \lambda,\ \frac{\partial\phi}{\partial z}\Big)=0. \tag{5.7}$$

According to (5.4) and (5.6), $l_1\Big(z,\ \lambda,\ \frac{\partial\phi}{\partial z}\Big)$ tends to 0 when $|\lambda|\to\infty$. By Assumption $l_{\text{sub}}\Big(p;\ z,\ \frac{\partial\varphi}{\partial z},\ \eta\Big)=0$ has distinct k roots near $z=0$, so we conclude that equation (5.7) has distinct k roots $\eta_i\Big(z,\ \lambda,\ \frac{\partial\phi}{\partial z'}\Big)$ for large λ. Thus by solving the equation

$$\text{(5.8)}\qquad \begin{cases} \dfrac{\partial\phi}{\partial z_0}=\eta_i\Big(z,\lambda,\dfrac{\partial\phi}{\partial z'}\Big) \\ \phi(0,z',\lambda)=0\,, \end{cases}$$

we get characteristic functions of operator (5.3) in the form $\zeta+\phi_i(z,\lambda)$ $(1\leqq i\leqq k)$. $\phi_i(z,\lambda)$ is a holomorphic function of $\lambda^{1/q}$ in $\{|\lambda|\geqq\Lambda_1\}$ $(\Lambda_1>\Lambda,\ i=1,2,\cdots,k)$.

Now we construct another solution of (5.1) by the method due to Hamada [3] and Wagschal [9]. Their result is as follows:

PROPOSITION 5.1. *Let $M(z,\partial)$ be a linear partial differential operator of order m with coefficients holomorphic on Ω $(\Omega\ni 0)$. Let $m(z,\xi)$ be its principal symbol and $K_i=\{z\in\Omega;\ \varphi_i(z)=0\}$ $(1\leqq i\leqq k,\ k\leqq m)$ be characteristic surfaces of $M(z,\partial)$ such that $\varphi_i(0,z)=z_1$,*

$$\text{(5.9)}\qquad \frac{\partial m\Big(z,\dfrac{\partial\varphi_i}{\partial z}\Big)}{\partial\xi_0}\neq 0\quad at\ z=0$$

and if $i\neq j$

$$\text{(5.10)}\qquad \frac{\partial\varphi_i(0)}{\partial z_0}\neq\frac{\partial\varphi_j(0)}{\partial z_0}.$$

Consider the problem

$$\text{(5.11)}\qquad \begin{cases} M(z,\partial)u(z)=\sum_{i=1}^{k}\{\varphi_i(z)^{-j_i}G_i(z)+\log\varphi_i(z)H_i(z)\}\,, \\ \Big(\dfrac{\partial}{\partial z_0}\Big)^s u(z)\Big|_{z_0=0}=u_s(z')\qquad (0\leqq s\leqq k-1)\,, \end{cases}$$

where $G_i(z)$ and $H_i(z)$ $(1\leqq i\leqq k)$ are holomorphic in Ω and $u_s(z')$ $(0\leqq s\leqq k-1)$ are holomorphic except at $z_1=0$ and have a pole or logarithmic singularity at $z_1=0$. Then there exists a solution $u(z)$ of (5.11) which is represented in the form

$$\text{(5.12)}\qquad u(z)=\sum_{i=1}^{k}\sum_{j\geqq J_i}a_{i,j}(z)f_j(\varphi_i(z))\,,$$

where $a_{i,j}(z)$ are holomorphic in Ω'' $(\subset\Omega)$ and $f_j(\zeta)$ $(j\in Z)$ are those defined in (2.22) and (2.23).

We refer the proof and details to Hamada [3] and Wagschal [9].

We apply Proposition 5.1 to the problem (5.1) containing a parameter λ. Set

$$\text{(5.13)}\qquad K_i(\lambda)=\{(z,\lambda);\ z\in\Omega,\ |\zeta|<r,\ |\lambda|>\Lambda_1,\ \zeta+\phi_i(z,\lambda)=0\}.$$

Initial data given in (5.1) have logarithmic singularity at $\zeta=0$. So by Proposition 5.1 there is a solution $\tilde{w}(z, \lambda, \zeta)$ holomorphic except on $\bigcup_{i=1}^{k} K_i(\lambda)$ and has the form

$$\tilde{w}(z, \lambda, \zeta)=\lambda^{k-t-1+\alpha}\sum_{i=1}^{k}\sum_{j\geqq J_i}\tilde{w}_{i,j}(z, \lambda)f_j(\zeta+\psi_i(z, \lambda)). \tag{5.14}$$

By the method of construction of $\tilde{w}_{i,j}(z, \lambda)$, they are independent of ζ and holomorphic functions of $\lambda^{1/q}$ when $(|\lambda|>\Lambda_2)$. The integer J_i in (5.14) depends on the initial values in (5.1).

§ 6. In this section we show that the function

$$\beta(z, \lambda, \zeta)=w(z, \lambda, \zeta)-\tilde{w}(z, \lambda, \zeta) \tag{6.1}$$

is holomorphic in some neighbourhood of $z=\zeta=0$. Since $\tilde{w}(z, \lambda, \zeta)$ is holomorphic in $\{(z, \zeta);\ (z, \zeta)\notin K_i,\ |z|+|\zeta|<\tilde{\gamma}\}$ and $w(z, \lambda, \zeta)$ is holomorphic in $\{(z, \zeta);\ z\in\Omega',\ 2\tilde{\gamma}>|\zeta|>c_0|z|\}$ for $|\lambda|>\Lambda$ by Proposition 3.7, $\beta(z, \lambda, \zeta)$ is holomorphic in some neighbourhood of $z=0$, $\zeta=\hat{\zeta}$ for any $\hat{\zeta}$ with $0<|\hat{\zeta}|<\tilde{\gamma}$.

We note that $\beta(z, \lambda, \zeta)$ satisfies

$$\begin{cases} \mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)\beta(z, \lambda, \zeta)=0\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s\beta(z, \lambda, \zeta)\Big|_{z_0=0}=0 \qquad (0\leqq s\leqq k-1)\,. \end{cases} \tag{6.2}$$

So in order to study $\beta(z, \lambda, \zeta)$, let us consider Goursat's problem

$$\begin{cases} \mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)u(z, \lambda, \zeta)=f(z, \lambda, \zeta)\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^p(u(z, \lambda, \zeta)-g(z, \lambda, \zeta))=0\,, \quad \text{when } z_0=0 \ \text{ if } 0\leqq p\leqq k-1, \\ \left(\dfrac{\partial}{\partial \zeta}\right)^q(u(z, \lambda, \zeta)-g(z, \lambda, \zeta))=0\,, \quad \text{when } \zeta=\hat{\zeta} \ \text{ if } 0\leqq q\leqq m-k-1\,. \end{cases} \tag{6.3}$$

From now on we assume that $H_p=\dfrac{\partial}{\partial z_0}$ and $|\hat{\zeta}|<\gamma$ and choose $\kappa>1$ so as to satisfy

$$\sup_{z\in\Omega}|\nu_i^0(z)|<\kappa/2\,, \tag{6.4}$$

where $\nu_i^0(z)$ $(i=1, 2, \cdots, k)$ are solutions of $l_{\text{sub}}\left(p\,;\ z, \dfrac{\partial\varphi}{\partial z}, \nu^0\right)=0$. Put

$$V_\gamma=\{(z, \zeta)\,;\ |z_i|<2\gamma,\ \kappa|z_0|+|\zeta|<3\kappa\gamma\} \tag{6.5}$$

and set

$$\bar{t}=(\rho_0z_0+z_1+z_2+\cdots+z_n+\zeta-\hat{\zeta})\,. \tag{6.6}$$

To solve equation (6.3) we give a lemma.

LEMMA 6.1. *Consider the equation with boundary conditions*

$$(6.7)\qquad \begin{cases} \left(\dfrac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))u(z,\lambda,\zeta)=f(z,\lambda,\zeta), \\ \left(\dfrac{\partial}{\partial z_0}\right)^{p}u(z,\lambda,\zeta)=0, \quad \text{when } z_0=0, \text{ if } 0\leqq p\leqq k-1, \\ \left(\dfrac{\partial}{\partial\zeta}\right)^{q}u(z,\lambda,\zeta)=0, \quad \text{when } \zeta=\hat{\zeta} \text{ if } 0\leqq q\leqq m-k-1. \end{cases}$$

If $f(z,\lambda,\zeta)$ is holomorphic on $\bar{V}_\gamma$, then there is a unique solution $u(z,\lambda,\zeta)$ holomorphic on $\bar{V}_\gamma$ for large λ, $|\lambda|>\Lambda_3$. Moreover if

$$(6.8)\qquad f(z,\lambda,\zeta)\ll D\tilde{\Theta}^{(m)}(\bar{t}), \quad \tilde{\Theta}(\bar{t})=\frac{1}{2\gamma-\bar{t}},$$

then there exist E and $\rho_0>1$ such that

$$(6.9)\qquad u(z,\lambda,\zeta)\ll DE\tilde{\Theta}(\bar{t}).$$

PROOF. Set $\left(\dfrac{\partial}{\partial\zeta}\right)^{m-k}u(z,\lambda,\zeta)=v(z,\lambda,\zeta)$. The Cauchy problem

$$(6.10)\qquad \begin{cases} (\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))v(z,\lambda,\zeta)=f(z,\lambda,\zeta), \\ \left(\dfrac{\partial}{\partial z_0}\right)^{p}v(z,\lambda,\zeta)\Big|_{z_0=0}=0 \qquad (0\leqq p\leqq k-1) \end{cases}$$

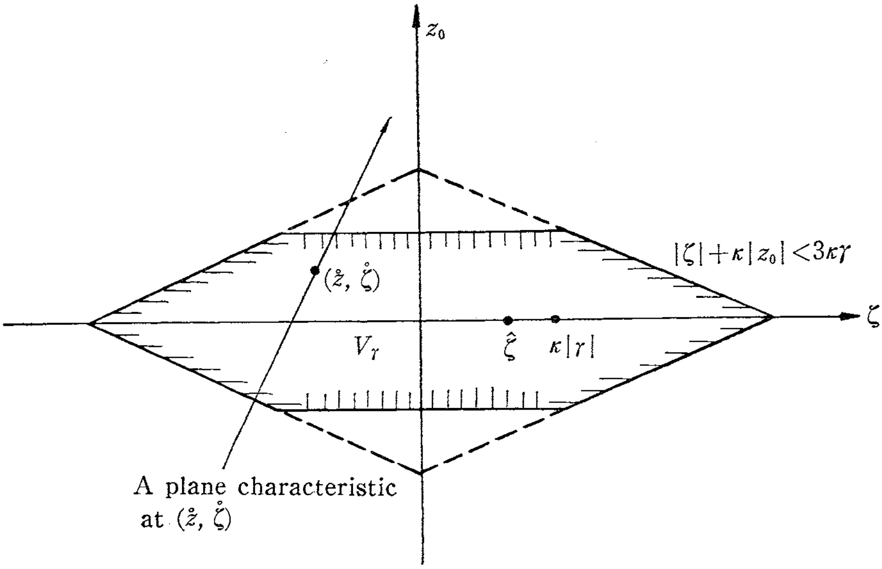
has a unique solution $v(z,\lambda,\zeta)$. Since the operator has the form

$$(6.11)\qquad \mathcal{L}_k^0+\mathcal{L}_k^1(\lambda)=\sum_{i\in\Delta_0}\tilde{q}_i\left(z,\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial}{\partial\zeta}\right)^{k-k_i}\left(\frac{\partial}{\partial z_0}\right)^{k_i}$$
$$+\sum_{i\in\Delta_0}\lambda^{-\beta_i}\tilde{q}_i\left(z,\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial}{\partial\zeta}\right)^{k-k_i}\left(\frac{\partial}{\partial z_0}\right)^{k_i}+\text{lower order terms,}$$

the principal part is an operator involving only $\left(\dfrac{\partial}{\partial z_0}\right)$ and $\left(\dfrac{\partial}{\partial\zeta}\right)$. When $|\lambda|\to\infty$, the characteristic root $\nu_i(z,\lambda,\xi_{n+1})$ with respect to ξ_0 (ξ_{n+1} is the dual variable of ζ) satisfies

$$(6.12)\qquad \sup_{z\in\bar{V}_\gamma}|\nu_i(z,\lambda,1)|<\kappa.$$

From a result due to Bony-Schapira [1], $v(z,\lambda,\zeta)$ is holomorphic on $\bar{V}_\gamma$.



Next let us consider the Cauchy problem

$$(6.13)\qquad \begin{cases} \left(\dfrac{\partial}{\partial\zeta}\right)^{m-k} u(z, \lambda, \zeta)=v(z, \lambda, \zeta), \\ \left(\dfrac{\partial}{\partial\zeta}\right)^{q} u(z, \lambda, \zeta)\Big|_{\zeta=\mathring{\zeta}}=0 \quad \text{if} \quad 0\leqq q\leqq m-k-1. \end{cases}$$

Obviously $u(z, \lambda, \zeta)$ exists on $\bar{V}_\gamma$ and it is a unique solution.

Suppose that (6.8) holds. From Lemma 3.3 it follows that there exist $\rho_0\geqq 1$ and F' such that

$$(6.14)\qquad v(z, \lambda, \zeta)\ll DF'\tilde{\Theta}^{(m-k)}(\bar{t}).$$

From (6.13) we have for some E

$$(6.15)\qquad u(z, \lambda, \zeta)\ll DE\tilde{\Theta}(\bar{t}).$$

PROPOSITION 6.2. *Suppose that the coefficients of $\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)$ are holomorphic on $\bar{V}_\gamma$. Suppose that $f(z, \lambda, \zeta)$ and $g(z, \lambda, \zeta)$ are also holomorphic on $\bar{V}_\gamma$. Then there is a constant A_4 such that there is a unique solution $u(z, \lambda, \zeta)$ of* (6.3) *holomorphic in a neighbourhood of $z=\zeta=0$ if $|\lambda|>A_4$.*

PROOF. The substitution $u=v+g$ reduces the proof to the case $g\equiv 0$, so we may as well assume that $g\equiv 0$ from the beginning. We solve the boundary value problem (6.3) by iteration. Define u_n by the recursion formula

$$\left(\frac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))u_0(z,\lambda,\zeta)=f(z,\lambda,\zeta), \tag{6.16}$$

$$\left(\frac{\partial}{\partial\zeta}\right)^{m-k}(\mathcal{L}_k^0+\mathcal{L}_k^1(\lambda))u_{n+1}(z,\lambda,\zeta) \tag{6.17}$$

$$+\sum_{l=k+1}^{m}\lambda^{-(1-\alpha)(l-k)}\left(\frac{\partial}{\partial\zeta}\right)^{m-l}\mathcal{L}_l(z,\lambda,\partial_z)u_n(z,\lambda,\zeta)=0$$

with the boundary conditions

$$\left(\frac{\partial}{\partial z_0}\right)^p u_n(z,\lambda,\zeta)\Big|_{z_0=0}=\left(\frac{\partial}{\partial\zeta}\right)^q u_n(z,\lambda,\zeta)\Big|_{\zeta=\hat{\zeta}}=0, \tag{6.18}$$

if $0\leqq p\leqq k-1$ and $0\leqq q\leqq m-k-1$.

By Lemma 6.1 each $u_n(z,\lambda,\zeta)$ exists on $\bar{V}_\gamma$. We show by induction on n that there are D and E such that

$$u_n(z,\lambda,\zeta)\ll\frac{DE^n\Theta(\hat{t})}{|\lambda|^{(1-\alpha)n}}. \tag{6.19}$$

Assume that (6.19) is valid when $n\leqq N$. We have by Lemma 3.2 for some E_1

$$\sum_{l=k+1}^{m}|\lambda|^{-(1-\alpha)(l-k)}\left(\frac{\partial}{\partial\zeta}\right)^{m-l}\mathcal{L}_l(z,\lambda,\partial_z)u_N(z,\lambda,\zeta) \tag{6.20}$$

$$\ll|\lambda|^{-(1-\alpha)(N+1)}DE^NE_1\tilde{\Theta}^{(m)}(\hat{t}).$$

Hence by Lemma 6.1 we have for some E_2

$$u_{N+1}(z,\lambda,\zeta)\ll|\lambda|^{-(1-\alpha)(N+1)}DE^NE_1E_2\tilde{\Theta}(\hat{t}). \tag{6.21}$$

Thus by choosing an E such that $E>E_1E_2$, we have (6.19) for $n=N+1$. It follows from (6.21) that $u(z,\lambda,\zeta)=\sum_{n=0}^{\infty}u_n(z,\lambda,\zeta)$ converges and is a solution of (6.3) if $E|\lambda|^{-(1-\alpha)}\leqq 1/2$ and $\rho_0|z_0|+|z'|+|\zeta-\hat{\zeta}|<2\gamma$. Uniqueness follows from Theorem 5.1.1 in Hörmander [5].

Thus as for $\beta(z,\lambda,\zeta)=w(z,\lambda,\zeta)-\tilde{w}(z,\lambda,\zeta)$ we have

PROPOSITION 6.3. *$\beta(z,\lambda,\zeta)$ is holomorphic in some neighbourhood of $z=\zeta=0$, when λ is large ($|\lambda|>\Lambda'$).*

PROOF. We expand $\beta(z,\lambda,\zeta)$ at $\zeta=\hat{\zeta}$:

$$\beta(z,\lambda,\zeta)=\sum_{q=0}^{m-k-1}(\zeta-\hat{\zeta})^q\beta_q(z,\lambda,\hat{\zeta})+O(|\zeta-\hat{\zeta}|^{m-k}). \tag{6.22}$$

$\beta_q(z,\lambda,\hat{\zeta})$ satisfies $\left(\frac{\partial}{\partial z_0}\right)^p\beta_q(z,\lambda,\hat{\zeta})\Big|_{z_0=0}=0$ $(0\leqq p\leqq k-1)$. Set $g(z,\lambda,\zeta)=$

$\sum_{q=0}^{m-k-1}(\zeta-\hat{\zeta})^q\beta_q(z, \lambda, \hat{\zeta})$ and $f(z, \lambda, \zeta)=0$ in (6.3). From Proposition 6.2 there is a unique solution $\tilde{\beta}(z, \lambda, \zeta)$ which is holomorphic in some neighbourhood of $z=\zeta=0$. Uniqueness implies that $\beta(z, \lambda, \zeta)=\tilde{\beta}(z, \lambda, \zeta)$. This completes the proof.

Thus we conclude that the difference of the functions $w(z, \lambda, \zeta)$ and $\tilde{w}(z, \lambda, \zeta)$ which satisfy the same underdetermined conditions is a holomorphic function of (z, ζ) near $z=\zeta=0$ and may be multi-valued with respect to λ.

§7. In §4 we constructed a solution $u(z)$ of the following problem (see (4.1))

$$\begin{cases} L(z, \partial)u(z)=0, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s u(0, z')=\delta_{s,t}u_t(z') \qquad (0\leqq s\leqq k-1), \end{cases} \tag{7.1}$$

where the initial function $u_t(z')$ is

$$u_t(z')=\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)}\exp(-\lambda z_1)\hat{v}(z', \lambda)(\log\lambda)d\lambda. \tag{7.2}$$

$u(z)$ is represented in the form

$$u(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)}\exp(-\lambda\varphi(z))W_n(z, \lambda)(\log\lambda)d\lambda. \tag{7.3}$$

In this section we shall give a representation of $u(z)$ by making use of the function $\tilde{w}(z, \lambda, \zeta)$ constructed in §5 and investigate integrals such as (7.2).

Now we know that $u(z)$ is also expressed in the form

$$\begin{aligned} u(z)=&\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)}\exp(-\lambda\varphi(z))(\log\lambda)d\lambda\int_{C^0(d,\theta)}\exp(-\lambda^\alpha\zeta)w^-(z, \lambda, \zeta)d\zeta \\ &+\sum_{n=0}^{\infty}\int_{\Gamma(\theta_0)}\exp(-\lambda\varphi(z))(\log\lambda)d\lambda\int_{C(\theta)}\exp(-\lambda^\alpha\zeta)w_n^+(z, \lambda, \zeta)d\zeta, \end{aligned} \tag{7.4}$$

where

$$w^-(z, \lambda, \zeta)=\sum_{n=0}^{\infty}\sum_{j=-\infty}^{-1}w_{n,j}(z, \lambda)f_j(\zeta) \tag{7.5}$$

and

$$w_n^+(z, \lambda, \zeta)=\sum_{j=0}^{n}w_{n,j}(z, \lambda)f_j(\zeta). \tag{7.6}$$

Set

$$w^+(z, \lambda, \zeta)=\sum_{n=0}^{\infty}w_n^+(z, \lambda, \zeta) \tag{7.7}$$

and

$$\beta(z, \lambda, \zeta)=w(z, \lambda, \zeta)-\tilde{w}(z, \lambda, \zeta). \tag{7.8}$$

We have

$$w(z, \lambda, \zeta)=w^{+}(z, \lambda, \zeta)+w^{-}(z, \lambda, \zeta)=\tilde{w}(z, \lambda, \zeta)+\beta(z, \lambda, \zeta). \tag{7.9}$$

Since $\beta(z, \lambda, \zeta)$ is holomorphic in some neighbourhood of $z=\zeta=0$, for small d we have

$$\int_{C^0(d,\theta)} \exp(-\lambda^{\alpha}\zeta)\beta(z, \lambda, \zeta)d\zeta=0.$$

Thus, if we set

$$u_{\mathrm{I}}(z)=\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)} \exp(-\lambda\varphi(z))(\log\lambda)d\lambda\int_{C^0(d,\theta)} \exp(-\lambda^{\alpha}\zeta)\tilde{w}(z, \lambda, \zeta)d\zeta, \tag{7.10}$$

$$u_{\mathrm{II}}(z)=\frac{-1}{2\pi qi}\int_{\Gamma(\theta_0)} \exp(-\lambda\varphi(z))(\log\lambda)d\lambda\int_{C^0(d,\theta)} \exp(-\lambda^{\alpha}\zeta)w^{+}(z, \lambda, \zeta)d\zeta, \tag{7.11}$$

$$u_{\mathrm{III}}(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)} \exp(-\lambda\varphi(z))(\log\lambda)d\lambda\int_{C(\theta)} \exp(-\lambda^{\alpha}\zeta)w_n^{+}(z, \lambda, \zeta)d\zeta, \tag{7.12}$$

then we have

THEOREM 7.1. *There is a solution $u(z)$ of (7.1) which represented in the form*

$$u(z)=u_{\mathrm{I}}(z)+u_{\mathrm{II}}(z)+u_{\mathrm{III}}(z) \tag{7.13}$$

in $\Omega^{\theta_0}_{0,\varepsilon_0}$ (see Theorem 4.7).

REMARK 7.2. In (7.13), $u_{\mathrm{I}}(z)$ is the most important part of $u(z)$ in order to investigate its behaviour near the singularity $K=\{z; \varphi(z)=0\}$. This will be clarified in the following section.

By taking account of the proof of Lemma 8.2 in Ōuchi [7], if $\tilde{u}(z)$ is another solution of (7.1) holomorphic in $\Omega^{\theta_0}_{0,\varepsilon_0}$, then $u(z)-\tilde{u}(z)$ is holomorphic near $z=0$. Thus we have

THEOREM 7.3. *Every solution $u(z)$ of (7.1) holomorphic in $\Omega^{\theta_0}_{0,\varepsilon_0}$ has the form*

$$u(z)=u_{\mathrm{I}}(z)+u_{\mathrm{II}}(z)+u_{\mathrm{III}}(z)+\bar{u}(z), \tag{7.14}$$

where $\bar{u}(z)$ is a holomorphic function in a neighbourhood of $z=0$.

Now let us investigate functions with an integral representation such as (7.2). Set

$$\omega=\{z'\in C^n;\ 0<|z_1|\leqq R,\ |z_i|\leqq R\ (2\leqq i\leqq n)\} \tag{7.15}$$

and let $\tilde{\omega}$ be its universal covering space. We denote the totality of functions holomorphic in $\tilde{\omega}$ by $\mathcal{O}(\tilde{\omega})$. Set

$$\omega_\theta=\{z'\in C^n;\ 0<|z_1|\leqq R,\ |\arg z_1+\theta|<\pi,\ |z_i|\leqq R\ (2\leqq i\leqq n)\}. \tag{7.16}$$

Every $v(z')\in\mathcal{O}(\tilde{\omega})$ defines a function $v(z',\theta)$ holomorphic in ω_θ. Put for $v(z')\in\mathcal{O}(\tilde{\omega})$

$$\hat{v}(z'',\lambda,\theta)=\frac{1}{2\pi i}\int_{\Xi(\theta)}\exp(\lambda z_1)v(z_1,z'',\theta)dz_1, \tag{7.17}$$

where the path $\Xi(\theta)$ is the union $\Xi(\theta)=\Xi^-(\theta)\cup\Xi^\varepsilon(\theta)\cup\Xi^+(\theta)$, where

$$\begin{cases}\Xi^-(\theta)=\{z_1=\xi\exp(-i(\theta+\pi)),\ \varepsilon\leqq\xi\leqq d\},\\ \Xi^\varepsilon(\theta)=\{z_1=\varepsilon\exp(i\rho),\ -\theta-\pi\leqq\rho\leqq-\theta+\pi\},\\ \Xi^+(\theta)=\{z_1=\xi\exp(-i(\theta-\pi)),\ \varepsilon\leqq\xi\leqq d\},\qquad(0<\varepsilon<d<R).\end{cases} \tag{7.18}$$

$\Xi(\theta)$ is a path which starts at $d\exp(-i(\theta-\pi))$ and goes around the origin on $\Xi^\varepsilon(\theta)$ in the negative direction and ends to $d\exp(-i(\theta+\pi))$ on $\Xi^-(\theta)$. Obviously $\hat{v}(z'',\lambda,\theta)$ is an entire function of λ and independent of ε. Hence we can show that for any $\varepsilon>0$ there is a constant C_ε such that

$$|\hat{v}(z'',\lambda,\theta)|\leqq C_\varepsilon\exp(\varepsilon|\lambda|) \tag{7.19}$$

if $|\arg\lambda-\theta+2m\pi|\leqq\pi/2$ $(m\in\boldsymbol{Z})$.

Conversely let us define

$$\tilde{v}(z')=\frac{1}{2\pi qi}\int_{\Gamma(\theta)}\exp(-\lambda z_1)\hat{v}(z'',\lambda,\theta)(\log\lambda)d\lambda. \tag{7.20}$$

Here $\Gamma(\theta)$ is the path defined by (4.4). Then we have

$$\tilde{v}(z')=\frac{1}{2\pi qi}\int_{\Gamma(\theta)}\exp(-\lambda z_1)(\log\lambda)d\lambda\frac{1}{2\pi i}\int_{\Xi(\theta)}\exp(\lambda s)v(s,z'',\theta)ds. \tag{7.21}$$

If $\varepsilon<|z_1|<d$ and $\arg z_1=-\theta$, then we have

$$\begin{aligned}v(z')&=\frac{1}{2\pi i}\int_{\Xi(\theta)}v(s,z'',\theta)ds\frac{1}{2\pi qi}\int_{\Gamma(\theta)}\exp(\lambda(s-z_1))(\log\lambda)d\lambda\\&=\frac{1}{2\pi i}\int_{\Xi(\theta)}\frac{v(s,z'',\theta)}{z_1-s}ds\\&=\frac{1}{2\pi i}\int_{\Xi(\theta)\cup\Xi^c(\theta)}\frac{v(s,z'',\theta)}{z_1-s}ds-\frac{1}{2\pi i}\int_{\Xi^c(\theta)}\frac{v(s,z'',\theta)}{z_1-s}ds,\end{aligned} \tag{7.22}$$

where $\Xi^c(\theta)$ is a path that starts at $d\exp(-i(\theta+\pi))$ and goes around the origin once on the circle $|s|=d$ and ends at $d\exp(-i(\theta-\pi))$. Thus from (7.22) we have

$$\tilde{v}(z')=v(z',\theta)+\bar{v}(z'), \tag{7.23}$$

where $\bar{v}(z')$ is holomorphic in $\{|z_1|<d,\ |z_i|\leqq R\ (2\leqq i\leqq n)\}$. (7.23) implies that the difference of $\tilde{v}(z')$ and $v(z',\theta)$ is holomorphic in a neighbourhood of $z=0$.

Now let us consider the equation

$$\begin{cases} L(z,\partial)u(z)=0, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s u(0,z')=\delta_{s,t}u_t(z') \quad (0\leqq s\leqq k-1), \end{cases} \tag{7.24}$$

where $u_t(z')\in\mathcal{O}(\tilde{\omega})$. We denote the universal covering space of $\Omega-K$ by $\widetilde{\Omega-K}$ and the set of all holomorphic functions on $\widetilde{\Omega-K}$ by $\mathcal{O}(\widetilde{\Omega-K})$. By summing up the above results, we have

THEOREM 7.4. *Each solution $u(z)$ of the equation* (7.24) *which belongs to $\mathcal{O}(\widetilde{\Omega-K})$ is represented on the set*

$$\Omega^{\theta_0}_{0,\varepsilon'}=\{z\in\Omega\,;\ |\arg\varphi(z)+\theta_0|<\pi/2,\ 0<|\varphi(z)|<\varepsilon'\}$$

in the form

$$u(z)=u_{\mathrm{I}}(z)+u_{\mathrm{II}}(z)+u_{\mathrm{III}}(z)+\tilde{u}(z), \tag{7.25}$$

where $u_{\mathrm{I}}(z)$, $u_{\mathrm{II}}(z)$ and $u_{\mathrm{III}}(z)$ are defined by (7.10), (7.11) *and* (7.12) *respectively and $\tilde{u}(z)$ is holomorphic on Ω' ($\Omega'\subset\Omega$).*

PROOF. We can decompose $u_t(z')=v(z')$ as

$$u_t(z')=v(z')=\frac{1}{2\pi qi}\int_{\Gamma(\theta_0)}\exp(-\lambda z_1)\hat{v}(z'',\lambda)(\log\lambda)d\lambda+\bar{v}(z'), \tag{7.26}$$

where $\bar{v}(z')$ is holomorphic near $z=0$. Then construct $u_{\mathrm{I}}(z)$, $u_{\mathrm{II}}(z)$ and $u_{\mathrm{III}}(z)$ by prescribed method with the aid of $\hat{v}(z'',\lambda)$ in (7.26). Then $\tilde{u}(z)=u(z)-(u_{\mathrm{I}}(z)+u_{\mathrm{II}}(z)+u_{\mathrm{III}}(z))$ satisfies

$$\begin{cases} L(z,\partial)\tilde{u}(z)=0 \\ \left(\dfrac{\partial}{\partial z_0}\right)^s u(0,z')=\delta_{s,t}\bar{v}(z') \quad (0\leqq s\leqq k-1). \end{cases} \tag{7.27}$$

Since $\tilde{u}(z)$ is holomorphic in $\Omega^{\theta_0}_{0,\varepsilon_0}$ and $\bar{v}(z')$ is also holomorphic on a neighbourhood of $z'=0$. Hence a modification of Lemma 8.2 in Ōuchi [7] leads that $\tilde{u}(z)$ is holomorphic in a neighbourhood of $z=0$. This completes the proof.

§ 8. In this section we shall study a function $u(z)$ which satisfies equation (7.1) (or (7.24)), when the trace $u_t(z')$ in (7.1), on $S=\{z_0=0\}$, has a pole at $z_1=0$. We shall investigate the behaviour of $u(z)$, when z tends to the characteristic surface $K=\{z;\ \varphi(z)=0\}$, by making use of the integral representation obtained in § 7. It will be clarified that $u_{\mathrm{I}}(z)$ is the most essential part of $u(z)$. To do so we need asymptotic behaviour of functions defined by some integrals, which appear in the integral represention of $u(z)$ (see (7.10)~(7.12)).

Now set

$$\hat{F}(\lambda, \theta)=\int_{C^0(d,\theta)} \exp(-\lambda^\alpha\zeta)F(\lambda, \zeta)d\zeta. \tag{8.1}$$

Here

$$F(\lambda, \zeta)=\sum_{j=J}^{\infty} a_j(\lambda)f_j(\zeta+b(\lambda)), \tag{8.2}$$

where $a_j(\lambda)$ and $b(\lambda)$ are holomorphic functions of $\lambda^{-1/q}$ on $\{|\lambda|\geqq \Lambda\}$:

$$\begin{cases} a_j(\lambda)=\sum_{i=0}^{\infty} a_{j,i}\lambda^{-i/q}, \\ b(\lambda)=\sum_{i=0}^{\infty} b_i\lambda^{-i/q}. \end{cases} \tag{8.3}$$

We assume that there are constants d and d' such that

$$\sup_{|\lambda|\geqq\Lambda} |a_j(\lambda)|\leqq A\Gamma(|j|+1)(2d)^{-|j|}, \tag{8.4}$$

$$\sup_{|\lambda|\geqq\Lambda} |b(\lambda)|\leqq d'<d. \tag{8.5}$$

Recall that $\alpha=l/q$ (see (2.10)).

LEMMA 8.1. *For any δ with $0<\delta<\left(\frac{1-\alpha}{\alpha}\right)\frac{\pi}{2}=\left(\frac{1}{\sigma-1}\right)\frac{\pi}{2}$, there is a d' such that*

$$\hat{F}(\lambda, \theta)\sim\exp(b(\lambda)\lambda^\alpha)\left(\sum_{j=J}^{\infty} a_j(\lambda)\lambda^{-\alpha(j+1)}\right) \tag{8.6}$$

as $|\lambda|\to\infty$ on the sector $|\arg\lambda^\alpha+\theta|\leqq(\pi/2+\delta)\alpha$.

PROOF. First we note that $F(\lambda, \zeta)$ is convergent at $\zeta+b(\lambda)\neq 0$ by (8.4). For $j<0$ we have

$$\begin{aligned} \oint_{|\zeta|=d} \exp(-\lambda^\alpha\zeta)f_j(\zeta+b(\lambda))d\zeta &= \frac{1}{2\pi i}\oint_{|\zeta|=d} \exp(-\lambda^\alpha\zeta)\frac{(-1)^{|j|+1}\Gamma(|j|)}{(\zeta+b(\lambda))^{|j|}}d\zeta \\ &=\exp(b(\lambda)\lambda^\alpha)\lambda^{-\alpha(j+1)}. \end{aligned} \tag{8.7}$$

For $j\geqq 0$, we have

$$\int_{C^0(d,\theta)} \exp(-\lambda^\alpha\zeta) f_j(\zeta+b(\lambda))d\zeta = \int_{-b(\lambda)}^{d\exp(i\theta)} \exp(-\lambda^\alpha\zeta)\frac{(\zeta+b(\lambda))^j}{\Gamma(j+1)}d\zeta \tag{8.8}$$

$$= \int_0^{\tilde{d}} \exp(b(\lambda)\lambda^\alpha - \lambda^\alpha\eta)\frac{\eta^j}{\Gamma(j+1)}d\eta \qquad (\tilde{d} = d\exp(i\theta)+b(\lambda))$$

$$= \exp(b(\lambda)\lambda^\alpha)\int_0^{\tilde{d}} \exp(-\lambda^\alpha\eta)\frac{\eta^j}{\Gamma(j+1)}d\eta\,.$$

Hence if $-\mathrm{Re}\,\lambda^\alpha\tilde{d} \leqq -c|\lambda|^\alpha$ for some $c>0$,

$$\int_{C^0(d,\theta)} \exp(-\lambda^\alpha\zeta) f_j(\zeta+b(\lambda))d\zeta \sim \exp(b(\lambda)\lambda^\alpha)\{\lambda^{-\alpha(j+1)} + O(\exp(-c'|\lambda|^\alpha))\}\,.$$

For δ with $0<\delta<\left(\frac{1-\alpha}{\alpha}\right)\frac{\pi}{2}$ there is d' such that $-\mathrm{Re}\,\lambda^\alpha\tilde{d} \leqq -c|\lambda|^\alpha$ for some c, if $|\arg\lambda^\alpha+\theta| \leqq (\pi/2+\delta)\alpha$. Hence (8.6) follows.

Hereafter we fix d' and δ. Put

$$\hat{G}(\lambda,\theta) = \int_{C^0(d,\theta)} \exp(-\lambda^\alpha\zeta)G(\lambda,\zeta)d\zeta\,, \tag{8.9}$$

where

$$G(\lambda,\zeta) = \sum_{j=0}^{\infty} \beta_j(\lambda)f_j(\zeta) \tag{8.10}$$

and $\beta_j(\lambda)$ is a holomorphic function of $\lambda^{-1/q}$ $(|\lambda| \geqq \Lambda)$ such that

$$\sup_{|\lambda|\geqq\Lambda} |\beta_j(\lambda)| \leqq A\Gamma(j+1)(1/2d)^j\,. \tag{8.11}$$

LEMMA 8.2. *We have*

$$\hat{G}(\lambda,\theta) \sim \sum_{j=0}^{\infty} \beta_j(\lambda)\lambda^{-\alpha(j+1)} \tag{8.12}$$

as $|\lambda|\to\infty$ *in the sector* $|\arg\lambda^\alpha+\theta| < \pi/2-\varepsilon$ *for any* $\varepsilon>0$.

By putting $b(\lambda)=0$ in Lemma 8.1, we get this lemma.

Finally set

$$\hat{H}_n(\lambda,\theta) = \int_{C(\theta)} \exp(-\lambda^\alpha\zeta)H_n(\lambda,\zeta)d\zeta \qquad (n=0,1,2,\cdots)\,, \tag{8.13}$$

where

$$H_n(\lambda,\zeta) = \sum_{j=0}^{n} \gamma_j^n(\lambda)f_j(\zeta) \tag{8.14}$$

and $\gamma_j^n(\lambda)$ is a holomorphic function of $\lambda^{-1/q}$ $(|\lambda| \geqq \Lambda)$. Then we have

(8.15) $$\hat{H}_n(\lambda,\theta)=\sum_{j=0}^{n}\gamma_j^n(\lambda)\lambda^{-\alpha(j+1)},$$

which is independent of θ. So we denote $\hat{H}_n(\lambda)=\hat{H}_n(\lambda,\theta)$.

Next we consider integrals in λ:

(8.16) $$\mathcal{F}(a,\theta)=\int_{\Gamma(\theta_1)}\exp(-\lambda a)\hat{F}(\lambda,\theta)h_r(\lambda)d\lambda,$$

(8.17) $$\mathcal{G}(a,\theta)=\int_{\Gamma(\theta_1)}\exp(-\lambda a)\hat{G}(\lambda,\theta)h_r(\lambda)d\lambda,$$

(8.18) $$\mathcal{H}(a)=\sum_{n=0}^{\infty}\int_{\Gamma(\theta_1)}\exp(-\lambda a)\hat{H}_n(\lambda)h_r(\lambda)d\lambda,$$

where

(8.19) $$h_r(\lambda)=\frac{\lambda^r}{2\pi qi}(\log\lambda).$$

Let $\gamma_j^n(\lambda)$ satisfy

(8.20) $$|\gamma_j^n(\lambda)|\leq MA^nB^{n-j}|\lambda|^{-(1-\alpha)n+\iota}j!,\ \iota\ \text{constant}.$$

Condition (8.20) is similar to (3.37). Then by the same arguments as in §4 we can show that $\mathcal{H}(a)$ is holomorphic in $0<|a|<c'$ for some c'.

Put $\beta_j(\lambda)=\sum_{n=j}^{\infty}\gamma_j^n(\lambda)$. We have for any $N>0$

(8.21) $$|\lambda|^N|\hat{G}(\lambda,\theta)-\sum_{n=0}^{\infty}\hat{H}_n(\lambda)|\to 0$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda^\alpha+\theta|<\pi/2-\varepsilon$. All the expansions hold, even if we replace $\arg\lambda^\alpha$ by $\arg\lambda^\alpha+2m\pi$ ($m\in\boldsymbol{Z}$).

It follows from Lemma 8.2 that $\mathcal{G}(a,\theta)$ is holomorphic, if $|\alpha\theta_1+\theta|<\pi/2$ and $|\arg a+\theta_1|<\pi/2$. Hence $\mathcal{G}(a,\theta)$ is holomorphic in $\{a\,;\,|\arg a-\theta/\alpha|<\pi(\alpha+1)/2\alpha\}$.

Next let us study $\mathcal{F}(a,\theta)$. According to Lemma 8.1,

(8.22) $$\hat{F}(\lambda,\theta)\sim\exp(b(\lambda)\lambda^\alpha)\sum_{j=J}^{\infty}a_j(\lambda)\lambda^{-\alpha(j+1)},$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda^\alpha+\theta|\leq(\pi/2+\delta)\alpha$. Hence from (8.3)

(8.23) $$\begin{aligned}\hat{F}(\lambda,\theta)h_r(\lambda)&\sim\frac{\exp(b(\lambda)\lambda^\alpha)}{2\pi qi}\sum_{j=J}^{\infty}a_j(\lambda)\lambda^{r-\alpha(j+1)}(\log\lambda)\\&\sim(2\pi qi)^{-1}\exp(b_0\lambda^\alpha+b_1\lambda^{\alpha-1/q}+\cdots+b_{l-1}\lambda^{1/q})\\&\quad\times\lambda^r(\log\lambda)\sum_{s=l(J+1)}^{\infty}\bar{a}_s\lambda^{-s/q}\end{aligned}$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda^\alpha+\theta|\leqq(\pi/2+\delta)\alpha$.

Further let us assume that r is a rational number such that qr is an integer. Under these hypotheses we study the function $\mathcal{F}(a, \theta)$. To do so let us consider

$$\mathcal{F}(a, b, \theta)=\frac{1}{2\pi qi}\int_{\Gamma(\theta_1)}\exp\Big(-\lambda a+\sum_{i=0}^{l-1} b_i\lambda^{\alpha-i/q}\Big)W(\lambda)(\log\lambda)d\lambda\,, \tag{8.24}$$

where $b=(b_0, b_1, \cdots, b_{l-1})$ and $W(\lambda)$ is holomorphic in $\{|\lambda|\geqq\Lambda\}$ such that

$$W(\lambda)\sim\sum_{j=m}^{-\infty} w_j\lambda^{j/q}\,,\quad w_m\neq 0 \tag{8.25}$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda^\alpha+\theta|\leqq(\pi/2+\delta)\alpha$. We may assume that $\theta=0$. Hereafter we denote $\mathcal{F}(a, b, \theta)$ by $\mathcal{F}(a, b)$. The path $\Gamma(\theta_1)$ and $C(\theta_1)$ are defined by (4.4) and (4.7) respectively. We also denote $C(\theta_1)$ by $C(\theta_1, d)$ to emphasize d.

Now we investigate the asymptotic behaviour of $\mathcal{F}(a, b)$ in domains determined by (a, b_0). It follows from (8.25) that $\mathcal{F}(a, b)$ is holomorphic in $|\arg a|<\pi+\delta$. Let $a>0$ and set $(\lambda a)^{1/q}=\mu$. Then

$$\begin{aligned}\mathcal{F}(a, b)=&\frac{q}{(2\pi i)a}\int_{C(0)}\exp(-\mu^q+b_0a^{-l/q}\mu^l+\cdots+b_{l-j}a^{-j/q}\mu^j+\cdots+b_{l-1}a^{-1/q}\mu)\\ &\times W(\mu^q a^{-1})\mu^{q-1}(\log\mu)d\mu-\frac{(\log a)}{(2\pi i)a}\int_{C(0)}\exp(-\mu^q+b_0a^{-l/q}\mu^l+\cdots+b_{l-1}a^{-1/q}\mu)\\ &\times W(\mu^q a^{-1})\mu^{q-1}d\mu=\mathcal{F}_1(a, b)-(\log a)\mathcal{F}_2(a, b)\,.\end{aligned} \tag{8.26}$$

Set

$$\tau=b_0a^{-l/q},\quad c_j=b_{l-j}b_0^{-j/l},\quad \tilde{c}_j=b_{l-j}b_0^{-1}, \tag{8.27}$$

$$\rho(\mu, a, b)=-\mu^q+b_0a^{-l/q}\mu^l+\sum_{j=1}^{l-1} b_{l-j}a^{-j/q}\mu^j\,. \tag{8.28}$$

Let us study the behaviour of $\mathcal{F}(a, b)$ when $|\tau|\to\infty$ under the condition that c_j is bounded, $|c_j|\leqq C$ $(j=1, 2, \cdots, l-1)$. We have

$$\rho(\mu, a, b)=-\mu^q+\tau\mu^l+\sum_{j=1}^{l-1} c_j\tau^{j/l}\mu^j\,. \tag{8.29}$$

If we set $\mu=\tau^p\xi$, $p=1/(q-l)$, then

$$\rho(\mu, a, b)=-\tau^{pq}(\xi^q-\xi^l)+\sum_{j=1}^{l-1} c_j\tau^{(jpq)/l}\xi^j\,. \tag{8.30}$$

Hence by putting $\tau^{pq}=\kappa$, we have

$$\begin{aligned}\rho(\mu, a, b)&=-\kappa\Big(\xi^q-\xi^l-\sum_{j=1}^{l-1} c_j\kappa^{j/l-1}\xi^j\Big)\\ &=-\kappa\phi(\xi, \kappa)\,.\end{aligned} \tag{8.31}$$

Consequently we get, if $\tau>0$ and $a>0$,

$$\mathcal{F}_1(a,\ b)=\frac{q\tau^{pq}}{(2\pi i)a}\int_{C(0)}\exp(-\kappa\phi(\xi,\ \kappa))W((\kappa/a)\xi^q)\xi^{q-1}(\log\xi)d\xi \tag{8.32}$$
$$+\frac{q\tau^{pq}(\log\tau^p)}{(2\pi i)a}\int_{C(0)}\exp(-\kappa\phi(\xi,\ \kappa))W((\kappa/a)\xi^q)\xi^{q-1}d\xi$$

and

$$\mathcal{F}_2(a,\ b)=\frac{\tau^{pq}}{(2\pi i)a}\int_{C(0)}\exp(-\kappa\phi(\xi,\ \kappa))W((\kappa/a)\xi^q)\xi^{q-1}d\xi\,. \tag{8.33}$$

Let us give two lemmas in order to study $\mathcal{F}(a,\ b)$, when $|\tau|\to\infty$, by the method of the saddle points. For stationary points of $\phi(\xi,\ \kappa)$ we have

LEMMA 8.3. *There exists a point* $\hat{\xi}(\kappa)$ *such that*

$$\frac{\partial\phi(\hat{\xi}(\kappa),\ \kappa)}{\partial\xi}=0\,, \tag{8.34}$$

which is expressed as follows:

$$\hat{\xi}(\kappa)=\hat{\xi}_0+\sum_{s=1}^{\infty}\kappa^{-s/l}\hat{\xi}_s(c)\,,\quad \hat{\xi}_0=(l/q)^p\qquad(0<\hat{\xi}_0<1)\,, \tag{8.35}$$

where $c=(c_1,\ c_2,\ \cdots,\ c_{l-1})$.

PROOF. Since

$$\frac{\partial\phi}{\partial\xi}(\xi,\ \kappa)=q\xi^{q-1}-l\xi^{l-1}-\sum_{j=1}^{l-1}jc_j\kappa^{j/l-1}\xi^{j-1}\,, \tag{8.36}$$

for large κ, there is a solution $\hat{\xi}(\kappa)$ of (8.34) such as (8.35). (8.35) converges for $|\kappa|>\kappa_0$.

Set

$$\phi^{(s)}(\xi,\ \kappa)=\left(\frac{\partial}{\partial\xi}\right)^s\phi(\xi,\ \kappa) \tag{8.37}$$

and

$$\Delta(\kappa)=\sqrt{\phi^{(2)}(\hat{\xi}(\kappa),\ \kappa)/2}\,. \tag{8.38}$$

Then from Lemma 8.3 it follows

LEMMA 8.4. (i) *The following expansions hold for* $|\kappa|>\kappa_1$:

$$(\sqrt{\kappa}\Delta(\kappa))^{-1}=\frac{1}{\sqrt{\kappa}}\left(\delta_0+\sum_{s=1}^{\infty}\delta_s\kappa^{-s/l}\right),\quad(\delta_0>0)\,, \tag{8.39}$$

(8.40) $$(\hat{\xi}_0(\kappa)+t/(\sqrt{\kappa}\,\Delta(\kappa)))^{m'}=(\xi_0)^{m'}+\sum_{s=\tilde{s}}^{\infty}\tilde{\xi}_s(t)\kappa^{-s/2l},\quad \tilde{s}=\min(l, 2).$$

(ii) *There is a coordinate transformation* $x=x(t, \kappa)$ *such that*

(8.41) $$x^2=\sum_{s=2}^{q}\kappa\Big(\frac{t}{\sqrt{\kappa}\,\Delta(\kappa)}\Big)^s\frac{\phi^{(s)}(\hat{\xi}(\kappa), \kappa)}{s\,!}.$$

PROOF. (8.39) and (8.40) follow easily from Lemma 8.3. By putting $x=t\Big(1+\sum_{s=3}^{q}\Big(\frac{t}{\sqrt{\kappa}}\Big)^{s-2}\frac{\phi^{(s)}(\hat{\xi}(\kappa), \kappa)}{(\Delta(\kappa))^s s\,!}\Big)^{1/2}$, we have (8.41).

Now set

(8.42) $$u(a, b)=\int_{C(0)}\exp(-\kappa\phi(\xi, \kappa))W((\kappa/a)\xi^q)\xi^{q-1}(\log\xi)d\xi$$

and

(8.43) $$v(a, b)=\int_{C(0)}\exp(-\kappa\phi(\xi, \kappa))W((\kappa/a)\xi^q)\xi^{q-1}d\xi.$$

Then we have

(8.44) $$\mathcal{F}_1(a, b)=\frac{q\tau^{pq}}{(2\pi i)a}u(a, b)+\frac{q\tau^{pq}(\log\tau^p)}{(2\pi i)a}v(a, b)$$

and

(8.45) $$\mathcal{F}_2(a, b)=\frac{\tau^{pq}}{(2\pi i)a}v(a, b).$$

Let us consider $u(a, b)$. Let $|\arg\kappa|<\pi/2-\varepsilon$ and $|\arg(\kappa/a)|\leqq\pi/2+\delta$. Then $u(a, b)$ is well defined. We have for large $c>0$

(8.46) $$\begin{aligned}u(a, b)=&\int_{C^0(0,\, c(|\kappa|^{-1}|a|)^{1/q})}\exp(-\kappa\phi(\xi, \kappa))W((\kappa/a)\xi^q)\xi^{q-1}(\log\xi)d\xi\\&+\int_{C^{\pm}(0,\, c(|\kappa|^{-1}|a|)^{1/q})}\exp(-\kappa\phi(\xi, \kappa))w_m(\kappa/a)^{m/q}(\xi)^{m'}(\log\xi)d\xi\\&+\int_{C^{\pm}(0,\, c(|\kappa|^{-1}|a|)^{1/q})}\exp(-\kappa\phi(\xi, \kappa))\widetilde{W}((\kappa/a)\xi^q)\xi^{q-1}(\log\xi)d\xi,\end{aligned}$$

where $\widetilde{W}(\lambda)$ satisfies, if $|\arg\lambda|\leqq\pi/2+\delta$

(8.47) $$|\widetilde{W}(\lambda)|\leqq C|\lambda|^{(m-1)/q}$$

and $m'=m+q-1$.

From now on we assume that $c_j=b_{l-j}b_0^{-j/l}$ are all bounded:

(8.48) $$|c_j|\leqq C\qquad (j=1, 2, \cdots, l-1).$$

The first term in (8.46) is $O(\exp(c'|\kappa|^{(1-\alpha)}))$ for some $c'>0$. For the second term of (8.46), we have

(8.49)
$$I=\int_{C^{\pm}(0,\,c(|\kappa|^{-1}|a|)^{1/q})}\exp(-\kappa\phi(\xi,\,\kappa))w_m(\kappa/a)^{m/q}(\xi)^{m'}(\log\xi)d\xi$$
$$=2\pi i\int_{c(|\kappa|^{-1}|a|)^{1/q}}^{\infty}\exp(-\kappa\phi(\xi,\,\kappa))w_m(\kappa/a)^{m/q}(\xi)^{m'}d\xi$$
$$=2\pi i\Big(\int_{c(|\kappa|^{-1}|a|)^{1/q}}^{\hat{\xi}_0/2}+\int_{\hat{\xi}_0/2}^{2}+\int_{2}^{\infty}\cdots d\xi\Big)=I_1+I_2+I_3\,.$$

I_3 decays exponential when $|\kappa|\to\infty$ in the sector $|\arg\kappa|<\pi/2-\varepsilon$, $|I_3|\leqq C_\varepsilon\times\exp(-c_\varepsilon|\kappa|)$. For I_2 we have

(8.50)
$$I_2=2\pi i\int_{\hat{\xi}_0/2}^{2}\exp(-\kappa\phi(\xi,\,\kappa))w_m(\kappa/a)^{m/q}(\xi)^{m'}d\xi$$
$$=2\pi i\int_{-\hat{\xi}(\kappa)+\hat{\xi}_0/2}^{2-\hat{\xi}(\kappa)}\exp(-\kappa\phi(\eta+\hat{\xi}(\kappa)),\,\kappa)(\eta+\hat{\xi}(\kappa))^{m'}w_m(\kappa/a)^{m/q}d\eta$$
$$=\frac{(2\pi i)\exp(-\kappa\phi(\hat{\xi}(\kappa),\,\kappa))}{\sqrt{\kappa\,\Delta(\kappa)}}$$
$$\times\int_{-\sqrt{\kappa}\Delta(\kappa)(\hat{\xi}(\kappa)-\hat{\xi}_0/2)}^{\sqrt{\kappa}\Delta(\kappa)(2-\hat{\xi}(\kappa))}\exp\Big\{-\sum_{s=2}^{q}\kappa\Big(\frac{t}{\sqrt{\kappa\,\Delta(\kappa)}}\Big)^{s}\frac{\phi^{(s)}(\hat{\xi}(\kappa),\,\kappa)}{s\,!}\Big\}$$
$$\times\Big(\hat{\xi}(\kappa)+\frac{t}{\sqrt{\kappa\,\Delta(\kappa)}}\Big)^{m'}w_m(\kappa/a)^{m/q}dt\,.$$

From Lemma 8.4 it follows that

(8.51)
$$\exp\Big(-\sum_{s=2}^{q}\kappa\Big(\frac{t}{\sqrt{\kappa\,\Delta(\kappa)}}\Big)^{s}\frac{\phi^{(s)}(\hat{\xi}(\kappa),\,\kappa)}{s\,!}\Big)\Big(\hat{\xi}(\kappa)+\frac{t}{\sqrt{\kappa\,\Delta(\kappa)}}\Big)^{m'}$$
$$=\exp(-x^2)((\hat{\xi}_0)^{m'}+O(\kappa^{-1/l'})),\quad l'=\max(l,\,2)\,.$$

Thus we have

(8.52)
$$I_2=\frac{2\pi i\exp(-\kappa\phi(\hat{\xi}(\kappa),\,\kappa))}{\sqrt{\kappa\,\Delta(\kappa)}}$$
$$\times\int_{x(\sqrt{\kappa}\Delta(\kappa)(\hat{\xi}(\kappa)-\hat{\xi}_0/2))}^{x(\sqrt{\kappa}\Delta(\kappa)(2-\hat{\xi}(\kappa)))}\exp(-x^2)(\kappa/a)^{m/q}(\hat{\xi}_0)^{m'}(w_m+O(\kappa^{-1/l'}))dx\,.$$

Since $\sqrt{\kappa}\,\Delta(\kappa)\hat{\xi}(\kappa)\sim(\hat{\xi}_0)\delta_0^{-1}\sqrt{\kappa}$ when $|\kappa|\to\infty$, we have

(8.53)
$$\lim_{|\kappa|\to\infty}I_2=\frac{(2\pi i)\exp(-\kappa\phi(\hat{\xi}(\kappa),\,\kappa))}{\sqrt{\kappa\,\Delta(\kappa)}}(\kappa/a)^{m/q}(\hat{\xi}_0)^{m'}\sqrt{\pi}(w_m+O(\kappa^{-1/l'}))\,.$$

For I_1 we note that $|(\kappa/a)^{m/q}(|\kappa|^{-1}|a|)^{m'/q}|=|\kappa/a|^{(1-q)/q}$. Hence

(8.54)
$$|\exp\{\kappa(\phi(\hat{\xi}_0/2,\,\kappa)-\tilde{\varepsilon})\}\quad I_1|=O(1)$$

for small $\tilde{\varepsilon}>0$.

Consequently

$$I=\frac{(2\pi i)\exp(-\kappa\phi(\hat{\xi}(\kappa),\kappa))}{\sqrt{\kappa}\Delta(\kappa)}(\kappa/a)^{m/q}\sqrt{\pi}(\hat{\xi}_0)^{m'}(w_m+O(\kappa^{-1/l'})). \tag{8.55}$$

By the same method, it follows from (8.47) that the third term J_1 of (8.46) satisfies

$$J_1=\frac{\exp(-\kappa\phi(\hat{\xi}(\kappa),\kappa))}{\sqrt{\kappa}\Delta(\kappa)}O((\kappa/a)^{(m-1)/q}). \tag{8.56}$$

Thus summing up the results we have

PROPOSITION 8.5. *Let $|\arg a|<\delta$ and $|c_j|\leqq C$. Then for any $\varepsilon>0$, we have*

$$u(a,b)=\exp\Big(\sum_{s=0}^{l-1}\hat{\phi}_s(c)\tau^{q(l-s)/l(q-l)}\Big)A'a^{-m/q}\tau^{(2m-q)/2(q-l)}(w_m+O(\tau^{-p})) \tag{8.57}$$

as $|\tau|\to\infty$ in the sector $|\arg\tau|<\dfrac{(q-l)\pi}{2q}-\varepsilon$, where $p=\dfrac{1}{q-l}$ and $\hat{\phi}_s(c)$ $(c=(c_1,c_2,\cdots,c_{l-1}))$ is holomorphic in c $(s=0,1,\cdots,l-1)$ and $\hat{\phi}_0(c)$ is a positive constant.

PROOF. We can get (8.57) by expanding $\exp(-\kappa\phi(\hat{\xi}(\kappa),\kappa))$ and $\Delta(\kappa)$ with respect to $\kappa^{-1/l}$. The domain of a where the asymptotic expansion (8.57) is valid can be extended by analytic continuation in a.

Now let us consider $v(a,b)$ ((8.43)). The integral (8.43) does not contain $\log\xi$. From (8.25) we have for $|\arg\lambda^\alpha|\leqq(\pi/2+\delta)\alpha$

$$W(\lambda)=\sum_{j=m}^{-N}w_j\lambda^{j/q}+W''(\lambda)\qquad(w_m\neq 0), \tag{8.58}$$

where

$$W''(\lambda)=O(|\lambda|^{-(N+1)/q}). \tag{8.59}$$

So we have for $\kappa>0$ and $a>0$

$$\begin{aligned}v(a,b)&=\int_{C(0)}\exp(-\kappa\phi(\xi,\kappa))\Big\{\sum_{j=m}^{-N}w_j(\kappa/a)^{j/q}\xi^{j+q-1}+W''((\kappa/a)\xi^q)\xi^{q-1}\Big\}d\xi\\&=\sum_{j=-q}^{-N}\int_{C(0)}\exp(-\kappa\phi(\xi,\kappa))w_j(\kappa/a)^{j/q}\xi^{j+q-1}d\xi\\&\quad+\int_{C(0)}\exp(-\kappa\phi(\xi,\kappa))W''((\kappa/a)\xi^q)\xi^{q-1}d\xi.\end{aligned} \tag{8.60}$$

By a method similar to the above we can show that

$$\int_{C(0)}\exp(-\kappa\phi(\xi,\kappa))W''((\kappa/a)\xi^q)\xi^{q-1}d\xi \tag{8.61}$$
$$=\exp(-\kappa\phi(\hat{\xi}(\kappa),\kappa))(\sqrt{\kappa}\,\Delta(\kappa))^{-1}O((\kappa/a)^{-(N+1)/q})$$

when $|\kappa|\to\infty$ in $|\arg\kappa|<\pi/2-\varepsilon$ and $|\arg a|<\delta$. The remainder terms in (8.60) have polynomial growth in κ. Hence, when $|\arg a|<\delta$, $u(a, b)$ majorates $v(a, b)$, if $|\kappa|\to\infty$ in $|\arg\kappa|<\pi/2-\varepsilon$. Thus we have

PROPOSITION 8.6. *Let $|\arg a-\theta/\alpha|<\delta$ and $|c_j|\leqq C$. Then for any $\varepsilon>0$, we have*

$$\mathcal{F}(a, b, \theta)=\exp\Big(\sum_{s=0}^{l-1}\hat{\phi}_s(c)\tau^{q(l-s)/l(q-l)}\Big)A a^{-(m+q)/q}\tau^{(2m+q)/2(q-l)}(w_m+O(\tau^{-p'})) \tag{8.62}$$

as $|\tau|\to\infty$ in the sector $|\arg\tau|<\dfrac{(q-l)\pi}{2q}-\varepsilon$.

PROOF. If $\theta=0$, (8.62) follows from Proposition 8.5, (8.26), (8.44) and (8.45). For general θ, (8.62) follows from the case $\theta=0$.

Now let us consider the behaviour of $\mathcal{F}(a, b)$ when $\mathrm{Re}\,\kappa<0$. To do so let us return to (8.24). We have, by setting $\lambda=\mu^q$,

$$\mathcal{F}(a, b)=\frac{1}{2\pi i}\int_{C(0)}\exp\Big(-a\mu^q+b_0\mu^l+\sum_{i=1}^{l-1}b_{l-i}\mu^i\Big)W(\mu^q)\mu^{q-1}(\log\mu^q)d\mu\,. \tag{8.63}$$

If $|\arg a|<\delta$ and

$$|\arg(-\tau)|<\frac{\pi}{2}(1+l/q)\,, \tag{8.64}$$

then we can choose an $\arg\mu=\theta_1$ such that $|\arg a\mu^q|<\pi/2$, $|\arg(-b_0\mu^l)|<\pi/2$ and $|q\theta_1|<\pi/2+\delta$. So

$$e(\mu)=\exp\Big(-a\mu^q-(-b_0\mu^l)\Big(1+\sum_{i=1}^{l-1}\tilde{c}_i\mu^{i-l}\Big)\Big) \tag{8.65}$$

is bounded, if $|\tilde{c}_j|\leqq C$.

On the other hand $W(\mu^q)=\sum_{j=m}^{-q}w_j\mu^j+\hat{W}(\mu)$, where $|\hat{W}(\mu)|\leqq C|\mu|^{-q-1}$ in $|\arg\mu^q|<\pi/2+\delta$. Thus we have

$$\mathcal{F}(a, b)=\hat{\mathcal{F}}_1(a, b)+\hat{\mathcal{F}}_2(a, b)\,, \tag{8.66}$$

where

(8.67) $$\hat{\mathcal{F}}_1(a,b)=\sum_{j=m}^{-q}\frac{1}{2\pi i}\int_{C(0)}e(\mu)w_j\mu^{j+q-1}(\log\mu^q)d\mu$$

and

(8.68) $$\hat{\mathcal{F}}_2(a,b)=\frac{1}{2\pi i}\int_{C(0)}e(\mu)\hat{W}(\mu)\mu^{q-1}(\log\mu^q)d\mu\,.$$

$\hat{\mathcal{F}}_2(a,b)$ is bounded. $\hat{\mathcal{F}}_1(a,b)=O((b_0)^{-(m+q)/l})$ if $m+q\geqq 1$, $=O(\log(-b_0))$ if $(m+q=0)$, in $|\arg(-\tau)|<\frac{\pi}{2}(1+l/q)-\varepsilon$ for any $\varepsilon>0$.

PROPOSITION 8.7. *Let* $|\arg a-\theta/\alpha|<\delta$, $|\arg(-\tau)|<\frac{\pi}{2}(1+l/q)$ $|\tilde{c}_j|\leqq C$. *Then* $\mathcal{F}(a,b)$ *is expressed in the form*

(8.69) $$\mathcal{F}(a,b)=\hat{\mathcal{F}}_1(a,b)+\hat{\mathcal{F}}_2(a,b)\,,$$

where $\mathcal{F}_1(a,b)=O((b_0)^{-(m+q)/l})$ $(m+q\geqq 1)$, $=O(\log(-b_0))$ $(m+q=0)$ *in* $|\arg(-\tau)|<\frac{\pi}{2}(1+l/q)-\varepsilon$ *for any* $\varepsilon>0$ *and* $\hat{\mathcal{F}}_2(a,b)$ *is bounded, as* $b\to 0$ *and* $\tau\to\infty$.

REMARK 8.8. We can get more precise informations concerning asymptotic behaviour of $\mathcal{F}(a,b)$ by making full use of the expansion of $W(\lambda)$. We impose the condition $|c_j|$, $|\tilde{c}_j|\leqq C$ in order to analyze. We can impose other conditions and expand by another parameter. When $b_0=0$, we should expand by another parameter.

From Propositions 8.6 and 8.7 we can find that the asymptotic behaviour of $\mathcal{F}(a,b)$ in the sector $\mathrm{Re}\,\kappa>0$ differs from that in the sector $\mathrm{Re}\,\kappa<0$ $(\kappa=\tau^{q/(q-l)})$.

For $\mathcal{G}(a,\theta)$ and $\mathcal{H}(a)$ we have

PROPOSITION 8.9. $\mathcal{G}(a,\theta)-\mathcal{H}(a)$ *is holomorphic in* $A=\bigcup_{\varepsilon>0}A_\varepsilon$, *where* $A_\varepsilon=\left\{a\in C^1;\ 0<|a|<c',\ |\arg a-\theta/\alpha|<\frac{\pi(\alpha+1)}{2\alpha}-\varepsilon\right\}$, *and* C^∞ *up to* $a=0$ *in* A_ε.

PROOF. We showed that $\mathcal{G}(a,\theta)$ is holomorphic in A and $\mathcal{H}(a)$ is holomorphic $0<|a|<c'$. It follows from (8.21) that $\mathcal{G}(a,\theta)-\mathcal{H}(a)$ is smooth up to $\bar{A}_\varepsilon$.

Now let us apply Propositions 8.6, 8.7 and 8.9 to the function $u(z)\in\mathcal{O}(\widetilde{\Omega-K})$ that satisfies

(8.70) $$\begin{cases} L(z,\partial)u(z)=0\,, \\ \left(\dfrac{\partial}{\partial z_0}\right)^s u(0,z)=\delta_{s,t}u_t(z')\,, \quad 0\leqq s\leqq k-1\,, \end{cases}$$

where $u_t(z')$ has a pole on $z_1=0$:

(8.71) $$u_t(z')=\frac{\tilde{u}_t(z'')}{z_1^{r'+1}}=\frac{1}{2\pi qi}\,\frac{\tilde{u}_t(z'')}{\Gamma(r'+1)}\int_{\Gamma(0)}\exp(-\lambda z_1)\lambda^{r'}(\log\lambda)d\lambda\,.$$

Hence by putting in (2.1)

$$\hat{v}(z', \lambda)=\frac{\tilde{u}_t(z'')}{\Gamma(r'+1)}, \tag{8.72}$$

we can construct a formal solution $U(z, \lambda)=\exp(-\lambda\varphi(z))W(z, \lambda)$ of (2.1). We note that $\hat{v}(z', \lambda)$ does not depend on λ and θ. On the other hand we get $\tilde{w}(z, \lambda, \zeta)$ and $\tilde{w}_{i,j}(z, \lambda)$ from the results in §5. $\tilde{w}_{i,j}(z, \lambda)$ are all holomorphic functions of $\lambda^{-1/q}$ in $|\lambda|>\Lambda_1$. Set $U_{r'}(z, \lambda)=\lambda^{r'}U(z, \lambda)$. After integrating $U_{r'}(z, \lambda)$ with respect to λ as in §4, we have a solution of (8.70). We also have a representation by making use of $w_{r'}(z, \lambda, \zeta)=\lambda^{r'}\tilde{w}(z, \lambda, \zeta)$.

Now let us apply above propositions. Set

$$F_i(z, \lambda, \theta)=\int_{C^0(d,\theta)}\exp(-\lambda^\alpha\zeta)\sum_{j=J_i}^{\infty}\tilde{w}_{i,j}(z, \lambda)f_j(\zeta+\psi_i(z, \lambda))d\zeta, \tag{8.73}$$

$$G(z, \lambda, \theta)=\int_{C^0(d,\theta)}\exp(-\lambda^\alpha\zeta)w^+(z, \lambda, \zeta)d\zeta, \tag{8.74}$$

$$H_n(z)=\int_{C(\theta)}\exp(-\lambda^\alpha\zeta)w_n^+(z, \lambda, \zeta)d\zeta \tag{8.75}$$

and

$$r=q-1+t-\alpha+r'. \tag{8.76}$$

We expand $\psi_i(z, \lambda)$:

$$\psi_i(z, \lambda)=\psi_{i,0}(z)+\psi_{i,1}(z)\lambda^{-1/q}+\cdots+\psi_{i,l-1}\lambda^{-(l-1)/q}+\cdots \tag{8.77}$$

and set

$$\tau^i(z)=\psi_{i,0}(z)(\varphi(z))^{-l/q} \tag{8.78}$$

and

$$c_j^i(z)=\psi_{i,l-j}(z)(\psi_{i,0}(z))^{-j/l}, \quad \tilde{c}_j^i(z)=\psi_{i,l-j}(z)(\psi_{i,0}(z))^{-1}. \tag{8.79}$$

Let us apply Lemmas 8.1 and 8.2 to $F_i(z)$ and $G(z)$. Since $\psi_i(0, \lambda)=0$, $|\psi_i(z, \lambda)|<d'$ in a small neighbourhood of $z=0$ which is independent of λ by (8.77). Therefore we have

$$\begin{aligned} F_i(z, \lambda, \theta) &\sim \exp(\lambda^\alpha\psi_i(z, \lambda))\sum_{j=J_i}^{\infty}w_{i,j}(z, \lambda)\lambda^{-\alpha(j+1)} \\ &\sim \exp(\psi_{i,0}(z)\lambda^\alpha+\psi_{i,1}(z)\lambda^{\alpha-1/q}+\cdots+\psi_{i,l-1}(z)\lambda^{1/q})\sum_{s=s_i}^{\infty}\overline{w}_{i,s}(z)\lambda^{-s/q} \end{aligned} \tag{8.80}$$

as $|\lambda|\to\infty$ in $|\arg\lambda^\alpha+\theta|\leq(\pi/2+\delta)\alpha$ and $\{z;\ |\psi_i(z, \lambda)|<d'\}$. We also have

(8.81) $$G(z, \lambda, \theta)\sim \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} w_{n,j}(z, \lambda)\lambda^{-\alpha(j+1)}$$

as $|\lambda|\to\infty$ in the sector $|\arg\lambda^{\alpha}+\theta|<\pi/2-\varepsilon$, and

(8.82) $$H_n(z, \lambda)=\sum_{n\geqq j\geqq 0} w_{n,j}(z, \lambda)\lambda^{-\alpha(j+1)}.$$

Thus we have a theorem by which we know that the behaviours of solutions of (8.70) depend on $\arg\tau^i(z)$. This is the Stokes phenomenon of solutions of (8.70).

THEOREM 8.10. *Any solution* $u(z)\in\mathcal{O}(\widetilde{\Omega-K})$ *of* (8.70) *is expressed in a small neighbourhood* Ω' *of* $z=0$ *in the form*

(8.83) $$u(z)=\sum_{i=1}^{k} u^{\theta}_{\mathrm{I},i}(z)+u^{\theta}_{\mathrm{II}}(z)+u^{\theta}_{\mathrm{III}}(z)+\bar{u}^{\theta}(z),$$

where

(8.84) $$u^{\theta}_{\mathrm{I},i}(z)=\frac{1}{2\pi qi}\int_{\Gamma(\theta_1)}\exp(-\lambda\varphi(z))F_i(z, \lambda, \theta)\lambda^{r}(\log\lambda)d\lambda,$$

(8.85) $$u^{\theta}_{\mathrm{II}}(z)=\frac{-1}{2\pi qi}\int_{\Gamma(\theta_1)}\exp(-\lambda\varphi(z))G(z, \lambda, \theta)\lambda^{r}(\log\lambda)d\lambda,$$

(8.86) $$u^{\theta}_{\mathrm{III}}(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi qi}\int_{\Gamma(\theta_1)}\exp(-\lambda\varphi(z))H_n(z, \lambda)\lambda^{r}(\log\lambda)d\lambda.$$

$\bar{u}^{\theta}(z)$ *is holomorphic in* Ω' *and* $u^{\theta}_{\mathrm{I},i}(z)$, $u^{\theta}_{\mathrm{II}}(z)$ *and* $u^{\theta}_{\mathrm{III}}(z)$ *belong to* $\mathcal{O}(\widetilde{\Omega-K})$. *The function* $(u^{\theta}_{\mathrm{II}}(z)+u^{\theta}_{\mathrm{III}}(z))$ *is* C^{∞} *up to the surface* K *when* $|\arg\varphi(z)-\theta/\alpha|<\frac{(\alpha+1)\pi}{2\alpha}-\varepsilon$ *for any* $\varepsilon>0$.

When $|\arg\varphi(z)-\theta/\alpha|<\delta$, *the function* $u_{\mathrm{I},i}(z)$ *behaves asymptotically as follows*:

For any $\varepsilon>0$, if $|\arg\tau^i(z)+2m\pi|<\frac{(q-l)\pi}{2q}-\varepsilon$ $(m\in\boldsymbol{Z})$ *and* $|c^i_j(z)|\leqq C$,

(8.87) $$u^{\theta}_{\mathrm{I},i}(z)=\exp\Big(\sum_{s=0}^{l-1}\hat{\phi}_{i,s}(c^i(z))\tau^i(z)^{q(l-s)/l(q-l)}\Big)$$
$$\times A_i\varphi(z)^{-(m_i+q)/q}\tau^i(z)^{(2m_i+q)/2(q-l)}(w_{m_i}(z)+O(\tau^{-p'}))$$

as $|\tau^i(z)|\to\infty$, *and if* $|\arg(-\tau^i(z))+2m\pi|<\frac{\pi}{2}(1+l/q)-\varepsilon$ *and* $|\tilde{c}^i_j(z)|\leqq C$,

(8.88) $$\begin{aligned}u^{\theta}_{\mathrm{I},i}(z)&=O((\psi_{i,0}(z))^{-(m_i+q)/l}) \qquad (m_i+q\geqq 1)\\ &=O(\log(-\psi_{i,0}(z))) \qquad (m_i+q=0)\\ &=O(1) \qquad (m_i+q\leqq -1)\end{aligned}$$

as $\psi_{i,0}(z)\to 0$ *and* $|\tau^i(z)|\to\infty$, *where* $m_i=qr+s_i$.

PROOF. We first note that $\hat{v}(z', \lambda, \theta)$ does not depend on θ, $\hat{v}(z', \lambda, \theta)=\hat{v}(z', \lambda)$. From Theorem 7.4 and by varying θ_1, we can show that $u^{\theta}_{\mathrm{I},i}(z)$, $u^{\theta}_{\mathrm{II}}(z)$ and $u^{\theta}_{\mathrm{III}}(z)$ belong to $\mathcal{O}(\widetilde{\Omega-K})$. Put $\theta_1=-\theta/\alpha$. Then the statements of the proposition follow from Theorem 7.3 and Propositions 8.6, 8.7 and 8.9.

REMARK 8.11. (i) If $\psi_i(z, \lambda)$ is a solution corresponding to a non-zero root of the subcharacteristic equation, then $\psi_i(z, \lambda)\not\equiv 0$. In addition, if $w_{m_i}(z)\not\equiv 0$, then $u^{0}_{\mathrm{I},i}(z)$ has exponential growth in some domain when z tends to the characterisitc surface K.

(ii) Our methods are applicable directly to the case when $u_t(z')$ in (8.70) has singularity of the form z_1^{γ} or $\log z_1$. Moreover, if we know asymptotic behaviour of $u_t(z')$ near $z_1=0$, we can obtain asymptotic behaviours of $u(z)$.

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(Received September 1, 1978)

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