Standard subgroups of type $Sp_6(2)$, I

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Introduction.

A group, L, is said to be quasisimple if L is simple modulo its center $Z(L)$ and L is perfect, that is to say L is its own commutator group. A quasisimple subgroup, L, of a finite group, G, is said to be standard if the centralizer of L in G has even order, L is normal in the centralizer of every involution centralizing L, and L commutes with none of its conjugates.

In 1973, M. Aschbacher proved [2] theorems which definitely showed the fundamental importance of standard subgroups in the study of finite simple groups, and since then research of several authors has focused on the determination of finite groups with a standard subgroup of known type. In particular, Seitz [10] has dealt with the groups, G, containing a standard subgroup, L, such that $L/Z(L)$ is a Chevalley type group of characteristic 2 and of rank greater than 2 with $|Z(L)|$ odd and such that the centralizer of L in G has cyclic Sylow 2-subgroups. Using the "induction" method, he has reduced the problem to the four special cases $L/Z(L)\cong Sp_6(2)$, $U_6(2)$, $O_8^-(2)$, and $O_8^+(2)$: these are the only Chevalley type groups of characteristic 2 and of rank greater than 2 for which the induction method breaks down. The purpose of this half of the paper is to settle certain problems which arise in the analysis of the case that $L/Z(L)\cong Sp_6(2)$. In the appendix, we discuss a similar problem in the case that $L/Z(L)\cong U_6(2)$, which is handled by Yamada [14].

The induction method also breaks down for Chevalley type groups of rank 1 and 2, and these have been treated by the so-called "pushing up" method (see, for example, [8] and [9]). As in those cases, the analysis of the cases that $L/Z(L)\cong Sp_6(2)$ and $U_6(2)$ divides into two parts:

A: For suitably chosen maximal parabolic subgroups P_1 and P_2 of L, we construct a 2-local subgroup N_i, $i\in\{1, 2\}$, which contains P_i and has appropriate properties.

B: Using N_1 and N_2, we construct a subgroup which is a central extension of a certain Chevalley type group.

For example, when $L/Z(L)\cong U_6(2)$, our task in Part A is to construct 2-local subgroups which resemble certain of the parabolic subgroups of $L_6(4)$ or $U_6(2)\times U_6(2)$. Then our task in Part B is to construct a central extension of $L_6(4)$ or $U_6(2)\times U_6(2)$.

In this paper we concentrate on Part B. The arguments in Part A involve a detailed analysis of 2-local subgroups using individual properties of L and vary with the type of L. On the contrary, the arguments in Part B are formal applications of general properties of Chevalley type groups. Although the results of this paper focus on $Sp_6(2)$ and $U_6(2)$, it seems that analogous theorems can be obtained for the groups $O_8^-(2)$ and $O_8^+(2)$ as well. However, the shape of such theorems would depend on the outcome of the analysis in Part A (see Theorem 2 of Section 4 in this connection). Since work on the part is in progress, we will not discuss the groups here.

The organization of the paper is as follows. Section 1 contains certain results on Chevalley type groups. The main result there is (1.8), which is a consequence of a theorem of Curtis [7]. Section 2 contains the statement of the basic hypotheses of the paper. Sections 3-7 contain the statement and proof of the main results of the paper.

We shall follow Artin's notation [1], Section 1, for the finite classical groups except that we denote by $Sp_n(q)$ the symplectic groups. Otherwise our notation is standard. Thus if q is a prime power, E_q denotes elementary abelian groups of order q, and an E_q-subgroup is a subgroup isomorphic to E_q.

For the background of the theory of Chevalley type groups, we refer the reader to Carter [4] and Steinberg [13].

1. Preliminaries.

The first four lemmas of this section are concerned with the linear, unitary, symplectic, and orthogonal groups in the sense of Artin [1]. They are identified with certain of the groups defined by Chevalley [5] and Steinberg [12] and so they have "natural" BN-pairs.

(1.1) LEMMA. *Let M be one of the groups $U_4(2)$, $L_4(2)$, $U_5(2)$, $L_5(2)$, $Sp_4(4)$, $Sp_4(2)\times Sp_4(2)$, and z an involutive automorphism of M such that $K=C_M(z)$ is isomorphic to $Sp_4(2)$. Let S be a z-invariant S_2-subgroup of M, (B, N) a natural BN-pair of K such that $B=C_S(z)$, and s, t the distinguished generators of the Weyl group $W=N/B\cap N$. Then there exists a natural BN-pair (B_1, N_1) of M satisfying the following conditions:*

(1) $S\leq B_1$;

(2) *z normalizes B_1 and N_1;*

(3) $C_{B_1}(z)=B$ *and* $C_{N_1}(z)=N$;

(Under the condition (3), W is canonically embedded in the Weyl group $W_1=$

$N_1/B_1 \cap N_1$, *so we regard* $W \leq W_1$. *Also*, z *acts on* W_1 *by the condition* (2).)

(4) *if* $M \cong U_4(2)$, $U_5(2)$ *or* $Sp_4(4)$, *then* s, t *are the distinguished generators of* W_1;

(5) *if* $M \cong L_4(2)$ *and the distinguished generators of* W_1 *are labeled*

$$\begin{array}{ccccc} \circ & - & \circ & - & \circ \\ r_1 & & r_2 & & r_3 \end{array}$$

then $r_1^z = r_3$ *and* $\{s, t\} = \{r_2, r_1r_3\}$;

(6) *if* $M \cong L_5(2)$ *and the distinguished generators of* W_1 *are labeled*

$$\begin{array}{ccccccc} \circ & - & \circ & - & \circ & - & \circ \\ r_1 & & r_2 & & r_3 & & r_4 \end{array}$$

then $r_1^z = r_4$, $r_2^z = r_3$, *and* $\{s, t\} = \{r_1r_4, r_2r_3r_2\}$;

(7) *if* $M \cong Sp_4(2) \times Sp_4(2)$ *and the distinguished generators of* W_1 *are labeled*

$$\begin{array}{ccccccc} \circ & = & \circ & & \circ & = & \circ \\ r_1 & & r_2 & & r_3 & & r_4 \end{array}$$

then $\{r_1, r_2\}^z = \{r_3, r_4\}$ *and* $\{s, t\} = \{r_1r_3, r_2r_4\}$ *or* $\{r_1r_4, r_2r_3\}$.

PROOF. Assume $M \cong Sp_4(2) \times Sp_4(2)$. Then $M = M_1 \times M_2$, $M_2 = M_1^z \cong Sp_4(2)$, and the mapping $x \to xx^z$, $x \in M_1$, is an isomorphism from M_1 onto K. Let B_0 and N_0, respectively, be the inverse images of B and N under this isomorphism and define

$$B_1 = B_0 B_0^z, \qquad N_1 = N_0 N_0^z.$$

Then (B_1, N_1) meets all the requirements.

Therefore, assume $M \ncong Sp_4(2) \times Sp_4(2)$. Let $n = 4$ or 5, τ the automorphism of $GL_n(4)$ induced by the transpose-inverse mapping followed by conjugation by the matrix

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix},$$

(blank spaces denote zeros) and σ the automorphism of $GL_n(4)$ induced by the nontrivial automorphism of $GF(4)$. We shall use the following identification:

$$U_n(2) = C(\sigma\tau) \cap SL_n(4),$$

$$Sp_4(4) = C(\tau) \cap GL_4(4).$$

Then we may assume that the action of z on M is induced by τ if $M \cong U_n(2)$ or $L_n(2)$ and by σ if $M \cong Sp_4(4)$ by Section 19 of [3]. Let S_0 be the subset of lower triangular matrices with diagonal entries equal to 1, so S_0 is a z-invariant

S_2-subgroup of M and $C_{S_0}(z)\in \mathrm{Syl}_2(K)$. Considering the semidirect product of M by $\langle z\rangle$, we prove the following:

(1.2) *If* $z_1\in I(zS_0)$ *and* $C_M(z_1)\cong Sp_4(2)$, *then* $z_1\in z^{S_0}$.

PROOF. Assume $M\cong U_4(2)$ or $L_4(2)$. Let A_0 be the unique E_{16}-subgroup of S_0. Then $S_0/A_0\cong E_4$ and $|C_{S_0/A_0}(z)|=2$; so z_1A_0 is conjugate to zA_0 under S_0. Hence we may assume $z_1\in zA_0$. Then $z_1\in zA$, where $A=C_{A_0}(z)$. Let

$$u=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}, \qquad v=\begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix}.$$

If $a\in A_0-A$, then an easy computation shows $z^a=zv$ and $(zu)^a=zvu$, Now $z_1\in z^M$, but $zu\notin z^M$ by [3], Section 19. Hence $zvu\notin z^M$, and since $N_K(A)$ acts transitively on $(A/\langle v\rangle)^{\#}$, it follows that $z^M\cap zA=\{z, zv\}$. Therefore, $z_1\in z^{S_0}$.

Assume $M\cong U_5(2)$ or $L_5(2)$. Let u be the involution of $Z(S_0)$ and let $Q=O_2(C_M(u))$. Then an easy computation shows that the number of elements of S_0/Q that are inverted by z is equal to $|S_0/Q:C_{S_0/Q}(z)|$; so z_1Q is conjugate to zQ under S_0. Hence we may assume $z_1\in zQ$. Then since $Q/\langle u\rangle\cong E_{64}$ and $|C_{Q/\langle u\rangle}(z)|=8$, it follows that $z_1\langle u\rangle$ is conjugate to $z\langle u\rangle$ under Q. Hence we may assume $z_1\in z\langle u\rangle$. As $zu\in z^Q$, we have $z_1\in z^{S_0}$.

Finally, assume $M\cong Sp_4(4)$. Let A be one of the two E_{64}-subgroups of S_0. Then $S_0/A\cong E_4$, $|C_{S_0/A}(z)|=2$, and $|C_A(z)|=8$. Hence $z_1\in z^{S_0}$. Thus (1.2) holds in all cases.

Now let S be a z-invariant S_2-subgroup of M. We argue that S is conjugate to S_0 under K. There exists an element $m\in M$ such that $S^m=S_0$ by Sylow's theorem. Set $z_1=z^m$. Then $z_1\in zN_M(S_0)$, so we may assume $z_1\in zS_0$ again by Sylow's theorem. By (1.2) there exists an element $x\in S_0$ such that $z_1^x=z$. Thus $mx\in K$ and $S^{mx}=S_0$.

Therefore, in proving (1.1), it suffices to consider the case that $S=S_0$. Assume $S=S_0$ and set $U=C_S(z)$. The following is a standard knowledge:

(1.3) *There exists a natural BN-pair* (B, N) *of* K *such that* $B=U$ *and a natural BN-pair* (B_1, N_1) *of* M *satisfying the condition* (1)—(6) *of* (1.1).

Furthermore, we have

(1.4) *If* (B, N^*) *is another natural BN-pair of* K, *then* N^* *is conjugate to* N *under* B.

This follows from the fact that N^* is determined canonically by an S_2-subgroup U^* of K such that $U\cap U^*=1$ and that U acts, by conjugation, transitively on the set of such S_2-subgroups. (1.1) now follows from (1.3) and (1.4).

Essentially the same proof shows the following:

(1.5) LEMMA. *Let M be one of the groups $L_3(4)$, $SL_3(4)$, $L_3(2)\times L_3(2)$, and z an involutive automorphism of M such that $K=C_M(z)$ is isomorphic to $L_3(2)$. Let S be a z-invariant S_2-subgroup of M, (B, N) a natural BN-pair of K such that $B=C_S(z)$, and r, s the distinguished generators of the Weyl group $W=N/B\cap N$. Then there exists a natural BN-pair (B_1, N_1) of M satisfying the following conditions:*

(1) $S\leq B_1$;

(2) *z normalizes B_1 and N_1;*

(3) $C_{B_1}(z)=B$ *and* $C_{N_1}(z)=N$;

(4) *if $M\cong L_3(4)$ or $SL_3(4)$, then r, s are the distinguished generators of the Weyl group $W_1=N_1/B_1\cap N_1$;*

(5) *if $M\cong L_3(2)\times L_3(2)$ and the distinguished generators of W_1 are labeled*

$$\begin{array}{ccccccc} \circ & \text{———} & \circ & \qquad & \circ & \text{———} & \circ \\ r_1 & & r_2 & & r_3 & & r_4 \end{array}$$

then $\{r_1, r_2\}^z=\{r_3, r_4\}$ *and* $\{r, s\}=\{r_1r_3, r_2r_4\}$ *or* $\{r_1r_4, r_2r_3\}$.

(1.6) LEMMA. *Let the notation be as in (1.1). Then $M=\langle S, K\rangle$.*

PROOF. By (1.1) $M=\langle S, s, t\rangle$, hence the result follows.

Similarly, we have

(1.7) LEMMA. *Let the notation be as in (1.5). Then $M=\langle S, K\rangle$.*

(1.8) LEMMA. *Let G_0^* be a group with a Bruhat decomposition in the sense of [6]. Then G_0^* has a BN-pair (B^*, N^*) such that $B^*\cap N^*$ has a normal complement S^* in B^*. Assume that the Weyl group $W^*=N^*/B^*\cap N^*$ is isomorphic to the Weyl group $W(\Sigma)$ of a root system Σ of rank $l>1$ and that the isomorphism carries the distinguished generators of W^* onto the reflection $w_1, \cdots, w_l$ associated with the fundamental roots $\Pi=\{r_1, \cdots, r_l\}$: see [4], Chapter 2. Identify W^* with $W(\Sigma)$ and choose an element $w_0\in W^*$ which carries the set Σ^+ of positive roots onto the set of negative roots. For each $w\in W^*$, choose a representative $n(w)$ of w in N^*, and define*

$$X_{r_i}^*=S^*\cap n(w_0w_i)^{-1}S^*n(w_0w_i)\,.$$

A root subgroup is a conjugate under N^* of $X_{r_i}^*$, $1\leq i\leq l$. Let $r\rightarrow X_r^*$, $r\in\Sigma$, be the one to one mapping of Σ onto the set of root subgroups given by [6], Theorem 3.8, such that

$$n(w)X_r^*n(w)^{-1}=X_{w(r)}^*\,.$$

Furthermore, assume the following:

(1) $G_0^*=\langle X_r^*\,;\, r\in\Sigma\rangle$ *and* G_0^* *is quasisimple;*

(2) *for all independent roots* $r, s\in\Sigma$, *it is possible to arrange the roots of the form* $\lambda r+\mu s$, λ, $\mu>0$, *in some order so that*

$$[X_r^*, X_s^*]\leq\prod X_{\lambda r+\mu s}^*;$$

(3) *if* A_{ij} *is the set of all roots of the form* $\lambda r_i+\mu r_j$, $i\neq j$, *and* $A=\cup A_{ij}$, *then for each* $r\in A$, *there exists a subset* S *of* A_{ij} *for some* $\{i, j\}$ *such that* $r\in S$, $w(S)\leq\Sigma^+$ *for some* $w\in W^*$, *and* $X_r^*\leq\langle X_s^*; r\neq s\in S\rangle$.

Now let G_0 *be a group and suppose there exist subgroups* X_r, $r\in\pm\Pi$, *satisfying the following conditions*:

(i) $G_0=\langle X_r; r\in\pm\Pi\rangle$ *and* G_0 *is perfect*;

(ii) *for each* $r\in\pm\Pi$, *there exists an isomorphism* $X_r^*\rightarrow X_r$;

(iii) *for each* $\{i, j\}$, $i\neq j$, *there exists an epimorphism*

$$\langle X_{\pm r_i}^*, X_{\pm r_j}^*\rangle\longrightarrow\langle X_{\pm r_i}, X_{\pm r_j}\rangle$$

which extends the isomorphisms given in (ii).

Then there exists an epimorphism $\phi: G_0\rightarrow G_0^*/Z(G_0^*)$ *such that* $\ker\phi=Z(G_0)$ *and* $(X_{\pm r_i})^\phi=(X_{\pm r_i}^*)^\sigma$, *where* σ *is the natural epimorphism* $G_0^*\rightarrow G_0^*/Z(G_0^*)$.

PROOF. Suppose $r\in A-\Pi\cup(-\Pi)$. Then there exists a unique set $\{i, j\}$ such that $r\in A_{ij}$. There exists an element $w\in\langle w_i, w_j\rangle$ such that $r\in w(\{r_i, r_j\})$ and we may choose $n(w_i)\in\langle X_{\pm r_i}^*\rangle$ for each i. Hence it follows that

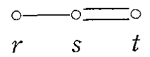
$$X_r^*\leq\langle X_{\pm r_i}^*, X_{\pm r_j}^*\rangle$$

and that the epimorphism given in (iii) is faithful on X_r^*. Define X_r to be the image of X_r^* under the epimorphism. Thus we have defined the subgroups X_r, $r\in A$, and isomorphisms $X_r^*\rightarrow X_r$. It is clear that G_0 is generated by the sets X_r, $r\in A$, and that the generators satisfy all the commutator relations implied by (2) for pairs of independent roots r, s belonging to some A_{ij}. By [7], Theorem 1.4, there exists a central extension $(\hat{G}, \pi)$ of G_0^*, an epimorphism $\rho:\hat{G}\rightarrow G_0$, and subgroups $\hat{X}_r$, $r\in A$, of $\hat{G}$ such that $(\hat{X}_r)^\pi=X_r^*$ and $(\hat{X}_r)^\rho=X_r$. Let $\hat{N}=\ker\pi$, $Z=\ker\pi\sigma$, and $N=\ker\rho$. Then since $\hat{G}/Z$ is simple and G_0 is perfect, it follows that $N\leq Z$. Thus there exists an epimorphism $\phi: G_0\rightarrow G_0^*/Z(G_0^*)$ such that $\pi\sigma=\rho\phi$. Clearly, $\ker\phi=Z^\rho$ and $(X_r)^\phi=(X_r^*)^\sigma$. Now $[\hat{G}, Z]\leq\hat{N}\leq Z(\hat{G})$; so $[\hat{G}', Z]=1$ by the three-subgroup lemma. Since $G_0=G_0'$, it follows that $G_0=(\hat{G}')^\rho$, and therefore $[G_0, Z^\rho]=1$. Thus $\ker\phi\leq Z(G_0)$ and, as $G_0^*/Z(G_0^*)$ is simple, $\ker\phi=Z(G_0)$. This completes the proof.

2. Notation and hypothesis.

In this section we fix notation for $Sp_6(2)$, state the basic hypothesis, and define certain subgroups of the group satisfying the hypothesis.

(2.1) *Notation.* Let $L=Sp_6(2)$. Then L is a Chevalley group of type (C_3) over $GF(2)$. Let Λ be a root system of type (C_3) and $\Phi=\{r, s, t\}$ a system of fundamental roots, where the Dynkin diagram is as follows:



Let (B, N) be a natural BN-pair of L and let $U\in \mathrm{Syl}_2(B)$. In fact, $B=U\in \mathrm{Syl}_2(L)$ and $B\cap N=1$. There exists an isomorphism $W=N/B\cap N\to W(\Lambda)$ which carries the distinguished generators of W onto the fundamental reflections. We shall denote by r, s, t the distinguished generators of W also. Thus r, s, t are involutions generating N and satisfying the relations

$$(rs)^3=(st)^4=(tr)^2=1\,.$$

Let w be the element of W such that $U\cap U^w=1$. Then

$$w=rstsr\cdot stst=tstrst\cdot rsr\,.$$

Notice that $stst=tsts$ and $rsr=srs$ by the above relations. Let

$$P_1=\langle U, s, t\rangle\,, \qquad P_2=\langle U, r, s\rangle\,,$$

$$A_1=O_2(P_1)\,, \qquad A_2=O_2(P_2)\,.$$

Then $P_1/A_1\cong Sp_4(2)$ and $P_2/A_2\cong L_3(2)$. Both A_1 and A_2 are elementary abelian and have order 2^5 and 2^6, respectively. Let

$$K_1=P_1\cap P_1^{rstsr}\,, \qquad K_2=P_2\cap P_2^{tstrst}\,,$$

$$U_1=U\cap U^{rstsr}\,, \qquad U_2=U\cap U^{tstrst}\,.$$

Then for each $i\in\{1, 2\}$, we have $P_i=A_iK_i$, $A_i\cap K_i=1$, and $U_i\in \mathrm{Syl}_2(K_i)$. Also, $\{s, t\}\leq K_1$ and $\{r, s\}\leq K_2$, as $rstsr$ centralizes $\langle s, t\rangle$ and $tstrst$ permutes $\{r, s\}$. The groups K_1, K_2 will not appear in the statement of the main theorems, but play an important role in their proof.

It will be helpful for computations to represent L as a group of matrices. We identify L with the group of matrices X with entries in $GF(2)$ satisfying

$${}^tX\begin{pmatrix} & & & & & 1\\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}X=\begin{pmatrix} & & & & & 1\\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix},$$

where ${}^tX$ is the transposed matrix of X. Under this identification, we may choose U to be the subset of L consisting of lower triangular matrices, and

$$r=\begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix}, \quad s=\begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & & & & 1 \end{pmatrix}$$

$$t=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad w=\begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}.$$

We shall consider the following situation:

(2.2) HYPOTHESIS. G is a finite group, z is an involution of G, L is a subgroup of the centralizer of z in G, and $L \cong Sp_6(2)$. Furthermore, G satisfies the following conditions:

(1) for $i \in \{1, 2\}$, there exists a z-invariant 2-subgroup C_i such that $C_{C_i}(z) = A_i$;

(2) for $i \in \{1, 2\}$, there exists a z-invariant subgroup M_i of $N_G(C_i)$ containing C_i such that $C_{M_i/C_i}(z) = P_i C_i/C_i$;

(3) there exists a z-invariant 2-subgroup S such that $C_S(z) = U$ and $S \in \mathrm{Syl}_2(M_i)$ for each i.

We remark that conditions (2) and (3) imply that each C_i is a normal subgroup of S.

(2.3) DEFINITION. Under the hypothesis (2.2) we set

$$L_1 = M_1 \cap M_1^{rstsr}, \qquad L_2 = M_2 \cap M_2^{tstrst},$$

$$S_1 = S \cap S^{rstsr}, \qquad S_2 = S \cap S^{tstrst}.$$

Thus $K_i \leq L_i$ and $U_i \leq S_i \leq L_i$ for $i \in \{1, 2\}$. Let

$$X_r = S_2 \cap S_2^{rs}, \qquad X_s = S_2 \cap S_2^{sr}, \qquad X_t = S_1 \cap S_1^{ts}.$$

For $u \in \{r, s, t\}$, let

$$X_{-u} = (X_u)^u, \qquad Y_u = \langle X_{\pm u} \rangle.$$

The groups $X_{\pm u}$ will be called X-subgroups. Finally, we set

$$G_0 = \langle Y_r, Y_s, Y_t \rangle.$$

(2.4) COROLLARY. *Under the hypothesis and definition above, the following inclusions hold:*

$$L \leq G_0 \leq \langle L_1, L_2 \rangle \leq \langle M_1, M_2 \rangle.$$

PROOF. Since $S_1 \leq L_1$ and $s, t \in K_1 \leq L_1$, it follows that $X_t \leq L_1$; so $Y_t \leq L_1$.

Similarly, Y_r and $Y_s \leq L_2$. Therefore, $G_0 \leq \langle L_1, L_2 \rangle$. Since $L_i \leq M_i$ for each i, it follows that $\langle L_1, L_2 \rangle \leq \langle M_1, M_2 \rangle$. Now let

$$U_r = U_2 \cap U_2^{rs}, \qquad U_s = U_2 \cap U_2^{sr}, \qquad U_t = U_1 \cap U_1^{sts},$$

and for $u \in \{r, s, t\}$ let $V_u = \langle U_u, (U_u)^u \rangle$. Then $L = \langle V_r, V_s, V_t \rangle$ and hence $L \leq G_0$.

3. $O_8^-(2)$ and $O_8^+(2)$.

We continue with the notation and hypothesis of Section 2. In this section, we prove the following:

THEOREM 1. *Suppose the following conditions hold:*

(1) $M_1/C_1 \cong U_4(2)$ *or* $L_4(2)$ *and* $M_2/C_2 \cong L_3(2)$;

(2) $|S| \geq 2^{12}$;

(3) $[S, (U_2 \cap U_2^{rs})^{sts}] = 1$.

Then G_0 *is a perfect central extension of* $O_8^-(2)$ *or* $O_8^+(2)$.

PROOF. For each $i \in \{1, 2\}$, let $\bar{M}_i = M_i/C_i$ and let bars denote images in $\bar{M}_i$. We will discuss the structure of $\bar{M}_i$ more carefully. The situation in $\bar{M}_2$ is clear. Since $\bar{M}_2 \cong K_2 \cong L_3(2)$ and $\bar{S} \cong U_2$, we have

(3.1) *The mapping* $x \to \bar{x}$, $x \in K_2$, *is an isomorphism of* K_2 *onto* $\bar{M}_2$, *and it carries* U_2 *onto* $\bar{S}$.

Hence

(3.2) $|S : S \cap S^r| = 2$,

(3.3) $|S : S \cap S^s| = 2$.

Next, consider $\bar{M}_1$. As $|\bar{S} : \bar{S} \cap \bar{S}^s| = 2$ by (3.3), (1.1) shows that the following holds:

(3.4) *There exists a natural BN-pair* $(\bar{B}_1, \bar{N}_1)$ *of* $\bar{M}_1$ *such that* $\bar{S} \leq \bar{B}_1$ *and either*

(1) $\bar{M}_1 \cong U_4(2)$ *and* $\bar{s}$, $\bar{t}$ *are the distinguished generators of the Weyl group* $\bar{W}_1 = \bar{N}_1 / \bar{B}_1 \cap \bar{N}_1$, *or*

(2) $\bar{M}_1 \cong L_4(2)$ *and if the distinguished generators of* $\bar{W}_1$ *are labeled*

$$\underset{\bar{t}_1}{\circ} \!-\!\!-\!\!-\! \underset{\bar{t}_3}{\circ} \!-\!\!-\!\!-\! \underset{\bar{t}_2}{\circ}$$

then $(\bar{s}, \bar{t}) = (\bar{t}_3, \bar{t}_1\bar{t}_2)$ *and* $\bar{t}_1^{\bar{s}} = \bar{t}_2$.

As a consequence, we have

(3.5) $|S : S \cap S^t| = 4$.

Next, we prove

(3.6) *For* $i\in\{1, 2\}$, *we have* $M_i=C_iL_i$, $C_i\cap L_i=1$, *and* $S_i=S\cap L_i$.

PROOF. First, assume $i=1$. Consider the following chain of conjugates of S:

$$S,\ S^r,\ S^{sr},\ S^{tsr},\ S^{stsr},\ S^{rstsr}.$$

The orders of the intersections of two consecutive conjugates are known by (3.2), (3.3), and (3.5), and hence we can estimate the order of the intersection of all the conjugates. As $|S|\geq 2^{12}$ by hypothesis, we have $|S_1|\geq 2^6$. Now $C_{M_1}(z)=P_1$ by hypothesis and so

$$C_1\cap M_1^{rstsr}\cap C(z)=A_1\cap P_1^{rstsr}=A_1\cap K_1=1.$$

Hence $C_1\cap L_1=1$. Let $\bar{M}_1=M_1/C_1$. Then since $\bar{S}_1\leq\bar{S}$ and $|\bar{S}_1|=|S_1|\geq 2^6=|\bar{S}|$, it follows that $\bar{S}=\bar{S}_1$. Thus $\langle\bar{S}, \bar{K}_1\rangle\leq\bar{L}_1\leq\bar{M}_1$ and so $\bar{L}_1=\bar{M}_1$ by (1.6). As $S=C_1S_1$, $S\cap L_1=S_1$.

Next, assume $i=2$. As above, we have $C_2\cap L_2=1$. Since $K_2\leq L_2$ and $M_2/C_2\cong K_2$, it follows that $M_2=C_2L_2$. Similarly, since $U_2\leq S_2$ and $S/C_2\cong U_2$, we have $S_2=S\cap L_2$.

We also have the following:

(3.7) $S_2=U_2$.

Next, we prove

(3.8) *The following conditions hold*:
(1) $S\cap S^w=1$;
(2) $X_r=S\cap S^{wr}$;
(3) $X_s=S\cap S^{ws}=S_1\cap S_1^{tst}$;
(4) $X_t=S\cap S^{wt}$;
(5) $X_t=(C_2\cap C_2^{tst})^{rs}$;
(6) $S\cap S^s=C_1C_2$;
(7) $|S|=2^{12}$.

PROOF. As $S\cap S^w\cap C(z)=U\cap U^w=1$, (1) holds. By definition, $X_r\leq S\cap S^{tstrstrs}=S\cap S^{wr}$. Now (3.6) shows that there is a natural isomorphism $\bar{M}_2\rightarrow L_2$ which carries $\bar{S}$, $\bar{r}$, and $\bar{s}$ onto S_2, r, and s, respectively. Thus (3.1) yields that $|X_r|=2$. Also, since $S^r\cap S^{wr}=1$ and $|S:S\cap S^r|=2$, it follows that $|S\cap S^{wr}|\leq 2$. Comparing orders, we have $X_r=S\cap S^{wr}$.

By definition, $S_1\cap S_1^{tst}\leq S\cap S^{rstsrtst}=S\cap S^{ws}$ and $X_s\leq S\cap S^{tstrstsr}=S\cap S^{ws}$. Arguing as above, we have $|X_s|=2$ and $|S\cap S^{ws}|\leq 2$. Furthermore, using (3.4), (3.5), and the natural isomorphism $\bar{M}_1\rightarrow L_1$ given by (3.6), we have $|S_1\cap S_1^{tst}|=2$. Comparing orders, we thus obtain that $X_s=S\cap S^{ws}=S_1\cap S_1^{tst}$.

Arguing just as above, we have that $X_t=S\cap S^{wt}$ and that $|X_t|=4$. Now (3.1) shows that $C_2=S\cap S^{rsr}$; so

$$\begin{aligned}(C_2\cap C_2^{tst})^{rs}&=C_2\cap C_2^{tstrs}\\ &\le S\cap S^{rsrtstrs}\\ &=S\cap S^{wt}.\end{aligned}$$

Consequently, $|C_2\cap C_2^{tst}|\le 4$. Now $|S:S\cap S^s|=2$ and $C_1C_2\le S\cap S^s$, which shows that $|C_1C_2:C_2|\le 4$. Hence $|C_1:C_1\cap C_2|\le 4$ and consequently $|C_1:C_1\cap C_2\cap C_2^{tst}|\le 16$. Hence $|C_1|\le 2^6$. Since $|S|\ge 2^{12}$ by hypothesis, we must have that $|C_1|=2^6$, $|C_2\cap C_2^{tst}|=4$, and $|C_1C_2:C_2|=4$. (5), (6), and (7) now follow immediately.

Next, we establish a crucial commutator relation.

(3.9) $[Y_r, Y_t]=1$.

PROOF. From the definition of X_r and (3.7), it follows that $X_r=U_2\cap U_2^{rs}$; so by hypothesis $[S, X_r^{sts}]=1$. Since $X_s\le S^{sts}\cap S^{rstsr}$, it follows that

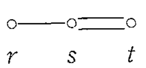
$$X_t\le C(X_r)\cap C(X_r^{stsrstsr}).$$

Computing in L, we have that $X_r^{stsrstsr}=(X_r)^r$ and that $[Y_r, t]=1$. Therefore, $[Y_r, Y_t]=1$.

We can now identify G_0. We distinguish two cases.

(3.10) *If $M_1/C_1\cong U_4(2)$, then G_0 is a perfect central extension of $O_8^-(2)$.*

PROOF. Let $G_0^*=O_8^-(2)$ and let B^*, N^*, S^*, and W^* be as in (1.8). Then W^* is isomorphic to the Weyl group of a root system of type (B_3). Let Π be a system of fundamental roots and choose notation so that the labeling of the Dynkin diagram is as follows:



For $u\in\pm\Pi$, define X_u^* as in (1.8) and set $Y_u^*=\langle X_{\pm u}^*\rangle$. We remark that the extension G^* of G_0^* by a field automorphism satisfies the hypothesis of Theorem 1. Hence the conditions which hold in G also hold in G^*. Thus (3.1), (3.6), (1)-(3) of (3.8) and their analogues for G^* show that there exists an isomorphism

$$\alpha:\langle Y_r^*, Y_s^*\rangle\longrightarrow\langle Y_r, Y_s\rangle$$

such that $(X_{\pm r}^*)^\alpha=X_{\pm r}$ and $(X_{\pm s}^*)^\alpha=X_{\pm s}$. Also, it follows from (3.4), (3.6), and (1), (3), (4) of (3.8) that there exists an isomorphism

$$\beta: \langle Y_s^*, Y_t^* \rangle \longrightarrow \langle Y_s, Y_t \rangle$$

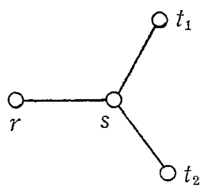
such that $(X_{\pm s}^*)^\beta = X_{\pm s}$ and $(X_{\pm t}^*)^\beta = X_{\pm t}$. As $|X_{\pm s}^*|=2$, α and β are equal on Y_s^*. Also, since $[Y_r, Y_t]=1$, it follows that there exists an isomorphism

$$\langle Y_r^*, Y_t^* \rangle \longrightarrow \langle Y_r, Y_t \rangle$$

which is equal to α on Y_r^* and to β on Y_t^*. Now both $\langle Y_r, Y_s \rangle$ and $\langle Y_s, Y_t \rangle$ are perfect as they are homomorphic images of perfect groups. Therefore, G_0 is perfect. We can now apply (1.8) to prove that $G_0/Z(G_0) \cong G_0^*$.

(3.11) *If $M_1/C_1 \cong L_4(2)$, then G_0 is a perfect central extension of $O_8^+(2)$.*

PROOF. The argument is similar to that of (3.10). Let $G_0^* = O_8^+(2)$ and let B^*, N^*, S^*, and W^* be as in (1.8). Then W^* is isomorphic to the Weyl group of a root system Σ of type (D_4). Let Π be a system of fundamental roots and choose notation so that the labeling of the Dynkin diagram is as follows:



Define X_u^* and Y_u^*, $u \in \pm\Pi$, as in (3.10). Then there exists an isomorphism

$$\alpha: \langle Y_r^*, Y_s^* \rangle \longrightarrow \langle Y_r, Y_s \rangle$$

which carries each root subgroups onto the corresponding X-subgroups. Also, there exists an isomorphism

$$\beta: \langle Y_s^*, Y_{t_1}^*, Y_{t_2}^* \rangle \longrightarrow \langle Y_s, Y_t \rangle$$

which is equal to α on Y_s^* such that

$$(X_{\pm t_1}^* X_{\pm t_2}^*)^\beta = X_{\pm t}\,.$$

Indeed, if t_i, $i \in \{1, 2, 3\}$, denotes the image of $\bar{t}_i$ under the natural isomorphism $\bar{M}_1 \to L_1$, then $s = t_3$, $t = t_1 t_2$, and $t_1^s = t_2$. Hence if we define

$$X_{t_i} = S_1 \cap S_1^{stst_{3-i}}, \qquad X_{-t_i} = (X_{t_i})^{t_i},$$

for each i, then $X_{\pm t} = X_{\pm t_1} X_{\pm t_2}$. Furthermore, by the definition of S_1

$$S_1 \cap S_1^{stst_{3-i}} \leq S \cap S^{wt_i},$$

and comparing orders, we have that equality holds here in the extension G^* of G_0^* by a graph automorphism. So β may be chosen so that $(X_{\pm t_i}^*)^\beta = X_{\pm t_i}$. Now let

$$Y_{t_i}=\langle X_{\pm t_i}\rangle .$$

Then as $[Y_r, Y_{t_i}]\leq[Y_r, Y_t]=1$, there exists an isomorphism

$$\langle Y_r^*, Y_{t_i}^*\rangle \longrightarrow \langle Y_r, Y_{t_i}\rangle$$

which is equal to α on Y_r^* and to β on $Y_{t_i}^*$. Therefore, G_0 is perfect and $G_0/Z(G_0)\cong G_0^*$ by (1.8).

4. $U_6(2)$ and $L_6(2)$.

In this section, we prove the following:

THEOREM 2. *Suppose the following conditions hold:*
(1) $M_1/C_1\cong U_4(2)$ *or* $L_4(2)$ *and* $M_2/C_2\cong L_3(4)$, $SL_3(4)$ *or* $L_3(2)\times L_3(2)$;
(2) $|S|\geq 2^{15}$;
(3) *either* $[M_1, (U_1\cap U_1^{sts})^{sr}]=1$ *or* C_2 *is abelian.*
Then G_0 *is a perfect central extension of* $U_6(2)$ *or* $L_6(2)$.

PROOF. The argument is similar to that of Theorem 1. For each $i\in\{1, 2\}$, let $\bar{M}_i=M_i/C_i$. Using (1.5), we have

(4.1) *There exists a natural BN-pair* $(\bar{B}_2, \bar{N}_2)$ *of* $\bar{M}_2$ *such that* $\bar{S}\leq\bar{B}_2$ *and either*
(1) $\bar{M}_2\cong L_3(4)$ *or* $SL_3(4)$ *and* $\bar{r}$, $\bar{s}$ *are the distinguished generators of the Weyl group* $\bar{W}_2=\bar{N}_2/\bar{B}_2\cap\bar{N}_2$, *or*
(2) $\bar{M}_2\cong L_3(2)\times L_3(2)$ *and if the distinguished generators of* $\bar{W}_2$ *are labeled*

$$\overset{\circ}{\bar{r}_1}\!\!-\!\!-\!\!\overset{\circ}{\bar{s}_1}\qquad \overset{\circ}{\bar{r}_2}\!\!-\!\!-\!\!\overset{\circ}{\bar{s}_2}$$

then $\{\bar{r}_1, \bar{s}_1\}^z=\{\bar{r}_2, \bar{s}_2\}$ *and* $\{\bar{r}, \bar{s}\}=\{\bar{r}_1\bar{r}_2, \bar{s}_1\bar{s}_2\}$ *or* $\{\bar{r}_1\bar{s}_2, \bar{s}_1\bar{r}_2\}$.

Hence

(4.2) $|S:S\cap S^r|=4$,

(4.3) $|S:S\cap S^s|=4$.

Using (1.1) and (4.3), we have

(4.4) *There exists a natural BN-pair* $(\bar{B}_1, \bar{N}_1)$ *of* $\bar{M}_1$ *such that* $\bar{S}\leq\bar{B}_1$ *and either*
(1) $\bar{M}_1\cong U_4(2)$ *and* $\bar{s}$, $\bar{t}$ *are the distinguished generators of the Weyl group* $\bar{W}_1=\bar{N}_1/\bar{B}_1\cap\bar{N}_1$, *or*
(2) $\bar{M}_1\cong L_4(2)$ *and if the distinguished generators of* $\bar{W}_1$ *are labeled*

$$\begin{array}{ccccc} \circ & \text{—} & \circ & \text{—} & \circ \\ \bar{t}_1 & & \bar{t}_3 & & \bar{t}_2 \end{array}$$

then $(\bar{s}, \bar{t})=(\bar{t}_1\bar{t}_2, \bar{t}_3)$ *and* $\bar{t}_1^z=\bar{t}_2$.

Hence

(4.5) $|S:S\cap S^t|=2$.

Next, we prove

(4.6) *For* $i\in\{1, 2\}$, *we have* $M_i=C_iL_i$, $C_i\cap L_i=1$, *and* $S_i=S\cap L_i$.

PROOF. For the case $i=1$, the argument of (3.6) applies with obvious changes. A similar argument using (1.7) shows that the assertion for $i=2$ will follow, once we prove $|S_2|\geq 2^6$. As in the case of S_1, this is proved by consideration of the chain

$$S,\ S^t,\ S^{st},\ S^{rst},\ S^{trst},\ S^{strst},\ S^{tstrst}.$$

Next, the argument of (3.8) gives

(4.7) *The following conditions hold*:
(1) $S\cap S^w=1$;
(2) $X_r=S\cap S^{wr}$;
(3) $X_s=S\cap S^{ws}=S_1\cap S_1^{tst}$;
(4) $X_t=S\cap S^{wt}$;
(5) $X_t=(C_2\cap C_2^{tst})^{rs}$;
(6) $S\cap S^s=C_1C_2$;
(7) $|S|=2^{15}$.

Using the condition (3) of the theorem, we next prove

(4.8) $[Y_r, Y_t]=1$.

PROOF. By the definition of X_t, $U_1\cap U_1^{sts}\leq X_t$. Computing in L, we have $|U_1\cap U_1^{sts}|=2$. Also, it follows from (4.4), (4.5), and (4.6) that $X_t=S_1\cap S_1^{sts}$ has order 2: see the proof of (3.8). Therefore, $X_t=U_1\cap U_1^{sts}$.

First, assume $[M_1, (U_1\cap U_1^{sts})^{sr}]=1$. Then $M_1\leq C_G(X_t^{sr})$ by the above. Now $X_s=S_1\cap S_1^{tst}\leq L_1$ by (4.7); so $Y_s\leq L_1=M_1\cap M_1^{rstsr}$. Thus

$$Y_s\leq C_G(X_t^{sr})\cap C_G(X_t^{tsr}).$$

Transforming by rs, we have

$$Y_r\leq C_G(X_t)\cap C_G(X_t^t)=C_G(Y_t).$$

Next, assume that C_2 is abelian. Then (5) of (4.7) shows that the group $M=\langle C_2, C_2^{tst}\rangle$ centralizes X_t^{sr}. As C_1 normalizes C_2 and as $C_2\cap C_2^{tst}=X_t^{sr}$ has order 2, C_1 centralizes X_t^{sr}. As $S\cap S^s=C_1C_2$ by (6) of (4.7), the structure of M_1/C_1 shows that M covers M_1/C_1. Thus $MC_1=M_1$ and it follows that M_1 centralizes X_t^{sr}. Therefore, $[Y_r, Y_t]=1$ by the previous discussion.

We need one more result before identifying G_0:

(4.9) *One of the following holds:*
(1) *$M_1/C_1\cong U_4(2)$ and $M_2/C_2\cong L_3(4)$ or $SL_3(4)$;*
(2) *$M_1/C_1\cong L_4(2)$ and $M_2/C_2\cong L_3(2)\times L_3(2)$.*

PROOF. Let $X=\langle S, s\rangle/S\cap S^s$ and consider its structure. By (4.1)

$$X\cong\begin{cases} L_2(4) & \text{if } M_2/C_2\cong L_3(4) \text{ or } SL_3(4)\,, \\ L_2(2)\times L_2(2) & \text{if } M_2/C_2\cong L_3(2)\times L_3(2). \end{cases}$$

Also, by (4.3) and (4.4)

$$X\cong\begin{cases} L_2(4) & \text{if } M_1/C_1\cong U_4(2)\,, \\ L_2(2)\times L_2(2) & \text{if } M_1/C_1\cong L_4(2)\,. \end{cases}$$

Hence the result follows.

Now we identify G_0.

(4.10) *If $M_1/C_1\cong U_4(2)$, then G_0 is a perfect central extension of $U_6(2)$.*

PROOF. The argument is quite similar to that of (3.10); so we only indicate necessary changes. First of all, $M_2/C_2\cong L_3(4)$ or $SL_3(4)$ by (4.9). We let $G_0^*=SU_6(2)$. Arguing as in (3.10), we may find epimorphisms

$$\alpha: \langle Y_r^*, Y_s^*\rangle \longrightarrow \langle Y_r, Y_s\rangle\,,$$

$$\beta: \langle Y_s^*, Y_t^*\rangle \longrightarrow \langle Y_s, Y_t\rangle$$

which carry the root subgroups faithfully onto the corresponding X-subgroups. However, since $|X_s^*|=4$, it may happen that α is not equal to β on Y_s^*. In that case, replace α by $\gamma\alpha$, where γ is a suitable field automorphism of $\langle Y_r^*, Y_s^*\rangle$ such that $(X_{\pm r}^*)^\gamma=X_{\pm r}^*$, $(X_{\pm s}^*)^\gamma=X_{\pm s}^*$, and $\gamma\alpha=\beta$ on Y_s^*.

(4.11) *If $M_1/C_1\cong L_4(2)$, then G_0 is a perfect central extension of $L_6(2)$.*

PROOF. First, $M_2/C_2\cong L_3(2)\times L_3(2)$ by (4.9). We will proceed just as in (3.11). Let $G_0^*=L_6(2)$ and label a fundamental root system of type (A_5)

$$\underset{r_1}{\circ}\!-\!\!-\!\!-\!\underset{s_1}{\circ}\!-\!\!-\!\!-\!\underset{t}{\circ}\!-\!\!-\!\!-\!\underset{s_2}{\circ}\!-\!\!-\!\!-\!\underset{r_2}{\circ}$$

Then there is an isomorphism

$$\alpha: \langle Y_{r_i}^*, Y_{s_i}^*; i=1, 2\rangle \longrightarrow \langle Y_r, Y_s\rangle$$

such that $(X_{\pm r_1}^* X_{\pm r_2}^*)^\alpha = X_{\pm r}$ and $(X_{\pm s_1}^* X_{\pm s_2}^*)^\alpha = X_{\pm s}$. Also, there is an isomorphism

$$\beta: \langle Y_{s_1}^*, Y_{s_2}^*, Y_t^*\rangle \longrightarrow \langle Y_s, Y_t\rangle$$

such that $(X_{\pm s_1}^* X_{\pm s_2}^*)^\beta = X_{\pm s}$ and $(X_{\pm t}^*)^\beta = X_{\pm t}$. Hence

$$(X_{\pm s_1}^* X_{\pm s_2}^*)^\alpha = (X_{\pm s_1}^* X_{\pm s_2}^*)^\beta$$

and both α and β are isomorphisms from $Y_{s_1}^* \times Y_{s_2}^*$ onto Y_s. Thus Y_s is written, in two different ways, as a direct product of two subgroups each isomorphic to $L_2(2)$. An easy argument using Krull-Schmidt theorem shows that such a direct product decomposition is unique up to permutations of direct factors. Hence either $(Y_{s_i}^*)^\alpha = (Y_{s_i}^*)^\beta$ for each i or $(Y_{s_i}^*)^\alpha = (Y_{s_{3-i}}^*)^\beta$ for each i. In the latter case, replace α by $\gamma\alpha$, where γ is an automorphism of $\langle Y_{r_i}^*, Y_{s_i}^*; i=1, 2\rangle$ such that $(X_{\pm r_i}^*)^\gamma = X_{\pm r_{3-i}}^*$ and $(X_{\pm s_i}^*)^\gamma = X_{\pm s_{3-i}}^*$ for each i. Thus we may choose α and β so that $(Y_{s_i}^*)^\alpha = (Y_{s_i}^*)^\beta$ for each i. Then since

$$X_{\pm s_i}^* = Y_{s_i}^* \cap X_{\pm s_1}^* X_{\pm s_2}^*,$$

it follows that $(X_{\pm s_i}^*)^\alpha = (X_{\pm s_i}^*)^\beta$. Therefore, $\alpha = \beta$ on $\langle Y_{s_i}^*; i=1, 2\rangle$. Now define

$$Y_{r_i} = \langle (X_{\pm r_i}^*)^\alpha\rangle, \qquad Y_{s_i} = \langle (X_{\pm s_i}^*)^\alpha\rangle.$$

Then there exists an isomorphism

$$\langle Y_{r_i}^*, Y_t^*\rangle \longrightarrow \langle Y_{r_i}, Y_t\rangle$$

which is equal to α on $Y_{r_i}^*$ and to β on Y_t^*. Therefore, G_0 is perfect and $G_0/Z(G_0) \cong G_0^*$ by (1.8).

5. $U_7(2)$ and $L_7(2)$.

In this section, we prove the following:

THEOREM 3. *Suppose the following conditions hold:*
(1) $M_1/C_1 \cong U_5(2)$ *or* $L_5(2)$ *and* $M_2/C_2 \cong L_3(4)$, $SL_3(4)$ *or* $L_3(2) \times L_3(2)$;
(2) $|S| \geq 2^{21}$;
(3) $Z(C_1C_2/C_1) = Z(C_2)C_1/C_1$.
Then G_0 *is a perfect central extension of* $U_7(2)$ *or* $L_7(2)$.

PROOF. The argument is quite similar to that of Theorem 2. First, (4.1), (4.2), and (4.3) hold without changes. We will refer to these as (5.1), (5.2), and (5.3), respectively. Next, we have

(5.4) *There exists a natural BN-pair* $(\bar{B}_1, \bar{N}_1)$ *of* $\bar{M}_1$ *such that* $\bar{S} \leq \bar{B}_1$ *and either*

(1) $\bar{M}_1 \cong U_5(2)$ *and* $\bar{s}$, $\bar{t}$ *are the distinguished generators of the Weyl group* $\overline{W}_1 = \bar{N}_1/\bar{B}_1 \cap \bar{N}_1$, *or*

(2) $\bar{M}_1 \cong L_5(2)$ *and if the distinguished generators of* $\overline{W}_1$ *are labeled*

$$\underset{\bar{t}_1}{\circ} \text{---} \underset{\bar{t}_2}{\circ} \text{---} \underset{\bar{t}_3}{\circ} \text{---} \underset{\bar{t}_4}{\circ}$$

then $(\bar{s}, \bar{t}) = (\bar{t}_1 \bar{t}_4, \bar{t}_2 \bar{t}_3 \bar{t}_2)$ *and* $\bar{t}_i^{\bar{s}} = \bar{t}_{5-i}$.

Hence we have

(5.5) $|S : S \cap S^t| = 8$.

Next, the arguments of (4.6) and (4.7) show that the same holds in this case as well except for an obvious change in (7) of (4.7). We will refer to them as (5.6) and (5.7), respectively. Next, we prove

(5.8) $[Y_r, Y_t] = 1$.

PROOF. The argument is similar to that of (4.8). By (5.7)

$$X_t = (C_2 \cap C_2^{tst})^{rs},$$

and

$$S \cap S^s = C_1 C_2 .$$

Thus the group $M = \langle Z(C_2), Z(C_2)^{tst} \rangle$ centralizes X_t^{sr}. Also, since C_1 normalizes C_2, it follows that C_1 normalizes X_t^{sr}. Therefore, $MC_1 \leq N_G(X_t^{sr})$.

Now in the factor group $\bar{M}_1 = M_1/C_1$, $\bar{S} \cap \bar{S}^s = \bar{C}_2$ by the above, and $Z(\bar{C}_2) = \overline{Z(C_2)}$ by hypothesis. Hence

$$\bar{M} = \langle Z(\bar{S} \cap \bar{S}^s), Z(\bar{S} \cap \bar{S}^s)^{tst} \rangle ,$$

and

$$\bar{M} \cong U_4(2) \quad \text{or} \quad L_4(2)$$

by (5.4). Let

$$L_0 = \langle Y_s, Z(X_{\pm t}) \rangle .$$

By (5.6), there is a natural isomorphism $\bar{M}_1 \to L_1$ which carries $\bar{S}$, $\bar{s}$, and $\bar{t}$ onto S_1, s, and t, respectively. Using (5.4) and (5.7), and computing in L_1, we obtain

$$L_0 = \langle Z(S_1 \cap S_1^s), Z(S_1 \cap S_1^s)^{tst} \rangle .$$

Hence $\bar{L}_0 = \bar{M}$, which shows that L_0 is isomorphic to $U_4(2)$ or $L_4(2)$ and normalizes X_t^{sr}. Since X_t is a quaternion group or a dihedral group of order 8, it follows that $[L_0, X_t^{sr}] = 1$. An easy computation using (3) of (5.7) shows that $rstsr$ normalizes L_0; so $[L_0, X_t^{tsr}] = 1$. Therefore, $[L_0, Y_t^{sr}] = 1$ and, in particular, $[Y_s, Y_t^{sr}] = 1$. Transforming by rs, we obtain $[Y_r, Y_t] = 1$.

An argument of the proof of (4.9) yields the following:

(5.9) *One of the following holds*:
(1) $M_1/C_1 \cong U_5(2)$ *and* $M_2/C_2 \cong L_3(4)$ *or* $SL_3(4)$;
(2) $M_1/C_1 \cong L_5(2)$ *and* $M_2/C_2 \cong L_3(2) \times L_3(2)$.

Now, arguing just as in (4.10) and (4.11), we have the following:

(5.10) *If* $M_1/C_1 \cong U_5(2)$, *then* G_0 *is a perfect central extension of* $U_7(2)$.

(5.11) *If* $M_1/C_1 \cong L_5(2)$, *then* G_0 *is a perfect central extension of* $L_7(2)$.

6. $Sp_6(4)$ and $Sp_6(2) \times Sp_6(2)$.

In this section, we sketch a proof of the following:

THEOREM 4. *Suppose the following conditions hold*:
(1) $M_1/C_1 \cong Sp_4(4)$ *or* $Sp_4(2) \times Sp_4(2)$ *and* $M_2/C_2 \cong L_3(4)$, $SL_3(4)$ *or* $L_3(2) \times L_3(2)$;
(2) $|S| \geq 2^{18}$;
(3) C_2 *is abelian.*
Then either G_0 *is a perfect central extension of* $Sp_6(4)$ *or* $G_0/Z(G_0) \cong Sp_6(2) \times Sp_6(2)$ *and* z *interchanges the components of* $G_0/Z(G_0)$.

As in the previous sections, a crucial step of the proof of Theorem 4 is the proof of an analogue of (3.8) and the commutator relation $[Y_r, Y_t]=1$. These are proved by arguments parallel to those in the proof of (3.8) and (5.8). Once these are proved, the argument of (4.10) shows that if $M_1/C_1 \cong Sp_4(4)$, then G_0 is a perfect central extension of $Sp_6(4)$.

Assume $M_1/C_1 \cong Sp_4(2) \times Sp_4(2)$. Then $M_2/C_2 \cong L_3(2) \times L_3(2)$. Arguing as in (4.11), we have that there exist direct product decompositions

$$Y_r = Y_{r_1} \times Y_{r_2}, \qquad Y_s = Y_{s_1} \times Y_{s_2}, \qquad Y_t = Y_{t_1} \times Y_{t_2}$$

such that $[Y_{r_i}, Y_{s_{3-i}}]=1$ and $[Y_{s_i}, Y_{t_{3-i}}]=1$ and such that $Y_{r_i}^z = Y_{r_{3-i}}$, $Y_{s_i}^z = Y_{s_{3-i}}$, and $Y_{t_i}^z = Y_{t_{3-i}}$: see the proof of (3.11). Hence if we set

$$G_1 = \langle Y_{r_1}, Y_{s_1}, Y_{t_1} \rangle,$$

then G_0 is a central product of G_1 and G_1^z. The mapping $\phi: x \to xx^z$, $x \in G_1$, is a homomorphism from G_1 into $C_{G_0}(z)$ with $\ker \phi$ contained in $Z(G_1)$. Let V_u, $u \in \{r, s, t\}$, be as in the proof of (2.4). Then clearly $V_u \leq C_{Y_u}(z)$ and so, as $V_u \cong C_{Y_u}(z) \cong L_2(2)$, we have $V_u = C_{Y_u}(z)$. This shows $Y_{u_1}^{\phi} = V_u$ so

$$G_1^{\phi} = \langle V_r, V_s, V_t \rangle = L.$$

Hence $G_1/Z(G_1) \cong Sp_6(2)$, which completes the proof of Theorem 4.

7. $G_0=\langle M_1, M_2\rangle$.

In this section we supplement Theorems 1-4 by the following result:

THEOREM 5. *In Theorems* 1-4, *if* $G_0/Z(G_0)\cong Sp_6(2)\times Sp_6(2)$, *then the following conditions hold*:

(1) $G_0=\langle M_1, M_2\rangle$;

(2) $Z(G_0)$ *has odd order*;

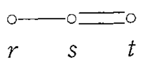
(3) $S\in \mathrm{Syl}_2(G_0)$.

PROOF. In the proof of Theorems 1-4, $G_0/Z(G_0)$ has been identified with $G_0^*/Z(G_0^*)$ by using an epimorphism $\phi: G_0\to G_0^*/Z(G_0^*)$ such that $\ker\phi=Z(G_0)$ and $(X_{\pm r_i})^{\phi}=(X_{\pm r_i}^*)^{\sigma}$, $1\leq i\leq l$, whose existence is guaranteed by (1.8). In the present situation, X_{r_i} is a subgroup of S and $Z(G_0^*)$ has odd order. Furthermore, $|S|=|G_0^*/Z(G_0^*)|_2$, as analogues of (7) of (3.8) show. Now let S^* be as in (1.8). Then $S^*\in \mathrm{Syl}_2(G_0^*)$ and $S^*=\langle X_{r_i}^*; 1\leq i\leq l\rangle$ by Lemma 4 of [11]. Hence if we set $T=\langle X_{r_i}; 1\leq i\leq l\rangle$, then $T^{\phi}=(S^*)^{\sigma}$. Since $T\leq S$ and $|S|=|(S^*)^{\sigma}|$, we must have $S=T$ and $S\cap Z(G_0)=1$. Consequently, $S\leq G_0$ and $SZ(G_0)/Z(G_0)\in \mathrm{Syl}_2(G_0/Z(G_0))$. Since G_0 is perfect, Gaschütz theorem implies that $Z(G_0)$ has odd order, and hence $S\in\mathrm{Syl}_2(G_0)$. Analogues of (4.1) and (4.4) show that $M_1=\langle S, S^{stst}\rangle$ and $M_2=\langle S, S^{rsr}\rangle$. Since $L\leq G_0\leq\langle M_1, M_2\rangle$ by (2.4), it follows that $G_0=\langle M_1, M_2\rangle$.

Appendix.

We will discuss Part B for the case that $L/Z(L)\cong U_6(2)$. By the discussions of the previous sections, it is now clear how we formulate and prove our result.

Let $L=U_6(2)$ or $SU_6(2)$. Let Λ be a root system of type (B_3) and $\Phi=\{r, s, t\}$ be a system of fundamental roots, where the Dynkin diagram is as follows:



Let (B, N) be a natural BN-pair of L and let $U\in\mathrm{Syl}_2(B)$. There exists an isomorphism $W=N/B\cap N\to W(\Lambda)$ which carries the distinguished generators of W onto the fundamental reflections. We shall denote by r, s, t the distinguished generators of W also. For simplicity of notation, we shall also denote by r, s, t the representatives in N of the distinguished generators, and choose them to be involutions satisfying the relations

$$(rs)^3=(st)^4=(tr)^2=1\,.$$

Define

$$P_1=\langle U, s, t\rangle\,,\qquad P_2=\langle U, r, s\rangle\,,$$

$$A_1=O_2(P_1)\,,\qquad A_2=O_2(P_2)\,.$$

Now consider the following situation: G is a group, z is an involution of G, L is a subgroup of the centralizer of z in G, and $L\cong U_6(2)$ or $SU_6(2)$. Assume that G satisfies the following conditions:

(1) for $i\in\{1, 2\}$, there exists a z-invariant 2-subgroup C_i such that $C_{C_i}(z)=A_i$;

(2) for $i\in\{1, 2\}$, there exists a z-invariant subgroup M_i of $N_G(C_i)$ containing C_i such that $C_{M_i/C_i}(z)=P_iC_i/C_i$;

(3) there exists a z-invariant 2-subgroup S such that $C_S(z)=U$ and $S\in \mathrm{Syl}_2(M_i)$ for each i.

Assume furthermore that the following conditions hold:

(i) $M_1/C_1\cong L_4(4)$ or $U_4(2)\times U_4(2)$ and $M_2/C_2\cong L_3(4)\times L_3(4)$, $SL_3(4)\times SL_3(4)$, or $SL_3(4)*SL_3(4)$, where $*$ denotes the central products with amalgamated centers;

(ii) $|S|\geq 2^{30}$;

(iii) C_2 is abelian.

The argument of the previous sections shows that under the above hypotheses, the group G_0 as defined in (2.3) is a z-invariant subgroup such that $L\leq G_0$ and $G_0/Z(G_0)\cong L_6(4)$ or $U_6(2)\times U_6(2)$. Furthermore, G_0 is perfect and the conditions (1)-(3) of Theorem 5 hold even if $G_0/Z(G_0)\cong U_6(2)\times U_6(2)$. To see this, consider the proof of Theorem 4. In the present case $\langle Y_{s_1}, Y_{t_1}\rangle$ is isomorphic to $U_4(2)$ and hence perfect. Similarly, $\langle Y_{r_1}, Y_{s_1}\rangle$ is perfect. Therefore, G_0 is perfect. Next, the homomorphism $x\rightarrow xx^z$, $x\in G_1$, carries $\langle X_{r_1}, X_{s_1}, X_{t_1}\rangle$ onto $\langle U_r, U_s, U_t\rangle$, where U_r, U_s, U_t are the same as in the proof of (2.4). We now proceed as in the proof of Theorem 5. As $U=\langle U_r, U_s, U_t\rangle$ by Lemma 4 of [11], $S\cap G_0$ covers an S_2-subgroup of $G_0/Z(G_0)$. As $|S|=2^{30}$ by an analogue of (7) of (3.8), we have $S\leq G_0$ and $S\cap Z(G_0)=1$. Hence (1), (2), and (3) of Theorem 5 follow. In the case that $L\cong Sp_6(2)$, this argument breaks down as $\langle Y_{s_1}, Y_{t_1}\rangle\cong Sp_4(2)$ is not perfect and $U\neq\langle U_r, U_s, U_t\rangle$.

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