Standard subgroups of type $Sp_6(2)$, II

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Introduction.

The purpose of this half of the paper is to complete our study of finite groups with a standard subgroup isomorphic to $Sp_6(2)$ begun in the first half of this paper [10]. Specifically, we prove

MAIN THEOREM. *Let* G *be a finite group and suppose* L *is a standard subgroup of* G *isomorphic to* $Sp_6(2)$. *Furthermore, assume that* $C_G(L)$ *has cyclic Sylow 2-subgroups and that if* $G \neq G'$ *then* $O(N_{G'}(X)) \leqq O(G')$ *for every 2-subgroup* X *of* G'. *Then if* $LO(G)$ *is not a normal subgroup of* G, *the normal closure* $\langle L^G \rangle$ *of* L *is isomorphic to one of the following Chevalley type groups:*

$$O_8^-(2),\ O_8^+(2),\ U_6(2),\ SU_6(2),\ L_6(2),$$

$$U_7(2),\ L_7(2),\ Sp_6(4),\ Sp_6(2)\times Sp_6(2).$$

Some remarks may be in order on the assumptions of the main theorem. The case that $C_G(L)$ has noncyclic Sylow 2-subgroups has been treated by Aschbacher [1] and Aschbacher and Seitz [2]. The second assumption is concerned with the Unbalanced Group Conjecture (see Harris [15]). If $O(N_{G'}(X)) \nleqq O(G')$ for some 2-subgroup X of G', then the conjecture implies that $\langle L^G \rangle$ is of known type. Thus if the conjecture turns out to be true, the second assumption becomes superfluous. Conversely, we may use the main theorem in the inductive proof of the Unbalanced Group Theorem (see [23]).

Our proof of the main theorem utilizes the so-called "pushing up" method and thus follows the same line of arguments as in the previous paper [8]. The bulk of the paper is devoted to the construction of 2-local subgroups that resemble certain of the parabolic subgroups of the Chevalley type groups listed in the main theorem. Once this is accomplished, the results of [10] enable us to construct a semisimple subgroup G_0 isomorphic to the Chevalley type groups in question with $L \leqq G_0$. In order to complete the proof of the main theorem, we must, of course, show that G_0 is a normal subgroup of G. Fortunately, this problem has been treated by Seitz [19, III] in a more general context. We will appeal to the results of his paper except in the case

$G_0 \cong Sp_6(2) \times Sp_6(2)$, where a more effective method is available thanks to a product fusion theorem of Shult [20].

As remarked in the introduction of [10], $Sp_6(2)$ is one of the four exceptional groups for which the "induction" method of Seitz [19] breaks down. What follows is my understanding of why $Sp_6(2)$ is an exceptional group. In studying groups, G, with a standard subgroup, L, isomorphic to $Sp_6(2^n)$, one is naturally lead to sections, $X=Y/Z$, of G containing a noncentral involution, z, such that

$$C_X(z) = \langle z \rangle \times K \times O(C_X(z)),$$

where K is the image of $L \cap Y$ in X and $K \cong Sp_4(2^n)$. If $n>1$, the result of [8] shows that $A = \langle K^X \rangle$ is isomorphic modulo center to

$$U_4(2^n),\ L_4(2^n),\ U_5(2^n),\ L_5(2^n),$$

$$Sp_4(2^{2n}),\ or\ Sp_4(2^n) \times Sp_4(2^n).$$

All of these groups actually occur in the known examples of G. When $n=1$, an unpleasant phenomenon occurs. The results of Harris and Solomon [16] and others indeed show that $B = \langle (K')^X \rangle$ is isomorphic modulo center to

$$U_4(2),\ L_4(2),\ U_5(2),\ L_5(2),$$

$$Sp_4(4),\ A_6 \times A_6,\ or\ U_4(3).$$

However, what we need is the structure of A, and there is a gap between A and B. Moreover, $U_4(3)$ does not occur in any known examples of G. Difficulties arise when we try to prove that $B/Z(B) \cong U_4(3)$ and that either $A=B$ or A is contained in a subgroup of X isomorphic to $Sp_4(2) \times Sp_4(2)$. Another difficulty arises when $n=1$ because we can not utilize the Cartan subgroups of L.

Our notation is standard and for the most part taken from [11]. Possible exceptions are the use of the following:

$m(X)$	the 2-rank of X.
$I(X)$	the set of involutions of X.
$\mathcal{E}^*(X)$	the set of maximal elementary abelian subgroups of X.
X^2	the subgroup of X generated by the squares of elements of X.
$E(X)$	the product of the quasisimple subnormal subgroups of X.
$F^*(X)=F(X)E(X)$	the generalized Fitting subgroup.
X wreath Y	the wreath product of X by Y.
$X*Y$	a central product of X and Y.

$f(X \bmod Y)$	the preimage in X of $f(X/Y)$, where f is a function from groups to groups.
Z_n, $n \geqq 2$	the cyclic group of order n.
E_{2^n}, $n \geqq 2$	the elementary abelian group of order 2^n.
$\mathcal{E}_{2^n}(X)$	the set elementary abelian subgroups of X of order 2^n.
D_{2n}, $n \geqq 3$	the dihedral group of order $2n$.
Σ_n, $n \geqq 3$	the symmetric group of degree n.

We use the "bar" convention for homomorphic images. Thus if G is a group, N is a normal subgroup, and $\bar{G}$ denotes the factor group G/N, then for any subset X of G, $\bar{X}$ will denote the image of X under the natural epimorphism $G \to \bar{G}$. A similar convention will be used when a group G has a permutation representation on a set Ω, where we write X^{Ω} instead of $\bar{X}$.

Contents

1. Preliminaries.

In this section, we collect some helpful preliminary results. In the first two lemmas, p will denote an arbitrary prime integer.

(1A) LEMMA. *Let A be a p'-group of automorphisms of a p-group G. Then the following holds:*

(1) *if A_1, A_2 are normal subgroups of A such that $G = C_G(A_1)C_G(A_2)$ and $C_G(A_1) \cap C_G(A_2) = 1$, then $G = C_G(A_1) \times C_G(A_2)$;*

(2) *if $C_G(A) \lhd G$, then $G = C_G(A) * [G, A]$.*

PROOF. For a proof of (1), see [8], (II). If $C_G(A) \lhd G$, then $[C_G(A), G, A] = 1 = [A, C_G(A), G]$; so $[G, A, C_G(A)] = 1$ by the three-subgroup lemma [11], Theorem 2.2.3. As $G = C_G(A)[G, A]$ by [11], Theorem 5.3.5, (2) holds.

(1B) LEMMA. *If P is a p-group of class at most two, the exponent of $P/Z(P)$ is at most equal to the exponent of $Z(P)$. In particular, if P is a p-group such that $|P'| = |Z(P)| = p$, then P is an extra-special p-group.*

PROOF. This follows from the commutator identity $[x^n, y]=[x, y]^n$, which holds in groups of class at most two: see [11], Lemma 2.2.2.

(1C) LEMMA. *Let z be an involution of a group G, A a subgroup of G, and B a $\langle A, z\rangle$-invariant subgroup of $[A, z]$ with $B=O^{2'}(B)$. Then if $[C_{O(G)}(z), B]=1$, $[O(G), B]=1$.*

PROOF. See (1J) of [8].

(1D) LEMMA. *Let R be an S_2-subgroup of a group G and S a normal subgroup of R with R/S abelian. Let x be an involution of $R-S$ and suppose that each extremal conjugate of x in R is contained in xS. Then $x\notin G'$.*

PROOF. This is Lemma (1E) of [9].

(1E) LEMMA. *Let E be an elementary abelian 2-subgroup of a group G and let z be an involution of $N_G(E)$. Then the following holds:*

(1) $\mathcal{E}^*(\langle E, z\rangle)=\{E, \langle C_E(z), z\rangle\}$;

(2) $|E: C_E(z)|\leqq|C_E(z)|$ *and equality holds if and only if* $I(zE)=z^E$;

(3) *if* $|E: C_E(z)|\geqq 4$, *then* $N_G(\langle E, z\rangle)\leqq N_G(E)\cap N_G(\langle C_E(z), z\rangle)$.

PROOF. See Lemma (1C) of [9] and its proof.

(1F) LEMMA. *Let Q be a 2-group of order 2^{2n} admitting an automorphism α of order 2 and a nontrivial automorphism ρ of order 2^n-1 such that*

(i) *α and ρ commute with each other under the action on Q, and*

(ii) $C_Q(\alpha)\cong E_{2^n}$.

Then Q is either elementary abelian or homocyclic of rank n.

PROOF. This is a consequence of Theorem B of [18].

(1G) LEMMA. *Let $A\leqq B\leqq C\leqq D$ be a chain of 2-groups such that $D^2\leqq B$, $C^2\leqq A$ and $A\leqq Z(D)$. Furthermore, assume that there is an involution $z\in B$ such that $C=C_D(z)$ and $B=\langle z\rangle A$. Then $D^2\leqq A$.*

PROOF. Suppose by way of contradiction that $D^2\not\leqq A$. Then D/A has a Z_4-subgroup X/A containing B/A. Since $A\leqq Z(X)$, it follows that X is abelian, hence $X\leqq C_D(z)=C$. This is a contradiction as C/A is elementary abelian.

(1H) LEMMA. *Let D be a 2-group, A an elementary abelian subgroup of index 4 and order at least 8, and z an involution of $C_D(A)-Z(D)\cup A$. Let H be a group of automorphisms of D leaving A and z invariant, and suppose $|C_A(H)|=2$ and H is irreducible on $A/C_A(H)$. Then $D^2=C_A(H)$ and $Z(D)=A$.*

PROOF. Let $B=\langle z\rangle A$, $I=C_A(H)$, and $J=\langle z\rangle I$. Then $B=C_D(z)$ and $|D:B|=2$, so $[D, z]$ is an H-invariant subgroup of A of order 2. Thus $I=[D, z]$ and $J=\langle z^D\rangle$, which shows that both I and J are H-invariant normal subgroups of D. Now D/J is not cyclic and so $(D/J)^2<B/J$. As H is irreducible on B/J, we have $D^2\leqq J$. Then (1G) applied to the chain $I\leqq J\leqq B\leqq D$ shows $D^2\leqq I$, and so $D^2=I$ as D is not abelian. As D can not be extra-special, we have $I<Z(D)$. Also, $J\not\leqq Z(D)\leqq B$ as $C_D(z)=B$. Since H is irreducible on B/J, it follows that $Z(D)$ is maximal in B. Thus $I<A\cap Z(D)$ and the irreducible action of H on A/I shows that $Z(D)=A$.

(1I) LEMMA. *Let z be an involution of a group G and suppose $C(z)=\langle z\rangle\times K\times O(C(z))$ with $K\cong Sp_4(2)$. Furthermore, assume that z is not a central involution of G and let $M=E(G)$. Tnen M satisfies one of the following conditions:*

(1) $M/O(M)\cong A_6\times A_6$ *and* $|O(M)|\leqq 3$;

(2) $M\cong U_4(2)$, $L_4(2)$, $U_5(2)$, $L_5(2)$, *or* $Sp_4(4)$;

(3) $M/O(M)\cong U_4(3)$.

Furthermore, we have $C(M)=O(G)$ and $O(C(z))=C_{O(G)}(z)$. If (1) *holds and if M_1, M_2 are the components of M, then $M_1^z=M_2$, $N(M_1)=N(M_2)$, $G=\langle z\rangle N(M_1)$, and one of the following holds:*

(1.a) $C(M_i)=M_{3-i}O(G)$ *for each i and* $N(M_1)/C(M_1)C(M_2)\cong Z_2$ *or* E_4;

(1.b) $C(M_i)/O(G)\cong Sp_4(2)$ *for each i and* $N(M_1)=C(M_1)C(M_2)$.

If (2) *or* (3) *holds, then $G=\langle z\rangle MO(G)$ and $\langle z\rangle KC_{O(M)}(z)=\langle z\rangle\times C_M(z)$.*

PROOF. Let $L=K'$ and $\bar{G}=G/O(G)$. Then $L\cong A_6$, $L\lhd C(z)$, $\langle z\rangle\in \mathrm{Syl}_2(C_{C(z)}(L))$ and $C(z)/LC_{C(z)}(L)\cong Z_2$. Furthermore, $z\notin Z^*(G)$ by hypothesis. Therefore, Theorem 2 of [16] shows that $F^*(\bar{G})=E(\bar{G})$ and that one of the following holds:

(i) $F^*(\bar{G})=L\times L$ and z interchanges the components of $F^*(\bar{G})$;

(ii) $F^*(\bar{G})$ is a simple group of sectional 2-rank less than 5;

(iii) $F^*(\bar{G})\cong U_5(2)$, $L_5(2)$, or $Sp_4(4)$.

If (ii) holds, the structure of $F^*(\bar{G})$ is known by the main theorem of [12]. Moreover, the structure of $C_{F^*(\bar{G})}(\bar{z})$ is known by Section 19 of [3], a table on p. 441 of [2], and Appendix 1 of [5] (when $F^*(\bar{G})$ is an alternating group, the knowledge is standard). Hence we have that $F^*(\bar{G})\cong U_4(2)$, $L_4(2)$, or $U_4(3)$ in Case (ii).

Let X be the complete inverse image of $F^*(\bar{G})$, A an S_2-subgroup of $C_X(z)$, B a z-invariant S_2-subgroup of $N_X(A)$, and $P/A=C_{B/A}(z)$. As $A<B$, we have $A<P$, so $1\neq[P, z]\leqq O^{2'}(C(z))$ and the structure of $C(z)$ shows $[O(C(z)), [P, z]]=1$. Thus $[O(G), [P, z]]=1$ by (1C) and, since $X/O(G)$ has no proper nontrivial subgroups normal in $G/O(G)$, it follows that $X=C_X(O(G))O(G)$. Therefore, $X=MO(G)$ and hence $C(M)=O(G)$.

Assume that (i) holds. Then $M/O(M)\cong A_6\times A_6$ and if M_1, M_2 are the components of M, then $M_1^z=M_2$, $N(M_1)=N(M_2)$, and $G=\langle z\rangle N(M_1)$. Inspecting the Schur multiplier of A_6, we have that $M_i\cong A_6$ or $\hat{A}_6$ for each i, where $\hat{A}_6$ is a 3-fold cover of A_6: see a table on p. 60 of [6]. If $M_i\cong\hat{A}_6$, then $O(M_1)=O(M_2)$, as otherwise $C_M(z)\cong\hat{A}_6$ against the structure of $C(z)$. Therefore, $|O(M)|\leqq 3$.

Now $C(\bar{M}_1)\cap C(\bar{M}_2)=1$ and, consequently, $\bar{M}_i\leqq C(\bar{M}_{3-i})\hookrightarrow \mathrm{Aut}(\bar{M}_i)\cong P\Gamma L_2(9)$ for each i. As $C(\bar{z})\cap C(\bar{M}_1)C(\bar{M}_2)\cong C(\bar{M}_1)$, the structure of $C(\bar{z})$ shows that either

(a) $C(\bar{M}_{3-i})=\bar{M}_i$ for each i, or

(b) $C(\bar{M}_i)\cong Sp_4(2)$ for each i.

Also, $N(\bar{M}_1)/\bar{M}C(\bar{M}_i)\hookrightarrow \mathrm{Out}(\bar{M}_i)\cong E_4$ and, since $\bar{M}C(\bar{M}_1)\cap\bar{M}C(\bar{M}_2)=\bar{M}$, it follows that $N(\bar{M}_1)/\bar{M}$ is an elementary abelian 2-group. Furthermore, since $I(\bar{z}\bar{M})=\bar{z}^{\bar{M}}$, it follows that

$$N(\langle\bar{z}\rangle\bar{M})\cap N(\bar{M}_1)=(C(\bar{z})\cap N(\bar{M}_1))\bar{M},$$

and hence

$$N(\langle\bar{z}\rangle\bar{M})\cap N(\bar{M}_1)=\langle\bar{a}\rangle\bar{M},$$

where $\bar{a}$ is an involution such that $\langle\bar{a}\rangle\bar{M}_i\cong Sp_4(2)$ for each i. This forces $|N(\bar{M}_1)/\bar{M}|\leqq 4$. Hence if (b) holds, then $N(\bar{M}_1)=C(\bar{M}_1)C(\bar{M}_2)$. As $N(\bar{M}_i)=\overline{N(M_i)}$, (1.b) holds. Clearly, (a) implies (1.a).

Now assume that (ii) or (iii) holds. Then $M/O(M)\cong U_4(3)$, $U_4(2)$, $L_4(2)$, $U_5(2)$, $L_5(2)$, or $Sp_4(4)$. Inspecting the Schur multipliers of these groups, we have that $O(M)=1$ unless $M/O(M)\cong U_4(3)$: see [6] and [14] (for $U_5(2)$ we have to use an unpublished result of Steinberg). Also, $\bar{z}$ induces an outer automorphism on $\bar{M}$ and $C_{\bar{M}}(\bar{z})\cong Sp_4(2)$ by [3] and [5]. Thus $\langle\bar{z}\rangle\bar{K}=\langle\bar{z}\rangle\times C_{\bar{M}}(\bar{z})$ and hence $\langle z\rangle KC_{O(M)}(z)=\langle z\rangle\times C_M(z)$. Unless $\bar{M}\cong U_4(3)$ or $Sp_4(4)$, $|\mathrm{Out}(\bar{M})|=2$, see [21], and hence we have $\bar{G}=\langle\bar{z}\rangle\bar{M}$. If $\bar{M}\cong Sp_4(4)$, then $\mathrm{Out}(M)\cong Z_4$ and the involutions of $\mathrm{Aut}(\bar{M})-\mathrm{Inn}(\bar{M})$ are all conjugate under $\mathrm{Inn}(\bar{M})$: see Section 19 of [3]. Hence $\bar{G}=\bar{M}C(\bar{z})$ and then $\bar{G}=\langle\bar{z}\rangle\bar{M}$. Assume $\bar{M}\cong U_4(3)$. Then $\bar{z}$ is a diagonal automorphism, so $\langle\bar{z}\rangle\bar{M}\lhd\bar{G}$. As $C_{\bar{M}}(\bar{z})$ contains an S_5-subgroup $\bar{S}$ of $\bar{M}$ and as $C_{\bar{M}}(\bar{S})$ has odd order, Sylow's theorem shows that $\bar{z}^{\bar{G}}=\bar{z}^{\bar{M}}$, hence $\bar{G}=\bar{M}C(\bar{z})$. Therefore, $\bar{G}=\langle\bar{z}\rangle\bar{M}$ in this case as well.

Now $|\bar{G}:\bar{M}|$ is a power of 2 and $C_{\bar{M}}(\bar{z})\cong A_6$ or $Sp_4(2)$, so $O(C_{\bar{G}}(\bar{z}))=1$ in all cases. Hence $O(C(z))=C_{O(G)}(z)$. Thus all parts of the lemma hold.

(1J) LEMMA. *Let G be a group satisfying the hypothesis of* (1I) *and A an E_8-subgroup of K. Assume that $O(G)=1$ and that there is an $N_K(A)$-invariant subgroup D of order 2^7 such that $C_D(z)=\langle z\rangle A$ and $D^2\leqq A$. Then $N(\langle z\rangle A)=N_K(A)D$ and there is a subgroup M of index 2 satisfying the following conditions*:

(1) $M\cong U_5(2)$, $L_5(2)$, $Sp_4(4)$, $Sp_4(2)\times Sp_4(2)$, *or* $U_4(3)$;

(2) $G=\langle z\rangle M$;

(3) $C_M(z)=K$;

(4) *either* $M\cong U_4(3)$ *or* $A\not\leqq Z(D)$.

PROOF. We first construct a subgroup M of index 2 satisfying (1) and (2). Let $E=E(G)$. Since $|G|_2\geqq|N_K(A)D|_2=2^8$, (1I) shows that one of the following holds:

(a) $E\cong U_5(2)$, $L_5(2)$, $Sp_4(4)$, or $U_4(3)$;

(b) $E\cong A_6\times A_6$.

Furthermore, if (a) holds, then $G=\langle z\rangle M$ and so $M=E$ satisfies (1) and (2).

Let H be a Z_3-subgroup of $N_K(A)$ and set $I=C_A(H)$. Then $|I|=2$ and H is transitive on $(A/I)^{\#}$. Let $B=\langle z\rangle A$ and $D_0/B=C_{D/B}(H)$. Our hypothesis implies that $[D, z]=A$. Hence $|D_0/B|=2$. Otherwise $D_0=D$, so $D=BC_D(H)$ and then $A=[C_D(H), z]$, which is a contradiction as $[A, H]\neq 1$. Hence if we set $Q=[D, H]A$, then $|Q/A|=4$ and H is transitive on $(Q/A)^{\#}$.

Assume that (b) holds and let $E=E_1\times E_2$ with $E_1^z=E_2\cong A_6$. (1I) shows that if $C(E_i)\neq E_{3-i}$ for each i, then $M=C(E_1)C(E_2)$ satisfies (1) and (2). Assume, therefore, that $C(E_i)=E_{3-i}$ for each i. As in (1I), we have that

$$N(\langle z\rangle E)\cap N(E_1)=\langle e\rangle E\,,$$

where e is an involution such that $\langle e\rangle E_i\cong Sp_4(2)$ for each i. Assume that $N(E_1)\neq\langle e\rangle E$. Then $N(E_1)/E\cong E_4$ and there is a subgroup X of $N(E_1)$ containing E such that X/E_1 has a semidihedral S_2-subgroup of order 16. As $I(X/E_1)\leqq E/E_1$ and as $X^z\neq X$, it follows that $\langle I(N(E_1))\rangle=\langle e\rangle E$. As $G/E\cong D_8$, we conclude that $\langle I(G)\rangle=\langle z, e\rangle E$. Of course, this is true even if $N(E_1)=\langle e\rangle E$. Now $D_0^2=I$ and $Z(D_0)=A$ by (1H). This implies that $D_0\cong E_4\times D_8$ and, consequently, $D_0=\Omega_1(D_0)\leqq\langle z, e\rangle E$. Also, $[D, H]\leqq[D, E]\leqq E$ as $H\leqq E$. Therefore, $D=[D, H]D_0\leqq\langle z, e\rangle E$, and hence $A=[D, z]\leqq(\langle z, e\rangle E)'=E$. However, this is a contradiction as $C_E(z)\cong A_6$. Thus we have shown that there always exists a subgroup M of index 2 satisfying (1) and (2). Furthermore, $C_M(z)\cong Sp_4(2)$ by construction of M.

Now $A=[D, z]\leqq M$. So $K=K'A\leqq M$ and then $C_M(z)=K$ by the above remark. Also, we have $A=B\cap M\lhd N(B)$ and so $z^{N(B)}=zA=z^D$. This implies that $N(B)=N_{C(z)}(B)D$. Since we are assuming $O(G)=1$, (1I) shows that $C(z)=\langle z\rangle\times K$ and consequently $N_{C(z)}(B)=\langle z\rangle N_K(A)$. Thus $N(B)=N_K(A)D$.

Assume that $M\cong U_4(3)$. Clearly, $Q\leqq M$; so let $Q\leqq S\in\mathrm{Syl}_2(M)$. Then S has exactly two E_{16}-subgroups A_1 and A_2, and $T=\langle A_1, A_2\rangle$ is of type $L_3(4)$. Furthermore, $S=\langle x\rangle T$, where x is an involution acting as a field automorphism of $L_3(4)$ on T. Assume that $A\leqq Z(D)$. Then $Q\leqq C_S(A)$ and, in particular, $|C_S(A)|\geqq 2^5$. This forces $A\leqq T$. We assume $A\leqq A_1$ without loss of generality. Then $C_T(A)=A_1$ and so $|C_S(A)|=2^5$. Comparing orders, we have that $C_S(A)=Q$. So $A<A_1<Q$ and since H is transitive on $(Q/A)^{\#}$, it follows that Q is elementary abelian. However, this is a contradiction as $m(M)=4$.

The proof of the following result is parallel to that of (1I) and is omitted.

(1K) LEMMA. *Let z be an involution of a group G and suppose $C(z)=\langle z\rangle\times K\times O(C(z))$ with $K\cong L_3(2)$. Furthermore, assume that z is not a central involution, and let $M=E(G)$. Then $M\cong L_3(4)$, $SL_3(4)$, or $L_3(2)\times L_3(2)$. Furthermore, $C(M)=O(G)$, $C_M(z)=K$, $O(C(z))=C_{O(G)}(z)$, and $|G:\langle z\rangle M|$ is odd.*

(1L) LEMMA. *Let $K\cong U_3(3)$ and Q a D_8-subgroup of K. Then the following holds:*

(1) *each elementary abelian 2-subgroup of K is conjugate to a subgroup of Q;*

(2) *each fours subgroup of K is normal in some S_2-subgroup of K.*

PROOF. Let $S\in \mathrm{Syl}_2(K)$, so that $S\cong Z_4$ wreath Z_2. Let M be the subgroup of S isomorphic to $Z_4\times Z_4$, z the involution of $Z(S)$, and $N=O_2(C_K(z))$. Then $N\cong Z_4 * D_8$, $Z(N)<M\cap N<N$, and there is an element $x\in C_K(z)$ such that $(M\cap N)^x\neq M\cap N$. As $M\cap N=Z(N)\Omega_1(M)$, $\Omega_1(M)^x\not\leq M$ and so we may choose an involution $t\in\Omega_1(M)^x-M$. Let $\{a, b\}$ be a basis of M such that $a^t=b$. By an easy computation, we have that each fours subgroup of S is conjugate to $\langle a^2, b^2\rangle$ or $\langle t, a^2b^2\rangle$. As $\langle a^2, b^2\rangle=\Omega_1(M)$ and $\langle t, a^2b^2\rangle=\Omega_1(M)^x$, the fours subgroups of K are all conjugate to $\Omega_1(M)$. This proves (1) and (2).

2. Properties of $Sp_6(2)$.

In this section, we fix notation for $L=Sp_6(2)$ following (2.1) of [10] and record some facts about L. We identify L with the group of matrices, X, with entries in $GF(2)$ satisfying

$$ {}^tX\begin{pmatrix} &&&&&1\\ &&&&1&\\ &&&1&&\\ &&1&&&\\ &1&&&&\\ 1&&&&& \end{pmatrix}X=\begin{pmatrix} &&&&&1\\ &&&&1&\\ &&&1&&\\ &&1&&&\\ &1&&&&\\ 1&&&&& \end{pmatrix}, $$

where ${}^tX$ is the transposed matrix of X and blank spaces denote zeros. Let U be the subgroup generated by the lower triangular matrices in L and set

$$ r=\begin{pmatrix} &1&&&&\\ 1&&&&&\\ &&1&&&\\ &&&1&&\\ &&&&&1\\ &&&&1& \end{pmatrix},\quad s=\begin{pmatrix} 1&&&&&\\ &&1&&&\\ &1&&&&\\ &&&&1&\\ &&&1&&\\ &&&&&1 \end{pmatrix},\quad t=\begin{pmatrix} 1&&&&&\\ &1&&&&\\ &&&1&&\\ &&1&&&\\ &&&&1&\\ &&&&&1 \end{pmatrix}. $$

Thus $U\in \mathrm{Syl}_2(L)$ and $(U, \langle r, s, t\rangle)$ is a BN-pair of L. We define

$$ P_1=\langle U, s, t\rangle,\quad P_2=\langle U, r, s\rangle, $$

$$A_1=O_2(P_1),\quad A_2=O_2(P_2),$$

$$K_1=P_1\cap P_1^{rstsr},\quad K_2=P_2\cap P_2^{tstrst},$$

$$U_1=U\cap U^{rstsr},\quad U_2=U\cap U^{tstrst}.$$

Furthermore, we define

$$a_1=\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & 1 & 1 \end{pmatrix},\quad a_2=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & 1 & 1 & & & \\ & & & 1 & & \\ & & & 1 & 1 & \\ & & & & & 1 \end{pmatrix}.$$

$$b_1=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 1 & & & & & 1 \end{pmatrix},\quad b_2=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ 1 & & & & 1 & \\ & 1 & & & & 1 \end{pmatrix},$$

$$b_3=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & 1 & & & 1 & \\ & & & & & 1 \end{pmatrix},\quad b_4=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ 1 & & & 1 & & \\ & & & & 1 & \\ & & 1 & & & 1 \end{pmatrix},$$

$$b_5=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & 1 & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix},\quad b_6=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & 1 & & 1 & & \\ & & 1 & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

Finally, we let $a_0=[a_1, a_2]$, $I_1=\langle b_1\rangle$, $I_2=\langle b_2, b_4, b_6\rangle$, and $H=\langle b_5b_5^t\rangle$.

Now we list those properties of L which we shall frequently use in later sections. We shall omit their proof whenever we feel them to be well known or they may be checked by easy computations involving matrices.

(2A) LEMMA. *The following conditions hold:*
(1) $N_L(A_i)=P_i$ *for each* i;
(2) $P_i=A_iK_i$ *and* $A_i\cap K_i=1$ *for each* i;
(3) $K_1\cong Sp_4(2)$ *and* $U_1=\langle a_2, b_5, b_6\rangle\in \mathrm{Syl}_2(K_1)$;
(4) $A_1\cong E_{32}$ *and* $A_1=\langle a_0, a_1, b_1, b_2, b_4\rangle$;
(5) $K_2\cong L_3(2)$ *and* $U_2=\langle a_1, a_2\rangle\in \mathrm{Syl}_2(K_2)$;
(6) $A_2\cong E_{64}$ *and* $A_2=\langle b_i;\ 1\leqq i\leqq 6\rangle$.

The following two tables show the action of elements of K_i on A_i.

Table 1.

x	x^{a_0}	x^{a_1}	x^{a_2}	x^r	x^s
b_1	b_1	b_1	b_1	b_3	b_1
b_2	b_2	b_2	b_2	b_2	b_4
b_3	b_3	$b_1b_2b_3$	b_3	b_1	b_5
b_4	b_4	b_4	b_2b_4	b_6	b_2
b_5	$b_1b_4b_5$	b_5	$b_3b_5b_6$	b_5	b_3
b_6	b_2b_6	b_4b_6	b_6	b_4	b_6

Table 2.

x	x^{a_2}	x^{b_3}	x^{b_5}	x^{b_6}	x^s	x^t
a_0	a_0	a_0	$a_0b_1b_4$	a_0b_2	a_1	b_4
a_1	a_0a_1	$a_1b_1b_2$	a_1	a_1b_4	a_0	a_1
b_1	b_1	b_1	b_1	b_1	b_1	b_1
b_2	b_2	b_2	b_2	b_2	b_4	b_2
b_4	b_2b_4	b_4	b_4	b_4	b_2	a_0

(2B) LEMMA. *The following conditions hold:*

(1) $Z(U)=\langle b_1, b_2\rangle$;

(2) A_2 *is the only* E_{64}*-subgroup of* U;

(3) *if* X *is an* E_{32}*-subgroup of* A_1A_2 *and* $XA_2=A_1A_2$, *then* $X=A_1$;

(4) A_i *is self-centralizing in* L *for each* i.

PROOF. (1), (2), and (4) easily follow from (2A) and Table 1. In order to prove (3), assume by way of contradiction that $X\neq A_1$. Choose an element $x\in X-A_1$. Since $XA_2=A_1A_2$, it follows that $X\cap A_2\leqq C_{A_2}(A_1)=A_1\cap A_2$. Hence $x\notin A_2$. Now $\langle a_2, s\rangle$ is contained in $N_L(A_1)\cap N_L(A_2)$ and Table 2 shows that it acts transitively on $(A_1A_2/A_2)^{\#}$. Hence we may assume $x\in a_0A_2$. Then $x\in a_0C_{A_2}(a_0)=a_0\langle b_3\rangle(A_1\cap A_2)$, and as $x\notin A_1$, $x\in a_0b_3(A_1\cap A_2)$. There exists an element $y\in X\cap a_1A_2$. Write $y=a_1b$ with $b\in A_2$. Then by Table 2,

$$a_1b=y=y^x=(a_1b)^{a_0b_3}=(a_1b^{a_0})^{b_3}=a_1b_1b_2b^{a_0}$$

and hence $b_1b_2=bb^{-a_0}\in[A_2, a_0]$. However, Table 1 shows $[A_2, a_0]=\langle b_1b_4, b_2\rangle$.

(2C) LEMMA. *L has four conjugacy classes of involutions and we may choose* b_1, b_2, b_1b_2, *and* b_3b_4 *as their representatives.*

PROOF. See Section 7 of [3].

(2D) LEMMA. *The following conditions hold:*

(1) $I_1 \lhd P_1$ *and* A_1/I_1 *is a natural module for* $K_1 \cong Sp_4(2)$;

(2) *involutions of* A_1 *are conjugate in* P_1 *to* b_1, b_2, *or* b_1b_2;

(3) I_1 *is the only* K_1*-invariant nontrivial proper subgroup of* A_1.

PROOF. As $K_1=\langle a_2, b_5, s, t\rangle$, Table 2 shows $I_1 \lhd P_1$. Let bars denote images in A_1/I_1. For two elements

$$x=\bar{a}_1^{\alpha_1}\bar{a}_0^{\beta_1}\bar{b}_4^{\gamma_1}\bar{b}_2^{\delta_1} \quad \text{and} \quad y=\bar{a}_1^{\alpha_2}\bar{a}_0^{\beta_2}\bar{b}_4^{\gamma_2}\bar{b}_2^{\delta_2},$$

where α_i, β_i, γ_i, δ_i, $i\in\{1, 2\}$, are integers read modulo 2, define $(x, y)=\alpha_1\delta_2+\beta_1\gamma_2+\gamma_1\beta_2+\delta_1\alpha_2$. This is a nonsingular symplectic form on $\bar{A}_1$ and K_1 preserves it. Hence (1) follows. Consequently, we have that K_1 acts transitively on $\bar{A}_1^{\#}$. Hence (2) follows.

Assume that X is a K_1-invariant nontrivial subgroup of A_1. If $X\neq I_1$, then $A_1=I_1X$ by (1); so b_2 or b_1b_2 is contained in X. Table 2 shows $b_2\sim a_1\sim a_1b_1b_2$ and $b_1b_2\sim b_1a_0\sim b_1a_0b_2$ under K_1. Hence $\langle b_1, b_2\rangle\leqq X$ and consequently $X=A_1$.

We shall denote by Δ_2 and Δ_{12} the conjugacy classes of b_2 and b_1b_2 in P_1, respectively.

(2E) LEMMA. *The following conditions hold:*

(1) $I_2 \lhd P_2$, A_2/I_2 *is a natural module for* $K_2 \cong L_3(2)$, *and* I_2 *is its dual module*;

(2) *involutions of* A_2 *are conjugate in* P_2 *to* b_1, b_2, b_1b_2, *or* b_3b_4;

(3) I_2 *is the only* K_2*-invariant nontrivial proper subgroup of* A_2.

PROOF. As $K_2=\langle a_1, a_2, r, s\rangle$, Table 1 shows $I_2 \lhd P_2$. As a basis of A_2/I_2 and I_2, choose $\{b_1I_2, b_3I_2, b_5I_2\}$ and $\{b_2, b_4, b_6\}$, respectively. Compute the matrices of a_1, a_2, r and s with respect to these bases. The remaining parts of (1) then follow immediately. Computing the centralizers in P_2 of b_1, b_2, b_1b_2, and b_3b_4, we have that they have 7, 7, 21, and 28 conjugates in P_2, respectively. Hence (2) follows.

Suppose that X is a K_2-invariant nontrivial subgroup of A_2. If $A_2\neq X\neq I_2$, then $A_2=I_2X$ and $I_2\cap X=1$ by (1). In particular, $|X|=8$ and so $b_1\in X$ by the above paragraph. However, Table 1 shows that $b_1\sim b_3\sim b_1b_2b_3$ under K_2, hence $b_2\in X$, a contradiction.

We shall denote by Ω_1, Ω_2, Ω_{12}, and Ω_{34} the conjugacy classes of b_1, b_2, b_1b_2, and b_3b_4 in P_2, respectively.

(2F) LEMMA. *The following conditions hold:*

(1) $C_L(b_1)=P_1$;

(2) *if* $b=b_1$, b_2, *or* b_1b_2, *then* $C_L(b)=O^{2'}(C_L(b))$ *and* $\langle b\rangle$ *has no complement in* $C_L(b)$.

PROOF. By (2D), $P_1 \leqq C_L(b_1)$. As P_1 is maximal parabolic, (1) holds. If $b=b_1$, (2) follows from (1) and (2D) (3). If $b=b_2$ or b_1b_2, then we check by direct computations that $C_L(b)=O^{2'}(C_L(b))$. Indeed, $C_L(b_2)$ is an extension of a 2-group by $L_2(2)\times L_2(2)$, and $C_L(b_1b_2)$ is an extension of a 2-group by $L_2(2)$. From Table 1, we obtain $(a_2b_4)^2=b_2$ and $(a_1b_3)^2=b_1b_2$. Hence the remaining part of (2) follows.

(2G) LEMMA. *We have* $A_1/I_1=C_{A_1/I_1}(H)\times C_{A_1/I_1}(H^s)$.

3. Initial reductions.

In this section we begin the proof of the main theorem of this paper. Let G be a finite group with a standard subgroup L isomorphic to $Sp_6(2)$. Assume that $C(L)$ has a cyclic S_2-subgroup and that $LO(G)$ is not a normal subgroup of G. Let z be an involution of $C(L)$ and set $C=C(z)$. The symbols used in Section 2 for various objects defined for $Sp_6(2)$ will retain their meaning for the balance of the paper. Let $V=U\langle z\rangle$, $B_i=A_i\langle z\rangle$, and $J_i=I_i\langle z\rangle$ for $i\in\{1, 2\}$. Furthermore, let $\Delta=z^{N(B_1)}$ and $\Omega=z^{N(B_2)}$. The main result of this section is Lemma (3F), in which we determine Ω.

(3A) LEMMA. *The following conditions hold*:
(1) $C=L\times C_C(L)$;
(2) $z^G\cap C\neq\{z\}$.

PROOF. By hypothesis $L\lhd C$. As $\mathrm{Out}(L)=1$ by [21], (1) holds. As $LO(G)$ is not a normal subgroup, $z\notin Z^*(G)$ and so the Z^*-theorem [7] yields that $z^G\cap C\neq\{z\}$.

(3B) LEMMA. *We have* $\langle z\rangle\in \mathrm{Syl}_2(C_C(L))$.

PROOF. Let $T\in\mathrm{Syl}_2(C_C(L))$ and choose an element $g\in G$ such that $z\neq z^g\in UT$. This is possible by (3A). Then $T\leqq C(z^g)=C^g$. As z^g induces an inner automorphism on L, $C_L(z^g)=L\cap C^g$ contains an E_{64}-subgroup A by (2C). The image of $A\times T$ in $C^g/C_C(L)^g$ is abelian of rank at least 6 and its exponent is equal to that of T as $T\cap C_C(L)^g=1$. As $C^g/C_C(L)^g\cong L$ by (3A), (2) and (4) of (2B) force $|T|=2$.

(3C) LEMMA. *We have* $z^G\cap A_1=\varnothing$.

PROOF. Suppose that $z^g=b$ for some $b\in A_1$ and $g\in G$. Then $C_L(b)=O^{2'}(C_L(b))$ by (2D) and (2F); so $C_L(b)\leqq O^{2'}(C^g)=L^g\times\langle z^g\rangle=L^g\times\langle b\rangle$. Hence $C_L(b)=C_{L\cap L^g}(b)\times\langle b\rangle$. This, however, is impossible by (2F).

(3D) LEMMA. *The following conditions hold:*
(1) *B_2 is the only E_{128}-subgroup of V;*
(2) $\Omega = z^G \cap B_2$.

PROOF. (1) is a direct consequence of (2B) (2). Let $g \in G$ and $z^g \in B_2$. Then $B_2^{g^{-1}} \leqq C$; so there is an element $c \in C$ such that $B_2^{g^{-1}} = B_2^c$ by (1). Thus $cg \in N(B_2)$ and $z^g = z^{cg} \in z^{N(B_2)}$. This proves (2).

(3E) LEMMA. *The following conditions hold:*
(1) $C(B_i) = B_i O(C)$ *for* $i \in \{1, 2\}$;
(2) $N(B_1 B_2) = N(B_1) \cap N(B_2)$.

PROOF. (1) is a direct consequence of (2B) (4). (3D) shows $N(B_1 B_2) \leqq N(B_2)$. Hence if $g \in N(B_1 B_2)$ and $X = B_1^g$, X is an E_{64}-subgroup of $B_1 B_2$ such that $XB_2 = B_1 B_2$. By (3D) (1), $B_1 \cap B_2 \leqq X$ and so $Y = X \cap L$ is an E_{32}-subgroup of $A_1 A_2$ such that $YA_2 = A_1 A_2$. Thus $Y = A_1$ by (2B) (3) and then $X = Y\langle z \rangle = B_1$, proving (2).

(3F) LEMMA. *We have* $|\Omega| = 8$ *or* 64. *If* $|\Omega| = 8$, *then* $\Omega = \{z\} \cup z\Omega_1$ *or* $\{z\} \cup z\Omega_2$. *If* $|\Omega| = 64$, *then* $\Omega = zA_2$, $\{z\} \cup z\Omega_1 \cup z\Omega_{34} \cup \Omega_{34}$, $\{z\} \cup z\Omega_2 \cup z\Omega_{34} \cup \Omega_{34}$, *or* $\{z\} \cup z\Omega_1 \cup z\Omega_2 \cup z\Omega_{12} \cup \Omega_{34}$. *If X is a subgroup such that* $N_C(B_2) < X < N(B_2)$, *then* $z^X = \{z\} \cup z\Omega_i$, $i = 1$ *or* 2.

PROOF. First of all, (3A) (2), (2C), and (3D) (2) show that $\Omega \neq \{z\}$. Hence Ω is a union of two or more conjugacy classes of $N_C(B_2)$ contained in $B_2^\#$; that is, $\{z\}$, $z\Omega_1$, $z\Omega_2$, $z\Omega_{12}$, $z\Omega_{34}$, Ω_1, Ω_2, Ω_{12}, and Ω_{34}. However, Ω_1, Ω_2, and Ω_{12} are not contained in Ω by (3C). Furthermore, $C(B_2) \leqq N_C(B_2)$ and $N(B_2)/C(B_2)$ is isomorphic to a subgroup of $\mathrm{Aut}(B_2)$; so $|\Omega|$ divides $|GL_7(2)| = 2^{21} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31 \times 127$. Hence we have that Ω is one of the sets shown in the lemma or $|\Omega| = 15$ or else $|\Omega| = 36$. The same holds for z^X when X is a subgroup such that $N_C(B_2) < X \leqq N(B_2)$.

Suppose that $|\Omega| = 15$ or 36. Then $N(B_2)^\Omega$ is a primitive permutation group and the stabilizer $N_C(B_2)^\Omega$ of z in $N(B_2)^\Omega$ is isomorphic to the simple group $L_3(2)$. Hence $N(B_2)^\Omega$ is a simple group and has order $2^3 \cdot 3^2 \cdot 5 \cdot 7$ or $2^5 \cdot 3^3 \cdot 7$. Inspecting the list of simple groups of such order given by [13] and [22], we see that $N(B_2)^\Omega \cong U_3(3)$ with $|\Omega| = 36$ is the only possibility.

Now $C(\Omega) = C(B_2)$, so $N(B_2)/C(B_2) \cong U_3(3)$. Let bars denote images in $N(B_2)/C(B_2)$. Then $\bar{B}_1$ is a fours subgroup of $\overline{N(B_2)}$ and so normal in some S_2-subgroup of $\overline{N(B_2)}$ by (1L). As $N(B_1 C(B_2)) \leqq N(B_1 B_2) \leqq N(B_1)$ by (3E), $|\varDelta| = |N(B_1) : N_C(B_1)|$ is divisible by 4. Moreover, (3C) shows that $\varDelta$ is a union of conjugacy classes of $N_C(B_1)$ contained in zA_1; that is, $\{z\}$, $\{zb_1\}$, $z\varDelta_2$, and $z\varDelta_{12}$. Hence $|\varDelta| = 16$ or 32 and so $|N(B_1)|_2 = 2^{14}$ or 2^{15}.

Now $\tilde{V}$ is a D_8-subgroup of $\overline{N(B_2)}$. Hence if X is an elementary abelian 2-subgroup of $N(B_2)$, then $X^g \leqq V$ for some $g \in N(B_2)$ by (1L) (1). Then (3D) (1) shows that B_2 is the only E_{128}-subgroup of $N(B_2)$. Thus $|G:N(B_2)|$ is odd and $|G|_2=2^{12}$. This, however, contradicts $|N(B_1)|_2 \geqq 2^{14}$.

It remains to prove the last statement of the lemma. It is, however, immediate from the last remark in the first paragraph, as we have already shown that $|z^X|$ is a power of 2.

It is not difficult to see that $|\Omega|=64$ implies $\Omega=zA_2$ (see Section 6). This leads to the following trichotomy:

1. $\Omega=\{z\} \cup z\Omega_1$;
2. $\Omega=\{z\} \cup z\Omega_2$;
3. $|\Omega|=64$.

We shall treat each of these cases in different sections.

4. $O_8^-(2)$ and $O_8^+(2)$.

In this section, we study the following situation:

HYPOTHESIS 1. $\Omega=\{z\} \cup z\Omega_1$.

We shall prove the following:

THEOREM 1. *Under Hypothesis* 1, $\langle L^G \rangle \cong O_8^-(2)$ *or* $O_8^+(2)$.

The proof involves a series of lemmas. We begin by studying the structure of $N(B_i)$. Let $D_i=O_2(N(B_i))$ and $V_i=VD_i$ for $i \in \{1, 2\}$.

(4A) LEMMA. *The following conditions hold*:
(1) $N(B_2)=N_C(B_2)D_2$ *and* $N_C(B_2) \cap D_2=B_2$;
(2) *commutation by* z *induces an isomorphism* $D_2/B_2 \rightarrow A_2/I_2$;
(3) $Z(D_2)=I_2$ *and* $Z_2(D_2)=A_2$.

PROOF. By Hypothesis 1, $N(B_2)/C(B_2)$ is a 2-transitive permutation group on Ω. The stabilizer $N_C(B_2)/C(B_2)$ of the point z is isomorphic to the simple group $L_3(2)$. As $N(B_2)/C(B_2)$ can not be simple by [22], $N(B_2)/C(B_2)$ has the regular normal subgroup $X/C(B_2)$. Set $Y=C_{N(B_2)}(O(C))O(C)$. Then $N_C(B_2) \leqq Y$ and, as $C(B_2)=B_2 \times O(C)$ by (3E), $Y \lhd N(B_2)$. Hence $Y=N(B_2)$ and $X=C_X(O(C))O(C)$. Thus X is 2-closed and (1) follows.

Now $A_2=\langle ab; a, b \in \Omega \rangle$ by Hypothesis 1. Hence $A_2 \lhd D_2$ and $Z(D_2) \cap A_2 \neq 1$. Moreover, we have $A_2 \not\leqq Z(D_2)$. Otherwise, commutation by z would induce an isomorphism $D_2/B_2 \rightarrow [D_2, z]$, while $[D_2, z]=A_2$ by Hypothesis 1, a contradiction.

Thus $Z(D_2)\cap A_2=I_2$ by (2E) (3) and, in particular, $I_2\lhd N(B_2)$. As $A_2/I_2\lhd D_2/I_2$ and as K_2 acts irreducibly on A_2/I_2 by (2E), we have $A_2/I_2\leqq Z(D_2/I_2)$. Furthermore,

$$C_{D_2/I_2}(z)=B_2/I_2 \tag{4.1}$$

by Hypothesis 1. Hence (2) and (3) follow.

(4B) LEMMA. *$N(B_2)$ has a normal subgroup C_2 satisfying the following conditions:*

(1) $D_2=B_2C_2$ *and* $B_2\cap C_2=A_2$;

(2) $Z(C_2)=C_2^2=I_2$.

PROOF. By (4A) (3), I_2 and A_2 are normal in $N(B_2)$. Let $Y\in \mathrm{Syl}_7(K_2)$ and set $C_2=[D_2, Y]$. Then C_2 satisfies the condition (1) above, as Y acts irreducibly on D_2/B_2 by (4A) (2). Furthermore, (1F) and (4.1) show that C_2/I_2 is abelian. This forces $C_2\lhd N(B_2)$, as $|D_2/I_2: Z(D_2/I_2)|>4$ by (4A) (3). The same lemma shows $A_2\not\leqq Z(C_2)$. Hence $Z(C_2)\leqq A_2$, as otherwise the irreducible action of Y on C_2/A_2 yields that $C_2=A_2Z(C_2)$, which is a contradiction. Therefore, $Z(C_2)=I_2$ by (2E). Then $C_2^2\leqq I_2$ by (1B) and so $C_2^2=I_2$ again by (2E).

(4C) LEMMA. *The following conditions hold:*

(1) $N(B_1)/B_1=N_C(B_1)/B_1\times D_1/B_1$;

(2) $D_1/B_1\cong Z_2$;

(3) $Z(D_1)=A_1$ *and* $D_1^2=I_1$.

PROOF. Since $N_{D_2}(B_1B_2)\leqq N(B_1)$ by (3E), it follows that $N(B_1)\not\leqq C$. Hence $\Delta\neq\{z\}$. On the other hand, Hypothesis 1, (2C), and (3D) (2) show that $\Delta=\{z\}$ or $\{z, zb_1\}$. Thus $\Delta=\{z, zb_1\}$. Hence $|N(B_1): N_C(B_1)|=2$, and both I_1 and J_1 are normal in $N(B_1)$. Let bars denote images in $N(B_1)/C(B_1)$. Then $\bar{K}_1=\overline{N_C(B_1)}$ is a subgroup of index 2 isomorphic to $Sp_4(2)$. If $C_{\overline{N(B_1)}}(\bar{K}_1)=1$, then $\overline{N(B_1)}\cong \mathrm{Aut}(Sp_4(2))\cong P\Gamma L_2(9)$. But then $C(B_1/J_1)=C(B_1)$ and $\overline{N(B_1)}$ is isomorphic to a subgroup of $\mathrm{Aut}(B_1/J_1)\cong GL_4(2)$, which is a contradiction. Therefore, $\overline{N(B_1)}=\overline{N_C(B_1)}\times\bar{X}$ with $\bar{X}\cong Z_2$. Now $O(C)$ centralizes $N_{D_2}(B_1B_2)$ by (4A) (2) and $N(B_1)=N_{D_2}(B_1B_2)N_C(B_1)$, hence $N(B_1)=C_{N(B_1)}(O(C))O(C)$. (1) and (2) now follow as in (4A). (3) is a direct consequence of (1H).

In order to prove an analogue of (4B) for $N(B_1)$, we require the following lemma:

(4D) LEMMA. *We have* $V_1\leqq V_2$.

PROOF. As $|V_1: V|=2$, $V_1\leqq N(V)\leqq N(B_2)$ by (3D) (1). Thus V_1D_2 is a 2-subgroup of $N(B_2)$ containing $V_2=VD_2$. As $V_2\in \mathrm{Syl}_2(N(B_2))$ by (4A), we must have $V_1D_2=V_2$ and hence $V_1\leqq V_2$.

Now let $S=UC_2$ and $C_1=D_1\cap S$. We prove the following:

(4E) LEMMA. *C_1 is a normal subgroup of $N(B_1)$ and the following conditions hold:*

(1) *$D_1=B_1C_1$ and $B_1\cap C_1=A$;*

(2) *C_1 is elementary abelian;*

(3) $V_2\leqq N(C_1)$.

PROOF. As $V_2=\langle z\rangle S$ and $\langle z\rangle\leqq D_1\leqq V_2$ by (4D), C_1 is a maximal subgroup of D_1 such that $D_1=\langle z\rangle C_1$. Hence (1) holds. Now $S/C_2\cong D_8$ and A_1C_2/C_2 is a fours subgroup of S/C_2. Since $A_1C_2/C_2\leqq C_1C_2/C_2\leqq S/C_2$ and since C_1C_2/C_2 is elementary abelian by (4C) (3), it follows that $A_1C_2=C_1C_2$. Consequently, $C_1=A_1(C_1\cap C_2)$ and so $C_1^2=(C_1\cap C_2)^2\leqq I_1\cap I_2=1$ by (4B) and (4C). As $\mathcal{E}^*(D_1)=\{B_1, C_1\}$, we have $C_1\lhd N(B_1)$.

It remains to prove (3). Since $C_2/A_2\cong A_2/I_2$ as A_1-modules by (4A), it follows that $C_1C_2/A_2=A_1C_2/A_2\cong D_8*D_8$. As $C_1\cap A_2=A_1\cap A_2$, C_1A_2/A_2 is an E_8-subgroup of C_1C_2/A_2 and so $C_1A_2\lhd C_1C_2$. Hence if $g\in C_2$ and $X=C_1^g$, then $X\in \mathcal{E}_{64}(C_1A_2)$ and moreover $|X\cap A_2|=8$. By (2B) (2) $X\not\leqq A_1A_2$, so as $|C_1A_2: A_1A_2|=2$, $Y=X\cap A_1A_2$ is an E_{32}-subgroup of A_1A_2 and $|Y\cap A_2|=8$. Then (2B) (3) forces $Y=A_1$. Therefore, $X\leqq C_{C_1A_2}(A_1)=C_1C_{A_2}(A_1)=C_1(A_1\cap A_2)=C_1$. This shows that $C_2\leqq N(C_1)$. Hence $V_2=VC_2\leqq N(C_1)$.

(4F) LEMMA. *B_2 is the only E_{128}-subgroup of V_2.*

PROOF. We first show $m(C_2)=6$. Let X be an elementary abelian subgroup of C_2 of maximal rank and assume, by way of contradiction, that $|X|\geqq 2^7$. Then $I_2<X\cap A_2$ by (4B) (2) and, as K_2 acts transitively on $(A_2/I_2)^{\#}$, we may assume $b_1\in X\cap A_2$. Furthermore, $4\leqq|XA_2: A_2|$, as otherwise $A_2\leqq X$ and then the irreducible action of K_2 on C_2/A_2 yields that $A_2\leqq Z(C_2)$, contrary to (4B) (2). Since $|N_C(B_2): C(b_1)\cap N_C(B_2)|=7$ and since $b_1\notin Z(C_2)$, we conclude that $|b_1^{N(B_2)}|=14$. However, as A_2 and I_2 are normal in $N(B_2)$ by (4A), $b_1^{N(B_2)}$ is a union of conjugacy classes of $N_C(B_2)$ contained in A_2-I_2 (that is, Ω_1, Ω_{12}, and Ω_{34}) and so $|b_1^{N(B_2)}|\neq 14$. This contradiction shows $m(C_2)=6$.

Now let X be an elementary abelian subgroup of V_2 of maximal rank and assume, by way of contradiction, that $X\neq B_2$. It follows from (4.1), (1E) (1), and (4B) (2) that $\mathcal{E}^*(D_2/I_2)=\{B_2/I_2, C_2/I_2\}$. Hence $X\not\leqq D_2$ as $X\not\leqq C_2$ by the last paragraph. Also, if we set $Y=X\cap D_2$, then $Y\leqq C_{B_2}(X)$ or $C_{C_2}(X)$. Conjugating in $N(B_2)$, we may assume that $\langle a_0\rangle D_2\leqq XD_2$. Then (4A) and (4B) show

$$C_{I_2}(X) \leqq C_{I_2}(a_0) = \langle b_2, b_4 \rangle,$$

$$C_{A_2/I_2}(X) \leqq C_{A_2/I_2}(a_0) = \langle b_1, b_3 \rangle I_2/I_2,$$

$$C_{C_2/A_2}(X) \leqq C_{C_2/A_2}(a_0) \cong C_{A_2/I_2}(a_0).$$

As a consequence, we have $|C_{B_2}(X)| \leqq 2^5$ and $|C_{C_2}(X)| \leqq 2^6$. Hence if $|Y| \geqq 2^6$, then $Y \leqq C_2$ and so $I_2 \leqq Y \leqq C_{C_2}(X)$ by the last paragraph, a contradiction. Therefore, $XD_2 = \langle a_0, a_1 \rangle D_2$ or $\langle a_0, a_2 \rangle D_2$. In either case $|Y| \geqq 2^5$ and consequently $|Y \cap I_2| \geqq 2^2$ by the last paragraph. However, in the former case we have $C_{C_2/A_2}(X) \cong C_{A_2/I_2}(X) = \langle b_1 \rangle I_2/I_2$, so $|C_{B_2}(X)| \leqq 2^4$ and $|C_{C_2}(X)| \leqq 2^4$. In the latter case, we have $C_{I_2}(X) = \langle b_2 \rangle$. Hence we have a contradiction in either case.

Using (4F), we next prove the following:

(4G) LEMMA. *The following conditions hold*:
(1) $V_2 \in \mathrm{Syl}_2(G)$;
(2) $S \in \mathrm{Syl}_2(G')$.

PROOF. As $V_2 \in \mathrm{Syl}_2(N(B_2))$, (1) is a direct consequence of (4F). Now $U \leqq L \leqq G'$ and $C_2 = [K_2, C_2] \leqq G'$; so $S = UC_2 \leqq G'$. We argue that Ω is the set of extremal conjugates of z in V_2. If u is an extremal conjugate of z in V_2, then there is an element $g \in G$ such that $z^g = u$ and $V^g = C_{V_2}(u)$. In particular, $B_2^g \leqq V_2$ and so $B_2^g = B_2$ by (4F). Thus $u \in z^{N(B_2)} = \Omega$. Since $S \cap \Omega = \emptyset$, we have $z \notin G'$ by (1D). As $V_2 = S\langle z \rangle$, (2) holds.

Next, we consider the structure of $N(C_1)$. We let $M_1 = E(N(C_1) \bmod C_1)$ and prove the following:

(4H) LEMMA. *The following conditions hold*:
(1) $M_1/C_1 \cong U_4(2)$ *or* $L_4(2)$;
(2) $C_{M_1/C_1}(z) = K_1C_1/C_1$;
(3) $S \in \mathrm{Syl}_2(M_1)$;
(4) $[M_1, O(C)] = 1$.

PROOF. By (4E), $\mathcal{E}^*(D_1) = \{B_1, C_1\}$ and so $N(D_1) \cap N(C_1) = N(B_1)$. Hence if bars denote images in $N(C_1)/C_1$, then

$$C(\bar{z}) = \overline{N(B_1)} = \langle \bar{z} \rangle \times \bar{K}_1 \times \overline{O(C)}.$$

Furthermore, $\bar{V}_2 \in \mathrm{Syl}_2(\overline{N(C_1)})$ by (4E) and (4G), and consequently, $\bar{z}$ is not a central involution. As $|\bar{V}_2| = 2^7$, (1I) yields that $\bar{M}_1 \cong U_4(2)$ or $L_4(2)$ and that $\langle \bar{z} \rangle \bar{K}_1 = \langle \bar{z} \rangle \times C_{\bar{M}_1}(\bar{z})$.

Now $C_1 \leqq S \leqq G'$ by (4G) (2), and hence $M_1 = C_1 M_1' \leqq G'$. Hence $M_1 \lhd N(C_1) \cap G'$. Since $S \in \mathrm{Syl}_2(N(C_1) \cap G')$ and $|S| = |M_1|_2$, it follows that $S \in \mathrm{Syl}_2(M_1)$. In particular, $A_2 \leqq M_1$ and so, as $\bar{K}_1 = \bar{K}_1' \bar{A}_2$ and as $\bar{K}_1' \leqq \bar{M}_1$ by the above paragraph, $\bar{K}_1 \leqq \bar{M}_1$. Therefore, $C_{\bar{M}_1}(\bar{z}) = \bar{K}_1$.

It remains to prove (4). By (1I), $\overline{O(C)} \leqq O(\overline{N(C_1)})$ and hence $[\bar{M}_1, \overline{O(C)}] = 1$. Furthermore, as $O(C)$ stabilizes the series $1 < A_1 < C_1$, $[C_1, O(C)] = 1$. Therefore, $[M_1, O(C)] = 1$.

We are now in a position to complete the proof of Theorem 1. Let $M_2 = K_2 C_2$. Then the lemmas (4A), (4B), (4C), (4E), and (4H) show that z, L, C_i, M_i, $i \in \{1, 2\}$, and S satisfy Hypothesis (2.2) of [10] with $M_1/C_1 \cong U_4(2)$ or $L_4(2)$, $M_2/C_2 \cong L_3(2)$, and $|S| = 2^{12}$. Furthermore, (4A) (3) shows that S centralizes $(U_2 \cap U_2^{rs})^{sts} = \langle b_2 \rangle$. We may, therefore, apply Theorems 1 and 5 of [10] to conclude that $G_0 = \langle M_1, M_2 \rangle$ is a z-invariant quasisimple subgroup with $L \leqq G_0$ and $G_0/Z(G_0) \cong O_8^{\pm}(2)$. Furthermore, $C(G_0)$ has odd order by (3B) and, in particular, $Z(G_0)$ has odd order. Hence $Z(G_0) = 1$ as the Schur multiplier of $O_8^+(2)$ is of type $(2, 2)$ and the Schur multiplier of $O_8^-(2)$ is trivial: see [6], a table on p. 60.

Now we prove the following:

(4I) LEMMA. *If $g \in G$ and $z^g \in N(G_0)$, then $g \in N(G_0)$.*

PROOF. By (4H) (4), $O(C)$ centralizes $G_0 = \langle M_1, M_2 \rangle$. Since $C = \langle z \rangle L O(C)$, it follows that $C \leqq N(G_0)$. Now suppose $g \in G$ and $z^g \in N(G_0)$. Then $z^g \notin G_0$ as $G_0 \leqq G'$ while $z \notin G'$ by (4G). Results in Section 8 of [3] show that $\mathrm{Aut}(G_0)$ has precisely two conjugacy classes of involutions outside $\mathrm{Inn}(G_0)$. Since $z^G \cap B_2 \neq z A_2$ by (3D) and Hypothesis 1, it follows that $z^{gh} = z$ for some $h \in N(G_0)$. Thus $g \in CN(G_0) = N(G_0)$.

Now since (4I) has been proved, results in [19, III] show that $G_0 O(G) \lhd G$. Hence $\langle L^G \rangle O(G) = G_0 O(G)$ and, since $[\langle L^G \rangle, O(G)] = 1$ by (1H) of [9], it follows that $\langle L^G \rangle = G_0$. Thus we have proved Theorem 1.

5. $U_6(2)$ and $L_6(2)$.

In this section, we consider the following situation:

HYPOTHESIS 2. $\Omega = \{z\} \cup z\Omega_2$.

We shall prove the following:

THEOREM 2. *Under Hypothesis 2, $\langle L^G \rangle \cong U_6(2)$, $SU_6(2)$, or $L_6(2)$.*

As in the case of Theorem 1, the proof begins with an analysis of the structure of $N(B_i)$. Let $D_i=O_2(N(B_i))$ and $V_i=VD_i$ for $i\in\{1, 2\}$.

(5A) LEMMA. *The following conditions hold:*
(1) $N(B_2)=N_C(B_2)D_2$ *and* $N_C(B_2)\cap D_2=B_2$;
(2) *commutation by* z *induces an isomorphism* $D_2/B_2\rightarrow I_2$;
(3) $I_2\leqq Z(D_2)$.

PROOF. (1) follows as in (4A), the first paragraph of the proof. Hypothesis 2 shows $I_2=\langle\Omega\rangle-\Omega$. Hence $I_2\triangleleft N(B_2)$ and consequently $I_2\cap Z(D_2)\neq 1$. Thus $I_2\leqq Z(D_2)$ by (2E) (3) and, as $[D_2, z]=I_2$ by Hypothesis 2, commutation by z induces an isomorphism $D_2/B_2\rightarrow I_2$.

(5B) LEMMA. *The following conditions hold:*
(1) $N(B_1)=N_C(B_1)D_1$ *and* $N_C(B_1)\cap D_1=B_1$;
(2) *commutation by* z *induces an isomorphism* $D_1/B_1\rightarrow A_1/I_1$;
(3) $Z(D_1)=I_1$ *and* $Z_2(D_1)=A_1$.

PROOF. Arguing as in the first part of the proof of (4C), we have that

$$\Delta=\{z\}\cup z\Delta_2. \tag{5.1}$$

Let bars denote images in $N(B_1)/C(B_1)$. Then $\overline{N(B_1)}$ is a 2-transitive permutation group on Δ, and the stabilizer $\overline{N_C(B_1)}$ of the point z is isomorphic to Σ_6. If $\overline{N(B_1)}$ has no non-trivial solvable normal subgroups, then $F^*(\overline{N(B_1)})$ is a simple group of order 2^a3^b5 with $a\leqq 8$ and $b\leqq 2$. By [4], $F^*(\overline{N(B_1)})\cong A_5$ or A_6. But then $\overline{N(B_1)}$ is isomorphic to a subgroup of $\mathrm{Aut}(A_5)\cong\Sigma_5$ or $\mathrm{Aut}(A_6)\cong P\Gamma L_2(9)$, which is impossible as $|\overline{N(B_1)}|_2=2^8$. Thus $\overline{N(B_1)}$ has the regular normal subgroup. (1) now follows by the argument in the first paragraph of the proof of (4A).

Now $A_1=\langle ab; a, b\in\Delta\rangle$ by (5.1), and hence $A_1\triangleleft D_1$. Therefore, (2D) (3) shows that $Z(D_1)\cap A_1=I_1$ or A_1 and that $A_1/I_1\leqq Z(D_1/I_1)$. Furthermore, (5.1) shows

$$N_{D_1/I_1}(z)=B_1/I_1. \tag{5.2}$$

Therefore, $Z(D_1/I_1)=A_1/I_1$ and, as $[D_1, z]=A_1$ by (5.1), commutation by z induces an isomorphism $D_1/B_1\rightarrow A_1/I_1$. The above discussion shows that $Z(D_1)=I_1$ or A_1 and that if $Z(D_1)=I_1$, then $Z_2(D_1)=A_1$. Suppose that $Z(D_1)=A_1$. Then commutation by z induces an isomorphism $D_1/B_1\rightarrow[D_1, z]=A_1$. Since this is impossible, it follows that $Z(D_1)\neq A_1$. The proof is complete.

(5C) LEMMA. $N(B_1)$ *has a normal subgroup* C_1 *satisfying the following* *conditions*:

(1) $D_1=B_1C_1$ *and* $B_1\cap C_1=A_1$;

(2) C_1 *is an extra-special group with* $Z(C_1)=I_1$.

PROOF. (5B) and (5.2) show that the chain $A_1/I_1\leqq B_1/I_1\leqq B_1/I_1\leqq D_1/I_1$ satisfies the hypothesis of (1G). Therefore, $D_1^2\leqq A_1$. We define $C_1=[D_1, HH^s]A_1$ (see Section 2 for the definition of H and s). Since HH^s acts fixed-point-free on A_1/I_1 by (2G) and since $D_1/B_1\cong A_1/I_1$ as HH^s-modules by (5B) (2), it follows that C_1 is a HH^s-invariant subgroup satisfying (1). Furthermore, $C_1/A_1\cong A_1/I_1$ as HH^s-modules.

For any subgroup X of G, set $X^*=X\cap C(H)$. Then $|C_1^*/I_1|=16$ by (2G), and $C_{C_1^*/I_1}(z)=A_1^*/I_1$ by (5.2). Moreover, H^s acts on C_1^* as $[H, H^s]=1$. Thus C_1^*/I_1 is abelian by (1F). Furthermore, $C_1^*\cap C_1^{*s}=I_1$ by (2G); so $C_1=C_1^*C_1^{*s}$ and hence $C_1/I_1=C_1^*/I_1\times C_1^{*s}/I_1$ by (1A). Thus C_1/I_1 is an abelian maximal subgroup of D_1/I_1 and, as $|D_1/I_1: Z(D_1/I_1)|>4$ by (5B) (3), it follows that $C_1\lhd N(B_1)$. Consequently, $C_1/A_1\cong A_1/I_1$ as $N_C(B_1)$-modules by (5B) (2). In particular, K_1 acts irreducibly on C_1/A_1, and so arguing as in (4B), we have $C_1'=Z(C_1)=I_1$. Therefore, C_1 is an extra-special group by (1B).

(5D) LEMMA. *$N(B_2)$ has a normal subgroup C_2 satisfying the following conditions*:

(1) $D_2=B_2C_2$ *and* $B_2\cap C_2=A_2$;

(2) C_2 *is elementary abelian.*

PROOF. By Hypothesis 2, $J_2=\langle\Omega\rangle\lhd N(B_2)$. K_2 acts transitively both on $(B_2/J_2)^\#$ and on $(D_2/B_2)^\#$ by (5A) (2). Hence $B_2/J_2\leqq Z(D_2/J_2)$, and D_2/J_2 is elementary abelian provided that $I(D_2/J_2)\not\leqq B_2/J_2$.

We show $I(D_2)\not\leqq B_2$. Assume that this is false and let X be an elementary abelian subgroup of V_2 of maximal rank. If $X\neq B_2$, then $X\not\leqq D_2$, and as D_2 stabilizes the series $1<I_2<J_2<B_2$ by the last paragraph, the argument of the proof of (4F) shows that $|C_{B_2}(X)|\leqq 2^5$ and that if $|XD_2: D_2|=4$, then $|C_{B_2}(X)|\leqq 2^4$. As $|X|\geqq 2^7$, this yields a contradiction. Hence B_2 is the only E_{27}-subgroup of V_2. However, this implies that $V_2\in \mathrm{Syl}_2(G)$, contrary to $|V_2|<|V_1|$. Therefore, $I(D_2)\not\leqq B_2$.

The above discussions show that D_2/J_2 is elementary abelian. Hence

$$D_2^2\leqq I_2 \tag{5.3}$$

by (1G) applied to the chain $I_2<J_2<B_2<D_2$. This, in particular, shows $A_2\lhd D_2$. We show that A_2 is in fact the center of D_2, using an argument in the first paragraph of the proof of (4F). (5A) (2) and (3E) (2) imply that

$$|N_{D_2}(B_1): B_2|=4. \tag{5.4}$$

As $b_1 \in Z(N(B_1))$ by (5B), it follows that $|b_1^{N(B_2)}| \leqq 14$. As in (4F), this is possible only if $b_1^{N(B_2)} = \Omega_1$, or if $|N(B_2): C(b_1) \cap N(B_2)| = 7$. Hence $b_1 \in Z(D_2)$ and (2E) (3) yields that $A_2 \leqq Z(D_2)$. As $C_{D_2}(z) = B_2$, we have $Z(D_2) = A_2$.

Now let X be an S_7-subgroup of K_2 and let Y be an X-invariant subgroup of D_2 such that $D_2 = B_2 Y$ and $B_2 \cap Y = I_2$. Such Y exists by Maschke's theorem. By (1F), Y is abelian and so $C_2 = A_2 Y$ is an abelian subgroup satisfying the condition (1). Furthermore, $C_2 \lhd N(B_2)$ and $C_2/A_2 \cong I_2$ as $N_C(B_2)$-modules by (5A).

We define $E = N_{C_2}(B_1)$. Then $|E: A_2| = 4$ by (5.4). Moreover, we have $E \leqq V_1$. For $V \leqq N(E)$ and $VE \leqq N(D_1)$, and so VED_1 is a 2-group containing $VD_1 = V_1$. As $V_1 \in \mathrm{Syl}_2(N(B_1))$, we must have $VED_1 = V_1$, hence $E \leqq V_1$. Thus ED_1/D_1 is an abelian subgroup of V_1/D_1 containing $A_2 D_1/D_1$. As $A_2 D_1/D_1$ is self-centralizing in V_1/D_1 by (5B) (1), it follows that $ED_1 = A_2 D_1$, hence $E = A_2(E \cap D_1)$ and $|E \cap D_1| = 2^5$. This shows $E \cap C_1 \nleqq A_2$ as $C_1 \cap A_2 = A_1 \cap A_2$. As $(E \cap C_1)^2 \leqq I_1 \cap I_2 = 1$ by (5C) (2) and (5.3), we have $A_2 < \Omega_1(C_2)$. As K_2 acts irreducibly on C_2/A_2, we conclude that C_2 is elementary abelian.

(5E) LEMMA. *The following conditions hold:*

(1) $|C_1 \cap C_2| = 2^5$;

(2) $C_{C_2/I_1}(a_0) = \langle b_3, C_1 \cap C_2 \rangle / I_1$;

(3) $|C_{C_2}(a_0)| = 2^5$.

PROOF. Let $E = N_{C_2}(B_1)$. As shown in the last paragraph of the proof of (5D), $E = A_2(E \cap D_1)$ and $|E \cap D_1| = 2^5$. Now

$$\mathcal{E}^*(D_1/I_1) = \{B_1/I_1, C_1/I_1\} \tag{5.5}$$

by (5.2) and (5C). $E \cap D_1/I_1$ is elementary abelian by (5D), and $E \cap D_1 \nleqq B_1$ as $B_1 \cap C_2 = A_1 \cap A_2$. Thus $E \cap D_1 = E \cap C_1$. As $C_1 \cap C_2 \leqq N_{C_2}(B_1) = E$, $C_1 \cap C_2 = E \cap C_1$, and (1) follows.

Now $[E, a_0] \leqq B_1 \cap C_2 \leqq A_2$. As $|C_{C_2/A_2}(a_0)| = 4$ by (5A) (2), $C_{C_2/A_2}(a_0) = E/A_2$ and hence $C_{C_2/I_1}(a_0) \leqq E$. Using the expression $E/I_1 = (A_2/I_1)(E \cap C_1/I_1)$ and noticing that C_1/I_1 is abelian, we obtain

$$C_{E/I_1}(a_0) = (\langle b_3, A_1 \cap A_2 \rangle / I_1)(E \cap C_1/I_1).$$

Hence (2) follows. Consequently, $C_{C_2}(a_0) = \langle b_3 \rangle C_{C_1 \cap C_2}(a_0)$. As C_1 is extra-special and $C_1 \cap C_2$ is a maximal abelian subgroup of C_1, we have $|C_1 \cap C_2 : C_{C_1 \cap C_2}(a_0)| = 2$ and hence (3) follows.

(5F) LEMMA. *C_2 is the only E_{2^9}-subgroup of V_2.*

PROOF. The argument is similar to that of the second paragraph of (4F). Let X be an elementary abelian subgroup of V_2 of maximal rank, and assume that $X \neq C_2$. Then $X \nleqq D_2$ as $\mathcal{E}^*(D_2) = \{B_2, C_2\}$ by (1E) (1) and (5D) (2). As in

(4F), we have that $|C_{D_2}(X)|\leqq 2^7$ and that if $|XD_2: D_2|=4$, then $|C_{D_2}(X)|\leqq 2^6$. Hence $|X|\leqq 2^8$, which is a contradiction.

Now let $M_i=E(N(C_i) \bmod C_i)$ for $i\in\{1, 2\}$. Using (5F), we next prove

(5G) LEMMA. *The following conditions hold*:
(1) $M_2/C_2\cong L_3(4)$, $SL_3(4)$, *or* $L_3(2)\times L_3(2)$;
(2) $C_{M_2/C_2}(z)=K_2C_2/C_2$;
(3) $\langle z\rangle M_2$ *contains an* S_2-*subgroup of* $N(C_2)$;
(4) $[M_2, O(C)]=1$.

PROOF. As remarked in the proof of (5F), $\mathcal{E}^*(D_2)=\{B_2, C_2\}$ and so $N(D_2)=N(B_2)\leqq N(C_2)$. Hence if bars denote images in $N(C_2)/C_2$, then $C(\bar{z})=\overline{N(B_2)}=\langle\bar{z}\rangle\times\bar{K}_2\times\overline{O(C)}$. In particular, $\bar{V}_2\in\mathrm{Syl}_2(C(\bar{z}))$. Now $|V_2|<|V_1|$, hence $V_2\notin\mathrm{Syl}_2(G)$. Then (5F) shows $V_2\notin\mathrm{Syl}_2(N(C_2))$; so $\bar{z}$ is a noncentral involution. (1), (2), and (3) now follow from (1K). For the proof of (4), see the proof of (4H) (4).

Using (5E) and (5G), we next sharpen (5F).

(5H) LEMMA. *C_2 is the only E_{2^9}-subgroup of $N(C_2)$.*

PROOF. Let bars denote images in $N(C_2)/C_2$ and let X be an elementary abelian 2-subgroup of $N(C_2)$ of maximal rank. Suppose that $\bar{X}\not\leqq\bar{M}_2$. By (5G), involutions outside $\bar{M}_2$ are all conjugate to $\bar{z}$ and $\bar{V}_2\in\mathrm{Syl}_2(C(\bar{z}))$, so $\bar{X}$ is conjugate to a subgroup of $\bar{V}_2$. But then $X=C_2$ by (5F), a contradiction. Therefore, $\bar{X}\leqq\bar{M}_2$. Consequently, $m(\bar{X})\leqq m(\bar{M}_2)=4$ and hence $|X\cap C_2|\geqq 2^5$.

Now suppose $\bar{X}$ contains an element $\bar{x}\neq 1$ that is conjugate to an element of $\bar{K}_2$. Then $|C_{C_2}(x)|=2^5$ by (5E) (3); so the above paragraph yields that $X\cap C_2=C_{C_2}(x)$ and that $|\bar{X}|=2^4$.

Assume, by way of contradiction, that $\bar{X}\neq 1$. If $\bar{M}_2\cong L_3(4)$ or $SL_3(4)$, then $\bar{M}_2$ has only one conjugacy class of involutions. Hence if $\bar{X}\leqq\bar{S}\in\mathrm{Syl}_2(\bar{M}_2)$, then $\bar{X}$ is one of the two members of $\mathcal{E}^*(\bar{S})$ and so $\langle\bar{a}_0, \bar{a}_i\rangle\leqq\bar{X}^m$ for some $i\in\{1, 2\}$ and $m\in M_2$. But then $X^m\cap C_2=C_{C_2}(a_0)=C_{C_2}(a_i)$ and hence $C_{A_2}(a_0)=C_{A_2}(a_i)$, a contradiction. Therefore, $\bar{M}_2=\bar{M}_{21}\times\bar{M}_{22}$ with $(\bar{M}_{21})^z=\bar{M}_{22}\cong L_3(2)$.

We assume, without loss of generality, that $\langle\bar{U}, \bar{X}\rangle\leqq\bar{S}\in\mathrm{Syl}_2(\bar{M}_2)$ and that $\bar{S}^z=\bar{S}$. Suppose $\bar{X}\not\leqq\bar{M}_{2i}$ for each i. Then $\bar{X}$ contains a nonidentity element that fuses into $\bar{K}_2$, hence $|\bar{X}|=2^4$. Thus $\bar{X}$ is one of the four members of $\mathcal{E}^*(\bar{S})$ and, since $\bar{U}=C_{\bar{S}}(\bar{z})$, it follows that $\bar{a}_0\in\bar{X}$. Then $X\cap C_2=C_{C_2}(a_0)$ and, in particular, $X\cap C_2=X^z\cap C_2$. It also follows that $\langle\bar{a}_0, \bar{a}_i\rangle\leqq\langle\bar{X}, \bar{X}^z\rangle^m$ for some $i\in\{1, 2\}$ and $m\in M_2$. But then we have $X^m\cap C_2=C_{C_2}(a_0)=C_{C_2}(a_i)$, a contradiction. Therefore, $\bar{X}\leqq\bar{M}_{21}$, say. Then $|\bar{X}|\leqq 4$, hence $|X\cap C_2|\geqq 2^7$ and $|X\cap C_2\cap X^z|\geqq 2^5$. Since $\bar{a}_0$ is conjugate to an element of $\langle\bar{X}, \bar{X}^z\rangle$ and since $\langle\bar{X}, \bar{X}^z\rangle$ centralizes $X\cap C_2\cap X^z$, (5E) (3) yields that $|\bar{X}|=4$ and that $|X\cap C\cap X^z|=2^5$. But then $\langle\bar{a}_0, \bar{a}_i\rangle\leqq\langle\bar{X}, \bar{X}^z\rangle^m$ for some $i\in\{1, 2\}$ and $m\in M_2$, and hence $X^m\cap C_2\cap X^{zm}=C_{C_2}(a_0)=C_{C_2}(a_i)$, a contradiction.

(5I) LEMMA. *If $C_i^x\leqq V_1$ for some $x\in C(b_1)$ and $i\in\{1, 2\}$, then $C_i^x=C_1$.*

PROOF. We argue that $z^G\cap C_i=\varnothing$ for $i\in\{1, 2\}$. This is obvious if $i=2$, since $m(C)=7$ whereas $C_2\cong E_{2^9}$. Suppose that $z^G\cap C_1\neq\varnothing$. Then since C_1 is extra-special of order 2^9, V must contain an extra-special subgroup Y of order 2^7. We have $m(Y\cap B_2)\leqq m(Y)=4$ and so, as $|V/B_2|=2^3$, $V/B_2\cong Y/Y\cap B_2$. However, this is impossible as $V/B_2\cong D_8$ whereas $Y/Y\cap B_2$ is elementary abelian. Therefore, $z^G\cap C_1=\varnothing$.

Let $X=C_i^x$ and suppose that $X\neq C_1$. Let bars denote images in $C(b_1)/\langle b_1\rangle$. Then $C_{\bar{C}_1}(\bar{z})=\bar{A}_1$ by (5.2), and so $\bar{B}_1-\bar{C}_1=\bar{z}^{\bar{C}_1}$ by (1E). Also, $\mathcal{E}^*(\bar{D}_1)=\{\bar{B}_1, \bar{C}_1\}$ by (5.5). As $z^G\cap X=\varnothing$, it follows that $\bar{X}\cap\bar{D}_1=\bar{X}\cap\bar{C}_1$. In particular, $X\not\leqq D_1$. Now let tildes denote images in $N(B_1)/D_1$. Then $1\neq\tilde{X}\leqq\tilde{K}_1\cong Sp_4(2)$, hence $|\tilde{X}|\leqq 2^3$ and $|\bar{X}\cap\bar{C}_1|\geqq 2^5$. Thus $\tilde{X}$ centralizes a 5-dimensional subspace of $\bar{C}_1$. Since $\bar{C}_1/\bar{A}_1\cong\bar{A}_1$ as K_1-modules by (5B) (2), it follows that $\tilde{X}^\#$ consists of conjugates of $\tilde{b}_3$ in $\tilde{K}_1$: see Table 2. As $\tilde{b}_3$ is a transvection of $\tilde{K}_1\cong Sp_4(2)$, this is possible only if $|\tilde{X}|=2$. But then $|\bar{X}\cap\bar{C}_1|\geqq 2^7$, which is a contradiction as $\bar{C}_1/\bar{A}_1\cong\bar{A}_1$ as $\langle\tilde{b}_3\rangle$-modules and so $\tilde{b}_3$ can not centralize a hyperplane of $\bar{C}_1$.

Now we prove an analogue of (5G).

(5J) LEMMA. *The following conditions hold:*
(1) $M_1/C_1\cong U_4(2)$ *or* $L_4(2)$;
(2) $C_{\langle z\rangle M_1/C_1}(z)=\langle z\rangle K_1C_1/C_1$;
(3) $\langle z\rangle M_1$ *contains an* S_2-*subgroup of* G;
(4) $[M_1, O(C)]=1$.

PROOF. By (5.5) $\mathcal{E}^*(D_1/I_1)=\{B_1/I_1, C_1/I_1\}$. As $N(D_1)\leqq N(I_1)$ by (5B) (3), it follows that $N(D_1)=N(B_1)\leqq N(C_1)$. Hence if bars denote images in $N(C_1)/C_1$, then $C(\bar{z})=\langle\bar{z}\rangle\times\bar{K}_1\times\overline{O(C)}$.

We argue that $\bar{z}$ is a noncentral involution. Indeed, $Z(V_1)\leqq Z(D_1)=I_1$, hence $Z(V_1)=I_1$. Hence $N(V_1)\leqq C(b_1)$ and (5I) yields that $N(V_1)\leqq N(C_1)$. Now, $N(C_2)$ contains an S_2-subgroup of G by (5H) and so $|G|_2=2^{16}$ by (5G) (3). In particular, $|N(V_1): V_1|$ is even. Hence $V_1\notin\mathrm{Syl}_2(N(C_1))$ and thus $\bar{z}$ is a noncentral involution. Furthermore, we have $|\overline{N(C_1)}|_2\leqq 2^7$. The result now follows from (1I).

We have arrived at the goal of our 2-local analysis.

(5K) LEMMA. *The following conditions hold:*

(1) *M_1 and M_2 have a common z-invariant S_2-subgroup S such that* $C_S(z)=U$;
(2) $C_{M_1/C_1}(z)=K_1C_1/C_1$.

PROOF. By (5J) (2), $V_1\leqq\langle z\rangle M_1$. Let $V_1\leqq T\in \mathrm{Syl}_2(\langle z\rangle M_1)$ and set $S=T\cap M_1$. Then $S\in\mathrm{Syl}_2(M_1)$ and S is z-invariant. Also, $T\in\mathrm{Syl}_2(G)$ by (5J) (3) and so, as $N(C_1)\leqq C(b_1)$ by (5C) (2), $T\in\mathrm{Syl}_2(C(b_1))$. As $C_2\leqq C(b_1)$ by (5D) (2), T contains a conjugate X of C_2 in $C(b_1)$. We show $X=C_2\leqq S$. Let bars denote images in $C(b_1)/\langle b_1\rangle$ and let tildes denote images in $N(C_1)/C_1$. Then $\tilde{M}_1\cong U_4(2)$ or $L_4(2)$ and $\langle\tilde{z}\rangle C_{\tilde{M}_1}(\tilde{z})=\langle\tilde{z}\rangle\tilde{K}_1$ by (5J). If $X\nleqq S$, then $\tilde{X}\cap\tilde{z}\tilde{M}_1\neq\varnothing$; so there is an element $b\in M_1$ and an element $n\in N(C_1)$ such that $\tilde{z}\tilde{b}\in\tilde{X}^n$ and $\tilde{V}_1\in\mathrm{Syl}_2(C(\tilde{z}\tilde{b}))$ by Section 19 of [3]. Then $\tilde{X}^{nm}\leqq\tilde{V}_1$ for some $m\in N(C_1)$ by Sylow's theorem, which however contradicts (5I). Therefore, $X\leqq S$.

Now $m(X\cap C_1)\leqq m(C_1)=5$, hence $|\tilde{X}|\geqq 2^4$. From the structure of $\langle\tilde{z}\rangle\tilde{M}_1$, it follows that $\tilde{X}$ is the unique E_{2^4}-subgroup of $\tilde{S}$ and that $|\tilde{X}\cap\tilde{V}_1|=8$. Now $\tilde{b}_3\tilde{b}_6=[\tilde{b}_5,\tilde{a}_2]\in\tilde{K}_1'\leqq\tilde{M}_1$, hence $\tilde{b}_3\tilde{b}_6\in\tilde{V}_1\cap\tilde{M}_1$. As $\tilde{b}_3\tilde{b}_6\in Z(\tilde{V}_1)$ and as $\tilde{X}\cap\tilde{V}_1$ is an E_8-subgroup of $\tilde{V}_1\cap\tilde{M}_1\cong Z_2\times D_8$, it follows that $\tilde{b}_3\tilde{b}_6\in\tilde{X}\cap\tilde{V}_1$. Take an element $x\in X$ such that $\tilde{x}=\tilde{b}_3\tilde{b}_6$. Now $\bar{b}_3\bar{b}_6$ centralizes the E_{2^4}-group $\bar{C}_1\cap\bar{C}_2$ while Table 2 and (5B) (2) show $|C_{\bar{C}_1}(\bar{b}_3\bar{b}_6)|\leqq 2^4$. Thus $C_{\bar{C}_1}(\bar{b}_3\bar{b}_6)=\bar{C}_1\cap\bar{C}_2$ and (1E) shows that $\bar{x}$ is conjugate to $\bar{b}_3\bar{b}_6$ under $\bar{C}_1$. Consequently, $C_{\bar{C}_1}(\bar{x})=\bar{C}_1\cap\bar{C}_2$ and, comparing orders, we have that $X\cap C_1=C_1\cap C_2$.

Now $C_{V_1}(A_1\cap A_2/I_1)=A_2D_1$ and $C_{A_2D_1}(C_1\cap C_2/I_1)=A_2C_1$, hence $C_{V_1}(C_1\cap C_2)=A_2C_{C_1}(C_1\cap C_2)=A_2(C_1\cap C_2)$. Since $X\cap V_1\leqq C_{V_1}(C_1\cap C_2)$, we conclude that $X\cap V_1=A_2(C_1\cap C_2)$. Now $z\in N(X)$ as X is weakly closed in T with respect to G by (5H). Also, $C_X(z)\leqq X\cap V_1=A_2(C_1\cap C_2)$ by the above, hence $C_X(z)=A_2$. Thus $\mathcal{E}^*(\langle z\rangle X)=\{B_2, X\}$ and then $X\leqq N(B_2)$ by (1E). (5H) now shows that $X=C_2$.

Since $A_2\leqq C_2\leqq S$ by the above, $\tilde{K}_1=\tilde{K}_1'\tilde{A}_2\leqq\tilde{M}_1$. Thus, $C_{\tilde{M}_1}(\tilde{z})=\tilde{K}_1$. Also, $U\leqq S$ and so as $z\notin S$, we have $C_S(z)=U$. Now C_1C_2/C_1 is an E_{16}-subgroup of T/C_1; so $T/C_1C_2\cong D_8$ and $S/C_1C_2\cong E_4$ by the structure of $\langle\tilde{z}\rangle\tilde{M}_1$. Similarly, $T\leqq N(C_2)$ by (5H) and C_1C_2/C_2 is an E_{16}-subgroup of T/C_2. So $C_1C_2\leqq M_2$ and $T\cap M_2/C_1C_2$ is an E_4-subgroup of T/C_1C_2 by (5G). If $S\neq T\cap M_2$, then S/C_1C_2 and $T\cap M_2/C_1C_2$ are the E_4-subgroups of $T/C_1C_2\cong D_8$. But as $z\notin S$ and $z\notin M_2$, this yields a contradiction. Therefore, $S\leqq M_2$. The proof is complete.

We are now in a position to complete the proof of Theorem 2. The argument is similar to that of Theorem 1. It follows from (5A), (5B), (5C), (5D), (5G), (5J), and (5K) that z, L, C_i, M_i, $i\in\{1, 2\}$, and S satisfy Hypothesis (2.2) of [10] with $M_1/C_1\cong U_4(2)$ or $L_4(2)$, $M_2/C_2\cong L_3(4)$, $SL_3(4)$, or $L_3(2)\times L_3(2)$, and $|S|=2^{15}$. Since C_2 is abelian, Theorems 2 and 5 of [10] show that $G_0=\langle M_1, M_2\rangle$ is a z-invariant quasisimple subgroup with $L\leqq G_0$ and $G_0/Z(G_0)\cong U_6(2)$ or $L_6(2)$. By (3B), $C(G_0)$ has odd order, and hence $G_0\cong U_6(2)$, $SU_6(2)$, or $L_6(2)$: see [6], pp. 59-60.

Suppose $z^g \in N(G_0)$ for some $g \in G$. Then $z^g \notin G_0$ as $m(C_{G_0}(x)) \geqq 9$ for each $x \in I(G_0)$. Results in Section 19 of [3] show that there exist precisely two conjugacy classes of involutions in $\mathrm{Aut}(G_0) - \mathrm{Inn}(G_0)$. The argument of (4I) then shows that $g \in N(G_0)$. Thus an analogue of (4I) holds in this case as well.

As in the case of Theorem 1, results in [19, III] show that $G_0 O(G) \triangleleft G$, and hence it follows that $\langle L^G \rangle = G_0$. Thus we have proved Theorem 2.

REMARK. At one place in [19, III], after showing that $O_2(C_{G_0}(b_1)) \triangleleft C(b_1)$, Seitz uses his basic induction hypothesis to prove that $O^{2'}(C_{G_0}(b_1)) \triangleleft C(b_1)$. In our notation, $O_2(C_{G_0}(b_1)) = C_1$ and $O^{2'}(C_{G_0}(b_1)) = M_1$. Therefore, we need not use the induction hypothesis here.

6. $U_7(2)$, $L_7(2)$, $Sp_6(4)$, and $Sp_6(2) \times Sp_6(2)$.

In the balance of this paper, we consider the following situation:

HYPOTHESIS. $|\Omega| = 64$.

As may be imagined, our 2-local analysis under this hypothesis is the hardest. However, the analysis follows the same line of arguments as in Sections 4 and 5. In this section, we study the structure of $N(B_i)$, $i = 1, 2$, and the relationship between them. Let $D_i = O_2(N(B_i))$, $V_i = VD_i$, and $F_i = N_{D_i}(J_i)$ for $i \in \{1, 2\}$.

(6A) LEMMA. *The following conditions hold*:
(1) $N(B_2) = N_C(B_2) D_2$ *and* $N_C(B_2) \cap D_2 = B_2$;
(2) $F_2 \triangleleft N(B_2)$ *and commutation by* z *induces isomorphisms* $F_2/B_2 \to I_2$ *and* $D_2/F_2 \to A_2/I_2$;
(3) $Z(D_2) = I_2$ *or* A_2 *and if* $Z(D_2) = I_2$, *then* $A_2 \leqq Z_2(D_2)$.

PROOF. Let bars denote images in $N(B_2)/C(B_2)$. Then $\overline{N(B_2)}$ is a transitive permutation group on Ω and the stabilizer $\overline{N_C(B_2)}$ of the point z is isomorphic to the simple group $L_3(2)$. Let $\overline{N(B_2)}/\bar{X}$ be a composition factor. Then $\overline{N_C(B_2)} \not\leqq \bar{X}$ by (3F) and so $\overline{N_C(B_2)} \cap \bar{X} = 1$. Also, $\overline{N(B_2)} = \overline{N_C(B_2)}\bar{X}$ by [22]. Set $Y = C_{N(B_2)}(O(C))O(C)$. Then $N_C(B_2) \leqq Y \triangleleft N(B_2)$. If $Y \neq N(B_2)$, then $\Gamma = z^Y$ is equal to $\{z\} \cup z\Omega_1$ or $\{z\} \cup z\Omega_2$ by (3F) and, for any $g \in N(B_2) - Y$, Γ^g is a Y-orbit such that $\Gamma^g \cap \Gamma = \varnothing$. However, the $N_C(B_2)$-orbit decomposition of Ω shows that such Y-orbits Γ and Γ^g do not exist. Hence $Y = N(B_2)$ and (1) follows.

Now $zz^x \in [D_2, z]$ for any $x \in D_2$ and so $|[D_2, z]| \geqq |\Omega| = 64$ while, on the other hand, $[D_2, z] < B_2$. Hence $[D_2, z]$ is a maximal subgroup of B_2 and, moreover, it is K_2-invariant. (2E) (3) forces $[D_2, z] = A_2$. Therefore, (6.1) $$\Omega = zA_2 .$$

Since $C_{D_2}(z)=B_2$ and since $\Omega=zA_2$, it follows that $1<Z(D_2)\leqq A_2$. By (2E) (3), $Z(D_2)=I_2$ or A_2. If $Z(D_2)=I_2$, then as $A_2=B_2-\Omega\lhd D_2$, $I_2<A_2\cap Z_2(D_2)$. Thus $A_2\leqq Z_2(D_2)$ again by (2E) (3). Let $\bar{D}_2=D_2/I_2$. Then since $[\bar{D}_2, \bar{z}]=\bar{A}_2\leqq Z(\bar{D}_2)$, it follows that commutation by $\bar{z}$ induces an epimorphism $\bar{D}_2\to\bar{A}_2$ with kernel $\bar{F}_2$. Consequently, $F_2\lhd N(B_2)$. An element x of D_2 is contained in F_2 if and only if $z^x\in zI_2$. Hence $[F_2, z]=I_2$ and, since $I_2\leqq Z(F_2)$, commutation by z induces an isomorphism $F_2/B_2\to I_2$.

(6B) LEMMA. *The following conditions hold:*
(1) $N(B_1)/C(B_1)=(N_C(B_1)/C(B_1))O_2(N(B_1)/C(B_1))$;
(2) $b_1\in Z(N(B_1))$.

PROOF. Set $P=N_{F_2}(B_1B_2)$. Then

(6.2) $$P\leqq N(B_1) \quad and \quad |P: B_2|=4$$

by (3E) and (6A) (2). Thus $|\Delta|$ is divisible by 4. Also, $\Delta\leqq zA_1$ by (3C), and Δ is a union of conjugacy classes of $N_C(B_1)$. Hence

(6.3) $$\Delta=\{z\}\cup z\Delta_2 , \quad \{z\}\cup z\Delta_{12} \quad or \quad zA_1 .$$

Let bars denote images in $N(B_1)/C(B_1)$. Then $\bar{K}_1=\overline{N_C(B_1)}$ is a subgroup of index 16 or 32 by (6.3) and isomorphic to Σ_6. In particular, $O(\overline{N(B_1)})=1$. If $O_2(\overline{N(B_1)})=1$, then $F^*(\overline{N(B_1)})\cong A_5$ or A_6 by [4], and then $\overline{N(B_1)}$ is isomorphic to a subgroup of $\mathrm{Aut}(A_5)\cong\Sigma_5$ or $\mathrm{Aut}(A_6)\cong P\Gamma L_2(9)$. Since this is not the case, we must have $O_2(\overline{N(B_1)})\neq 1$. A similar argument shows that either $\overline{N(B_1)}=\overline{N_C(B_1)}O_2(\overline{N(B_1)})$ or $\overline{N(B_1)}/O_2(\overline{N(B_1)})\cong P\Gamma L_2(9)$. Set $\bar{X}=O_2(\overline{N(B_1)})$. Then $[\bar{X}, \bar{K}_1']\neq 1$. For if $[\bar{X}, \bar{K}_1']=1$, then $\bar{K}_1'$ would centralize each element of z^X, which contains at least 8 elements, a contradiction. Hence if $\overline{N(B_1)}/\bar{X}\cong P\Gamma L_2(9)$, then $\bar{X}\cong E_{16}$ and $C(\bar{X})=\bar{X}$. However, this implies that $\overline{N(B_1)}/\bar{X}\cong P\Gamma L_2(9)$ is isomorphic to a subgroup of $\mathrm{Aut}(\bar{X})\cong GL_4(2)$, a contradiction. Therefore, $\overline{N(B_1)}=\overline{N_C(B_1)}\bar{X}$.

Now let $Y\in\mathrm{Syl}_2(X)$. Then $A_1\lhd Y$ as $A_1=\langle ab\,;\, a, b\in\Delta\rangle$ by (6.3). Hence $A_1\cap Z(Y)\neq 1$, and as $X=YO(C)$ and $[A_1, O(C)]=1$, it follows that $A_1\cap Z(X)\neq 1$. Then $b_1\in A_1\cap Z(X)$ by (2D) (3), and since $b_1\in Z(N_C(B_1))$, (2) follows.

(6C) LEMMA. *We have* $Z(F_2)=A_2$.

PROOF. Let $P=N_{F_2}(B_1B_2)$. Then $P\leqq N(B_1)$ and $|P: B_2|=4$ by (6.2). Since $P\leqq C(b_1)$ by (6B) (2), $\Gamma=b_1^{N_C(B_2)F_2}$ contains at most 14 elements. (6A) (3) shows that Γ is a union of conjugacy classes of $N_C(B_2)$ contained in A_2-I_2; that is,

Ω_1, Ω_{12}, and Ω_{34}. The only possibility is that $\Gamma=\Omega_1$. So b_1 is a central involution of $N_C(B_2)F_2$ and since $F_2 \lhd N_C(B_2)F_2$, it follows that $b_1 \in Z(F_2)$. Thus $I_2 < Z(F_2) \cap A_2$, and (2E) (3) and $C_{F_2}(z)=B_2$ imply that $Z(F_2)=A_2$.

(6D) LEMMA. *There is an element* $d \in N_{D_2}(B_1)$ *such that* $z^d=zb_1$.

PROOF. First of all, $F_2 \leqq C_{D_2}(A_2)$ by (6C). If $F_2 < C_{D_2}(b_1)$, choose $d \in C_{D_2}(b_1)-F_2$ so that $[V, d] \leqq F_2$. This is possible as V acts on $C_{D_2}(b_1)/F_2$. If $F_2=C_{D_2}(b_1)$, then D_2/F_2 acts regularly on b_1I_2 by (6A) (3); so we choose $d \in D_2-F_2$ so that $b_1^d=b_1b_2$. Then $[V, d] \leqq F_2$ in either case and since commutation by z induces an isomorphism $D_2/F_2 \to A_2/I_2$, it follows that $[d, z] \in b_1I_2$. Write $z^d=zb_1a$ with $a \in I_2$ and choose an element $f \in F_2$ so that $z^f=za$. Then $z^{df}=zb_1$; so replacing d by df, we have $z^d=zb_1$ and $b_1^d=b_1$ or b_1b_2. In particular, d normalizes $\langle z, b_1, b_2, b_4\rangle=B_1 \cap B_2$. As $C(A_1 \cap A_2) \cap N_L(A_2)=A_1A_2$ implies that $C(B_1 \cap B_2) \cap N(B_2) =B_1B_2 \times O(C)$ and as $N(B_1B_2) \leqq N(B_1)$, we obtain that $d \in N(B_1)$, proving the assertion.

(6E) LEMMA. *The following conditions hold*:
(1) $N(B_1)=N_C(B_1)D_1$ *and* $N_C(B_1) \cap D_1=B_1$;
(2) $F_1 \lhd N(B_1)$ *and commutation by* z *induces isomorphisms* $F_1/B_1 \to I_1$ *and* $D_1/F_1 \to A_1/I_1$;
(3) $Z(D_1)=I_1$ *or* A_1 *and if* $Z(D_1)=I_1$, *then* $A_1 \leqq Z_2(D_1)$;
(4) $F_1^2=I_1$ *and* $Z(F_1)=A_1$.

PROOF. First of all, (6D) together with (6.3) implies that $\Delta=zA_1$. Let bars denote images in $N(B_1)/C(B_1)$ and set $\bar{X}=O_2(\overline{N(B_1)})$. Then $\overline{N(B_1)}=\overline{N_C(B_1)X}$ by (6B) and so $|\bar{X}|=32$. As usual, (1) will follow once we prove $X=C_X(O(C))O(C)$. Set $Y=C_{N(B_1)}(O(C))O(C)$. Then $N_C(B_1) \leqq Y \lhd N(B_1)$. Choose an element d as in (6D). Then since $[D_2, O(C)]=1$ by (6A) (2), we have $d \in Y$ and so $\Gamma=z^Y$ contains zb_1. If $Y \neq N(B_1)$, then for any $g \in N(B_1)-Y$, Γ^g is a Y-orbit such that $\Gamma \cap \Gamma^g=\varnothing$. However, the $N_C(B_1)$-orbit decomposition of Δ shows that such Y-orbits Γ and Γ^g do not exist. Hence $Y=N(B_1)$ and (1) follows. (2) and (3) follow by the arguments parallel to those in the last paragraph of the proof of (6A). Finally, (4) is a consequence of (1H).

For any subgroup X of G, we let $X^*=X \cap C(H)$, where $H=\langle b_5b_6{}^t\rangle$ as defined in Section 2.

(6F) LEMMA. *Let* $Q=[D_1, HH^s]A_1$. *Then the following conditions hold*:
(1) $\langle D_1, H, s\rangle \leqq N(Q)$;
(2) $D_1=QF_1$ *and* $Q \cap F_1=A_1$;
(3) $Q^*=[Q^*, H^s]I_1$ *and* $Q^*/I_1 \cong E_{16}$ *or* $Z_4 \times Z_4$;

(4) $Q=Q^{*}*Q^{*s}$ *and* $Q^{*}\cap Q^{*s}=I_1$;
(5) $[D_1, s]\leqq Q$.

PROOF. (6E) (2) shows that $D_1^2\leqq F_1$ and that K_1 acts transitively on $(D_1/F_1)^{\#}$. Hence $D_1^2\leqq B_1$ and (1G) applied to the chain $A_1/I_1\leqq B_1/I_1\leqq F_1/I_1\leqq D_1/I_1$ yields that

$$D_1^2\leqq A_1. \tag{6.4}$$

(1) is now immediate from the definition. Also, since $C_{D_1/A_1}(HH^s)=F_1/A_1$ by (6E) (2) and (2G), (2) follows. Consequently, commutation by z induces an isomorphism $Q/A_1\to A_1/I_1$, and the argument in the last paragraph of the proof of (5C) shows that $Q^{*}/I_1\cong E_{16}$ or $Z_4\times Z_4$ and that $Q/I_1=Q^{*}/I_1\times Q^{*s}/I_1$. Hence $Q^{*}=[Q^{*}, H^s]I_1=[Q, H^s]I_1$, and (1A) (2) shows that $Q=Q^{*}*Q^{*s}$. (6E) (4) shows $F_1\cong E_{16}\times D_8$. So F_1 has precisely two elementary abelian maximal subgroups. Since B_1 is one of them and normalized by s, it follows that $[F_1, s]\leqq A_1$. Hence $[D_1, s]\leqq Q$.

We note that an analogue of (6E) holds for $N_{G^*}(B_1^*)$.

(6G) LEMMA. *The following conditions hold*:
(1) $N_{G^*}(B_1^*)=N_{C^*}(B_1^*)D_1^*$ *and* $N_{C^*}(B_1^*)\cap D_1^*=B_1^*$;
(2) $F_1^*\lhd N_{G^*}(B_1^*)$ *and commutation by* z *induces isomorphisms* $F_1^*/B_1^*\to I_1$ *and* $D_1^*/F_1^*\to A_1^*/I_1$;
(3) $Z(D_1^*)=I_1$ *or* A_1^* *and if* $Z(D_1^*)=I_1$, *then* $A_1^*\leqq Z_2(D_1^*)$;
(4) $F_1^{*2}=I_1$ *and* $Z(F_1^*)=A_1^*$;
(5) $D_1^*=Q^*F_1^*$ *and* $Q^*\cap F_1^*=A_1^*$.

PROOF. (6E) shows that an element $x\in D_1$ normalizes B_1^* if and only if $z^x\in zA_1^*$, or $[x, z]\in A_1^*$. Hence $|N_{D_1}(B_1^*): B_1|=8$ and $N_{D_1}(B_1^*)/B_1=C_{D_1/B_1}(H)=B_1D_1^*/B_1$. Since $|N(B_1^*): N_C(B_1^*)|\leqq 8$ by (3C), it follows that $N(B_1^*)=N_C(B_1^*)N_{D_1}(B_1^*)=N_C(B_1^*)D_1^*$. Hence (1) follows. (2), (3), and (4) may be derived from the corresponding assertions in (6E). Since $|Q^*: A_1^*|=4=|F_1^*: A_1^*|$ and $Q^*\cap F_1^*=A_1^*$ by (6F), (5) follows.

(6H) LEMMA. *Let* $\overline{G^*}=G^*/O(G^*)$. *Then* $\overline{G^*}$ *has a subgroup* $\overline{M^*}=M^*/O(G^*)$ *of index* 2 *satisfying the following conditions*:
(1) $\overline{M^*}\cong U_5(2)$, $L_5(2)$, $Sp_4(4)$, $Sp_4(2)\times Sp_4(2)$, *or* $U_4(3)$;
(2) $G^*=\langle z\rangle M^*$;
(3) $C_{\overline{M^*}}(\bar{z})=\overline{K_1^{*s}}$;
(4) *either* $\overline{M^*}\cong U_4(3)$ *or* $A_1^*\not\leqq Z(D_1^*)$;
(5) $Q^*\leqq M^*$.

PROOF. This is a consequence of (1J). We have that $C_{G^*}(z)=C^*=\langle z\rangle\times K_1^{rs}\times HO(C)$ and that A_1^* is an E_8-subgroup of K_1^{rs}. (6G) shows $|D_1^*|=2^7$ and $D_1^*\cap C^*=B_1^*$. Also, $D_1^{*2}\leqq A_1^*$ by (6.4). As $N(A_1^*)\cap K_1^{rs}\leqq N_{L^*}(A_1)\leqq N(D_1^*)$, (1J) implies that there is a subgroup $\overline{M^*}=M^*/O(G^*)$ of $\overline{G^*}$ of index 2 satisfying (1)—(4). As $\langle H^s, I_1\rangle\leqq K_1^{rs}\leqq M^*$, (6F) (3) shows $Q^*\leqq M^*$.

(6I) LEMMA. *Let the notation be as in* (6H) *and assume that* $\overline{M^*}\not\cong U_4(3)$. *Let* $\overline{X_i^*}=O_2(N_{\overline{G^*}}(\overline{B_i^*}))$ *and* $\overline{Y_i^*}=\overline{X_i^*}\cap\overline{M^*}$ *for* $i\in\{1, 2\}$. *Then the following conditions hold*:

(1) $N_{\overline{G^*}}(\overline{B_i^*})=N_{\overline{C^*}}(\overline{B_i^*})\overline{X_i^*}$, $N_{\overline{C^*}}(\overline{B_i^*})\cap\overline{X_i^*}=\overline{B_i^*}$, *and* $|\overline{X_i^*}:\overline{B_i^*}|=8$ *for each* i;

(2) *if* $\overline{M^*}\cong Sp_4(4)$ *or* $Sp_4(2)\times Sp_4(2)$, *then* $\overline{A_i^*}\leqq Z(\overline{X_i^*})$ *and* $\overline{Y_i^*}$ *is elementary abelian for each* i;

(3) *if* $\overline{M^*}\cong U_5(2)$ *or* $L_5(2)$, *then* $\overline{A_i^*}\leqq Z(\overline{X_i^*})$ *for exactly one value of* i, *and*

$$Z(\overline{Y_i^*})\cong\begin{cases} E_{16} & \text{if } \overline{A_i^*}\leqq Z(\overline{X_i^*}),\\ Z_4 & \text{if } \overline{A_i^*}\not\leqq Z(\overline{X_i^*});\end{cases}$$

(4) $\overline{X_1^*}=\overline{D_1^*}$.

PROOF. By our assumption and (6H), $\overline{G^*}=\langle\bar z\rangle\overline{M^*}$ and $\overline{M^*}\cong U_5(2)$, $L_5(2)$, $Sp_4(4)$, or $Sp_4(2)\times Sp_4(2)$. Furthermore, $C_{\overline{M^*}}(\bar z)=\overline{K_1^{rs}}\cong Sp_4(2)$, so both the action of $\bar z$ on $\overline{M^*}$ and the embedding of $\overline{K_1^{rs}}$ in $\overline{M^*}$ are uniquely determined up to conjugation by the elements of $\overline{M^*}$ and relabeling of elements of $\overline{K_1^{rs}}$ by its graph automorphism: see (1.1) of [10] and its proof. Thus (1)—(3) may be checked by direct computations involving matrices. (4) is a consequence of (6G) (1).

Now we state the main result of this section.

(6J) LEMMA. *For each* $i\in\{1, 2\}$, $N(B_i)$ *has a normal subgroup* C_i *satisfying the following conditions*:

(1) $D_i=B_iC_i$ *and* $B_i\cap C_i=A_i$;

(2) $|C_1\cap C_2|=2^6$;

(3) $C_i\cap V_{3-i}=A_i(C_1\cap C_2)$;

(4) $(C_i\cap V_{3-i})^2\leqq I_1$.

Furthermore, there are two possibilities for the structures of C_1 *and* C_2 *and the relationship between them*: *either*

(5) $C_1\cong E_{2^{10}}$ *or* $D_8*D_8*D_8*D_8*E_4$ *and* $C_2\cong E_{2^{12}}$, *or*

(6) $C_1\cong D_8*D_8*D_8*D_8*Z_4$, $C_2^2=A_2$, $\Omega_1(C_2)=Z(C_2)=C_2\cap F_2$, *and* $C_1\cap C_2=Z(C_1)\cdot(C_1\cap Z(C_2))$.

We divide the proof into four parts. First, we prove

(6K) Lemma. *Assume* $A_1^* \leqq Z(D_1^*)$. *Then* $N(B_1)$ *has a normal subgroup* C_1 *such that*

(1) $D_1 = B_1 C_1$ *and* $B_1 \cap C_1 = A_1$, *and either*

(2) $\overline{M^*} \cong Sp_4(4)$ *or* $Sp_4(2) \times Sp_4(2)$ *and* $C_1 \cong E_{2^{10}}$, *or*

(3) $\overline{M^*} \not\cong Sp_4(4)$ *or* $Sp_4(2) \times Sp_4(2)$, $C_1^2 = A_1$, $C_1' = I_1$, *and* $Z(C_1) = \Omega_1(C_1) = C_1 \cap F_1$.

Proof. First of all, $\overline{M^*} \cong U_5(2)$, $L_5(2)$, $Sp_4(4)$, or $Sp_4(2) \times Sp_4(2)$ by (6H) (4). We define $C_1 = Q(D_1^* \cap M^*)$. As $Q \lhd D_1$ by (6F), C_1 is a subgroup of D_1. Furthermore, $C_1^* = D_1^* \cap M^*$, $|C_1^* : Q^*| = 2$, and $D_1^* = \langle z \rangle C_1^*$ by (6G)—(6I). Thus C_1 satisfies (1). Now either C_1^* is elementary abelian, or $Z(C_1^*) \cong E_{16}$ by (6I). Furthermore, in either case $[C_1^*, H^s] \leqq Q^*$ and H^s acts irreducibly on Q^*/A_1^*, and in the latter case we have $C_1^*/A_1^* = Q^*/A_1^* \times Z(C_1^*)/A_1^*$ by (6G). Hence we deduce that $C_1^* = Q^* * C_{C_1^*}(H^s)$ and $C_{C_1^*}(H^s) \cong E_4$. Notice that $C_{C_1^*}(H^s) = C_{C_1}(HH^s)$. Now $s \in N(C_1)$ by (6F) (5) and so Q^{*s} centralizes $C_{C_1}(HH^s)$. This together with (6F) (4) shows that

$$C_1 = Q^* * Q^{*s} * C_{C_1}(HH^s)\,. \tag{6.5}$$

Then C_1/I_1 is abelian by (6F) (3) and, since $|D_1/I_1 : Z(D_1/I_1)| > 4$, it follows that $C_1 \lhd N(B_1)$. Now if $\overline{M^*} \cong Sp_4(4)$ or $Sp_4(2) \times Sp_4(2)$, then $C_1^{*2} = 1$ and so $C_1^2 = 1$. Assume $\overline{M^*} \cong U_5(2)$ or $L_5(2)$. Then $Z(C_1^*) = \Omega_1(C_1^*) \cong E_{16}$, $C_1^{*\prime} = I_1$, and $C_1^{*2} = A_1^*$. Hence $C_1^2 = A_1$, $C_1' = I_1$, and $Z(C_1) = \Omega_1(C_1) = A_1 C_{C_1}(HH^s)$. As $F_1 = A_1 C_{F_1}(HH^s)$ by (6E), we have $Z(C_1) = C_1 \cap F_1$. The proof is complete.

Next, we prove

(6L) Lemma. *Assume* $A_1^* \not\leqq Z(D_1^*)$. *Then* $N(B_1)$ *has a normal subgroup* C_1 *satisfying the following conditions*:

(1) $D_1 = B_1 C_1$ *and* $B_1 \cap C_1 = A_1$;

(2) $C_1 \cong D_8 * D_8 * D_8 * D_8 * Z_4$ *or* $D_8 * D_8 * D_8 * D_8 * E_4$ *with* $C_1^2 = I_1$ *and* $A_1 Z(C_1) = C_1 \cap F_1$.

Proof. As $D_1^* = Q^* F_1^*$ and $A_1^* = Z(F_1^*)$ by (6G), our assumption implies that $A_1^* \not\leqq Z(Q^*)$. It also follows from (6G) that H^s acts irreducibly on Q^*/A_1^* and on A_1^*/I_1. Thus we must have $Z(Q^*) = I_1$, and as Q^*/I_1 is abelian by (6F) (3), (1B) shows that Q^* is extra-special with $Q^{*2} = I_1$. As H^s is transitive on $(Q^*/A_1^*)^\#$, $Q^* \cong D_8 * D_8$ is the only possibility.

Let $Z^* = C_{D_1^*}(Q^*/I_1)$. Notice that $Q^* = [D_1^*, H^s] A_1^* \lhd N_{G^*}(B_1^*)$ as $H^s A_1^* \lhd N_{G^*}(B_1^*)$. If $Z^* = Q^*$, then (6G) yields that $C(Q^*/I_1) \cap N_{G^*}(B_1^*) = Q^* HO(C)$, so $N_{G^*}(B_1^*)/Q^* HO(C)$ is isomorphic to a subgroup of $\mathrm{Out}(Q^*) \cong L_2(2)$ wreath Z_2. However, (6G) (4) shows that $[F_1^*, N_{G^*}(B_1^*)] \leqq A_1^*$ and so $N_{G^*}(B_1^*)/Q^* HO(C) \cong E_4 \times L_2(2)$. This contradiction shows that $Q^* < Z^* < D_1^*$.

Now we define $C_1 = Q Z^*$. Then $C_1^* = Z^*$ and, as Z^*/I_1 is abelian, (1A) (2) shows $C_1^* = Q^* * C_{C_1^*}(H^s)$. Then an argument in the proof of (6K) shows that the condition (1) and (6.5) hold in this case as well. As $C_{C_1}(HH^s) = C_{C_1 \cap F_1}(HH^s) \cong Z_4$ or E_4 by (6E), (2) holds. Consequently, C_1/I_1 is abelian and then C_1 is normal in $N(B_1)$.

Next, we prove

(6M) LEMMA. *Let* $\overline{M^*}$ *be as in* (6H) *and assume* $\overline{M^*} \cong Sp_4(4)$ *or* $Sp_4(2) \times Sp_4(2)$. *Then* $Z(D_2) = A_2$.

REMARK. We show later that the assumption on $\overline{M^*}$ may be dropped.

PROOF. Set $\bar{X} = O_2(N_{\bar{G}^*}(\overline{B_2^*}))$ and let X be a preimage of $\bar{X}$ in $N_{G^*}(B_2^*)$ such that $B_2^* \leqq X \lhd N_{G^*}(B_2^*)$. Then $|X : C_X(B_2^*)| = |X : X \cap C| = 8$. Since $|N(B_2^*) : N_C(B_2^*)| \leqq 8$ by (3D) (2) and (6.1), it follows that $N(B_2^*) = XN_C(B_2^*) = XC(B_2^*)N_C(B_2^*)$. Also, $N_L(A_2^*) = C_L(A_2^*)N_{L^*}(A_2^*)$ implies $N_C(B_2^*) = C(B_2^*)N_{C^*}(B_2^*)$. Hence $XC(B_2^*) \lhd N(B_2^*)$.

Let $Y = \langle a_0, a_2, B_2 \rangle$ and $Z = N_{XC(B_2^*)}(Y)$. Then as $Y \in \mathrm{Syl}_2(C(B_2^*))$, a Frattini argument shows that $XC(B_2^*) = ZC(B_2^*)$ and, consequently, $|Z : Z \cap C(B_2^*)| = 8$. Also, $Z \leqq N(Y) \leqq N(B_2)$ by (3D), and $Z \cap C(B_2^*) = YO(C)$. Hence if bars denote images in $N(B_2)/D_2O(C)$, $\bar{Z}$ is a 2-subgroup containing $\langle \bar{a}_0, \bar{a}_2 \rangle$ and normalized by $\langle \bar{a}_1, \bar{a}_1^\tau \rangle \cong L_2(2)$. Since $\overline{N(B_2)} \cong L_3(2)$, we must have $\bar{Z} = \langle \bar{a}_0, \bar{a}_2 \rangle$, and since $D_2O(C) = D_2 \times O(C)$ by (6A), it follows that $|Z \cap D_2 : B_2| \geqq 8$. As $|N_{F_2}(B_2^*) : B_2| \leqq 2$, $|Z \cap D_2 \cap F_2 : B_2| \leqq 2$ and hence $|(Z \cap D_2)F_2 : B_2| \geqq 32$. Now since $\overline{A_2^*} \leqq Z(\bar{X})$ by (6I), it follows that $A_2^* \leqq Z(X)$ and hence $Z \leqq C(A_2^*)$. Also, $F_2 \leqq C(A_2^*)$ by (6C). Thus, $(Z \cap D_2)F_2 \leqq C_{D_2}(A_2^*)$ and, consequently, $|D_2 : C_{D_2}(b_1)| \leqq 2$. The argument of (6C) together with (6A) (3) now shows that $Z(D_2) = A_2$.

Now we come to the

PROOF OF (6J). We determine $|V_1 \cap V_2|$. As $z^{V_1 \cap V_2} \leqq z(A_1 \cap A_2)$, $|V_1 \cap V_2 : V| \leqq 8$. It easily follows from (6A) and (6E) that $V_1 \cap V_2 = N_{V_i}(B_{3-i}) = VN_{D_i}(B_{3-i})$ for each i. As $|N_{D_2}(B_1) : B_2| \geqq 8$ by (6.2) and (6D), we have that

$$|V_1 \cap V_2 : V| = 8. \tag{6.6}$$

Let C_1 be as in (6K) or (6L) according as $A_1^* \leqq Z(D_1^*)$ or $A_1^* \not\leqq Z(D_1^*)$. Then $|C_1 \cap V_2 : A_1| = 8$ by the above. As $(C_1 \cap V_2)' \leqq I_1 \leqq D_2$, $(C_1 \cap V_2)D_2/D_2$ is an abelian subgroup of V_2/D_2 containing A_1D_2/D_2, and so we must have $C_1 \cap V_2 \leqq A_1D_2$. Thus

$$C_1 \cap V_2 = A_1(C_1 \cap D_2) \quad \textit{and} \quad |C_1 \cap D_2 : A_1 \cap A_2| = 8.$$

Hence $z^{C_1 \cap D_2} = z(A_1 \cap A_2)$ and $[C_1 \cap D_2, z] = A_1 \cap A_2$. Now set

$$R = A_2(C_1 \cap D_2).$$

Then $R\cap B_2=A_2$ by the modular law and $|R:A_2|=8$ by the foregoing. Also, $z^R=z(A_1\cap A_2)$ and

$$[R, z]=A_1\cap A_2.$$

Hence

$$|R\cap F_2: A_2|=4 \quad \textit{and} \quad [R\cap F_2, z]=A_1\cap I_2.$$

Now $(C_1\cap D_2)^2\leqq A_1\cap D_2\leqq B_2$ and so RB_2/B_2 is an elementary abelian subgroup of D_2/B_2 not contained in F_2/B_2. As K_2 acts transitively on $(F_2/B_2)^{\#}$ and $(D_2/F_2)^{\#}$ by (6A), D_2/B_2 must be elementary abelian. Then using (1G) first to the chain $A_2\leqq B_2\leqq B_2\leqq F_2$ and next to the chain $A_2/I_2\leqq B_2/I_2\leqq F_2/I_2\leqq D_2/I_2$, we obtain

$$D_2^2\leqq A_2.$$

Now $I_i\leqq Z(D_i)$ for each i by (6A) and (6E), and so $I_1I_2\leqq C_{A_2}(C_1\cap D_2)$. Let $c=[s, a_2]$. Then c normalizes both $C_1\cap D_2$ and A_2, and c acts on A_2/I_1I_2 irreducibly. Hence $C_{A_2}(C_1\cap D_2)=I_1I_2$ or A_2. Also, by (6C) A_2 centralizes $C_1\cap F_2$, which is a maximal subgroup of $C_1\cap D_2$. Thus $[C_1\cap D_2, A_2]$ is a c-invariant subgroup of $A_1\cap A_2$ of order 1 or 4. Then the action of c on $A_1\cap A_2$ implies that

$$[C_1\cap D_2, A_2]\leqq A_1\cap I_2.$$

Now

$$R^2=\langle (C_1\cap D_2)^2, [C_1\cap D_2, A_2]\rangle.$$

Hence $R^2\leqq A_1\cap A_2$ and then $(RB_2)^2=(R\langle z\rangle)^2=\langle R^2, [R, z]\rangle\leqq A_1\cap A_2$. Let $a=[r, a_1][s, a_2]$ and $X=RB_2\cap R^aB_2\cap F_2$. Then $X^2\leqq(RB_2)^2\cap(R^aB_2)^2\leqq(A_1\cap A_2)\cap(A_1\cap A_2)^a=\langle b_4\rangle\leqq J_2$. Thus X/J_2 is elementary abelian and, as $|F_2: RB_2\cap F_2|=2$, $B_2/J_2<X/J_2$. As K_2 acts transitively on $(B_2/J_2)^{\#}$ and $(F_2/B_2)^{\#}$, F_2/J_2 must be elementary abelian, and then (1G) applied to the chain $I_2\leqq J_2\leqq B_2\leqq F_2$ shows that

$$F_2^2\leqq I_2.$$

Let Y be an S_7-subgroup of K_2 and take a Y-invariant subgroup Z of F_2 such that $F_2/I_2=B_2/I_2\times Z/I_2$. We define

$$E_2=A_2Z.$$

By (1F) Z is abelian and, as $A_2\leqq Z(F_2)$ by (6C),

$$E_2 \textit{ is an abelian subgroup of } F_2$$

such that $F_2=B_2E_2$ and $B_2\cap E_2=A_2$. As $|F_2: Z(F_2)|>4$,

$$E_2\lhd N(B_2).$$

Now commutation by z induces an isomorphism $F_2/B_2\rightarrow I_2$ which is commutable with the action of K_2. This in particular implies that c acts irreducibly on $R\cap F_2/A_2$ as $[R\cap F_2, z]=A_1\cap I_2$. As $[F_2/A_2, c]\leqq E_2/A_2$, it follows that

$$R\cap F_2=R\cap E_2.$$

Consequently,

$$C_1\cap F_2 \textit{ is abelian.}$$

Now define

$$E_1=C_1\cap F_1.$$

Then $E_1\leqq N_{C_1}(B_2)=C_1\cap V_2$ by (6E) and (3E), so $|E_1\cap D_2:A_1\cap A_2|=2$ and $[E_1\cap D_2, z]=I_1$. So $E_1\cap D_2\not\leqq C_1\cap F_2$ and, as $|C_1\cap D_2:C_1\cap F_2|=2$, we have that

$$C_1\cap D_2=(E_1\cap D_2)(C_1\cap F_2).$$

Now either $E_1\leqq Z(C_1)$ or $E_1=A_1Z(C_1)$ with $|Z(C_1)|=4$. If $|Z(C_1)|=4$, then $[Z(C_1), A_2]\leqq Z(C_1)\cap A_2=I_1$ and so $[Z(C_1), I_2]\leqq I_1\cap I_2=1$, which forces $Z(C_1)\leqq E_1\cap D_2$. Thus, if $|Z(C_1)|=4$, then

$$C_1\cap D_2=Z(C_1)(C_1\cap F_2) \quad \textit{and} \quad [Z(C_1), A_2]\leqq I_1.$$

In any case,

$$C_1\cap D_2 \textit{ is abelian.}$$

Suppose that (6K) (3) holds. Then $A_1\leqq Z(D_1)$ and so commutation by z induces an isomorphism $D_1/B_1\to A_1$. This implies $V_1/B_1\cong U$, so V_1/B_1 has only one E_{64}-subgroup by (2B). Now let $A=(C_1\cap V_2)R$ and $B=\langle z\rangle A$. Then $B/B_1=D_1\cap V_2/B_1\times B_1B_2/B_1$ and so $B/B_1\cong E_{64}$. Thus $V_1\leqq N(B)$ and, as $|V_1:UC_1|=2$ and $A=B\cap UC_1$, we have $V_1\leqq N(A)$. Let bars denote images in $N(A_1\cap I_2)/A_1\cap I_2$ and notice $V_i\leqq N(A_1\cap I_2)$. Then $\bar{A}=(\bar{C}_1\cap\bar{D}_2)\bar{I}_2* \bar{A}_1\bar{A}_2$ and $(\bar{C}_1\cap\bar{D}_2)\bar{I}_2\cap\bar{A}_1\bar{A}_2=\bar{I}_1\bar{I}_2$. As $(C_1\cap D_2)I_2$ is abelian and as $Z(\bar{A}_1\bar{A}_2)=\bar{I}_1\bar{I}_2$, it follows that $Z(\bar{A})=(\bar{C}_1\cap\bar{D}_2)\bar{I}_2$. Thus $V_1\leqq N((C_1\cap D_2)I_2)$ and then V_1 normalizes $\Omega_1((C_1\cap D_2)I_2)=(E_1\cap D_2)I_2$. But then $[C_1, I_2]\leqq C_1\cap(E_1\cap D_2)I_2=E_1\cap D_2\leqq E_1$, which implies that I_2 centralizes C_1/E_1. This is a contradiction as $C_1/E_1\cong A_1/I_1$ as modules for I_2. Therefore, (6K) (3) does not hold, and there are three possibilities for the structure of C_1:

1. $C_1\cong E_{2^{10}}$;
2. $C_1\cong D_8*D_8*D_8*D_8*E_4$;
3. $C_1\cong D_8*D_8*D_8*D_8*Z_4$.

In each of these cases, $(C_1\cap F_2)^2\leqq I_1\cap I_2=1$ so, as K_2 is irreducible on E_2/A_2,

$$E_2 \textit{ is elementary abelian.}$$

In Case 1, $(C_1\cap D_2)^2=1$ and $[C_1\cap D_2, A_2]=1$ by (6M). In Cases 2 and 3, $(C_1\cap D_2)^2=(Z(C_1)(C_1\cap F_2))^2=Z(C_1)^2\leqq I_1$ and $[C_1\cap D_2, A_2]=[Z(C_1), A_2]\leqq I_1$, while $[C_1\cap D_2, A_2]\leqq A_1\cap I_2$ as shown before. Thus

$$R^2=\begin{cases} 1 & \textit{in Cases 1 and 2,} \\ I_1 & \textit{in Case 3,} \end{cases}$$

and in any case, $[C_1\cap D_2, A_2]=1$. Hence $F_2<C_{D_2}(A_2)$ and, as K_2 is irreducible on D_2/F_2, it follows that

$$Z(D_2)=A_2.$$

Consequently, commutation by z induces an isomorphism $D_2/B_2\to A_2$. Also,

$$R \text{ is abelian}$$

and, as $|RE_2: R\cap E_2|=4$, $|[R, E_2]|=2$. Since $[R, E_2]$ is contained in $D_2'=A_2$ and normalized by $\langle U, s\rangle$, it follows that

$$[R, E_2]\leqq I_1.$$

Now we define

$$C_2=\langle R, R^a, R^b\rangle,$$

where $b=[r, a_1]$. As $R\cap F_2=R\cap E_2$, $R\cap E_2$ is a maximal subgroup of E_2 with $[R\cap E_2, z]=A_1\cap I_2=\langle b_2, b_4\rangle$, and hence $[R^a\cap E_2, z]=\langle b_4, b_6\rangle$ and $[R^b\cap E_2, z]=\langle b_2, b_6\rangle$. Therefore, there is an element $e\in E_2$ such that $E_2=\langle e\rangle(R\cap E_2)$, $R^a\cap R^b\cap E_2=\langle e\rangle A_2$, and $[e, z]=b_6$. Consequently

$$E_2\leqq C_2.$$

Since $[R, E_2]\leqq I_1$ and $R'=1$, it follows that $[C_2, e]\leqq I_1$ and hence $[D_2, \langle e\rangle A_2]\leqq\langle b_1, b_6\rangle$. Now a_1^r normalizes $\langle b_6\rangle$ and hence $[D_2, \langle e\rangle A_2]$, but a_1^r does not normalize $\langle b_1, b_6\rangle$. This forces $[D_2, \langle e\rangle A_2]\leqq\langle b_6\rangle$. Hence $[R, e]\leqq\langle b_1\rangle\cap\langle b_6\rangle=1$ and, since $E_2=\langle e\rangle(R\cap E_2)$, it follows that $[R, E_2]=1$. Therefore,

$$[C_2, E_2]=1.$$

Now $A_2\leqq C_2$ and $[C_2, z]=A_2$. Hence $D_2=B_2C_2$ and, as $[E_2, z]\neq 1$, we have $B_2\cap C_2=A_2$ and $C_2\cap F_2=E_2$. Moreover, $C_2=C_{D_2}(E_2)$ and so $C_2\lhd N(B_2)$. Clearly, $C_1\cap C_2=C_1\cap D_2$ and $C_2\cap V_1=R=A_2(C_1\cap C_2)$. Thus we have shown that C_i, $i\in\{1, 2\}$, is a normal subgroup of $N(B_i)$ satisfying (1)-(4) of (6J). It remains to prove that (5) or (6) of (6J) holds. In Cases 1 and 2, R is elementary abelian and hence so also is RE_2. As K_2 is transitive on $(C_2/E_2)^\#$, it follows that C_2 is elementary abelian. In Case 3, $R^2=I_1$ and hence $\Omega_1(RE_2)=E_2$, which implies $C_2^2=A_2$ and $\Omega_1(C_2)=E_2$. Also, C_2 can not be abelian and hence $Z(C_2)=E_2$. Finally, $C_1\cap C_2=C_1\cap D_2=Z(C_1)(C_1\cap F_2)=Z(C_1)(C_1\cap Z(C_2))$. Thus we have proved all parts of (6J).

(6N) LEMMA. *The following conditions hold*:
(1) *commutation by z induces an isomorphism $D_2/B_2\to A_2$*;
(2) $Z(D_2)=D_2^2=A_2$;
(3) $b_1\in Z(V_i)$ *for each* $i\in\{1, 2\}$.

PROOF. (1) and (2) are implicit in the proof of (6J). Since $b_1\in Z(V)\cap Z(D_i)$, (3) follows.

(6O) LEMMA. *Let* $A=(C_1\cap V_2)(C_2\cap V_1)$, $B=\langle z\rangle A$, *and* $N=N(B)\cap C(b_1)$. *Then the following conditions hold for each* i:

(1) $V_i\leqq N_N(B_i)\leqq N_N(D_i\cap V_{3-i})$;

(2) $N\leqq N(C_i\cap V_{3-i})$.

PROOF. By (6J) (3), $BD_i=A_{3-i}D_i$. We assert that B/B_i is the only $E_{2^{7-i}}$-subgroup of V_i/B_i whose product with D_i/B_i is equal to $A_{3-i}D_i/B_i$. Indeed, $B/B_i=D_i\cap V_{3-i}/B_i\times B_1B_2/B_i$ is elementary abelian of order 2^{7-i} by (6.4) and (6N) (2). If $i=1$, then $A_2D_1/F_1\cong A_1A_2/I_1$ by (6E), and the assertion follows from the fact that A_2/I_1 is the only E_{32}-subgroup of A_1A_2/I_1: see Table 1. If $i=2$, then $A_1D_2/B_2\cong A_1A_2$ by (6N) (1), and the assertion follows from (2B) (3). Since $A_{3-i}D_i\lhd V_i$ and $b_1\in Z(V_i)$, we conclude that $V_i\leqq N$. Since $D_i\cap V_{3-i}=D_i\cap B$, (1) follows.

Let bars denote images in $C(b_1)/\langle b_1\rangle$. Then $\bar C_1\cap\bar V_2\in\mathcal{E}_{2^7}(\bar B)$ and $\bar C_2\cap\bar V_1\in\mathcal{E}_{2^8}(\bar B)$ by (6J) (4). Hence $\bar A=(\bar A_1\bar A_2)*(\bar C_1\cap\bar C_2)$ and $\bar C_2\cap\bar V_1$ is the only E_{2^8}-subgroup of $\bar A$. Let $\bar X\in\mathcal{E}_{2^8}(\bar B)$ and suppose that $\bar X\not\leqq\bar A$. Then since $|(\bar X\cap\bar A)(\bar C_1\cap\bar C_2)|\leqq 2^8$, it follows that $|\bar X\cap\bar A:\bar X\cap\bar C_1\cap\bar C_2|\leqq 2^3$, hence $|\bar X\cap\bar C_1\cap\bar C_2|\geqq 2^4$. However, since $\bar B=\bar A\bar X$, $\bar X\cap\bar C_1\cap\bar C_2\leqq C_{\bar C_1\cap\bar C_2}(\bar z)\leqq\bar C_1\cap\bar C_2\cap\bar F_1$. This is a contradition as $|\bar C_1\cap\bar C_2\cap\bar F_1|\leqq 8$. Thus we have shown that $\bar C_2\cap\bar V_1$ is the only E_{2^8}-subgroup of $\bar B$. Consequently, $N\leqq N(C_2\cap V_1)$.

Let $\bar Y\in\mathcal{E}_{2^7}(\bar B)$ be such that $|\bar Y(\bar C_2\cap\bar V_1)|=2^{10}$, and suppose $\bar Y\not\leqq\bar A$. If $|\bar Y\cap\bar A:\bar Y\cap\bar C_1\cap\bar C_2|\leqq 2^2$, then $|\bar Y\cap\bar C_1\bar C_2|\geqq 2^4$, and this yields the same contradiction as before. If $|\bar Y\cap\bar A:\bar Y\cap\bar C_1\cap\bar C_2|\geqq 2^3$, then $(\bar Y\cap\bar A)(\bar C_1\cap\bar C_2)=\bar C_2\cap\bar V_1$ by the last paragraph and so $|\bar Y(\bar C_2\cap\bar V_1)|=2^9$, a contradiction. Therefore, $\bar Y\leqq\bar A$ and hence $\bar Y(\bar C_2\cap\bar V_1)=\bar A$ and $|\bar Y\cap\bar C_2\cap\bar V_1|=2^5$. As $Z(\bar A)=\bar C_1\cap\bar C_2$ has order 2^5, we must have $\bar Y\cap\bar C_2\cap\bar V_1=\bar C_1\cap\bar C_2$, and consequently $\bar A=\bar Y\bar A_1\bar A_2$ and $\bar Y\cap\bar A_2=\bar A_1\cap\bar A_2$. Thus $\bar Z=\bar Y\cap\bar A_1\bar A_2$ is an elementary abelian subgroup of $\bar A_1\bar A_2$ such that $\bar Z\bar A_2=\bar A_1\bar A_2$ and $\bar Z\cap\bar A_2=\bar A_1\cap\bar A_2$. An easy computation using Table 1 gives $\bar Z=\bar A_1$. Therefore, $\bar Y=\bar Z(\bar C_1\cap\bar C_2)=\bar C_1\cap\bar V_2$, which implies that $\bar C_1\cap\bar V_2$ is the only E_{2^7}-subgroup of $\bar B$ whose product with $\bar C_2\cap\bar V_1$ has order 2^{10}. Hence $N\leqq N(C_1\cap V_2)$.

7. $Sp_6(4)$ and $Sp_6(2)\times Sp_6(2)$.

In (6J) we have shown that the structure of C_2 is quite different according as $Z(C_1)^2=1$ or $Z(C_1)^2\neq 1$. Therefore, we shall separate these two cases from now on. In this section, we consider the following situation:

HYPOTHESIS 3. $|\Omega|=64$ and $Z(C_1)^2=1$.

This implies that either $A_1^*\leqq Z(D_1^*)$ and $C_1^2=1$ or $A_1^*\not\leqq Z(D_1^*)$ and $C_1\cong D_8*D_8*D_8*D_8*E_4$ with $C_1^2=I_1$: see (6J)-(6L). Also, $C_2^2=1$ by (6J) (5). We shall prove the following:

THEOREM 3. *Under Hypothesis* 3, *if either* $G=G'$ *or* $G\neq G'$ *and* $O(N_{G'}(X))=1$ *for every* 2-*subgroup* X *of* G', *then* $\langle L^G\rangle\cong Sp_6(4)$ *or* $Sp_6(2)\times Sp_6(2)$.

The argument is similar to that of Theorem 2. First, we prove an analogue of (5G).

(7A) LEMMA. *Let* $M_2=E(N(C_2)\bmod C_2)$. *Then the following conditions hold*:
(1) $M_2/C_2\cong L_3(4)$, $SL_3(4)$, *or* $L_3(2)\times L_3(2)$;
(2) $C_{M_2/C_2}(z)=K_2C_2/C_2$;
(3) $\langle z\rangle M_2$ *contains an* S_2-*subgroup of* $N(C_2)$;
(4) $[M_2, O(C)]=1$;
(5) $C_1\leqq M_2$.

PROOF. As $C_2^2=1$, $\mathcal{E}^*(D_2)=\{B_2, C_2\}$ by (1E) (1). Therefore, the assertions will follow from (1K) just as in (5G) once we verify that, in $\overline{N(C_2)}=N(C_2)/C_2$, $\bar{z}$ is a noncentral involution.

We show $N\leqq N(C_2)$. Notice that $V_2\leqq N\leqq N(C_2\cap V_1)$ by (6O). Clearly, $A_1D_2/C_2\cap V_1=B/C_2\cap V_1\times C_2/C_2\cap V_1$. Hence $A_1D_2\leqq C_N(B/C_2\cap V_1)$. As $C_2\cap V_1\leqq D_2\cap V_1\leqq B$ and $\mathcal{E}^*(D_2\cap V_1)=\{B_2, C_2\cap V_1\}$, we have $C_N(B/C_2\cap V_1)\leqq N_N(B_2)$ and hence $C_N(B/C_2\cap V_1)\leqq C_{N(B_2)}(A_1D_2/D_2)$. The structure of $N(B_2)/D_2$ shows that A_1D_2 is the only S_2-subgroup of $C_{N(B_2)}(A_1D_2/D_2)$. Therefore, A_1D_2 is the only S_2-subgroup of $C_N(B/C_2\cap V_1)$, and since $C_N(B/C_2\cap V_1)\lhd N$, it follows that $A_1D_2\lhd N$. Now $C_{V_2}(A_2)=C_V(A_2)D_2=B_2D_2=D_2$ by (6N), and hence $C_{A_1D_2}(C_2\cap V_1)=C_2$. Therefore, $N\leqq N(C_2)$.

As a consequence we have $V_1\leqq N(C_2)$, and since $|\bar{V}_1|=2^6$, it follows that $\bar{z}$ is a noncentral involution. Hence (1)-(4) follow. Also, $\bar{C}_1$ is an E_{16}-subgroup by (6J) (2), and so (1)-(3) show that $C_1\leqq M_2$.

Next, the argument of (5F) and (5H) yields

(7B) LEMMA. C_2 *is the only* $E_{2^{12}}$-*subgroup of* $N(C_2)$.

Our proof of the following result requires the hypothesis on G'.

(7C) LEMMA. $N(C_1)$ *has a* z-*invariant subgroup* M_1 *with* $C_1\leqq M_1$ *satisfying the following conditions*:

(1) *$M_1/C_1 \cong Sp_4(4)$ or $Sp_4(2)\times Sp_4(2)$, and $M_1 \lhd N(C_1)$ if $M_1/C_1\cong Sp_4(4)$;*
(2) *$C_{M_1/C_1}(z)=K_1C_1/C_1$;*
(3) *$[M_1, O(C)]=1$;*
(4) *$C_2\leqq M_1$.*

PROOF. If $C_1^2=1$, then $\mathcal{E}^*(D_1)=\{B_1, C_1\}$ by (1E) (1), so $N(D_1)=N(B_1)\leqq N(C_1)$. Suppose that $C_1^2\neq 1$. Then $A_1^*\not\leqq Z(D_1^*)$ by (6K) and so $Z(D_1)=I_1$ by (6E) (3). Furthermore, $\mathcal{E}^*(D_1/I_1)=\{C_1/I_1, F_1/I_1\}$ by (6E) (4) and (6L). Thus $N(D_1)\leqq N(C_1)\cap N(F_1)$. The same lemmas together with Hypothesis 3 imply that $\mathcal{E}^*(F_1)=\{B_1, C_1\cap F_1\}$. Hence $N(D_1)=N(B_1)\leqq N(C_1)$ in this case as well.

Let bars denote images in $N(C_1)/C_1$. Then $C(\bar{z})=\langle\bar{z}\rangle\times\bar{K}_1\times\overline{O(C)}$ by the last paragraph. Also, $\mathcal{E}^*(D_1\cap V_2/I_1)=\{C_1\cap V_2/I_1, F_1/I_1\}$; so the arguments of the second paragraph of the proof of (7A) show that $N\leqq N(C_1)$. In particular, $C_2\leqq N(C_1)$, and $\bar{C}_2$ is an E_{2^6}-subgroup of $\overline{N(C_1)}$ by (6J) (2). Now $N(C_2)$ contains an S_2-subgroup of G by (7B) and so

$$|G|_2=2^{19} \tag{7.1}$$

by (7A). Hence $|\overline{N(C_1)}|_2\leqq 2^9$. Now we let $\bar{X}=X/C_1=E(\overline{N(C_1)})$. Then (1I) shows that either $\bar{X}\cong Sp_4(4)$ or $\bar{X}/O(\bar{X})\cong A_6\times A_6$. Furthermore, if $\bar{X}\cong Sp_4(4)$ then $\bar{C}_2\leqq\bar{X}$ and $\bar{K}_1=\bar{K}_1'\bar{A}_2\leqq\bar{X}$, and so $M_1=X$ satisfies (1)-(4). Therefore, assume that $\bar{X}/O(\bar{X})\cong A_6\times A_6$. Set $Y/C_1=O(\overline{N(C_1)})$ and $Z/C_1=O(\bar{X})$. Since $\bar{C}_2\in\mathcal{E}_{2^6}(\overline{N(C_1)})$, (1.b) of (1I) must hold in $\overline{N(C_1)}$, and so there exists a subgroup $\bar{M}=M/C_1$ of $\overline{N(C_1)}$ containing $\langle\bar{Y}, \bar{C}_2\rangle$ such that $M/Y\cong Sp_4(2)\times Sp_4(2)$ and $N(C_1)=\langle z\rangle M$. If $\bar{Z}=1$, then $\bar{X}\bar{C}_2\cong Sp_4(2)\times Sp_4(2)$ and $\bar{K}_1=\bar{K}_1'\bar{A}_2\leqq\bar{X}\bar{C}_2$; so $M_1=XC_2$ satisfies (1)-(4) ($O(C)$ centralizes XC_2 by (1I) and (7A) (4)).

Therefore, assume $\bar{Z}\neq 1$. Then $|\bar{Z}|=3$ by (1I). We show that this case does not occur. As $\bar{A}_2=[\bar{C}_2, \bar{z}]$ centralizes $O(C(\bar{z}))=\overline{O(C)}$, $[\bar{A}_2, \bar{Y}]=1$ by (1C). This implies that $Y\leqq N(A_2C_1)$. Now Table 2 and (6E) show $|C_{C_1/I_1}(b_3b_6)|\leqq 2^5$, hence $C_{C_1/I_1}(b_3b_6)=C_1\cap C_2/I_1$ and then $C_{A_2C_1/I_1}(b_3b_6)=C_2\cap V_1/I_1$. Also, since $C_1^2\leqq I_1$ and $C_V(A_1/I_1)=B_1$, it follows that $C_{A_2C_1}(A_1/I_1)=C_1$. Hence if $C_1^2=1$, then $Z(A_2C_1)=C_1\cap C_2$, while if $C_1^2=I_1$, then $Z(A_2C_1/I_1)=C_1\cap C_2/I_1$. Therefore, $Y\leqq N(C_1\cap C_2)$. Let $y\in Y$. Then $A_1\cap(C_2\cap V_1)^y=A_1\cap C_1\cap(C_2\cap V_1)^y=A_1\cap(C_1\cap C_2\cap V_1)^y=A_1\cap(C_1\cap C_2)^y=A_1\cap C_1\cap C_2=A_1\cap A_2$. Thus $A_1(C_2\cap V_1)^y/A_1$ is an E_{2^6}-subgroup of A_2C_1/A_1. Now A/A_1 is the only E_{2^6}-subgroup of A_2C_1/A_1 and $C_2\cap V_1/I_1$ is the only E_{2^8}-subgroup of A/I_1: see the proof of (6O). Thus we conclude that $Y\leqq N(C_2\cap V_1)$.

Assume that $C_{ZC_2}(C_2\cap V_1)=C_2$. Then $Z\leqq N(C_2)$ by the last paragraph and so $[\bar{Z}, \bar{C}_2]=1$. Hence if we set $M_0=\langle X, C_2\rangle$, then $\bar{M}_0/\bar{Z}\cong Sp_4(2)\times Sp_4(2)$ and $\bar{Z}=Z(\bar{M}_0)$. However, this yields a contradiction as the Schur multiplier of $Sp_4(2)\cong\Sigma_6$ has order 2: see [17], Satz V. 25.12. Assume, therefore, that

$C_{ZC_2}(C_2\cap V_1)>C_2$. Since $C_{C_1}(b_3b_6)=C_1\cap C_2$, it follows that $C_{C_1C_2}(C_2\cap V_1)=C_2$. So $C_{ZC_2}(C_2\cap V_1)\not\leq C_1C_2$ and as $|ZC_2:C_1C_2|=3$, $ZC_2=C_1C_{ZC_2}(C_2\cap V_1)$. Thus $[C_1\cap C_2, Z]=1$ if $C_1^2=1$, and $[C_1\cap C_2/I_1, Z]=1$ if $C_1^2=I_1$. Hence if we set $C_0=\langle(C_1\cap C_2)^X\rangle$, then either $[C_0, Z]=1$ or $[C_0/I_1, Z]=1$. Now $A_1=\langle(A_1\cap A_2)^{K_1'}\rangle\leqq C_0$, hence $C_1\cap V_2=A_1(C_1\cap C_2)\leqq C_0$. Hence $C_1=\langle(C_1\cap V_2)^{K_1'}\rangle\leqq C_0$. Thus $C_1=C_0$ and either $[C_1, Z]=1$ or $[C_1/I_1, Z]=1$.

Now take an S_3-subgroup Z_0 of Z so that $Z_0\leqq X'$. This is possible as X/C_1 is perfect. Then $[C_1, Z_0]=1$, so $1<Z_0\leqq O(N_{G'}(C_1))$. Also, since $C_2=[C_2, K_2]\leqq G'$, $C_1=C_0\leqq G'$. Thus our hypothesis forces $G=G'$.

Let $\langle z\rangle C_1C_2\leqq T\in \mathrm{Syl}_2(\langle z\rangle M_2)$ and $S=T\cap M_2$. Then $T=\langle z\rangle S$, and $T\in\mathrm{Syl}_2(G)$ by (7A) and (7B). Since $z\in G'$, (1D) shows that S contains a conjugate x of z. Since C_1C_2/C_2 is an E_{16}-subgroup of M_2/C_2, (7A) (1) shows that we may choose $x\in C_1C_2$. Now $x\notin C_2$ and $|C_{C_2}(x)|\leqq 2^6$ as $m(C)=7$. Since $|C_2|=2^{12}$, (1E) shows that $I(xC_2)=x^{C_2}$. We assert that

$$C_1=(C_1\cap C_2)(C_1\cap C_2)^{tst}.$$

Indeed, $C_1\cap C_2\cap(C_1\cap C_2)^{tst}$ normalizes $B_1\cap B_2\cap(B_1\cap B_2)^{tst}=\langle b_1, z\rangle$ and so is contained in $C_1\cap F_1$. Moreover, it intersects with A_1 in I_1. Hence its order is at most 4 and, since $|C_1:C_1\cap C_2|=2^4$, the assertion follows. Consequently, $C_2^{tst}\cap xC_2$ contains an involution and then $C_2\cap z^G\neq\varnothing$, which is a contradiction. Thus we have proved the lemma.

As a final step of our 2-local analysis in this section, we prove the following:

(7D) LEMMA. *M_1 and M_2 have a common z-invariant S_2-subgroup S such that $C_S(z)=U$.*

PROOF. By (7C) (2), $V_1\leqq\langle z\rangle M_1$ and $UC_1C_2\leqq M_1$. Let $V_1C_1C_2\leqq T\in\mathrm{Syl}_2(\langle z\rangle M_1)$ and set $S=T\cap M_1$. Then $S\in\mathrm{Syl}_2(M_1)$ and S is z-invariant. The assertion is now verified by the argument in the last paragraph of the proof of (5K).

Now we complete the proof of Theorem 3. As in the cases of Theorems 1 and 2, results in [10] show that there exists a z-invariant semisimple subgroup G_0 containing L such that $G_0/Z(G_0)\cong Sp_6(4)$ or $Sp_6(2)\times Sp_6(2)$. Furthermore, $|C(G_0)|$ is odd by (3B). In particular, $|Z(G_0)|$ is odd and so $Z(G_0)=1$ by a table on p. 60 of [6].

Assume first that $G_0\cong Sp_6(4)$. In this case, we argue just as in the last paragraph of the proof of Theorem 2. Since $m(C_{G_0}(x))=12$ for every $x\in I(G_0)$ and since $\mathrm{Aut}(G_0)-\mathrm{Inn}(G_0)$ has only one conjugacy class of involutions, it follows that an analogue of (4I) holds. Then the results in [19] shows $G_0O(G)\lhd G$.

Hence $\langle L^G\rangle \cong Sp_6(4)$.

Therefore, assume that $G_0\cong Sp_6(2)\times Sp_6(2)$, and let $G_0=G_1\times G_2$, where $G_2=G_1^z\cong Sp_6(2)$. Let $z\in T\in \mathrm{Syl}_2(N(G_0))$ and $S_0=T\cap G_0$. As $|C(G_0)|$ is odd and $\mathrm{Out}(G_i)=1$, we have $|T|=2^{19}$ and so $T\in \mathrm{Syl}_2(G)$ by (7.1). Also, $T=\langle z\rangle S_0$ and $z^G\cap S_0=\varnothing$ as $m(C_{G_0}(x))=12$ for every $x\in I(G_0)$. Thus $z\notin G'$ by (1D), and since $S_0\leqq G'$, it follows that $S_0\in \mathrm{Syl}_2(G')$. Now S_0 contains a z-invariant $E_{2^{12}}$-subgroup E such that $N_{G_0}(E)/E\cong L_3(2)\times L_3(2)$. Let $N_{G_0}(E)/E=N_1/E\times N_2/E$ with $N_i/E\cong L_3(2)$. (7B) shows that E is conjugate to C_2, so the structure of $N(C_2)/C_2$ shows that $N_{G_0}(E)\lhd N(E)$, see (7A). Consequently, $N(E)$ permutes N_1 and N_2. Let $E_i=E\cap G_i$ for each i. Then $E_i=C_E(N_{3-i})$ and so $N(E)$ permutes E_1 and E_2. Since $S_0\in \mathrm{Syl}_2(N_{G'}(E))$ and $S_0\leqq N(E_i)$, we conclude that $N_{G'}(E)\leqq N(E_1)\cap N(E_2)$.

Let $S_i=S_0\cap G_i$ for each i, so that $S_0=S_1\times S_2$. We claim that S_i is strongly involution closed in S_0 with respect to G'. Suppose, by way of contradiction, that $x^g\in S_0-S_i$ for some $x\in I(S_i)$ and $g\in G'$. Conjugating in G_0, we may choose $x\in E_i$ and $x^g\in E-E_i$. But $S_0\in \mathrm{Syl}_2(G')$ and as E is the only $E_{2^{12}}$-subgroup of S_0, $x^g=x^h$ for some $h\in N_{G'}(E)$, and so $x^g\in E_i$ by the last paragraph. This is a contradiction proving the claim. We can now apply Corollary 2 of [20] to conclude that $\langle L^G\rangle\cong Sp_6(2)\times Sp_6(2)$: see (6R) of [9].

8. $U_7(2)$ and $L_7(2)$.

In this section we consider the following situation:

HYPOTHESIS 4. $|\Omega|=64$ and $Z(C_1)^2\neq 1$.

This implies that $A_1^*\nleqq Z(D_1^*)$ and $C_1\cong D_8*D_8*D_8*D_8*Z_4$ with $C_1^2=I_1$: see (6J) and (6K). For the structure of C_2, see (6J) (6). We shall prove the following:

THEOREM 4. *Under Hypothesis 4,* $\langle L^G\rangle\cong U_7(2)$ *or* $L_7(2)$.

First, we sharpen (6O).

(8A) LEMMA. *The following conditions hold for each* i:
(1) $N=V_iN_N(D_{3-i}\cap V_i)$ *and* $V_i\cap N_N(D_{3-i}\cap V_i)=V_1\cap V_2$;
(2) $|N_N(D_i\cap V_{3-i}):N_N(B_i)|=2$;
(3) $N_N(D_i\cap V_{3-i})\leqq N(D_i)\cap N(C_i)$.

PROOF. Fix $i\in\{1, 2\}$ and set $X=V_i\cap N_N(D_{3-i}\cap V_i)$. Then $V_1\cap V_2\leqq X$ and X normalizes $B_i\cap D_{3-i}\cap V_i=B_1\cap B_2$. (3C) shows that $V=C_X(z)$ has index at most 8 in X, while $|V_1\cap V_2: V|=8$ by (6.6). Thus we have $X=V_1\cap V_2$, proving the second equation in (1). Consider the action of N on $\bar{B}=B/C_{3-i}\cap V_i$. Since $\bar{B}=\langle\bar{z}\rangle\bar{A}$ and $N\leqq N(A)$, it follows that $|N:C_N(\bar{z})|\leqq|\bar{A}|=2^{i+1}$. As $C_N(\bar{z})=$

$N_N(D_{3-i}\cap V_i)$ and as $|V_i : C_{V_i}(\bar{z})| = |V_i : V_1\cap V_2| = 2^{i+1}$, the first equation of (1) holds.

Now $B = B_1B_2D_1\cap B_1B_2D_2$. It easily follows from this and the structure of $N(B_i)$ that $N_N(B_i) = D_iN_C(B_1B_2)$: see (6A) and (6E). Hence $|N_N(B_i): V_i| = 3|O(C)|$, and so (1) shows that

$$|N_N(D_1\cap V_2): N_N(B_1)| = |N_N(D_2\cap V_1): N_N(B_2)|.$$

Let n denote this index.

Assume $n=1$. Then $C_N(B/C_1\cap V_2)\leqq N_N(B_1)$, and the argument in the second paragraph of the proof of (7A) and of (7C) shows that $N\leqq N(C_1)$. Thus $Z(C_1)\lhd C_2$ by (6J) (6). It also follows from (6J) (6) that $b_1\in Z(C_1)\nleqq C_2\cap F_2$. Since commutation by z induces an isomorphism $C_2/A_2\to A_2$ that carries $C_2\cap F_2/A_2$ onto I_2, it follows that C_2 is generated by the conjugates of $Z(C_1)$ under K_2. As $C_{K_2}(b_1)\leqq K_1\leqq N(C_1)$, two distinct conjugates of $Z(C_1)$ under K_2 intersect in the identity element and so commute with each other. However, this implies that C_2 is abelian, a contradiction. Therefore, $n\neq 1$.

Now $Z(D_1)=I_1$ and $\mathcal{E}^*(D_1/I_1)=\{C_1/I_1, F_1/I_1\}$, so $N(B_1)\leqq N(D_1)\leqq N(C_1)\cap N(F_1)$. By (6E) (4), B_1 is one of the two members of $\mathcal{E}^*(F_1)$. Hence

$$|N(F_1): N(B_1)|\leqq 2.$$

In particular, $N(B_1)<N(F_1)$ and $N(F_1)$ normalizes $D_1=O_2(N(B_1))$. Thus

(8.1) $$N(D_1)=N(F_1)\leqq N(C_1).$$

Now $F_1\leqq N_{V_1}(B_2)\leqq V_2$ by (6E). So $\mathcal{E}^*(D_1\cap V_2/I_1)=\{C_1\cap V_2/I_1, F_1/I_1\}$, and hence $N_N(D_1\cap V_2)\leqq N_N(F_1)$. Thus $n=2$ and (3) holds for $i=1$.

By (6J) (6) and (6O) (2), $Z(C_2)\cap V_1=\Omega_1(C_2\cap V_1)\lhd N$. Also, $C_{V_2}(Z(C_2)\cap V_1)=C_2$ as $C_{V_2}(A_2)=D_2$ by (6N) (2). Thus $N_N(V_2)\leqq N_N(C_2)$ and, as $D_2=(D_2\cap V_1)C_2$, $N_N(V_2)\cap N_N(D_2\cap V_1)\leqq N(D_2)\cap N(C_2)$. Now $V_2\in \mathrm{Syl}_2(N_N(B_2))$ and $N_N(B_2)\lhd N_N(D_2\cap V_1)$ by (2). Therefore, a Frattini argument shows that $N_N(D_2\cap V_1)\leqq N(D_2)\cap N(C_2)$.

Now we let $\hat{D}_i=O_2(N(D_i)\cap N(C_i))$ and $\hat{V}_i=V\hat{D}_i$ for each $i\in\{1, 2\}$. We next study the structure of $N(D_i)\cap N(C_i)$. As shown above, $N(D_1)\cap N(C_1)=N(D_1)$.

(8B) LEMMA. *The following conditions hold*:
(1) $N(D_2)\cap N(C_2)=N(B_2)\hat{D}_2$ *and* $N(B_2)\cap\hat{D}_2=D_2$;
(2) *commutation by* z *induces an isomorphism* $\hat{D}_2/D_2\to C_2/Z(C_2)$;
(3) $Z(\hat{D}_2/Z(C_2))=C_2/Z(C_2)$.

PROOF. Let $X=N(D_2)\cap N(C_2)$ and let bars denote images in $X/Z(C_2)$. We consider the action of $\bar{X}$ on $\bar{D}_2=\langle\bar{z}\rangle\bar{C}_2$. Since $\langle\bar{z}\rangle=\bar{F}_2$ and since $\mathcal{E}^*(F_2)=\{B_2, Z(C_2)\}$ by (6J) (6), it follows that $C_{\bar{X}}(\bar{z})=\overline{N(B_2)}$. Consequently, $C_{\bar{X}}(\bar{D}_2)=$

$\bar{D}_2\overline{O(C)}$. Notice that $[\bar{D}_2, \overline{O(C)}]=1$ by (6A) (2). The same lemma shows that $\bar{K}_2$ is transitive on $(\bar{D}_2/\langle\bar{z}\rangle)^{\#}$. Since $C_{\bar{X}}(\bar{z})<\bar{X}$ by (8A), it follows that $\bar{z}^{\bar{X}}=\bar{z}\bar{C}_2$ and that $\bar{X}$ is 2-transitive on $\bar{z}^{\bar{X}}$. Arguing now as in (4A), we obtain the result.

(8C) LEMMA. *The following conditions hold*:
(1) $N(D_1)/D_1=N(B_1)/D_1\times\hat{D}_1/D_1$;
(2) $\hat{D}_1/D_1\cong Z_2$.

PROOF. As remarked in the proof of (8A), $|N(F_1):N(B_1)|\leqq 2$ and $N(D_1)=N(F_1)$. Thus $|N(D_1):N(B_1)|=2$ by (2) and (3) of (8A), and $N(D_1)$ acts on the E_{16}-group D_1/F_1. The argument of (4C) then shows that there is a normal subgroup X of $N(D_1)$ such that $N(D_1)=N(B_1)X$ and $N(B_1)\cap X=D_1O(C)$. It therefore suffices to prove that $N(D_1)\leqq C(O(C))O(C)$. Certainly, $N(B_1)\leqq C(O(C))O(C)$ by (6E). (1) and (2) of (8A) show that there is a 2-subgroup $Y\leqq N_N(D_1\cap V_2)\cap N_N(D_2\cap V_1)$ such that $V_1\cap V_2\leqq Y\not\leqq N_N(B_1)$. By (8A) (3), $Y\hat{D}_2$ is a 2-subgroup of $N(D_2)\cap N(C_2)$ containing $\hat{V}_2$ and so, as $\hat{V}_2\in \mathrm{Syl}_2(N(D_2)\cap N(C_2))$ by (8B) (1), $Y\hat{D}_2=\hat{V}_2$. As $[\hat{V}_2, O(C)]=1$ by (8B) (2) and as $N(D_1)=YN(B_1)$, we have that $N(D_1)\leqq C(O(C))O(C)$ as desired.

(8D) LEMMA. *$N(D_1)$ has a normal subgroup $\hat{C}_1$ satisfying the following conditions*:
(1) *$\hat{D}_1=D_1\hat{C}_1$ and $D_1\cap\hat{C}_1=C_1$*;
(2) *$\hat{C}_1$ is extra-special of order 2^{11} and $Z(\hat{C}_1)=I_1$.*

PROOF. This follows from consideration of the structure of $\overline{G^*}=G^*/O(G^*)$ and $N_{G^*}(B_1^*)$ discussed in (6G), (6H), and (6I). Since $\hat{D}_1/D_1\cong Z_2$, it follows that $\hat{D}_1^*/D_1^*\cong Z_2$, hence $2^9=|\hat{V}_1^*|\leqq|G^*|_2$. This implies that $\overline{M^*}\not\cong U_4(3)$. Also, since $A_1^*\not\leqq Z(D_1^*)$, it follows that $\overline{M^*}\not\cong Sp_4(4)$ or $Sp_4(2)\times Sp_4(2)$. Therefore, $\overline{M^*}\cong U_5(2)$ or $L_5(2)$, and by direct check, we obtain that $N_{\overline{G^*}}(\overline{D_1^*})$ is an extension of a group of order 2^8 by $L_2(2)$ and that $O_2(N_{\overline{G^*}}(\overline{D_1^*}))\cap\overline{M^*}$ is an extra-special group of order 2^7. (Here we have used the fact that the action of $\bar{z}$ on $\overline{M^*}$ and the embedding of $\overline{K_1^{*s}}\cong Sp_4(2)$ in $\overline{M^*}$ are unique up to conjugation by the elements of $\overline{M^*}$ and relabeling of the elements of $\overline{K_1^{*s}}$ by a graph automorphism of $\overline{K_1^{*s}}$: see (6I) and (1.1) of [10].) Now $\overline{N_{G^*}(D_1)}\leqq N_{\overline{G^*}}(\overline{D_1^*})$. Comparing orders, we have that equality holds here and that $\hat{D}_1^*\cap M^*$ is an extra-special group of order 2^7. As $\mathcal{E}^*(D_1^*/I_1)=\{C_1^*/I_1, F_1^*/I_1\}$, it then follows that $C_1^*\leqq\hat{D}_1^*\cap M_1^*$. Now we define

$$\hat{C}_1=C_1(\hat{D}_1^*\cap M^*).$$

Since $C_1\lhd N(D_1)$, $\hat{C}_1$ is a subgroup satisfying (1) and $\hat{C}_1^*=\hat{D}_1^*\cap M^*$. By (1A), $\hat{C}_1^*=[\hat{C}_1^*, H^s]*C_{\hat{C}_1^*}(H^s)$. Also, $[\hat{C}_1^*, H^s]\leqq Q^*$ and $Q^*=[Q^*, H^s]I_1$ by (6F). Therefore,

$[\hat{C}_1^*, H^s]=Q^*$ is extra-special of order 32 and $C_{\hat{C}_1^*}(H^s)$ is extra-special of order 8. Notice that $C_{\hat{C}_1^*}(H^s)=C_{\hat{C}_1}(HH^s)$. Now since $Z(C_1)\cong Z_4$, it follows that $\hat{D}_1/C_1 =D_1/C_1\times C_{\hat{D}_1}(Z(C_1))/C_1$. Consequently, $N(D_1)$ centralizes $\hat{D}_1/C_1$ and, in particular, $s\in N(\hat{C}_1)$. Thus $[Q^{*s}, C_{\hat{C}_1}(HH^s)]=1$ and as $C_1=Q^*Q^{*s}C_{C_1}(HH^s)$ by (6.5), we conclude that $\hat{C}_1=Q^* * Q^{*s} * C_{\hat{C}_1}(HH^s)$. Therefore, $\hat{C}_1$ is extra-special of order 2^{11} and $Z(\hat{C}_1)=I_1$.

(8E) LEMMA. *$N(D_2)\cap N(C_2)$ has a normal subgroup $\hat{C}_2$ satisfying the following conditions*:

(1) *$\hat{D}_2=D_2\hat{C}_2$ and $D_2\cap\hat{C}_2=C_2$*;

(2) *$\hat{C}_2/Z(C_2)$ is elementary abelian.*

PROOF. Consider the structure of $N(D_2)\cap N(C_2)/Z(C_2)$ discussed in (8B). Then (1G) shows that $\hat{D}_2/C_2$ is elementary abelian. Hence if X is an S_7-subgroup of K_2, there is an X-invariant subgroup $\hat{C}_2$ satisfying the condition (1). Since $\mathcal{E}^*(F_2)=\{B_2, Z(C_2)\}$, it follows that $N_{\hat{C}_2}(F_2)=N_{\hat{C}_2}(B_2)=C_2$, and then (1F) shows that $\hat{C}_2/Z(C_2)$ is either elementary abelian or homocyclic of rank 3. In any case, $\hat{C}_2/Z(C_2)$ is the unique abelian maximal subgroup of $\hat{D}_2/Z(C_2)$ and so $\hat{C}_2\lhd N(D_2)\cap N(C_2)$.

Now $A_2=D_2^2\lhd\hat{C}_2$ by (6N) (2). Since K_2 is irreducible on $Z(C_2)/A_2$, it follows that $Z(C_2)/A_2\leqq Z(\hat{C}_2/A_2)$. Suppose $\hat{C}_2/Z(C_2)$ is homocyclic of rank 3. Then $Z(C_2)/A_2<Z(\hat{C}_2/A_2)$ by (1B), and as K_2 is irreducible on $\Omega_1(\hat{C}_2/Z(C_2))=C_2/Z(C_2)$, we have $C_2/A_2\leqq Z(\hat{C}_2/A_2)$. Thus commutation by z induces a homomorphism $\hat{C}_2/C_2\to C_2/A_2$, and its image is a complement for $Z(C_2)/A_2$ in C_2/A_2 by (8B) (2). However, since $C_2/A_2\cong A_2$ as K_2-modules by (6N) (2), this yields a contradiction. Therefore, $\hat{C}_2/Z(C_2)$ is elementary abelian.

(8F) LEMMA. *The following conditions hold*:

(1) *$\hat{V}_1\leqq N_N(D_1\cap V_2)$*;

(2) *$\hat{V}_2\cap N\leqq N_N(D_2\cap V_1)$ and $|\hat{V}_2\cap N: V_2|=2$.*

PROOF. (8A) and (8C) show that $N_N(D_1\cap V_2)$ contains an S_2-subgroup of $N(D_1)$. Hence $\hat{V}_1=V\hat{D}_1\leqq N_N(D_1\cap V_2)$. Let $V_2\leqq X\in \mathrm{Syl}_2(N_N(D_2\cap V_1))$. Then $|X: V_2|=2$ and $X\leqq N(D_2)\cap N(C_2)$ by (8A). So $X\hat{D}_2$ is a 2-subgroup of $N(D_2)\cap N(C_2)$ containing $V\hat{D}_2=\hat{V}_2$ and, as $\hat{V}_2\in\mathrm{Syl}_2(N(D_2)\cap N(C_2))$ by (8B), we have that $X\leqq\hat{V}_2$. Thus $\hat{V}_2\cap N_N(D_2\cap V_1)=X$. As $B\cap D_2=D_2\cap V_1$, we have that $\hat{V}_2\cap N\leqq N_N(D_2\cap V_1)$ and hence $\hat{V}_2\cap N=X$. This proves (2).

(8G) LEMMA. *The following conditions hold*:

(1) *$\hat{C}_1\lhd N$*;

(2) *$\hat{C}_2\cap N\lhd N$.*

PROOF. Notice that $\hat{C}_1 \leqq N$ by (8F) (1). We argue as in the second paragraph of the proof of (7A). See also (7C). First of all, $\hat{V}_1 \leqq N \leqq N(C_1 \cap V_2)$. As $A_2\hat{D}_1/C_1 \cap V_2 = B/C_1 \cap V_2 \times \hat{C}_1/C_1 \cap V_2$, we have $A_2\hat{D}_1 \leqq C_N(B/C_1 \cap V_2)$. Next, as $C_N(B/C_1 \cap V_2) \leqq N_N(D_1 \cap V_2) \leqq N(D_1)$ by (8A) (3), we have $C_N(B/C_1 \cap V_2) \leqq C_{N(D_1)}(A_2\hat{D}_1/\hat{D}_1)$. Then the structure of $N(D_1)/\hat{D}_1$ shows that $A_2\hat{D}_1$ is the only S_2-subgroup of $C_N(B/C_1 \cap V_2)$: see (8C). Thus $A_2\hat{D}_1 \lhd N$ and, since $C_{A_2\hat{D}_1}(C_1 \cap V_2/I_1) = \hat{C}_1$, it follows that $\hat{C}_1 \lhd N$. Analogous arguments show that $A_1(\hat{C}_2 \cap N) \lhd N$. Also, $Z(C_2) \cap V_1 = \Omega_1(C_2 \cap V_1) \lhd N$ by (6J). Thus (2) will follow once we prove

$$C_{A_1(\hat{C}_2 \cap N)}(Z(C_2) \cap V_1) = \hat{C}_2 \cap N. \tag{8.2}$$

Let X denote the left-hand side of the above equation. Then $C_2 \leqq X \leqq C_{\hat{V}_2 \cap N}(I_2) = \hat{D}_2 \cap N$ as $I_2 \leqq Z(\hat{D}_2)$ by (2E) (3). However, $X \neq D_2$ or $\hat{D}_2 \cap N$ as $z \notin C(Z(C_2) \cap V_1)$. Notice that $|\hat{D}_2 \cap N : C_2| = 4$ by (8F) (2). Now $\mathcal{E}^*(\hat{D}_2 \cap N/Z(C_2)) = \{D_2/Z(C_2), \hat{C}_2 \cap N/Z(C_2)\}$ by (8B) and (8E), and $C_2^2 = A_2 \leqq Z(C_2) \cap V_1$ by (6J). Hence if $X \neq \hat{C}_2 \cap N$, then $\Omega_1(X/Z(C_2) \cap V_1) = C_2/Z(C_2) \cap V_1$ and so, as $X \lhd N$, $C_2 \lhd N$. This yields that $A_2 = C_2^2 \lhd N$, which is a contradiction because $[A_2, \hat{C}_1] \nleqq A_2 \cap \hat{C}_1$. Therefore, $X = \hat{C}_2 \cap N$ as desired.

(8H) LEMMA. *We have* $Z(C_2) \leqq Z(\hat{C}_2)$.

RROOF. We proceed as in the paragraph of the proof of (6J) in which C_2 is defined. We have that $C_2 = \langle R, R^a, R^b \rangle$, where $R = C_2 \cap V_1$, $a = [r, a_1][s, a_2]$, and $b = [r, a_1]$. Set $E_2 = Z(C_2)$. Then there is an element $e \in E_2$ such that $E_2 = \langle e \rangle (R \cap E_2)$, $R^a \cap R^b \cap E_2 = \langle e \rangle A_2$, and $[e, z] = b_6$. Also, $|E_2 : R \cap E_2| = 2$ by the first paragraph of the proof of (6J). Now set $X = \hat{C}_2 \cap N$. Then

$$R \cap E_2 \leqq Z(X)$$

by (8.2). This implies firstly that $D_2 < C_{\hat{D}_2}(A_2)$, and so the irreducible action of K_2 on $\hat{D}_2/D_2$ shows that

$$A_2 \leqq Z(\hat{D}_2).$$

Secondly, we have $|[X, E_2]| \leqq 2$ as $|E_2 : R \cap E_2| = 2$. As $A_2 = C_2^2 \lhd \hat{C}_2$, the irreducible action of K_2 on E_2/A_2 shows that $[\hat{C}_2, E_2] \leqq A_2$. Thus $[X, E_2]$ is a subgroup of A_2 and centralized by $\langle a_1, a_2, s \rangle$. Hence

$$[X, E_2] \leqq \langle b_1 \rangle.$$

Similarly, since $[X, z]E_2/E_2$ is a $\langle a_1, a_2 \rangle$-invariant Z_2-subgroup of C_2/E_2, it follows that $[X, z]E_2 = RE_2$. Then as $C_2 = \langle R, R^a, R^b \rangle$, (8B) (2) implies that $\hat{C}_2 = \langle X, X^a, X^b \rangle$. Therefore, $[\hat{C}_2, e] \leqq \langle b_1 \rangle$. Hence $[\hat{D}_2, \langle e \rangle A_2] \leqq \langle b_1, b_6 \rangle$ and then $[\hat{D}_2, \langle e \rangle A_2] \leqq \langle b_6 \rangle$ as a_1^r normalizes $\langle b_6 \rangle$, $\hat{D}_2$, and $\langle e \rangle A_2$ but does not normalize $\langle b_1, b_6 \rangle$. Hence $[X, e] \leqq \langle b_1 \rangle \cap \langle b_6 \rangle = 1$ and, since $E_2 = \langle e \rangle (R \cap E_2)$, it follows that

$[X, E_2]=1$. Thus $C_2<C_{\hat{C}_2}(E_2)$ and the irreducible action of K_2 on $\hat{C}_2/C_2$ shows that $E_2\leqq Z(\hat{C}_2)$.

(8I) LEMMA. *We have* $N\leqq N(\hat{C}_2)$.

PROOF. Set $X=N(\hat{D}_2\cap N)\cap N(\hat{C}_2\cap N)$. We argue that $X\leqq N(D_2)\cap N(C_2)$. As $D_2\cap(\hat{C}_2\cap N)=C_2$, it suffices to prove $X\leqq N(D_2)$. Since $\mathcal{E}^*(\hat{D}_2\cap N/Z(C_2))=\{D_2/Z(C_2), \hat{C}_2\cap N/Z(C_2)\}$, it follows that $\langle B_2^X\rangle\leqq D_2$. Let $Y=\langle B_2^{\hat{D}_2}\rangle$. Then Y is K_2-invariant and $B_2\leqq Y\leqq D_2$. Moreover, $F_2Y=D_2$ as $D_2=\langle F_2^{\hat{D}_2}\rangle=\langle (B_2Z(C_2))^{\hat{D}_2}\rangle=YZ(C_2)$. As commutation by z induces an isomorphism $D_2/B_2\to A_2$ that commutes with the action of K_2 and carries F_2/B_2 onto I_2, (2E) (3) forces $Y=D_2$. Thus $\langle B_2^X\rangle=D_2$ and, consequently, $X\leqq N(D_2)$.

Now since $A_1\hat{D}_2/\hat{C}_2\cap N=A_1(\hat{C}_2\cap N)/\hat{C}_2\cap N\times\hat{C}_2/\hat{C}_2\cap N$, it follows that $A_1\hat{D}_2\leqq C(A_1(\hat{C}_2\cap N)/\hat{C}_2\cap N)$. The above paragraph shows that

$$C(A_1(\hat{C}_2\cap N)/\hat{C}_2\cap N)\leqq C_{N(D_2)\cap N(C_2)}(A_1\hat{D}_2/\hat{D}_2).$$

Thus $A_1\hat{D}_2$ is the only S_2-subgroup of $C(A_1(\hat{C}_2\cap N)/\hat{C}_2\cap N)$ and, since N normalizes $\hat{C}_2\cap N$ and $A_1(\hat{C}_2\cap N)=B(\hat{C}_2\cap N)$ by (8G), it follows that N normalizes $A_1\hat{D}_2$. Since $C_{A_1\hat{D}_2}(Z(C_2)\cap V_1)=\hat{C}_2$ by (8H), the result follows.

Now we define $M_i=E(N(\hat{C}_i)\bmod \hat{C}_i)$ for each $i\in\{1, 2\}$.

(8J) LEMMA. *The following conditions hold:*
(1) $M_2/\hat{C}_2\cong L_3(4)$, $SL_3(4)$, *or* $L_3(2)\times L_3(2)$;
(2) $C_{M_2/\hat{C}_2}(z)=K_2\hat{C}_2/\hat{C}_2$;
(3) $\langle z\rangle M_2$ *contains an* S_2-*subgroup of* $N(\hat{C}_2)$;
(4) $[M_2, O(C)]=1$;
(5) $\hat{C}_1\leqq M_2$ *and* $|\hat{C}_1\hat{C}_2|=2^{19}$.

PROOF. The argument in the first paragraph of the proof of (8I) shows that $N(\hat{D}_2)\cap N(\hat{C}_2)=N(D_2)\cap N(C_2)$. Hence if bars denote images in $N(\hat{C}_2)/\hat{C}_2$, then $C(\bar{z})=\langle\bar{z}\rangle\times\bar{K}_2\times\overline{O(C)}$. Moreover, $\hat{C}_1\leqq N(\hat{C}_2)$ by (8I), and the image of $\hat{C}_1$ in $N(\hat{C}_2)/\hat{C}_2$ is elementary abelian of rank at least 4 as $\hat{C}_1^2=I_1$ and $\bar{C}_1\cong E_{16}$. In particular, $\bar{z}$ is a noncentral involution, and hence (1)-(4) follow as in (5G). (5) is a consequence of (1)-(4).

(8K) LEMMA. *The following conditions hold:*
(1) $M_1/\hat{C}_1\cong U_5(2)$ *or* $L_5(2)$;
(2) $C_{M_1/\hat{C}_1}(z)=K_1\hat{C}_1/\hat{C}_1$;
(3) $[M_1, O(C)]=1$.

PROOF. Our aim is to use (1I) in conjunction with (1J). Let bars denote images in $N(\hat{C}_1)/\hat{C}_1$. As $Z(\hat{D}_1)=I_1$ and $\mathcal{E}^*(\hat{D}_1/I_1)=\{\hat{C}_1/I_1, F_1/I_1\}$, (8.1) shows that $N(\hat{D}_1)=N(D_1)\leqq N(\hat{C}_1)$. Hence $C(\bar{z})=\langle\bar{z}\rangle\times\bar{K}_1\times\overline{O(C)}$. Furthermore, $N\leqq N(\hat{C}_1)$ by (8G) and, in particular, $\bar{z}$ is a noncentral involution. Thus $N(\hat{C}_1)/\hat{C}_1$ satisfies the hypothesis of (1I). As $A_2\cap\hat{C}_1=A_1\cap A_2$, $\bar{A}_2$ is an E_8-subgroup of $\bar{K}_1$. Similarly, as $D_2\cap\hat{C}_1=C_1\cap C_2$ by (8A) (1), $\bar{D}_2$ has order 2^7. As $B\cap\hat{D}_1=D_1\cap V_2$ and as $N_N(D_1\cap V_2)\leqq N(\hat{D}_1)$ by (8A) (3), we have $N\cap N(\hat{D}_1)=N_N(D_1\cap V_2)$. This together with (8A) (1) implies that $C_{\bar{D}_2}(\bar{z})=\overline{D_2\cap V_1}=\bar{B}_2=\langle\bar{z}\rangle\bar{A}_2$ as $D_2\cap V_1=B_2(C_1\cap C_2)$ by (6J) (3). Clearly, $\bar{D}_2$ is invariant under $N_{\bar{K}_1}(\bar{A}_2)=\overline{N_L(A_1A_2)}$, and $\bar{D}_2^2=\bar{A}_2\leqq Z(\bar{D}_2)$ by (6N) (2). Hence if we denote by tildes images in $N(\hat{C}_1)/O(N(\hat{C}_1) \bmod \hat{C}_1)$, then there is a subgroup $\tilde{X}$ of index 2 such that $\tilde{X}\cong U_5(2)$, $L_5(2)$, $Sp_4(4)$, or $Sp_4(2)\times Sp_4(2)$ and such that $C_{\tilde{X}}(\tilde{z})=\tilde{K}_1$ by (1J). Furthermore, the involutions outside $\tilde{X}$ are all conjugate to $\tilde{z}$ under $\tilde{X}$ by Section 19 of [3]. Set $E_2=Z(C_2)$. Then $\bar{E}_2\leqq Z(\bar{C}_2)$ and $\bar{C}_2$ is a subgroup of $\bar{D}_2$ such that $\bar{D}_2/\bar{A}_2=\bar{B}_2/\bar{A}_2\times\bar{C}_2/\bar{A}_2$. In particular, $|C(\tilde{E}_2)|_2\geqq 2^6$, hence we must have $\tilde{E}_2\leqq\tilde{X}$. Set $c=[s, a_2]$. Then $\langle\bar{c}\rangle$ is a Z_3-subgroup of $N_{\bar{K}_1}(\bar{A}_2)$, so $|C_{\bar{D}_2/\bar{B}_2}(\bar{c})|=2$ and $\bar{c}$ acts transitively on $((\bar{D}_2/\bar{B}_2)/C_{\bar{D}_2/\bar{B}_2}(\bar{c}))^{\#}$: see the second paragraph of the proof of (1J). Thus $\bar{c}$ acts transitively on $(\bar{C}_2/\bar{E}_2)^{\#}$. Consequently, $\tilde{C}_2=\tilde{E}_2[\tilde{C}_2, \tilde{c}]\leqq\tilde{X}$ and hence

$$\tilde{D}_2\cap\tilde{X}=\tilde{C}_2.$$

Since $\tilde{C}_2^2=\tilde{A}_2$ by (6J), we conclude that $X\cong U_5(2)$ or $L_5(2)$. The assertions now follow from (1I).

(8L) LEMMA. *The following conditions hold:*

(1) *M_1 and M_2 have a common z-invariant S_2-subgroup S such that $C_S(z)=U$;*

(2) $Z(\hat{C}_1\hat{C}_2/\hat{C}_1)=Z(\hat{C}_2)\hat{C}_1/\hat{C}_1$.

PROOF. We continue with the notation of the proof of (8K). By (1J), $\bar{D}_2=O_2(N_{\langle\bar{z}\rangle\bar{M}_1}(\langle\bar{z}\rangle\bar{A}_2))$, and we have shown that $\bar{C}_2=\bar{D}_2\cap\bar{M}_1$. Furthermore, $\bar{E}_2$ is an E_{16}-subgroup of $Z(\bar{C}_2)$. (1.1) of [10] shows that the action of $\bar{z}$ on $\bar{M}_1$ and the embedding of $\bar{K}_1$ in $\bar{M}_1$ are unique up to conjugation by the elements of $\bar{M}_1$ and relabeling of elements of $\bar{K}_1$ by a graph automorphism of $\bar{K}_1$. Thus we may identify $\bar{E}_2$ with the set of matrices of the form

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ * & * & & 1 & \\ * & * & & & 1 \end{pmatrix}$$

in $\bar{M}_1\cong U_5(2)$ or $L_5(2)$. Here we are using usual matrix representations of $\bar{M}_1$. Furthermore, $\bar{b}_3$ is a transvection of $\bar{M}_1$ and $\bar{b}_6$ is an involution that is not a transvection.

We argue that $\hat{C}_2 \leqq N(\hat{C}_1)$. First, notice that $\hat{C}_2 \leqq C(b_1)$ by (8H). Let $g \in \hat{C}_2$ and set $X = \hat{C}_1^g$. Since $\hat{C}_2/C_2 \cong A_2/I_2$ as modules for $c = [s, a_2]$, we have that c acts irreducibly on $\hat{C}_2/\hat{C}_2 \cap N$, and hence $\hat{C}_1(\hat{C}_2 \cap N) \lhd \hat{C}_1\hat{C}_2$. Thus $X \leqq \hat{C}_1(\hat{C}_2 \cap N)$. Now $|\hat{C}_1 \cap \hat{C}_2| = 2^7$ by (8J) (5), so $\hat{C}_1 \cap \hat{C}_2 \nleqq C_2$ and then we have $\hat{C}_2 \cap N = (\hat{C}_1 \cap \hat{C}_2)C_2$. Thus $X \leqq \hat{C}_1 C_2$ and $\bar{X} \leqq \bar{C}_2$. We have shown before that $\bar{c}$ is transitive on $(\bar{C}_2/\bar{E}_2)^{\#}$, and $\bar{C}_2^2 \neq 1$. Hence $\Omega_1(\bar{C}_2) = \bar{E}_2$ and $\bar{X} \leqq \bar{E}_2$. Now $\bar{b}_6$ centralizes $\hat{C}_1 \cap \hat{C}_2/I_1$ by (8H). On the other hand, as $|C_{A_1/I_1}(\bar{b}_6)| = 2^2$, (6E) (2) shows that $|C_{\hat{C}_1/I_1}(\bar{b}_6)| \leqq 2^6$. Thus $C_{\hat{C}_1/I_1}(\bar{b}_6) = \hat{C}_1 \cap \hat{C}_2/I_1$. Similarly, $|C_{\hat{C}_1/I_1}(\bar{b}_3)| \leqq 2^8$ as $|C_{A_1/I_1}(\bar{b}_3)| = 2^3$. Notice that if $\bar{x} \in \bar{X}$, then $X \cap \hat{C}_1/I_1 \leqq C_{\hat{C}_1/I_1}(\bar{x})$. Now assume that $\bar{X}$ contains a conjugate of $\bar{b}_6$. Then $|X \cap \hat{C}_1/I_1| \leqq |C_{\hat{C}_1/I_1}(\bar{b}_6)| \leqq 2^6$ and so $|\bar{X}| = 2^4$. So $\bar{X} = \bar{E}_2$ and, as $X^2 \leqq I_1$, X contains an element of $b_6(\hat{C}_1 \cap \hat{C}_2)$. But then $X \leqq C_{\hat{C}_1 E_2/I_1}(b_6) = (\hat{C}_1 \cap \hat{C}_2)E_2/I_1$ and so $X = (\hat{C}_1 \cap \hat{C}_2)E_2$, which is a contradiction. Therefore, if $\bar{X} \neq 1$, $\bar{X}$ consists of conjugates of $\bar{b}_3$. Since $\bar{b}_3$ is a transvection in $\bar{E}_2$, it follows that

$$|\bar{X}| \leqq \begin{cases} 2 \ \textit{if} \ \bar{M}_1 \cong U_5(2), \\ 4 \ \textit{if} \ \bar{M}_1 \cong L_5(2). \end{cases}$$

In any case, $|\bar{X}| \leqq 4$ and so $|X \cap \hat{C}_1/I_1| \geqq 2^8$. Now assume $X \neq \hat{C}_1$ and take an element $x \in X - \hat{C}_1$. Then $\bar{x}$ is conjugate to $\bar{b}_3$, so $|X \cap \hat{C}_1/I_1| \leqq |C_{\hat{C}_1/I_1}(\bar{x})| \leqq 2^8$. Thus $C_{\hat{C}_1/I_1}(\bar{x}) = X \cap \hat{C}_1/I_1$ and then $C_{X\hat{C}_1/I_1}(x) = X/I_1$. (1C) of [8] now shows that $\mathcal{E}^*(X\hat{C}_1/I_1) = \{\hat{C}_1/I_1, X/I_1\}$. Let $e \in X\hat{C}_1 \cap E_2 - \hat{C}_1$. Then $e \in X - \hat{C}_1$ and so $\langle \hat{C}_1 \cap E_2, e \rangle \leqq X \cap E_2$. However, this implies that $|\hat{C}_1 \cap E_2| < |X \cap E_2|$, a contradiction as $X \cap E_2 = (\hat{C}_1 \cap E_2)^g$. Therefore, we have $X = \hat{C}_1$ for all $g \in \hat{C}_2$, proving $\hat{C}_2 \leqq N(\hat{C}_1)$.

Now consider the image of $\hat{C}_2$ in $N(\hat{C}_1)/\hat{C}_1$. Since it centralizes $\bar{E}_2$ and has order 2^8, we obtain that it is identified with the group of matrices of the form

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ * & * & 1 & & \\ * & * & * & 1 & \\ * & * & * & & 1 \end{pmatrix}$$

in $\bar{M}_1 \cong U_5(2)$ or $L_5(2)$. In particular, we have $Z(\hat{C}_1\hat{C}_2/\hat{C}_1) = \hat{C}_1 E_2/\hat{C}_1$, proving (2). Also, $\hat{C}_1\hat{C}_2/\hat{C}_1$ is normal in an S_2-subgroup of $M_1/\hat{C}_1$. Hence we may choose a z-invariant S_2-subgroup S of M_1 such that $U \leqq S$ and $\hat{C}_1\hat{C}_2 \lhd S$. We argue that $S \leqq N(\hat{C}_2)$. Let $g \in S$. Then g normalizes $\hat{C}_1 E_2 = Z(\hat{C}_1\hat{C}_2 \bmod \hat{C}_1)$ and so normalizes $\hat{C}_1 \cap \hat{C}_2 = Z(\hat{C}_1 E_2 \bmod I_1)$. Now since $\hat{C}_1$ is extra-special of order 2^{11}, it easily follows that $Z(\hat{C}_1 \cap \hat{C}_2) = C_1 \cap E_2$ and that $C_{\hat{C}_1}(C_1 \cap E_2) = \hat{C}_1 \cap \hat{C}_2$. Thus g normalizes $\hat{C}_2 = C_{\hat{C}_1\hat{C}_2}(C_1 \cap E_2)$. The argument of the last paragraph of the proof of (5K) now shows that $S \leqq M_2$, proving (1).

Now, as in the cases of Theorems 1, 2, and 3, Lemma (8L) ends our 2-local analysis under Hypothesis 4 and therefore the proof of Theorem 4.

9. Conclusion of the proof of the main theorem.

Let G_0 be the normal closure of L in G and let bars denote images in $G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$ isomorphic to $Sp_6(2)$ and $C_{\bar{G}}(\bar{L})$ has cyclic Sylow 2-subgroups. Furthermore, if $\bar{G}\neq\bar{G}'$ then $O(N_{\bar{G}'}(\bar{X}))=1$ for every 2-subgroup $\bar{X}$ of $\bar{G}'$. As $\bar{L}$ is not normal in $\bar{G}$, Theorems 1-4 imply that $\bar{G}_0$ is isomorphic to $O_8^-(2)$, $O_8^+(2)$, $U_6(2)$, $L_6(2)$, $U_7(2)$, $L_7(2)$, $Sp_6(4)$ or $Sp_6(2)\times Sp_6(2)$. Now let A be a z-invariant Sylow 2-subgroup of $N_{G_0}(U)$ and let $B/U=C_{A/U}(z)$. Then $[B, z]\leqq U\leqq L$ and $[B, z]\neq 1$ as $U<B$. By (1C) $[B, z]\leqq C_{G_0}(O(G_0))$, and since $G_0/O(G_0)$ has no nontrivial proper subgroups normal in $G/O(G_0)$, it follows that $G_0=C_{G_0}(O(G_0))O(G_0)$. Thus $O(G_0)=Z(G_0)$ and, inspecting the Schur multipliers of the above-listed Chevalley type groups, we have that either $O(G_0)=1$ or $G_0\cong SU_6(2)$. This completes the proof of the main theorem.

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