On the uniqueness of the Dirichlet problem for harmonic maps

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§ 0. Introduction

Let (X, g) and (Y, h) be compact Riemannian manifolds with boundary. The *tention field* Δf of a C^∞ map $f: X \to Y$ is, by definition, a vector field along f which is written locally as

$$\Delta f = \left\{ \Delta f^\alpha + g^{ij} \Gamma^\alpha_{\beta\gamma}(f) \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \right\} \frac{\partial}{\partial y^\alpha}, \tag{0.1}$$

where Δ and Γ are the Laplacian on X and the Christoffel symbols on Y. A C^∞ map $f: X \to Y$ is called *harmonic* if it satisfies the equation

$$\Delta f = 0. \tag{0.2}$$

Harmonic maps arise naturally from the variational problem for the energy functional $E(f) = \int_X g^{ij} h_{\alpha\beta}(f) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$. To be more precise, for a given C^∞ map $\phi: \partial X \to Y$, we write $M_\phi(X, Y)$ for the set of all C^∞ maps $f: X \to Y$ with $f|_{\partial X} = \phi$, then $f \in M_\phi(X, Y)$ is harmonic iff it is a critical point of $E: M_\phi(X, Y) \to \boldsymbol{R}$. We put the C^0 topology on $M_\phi(X, Y)$. In this paper, we shall prove the following theorem.

THEOREM 1. *Let X and Y be compact Riemannian manifolds with boundary, and $\phi: \partial X \to Y$ be a C^∞ map. Assume that Y has nonpositive sectional curvatures and that ∂Y is convex.*[1] *If there exist two harmonic maps f_0 and f_1 in the same connected component of $M_\phi(X, Y)$, then we have $f_0 = f_1$.*

Note that, in Theorem 1, the existence of a harmonic map in each connected component of $M_\phi(X, Y)$ has been proved in [3]. The argument in the proof of Theorem 1 implies the following two corollaries.

COROLLARY 2. *Let X, Y and $\phi: \partial X \to Y$ be as in Theorem 1. Then each connected component of $M_\phi(X, Y)$ is contractible.*

1) ∂Y is called convex if its inner second fundamental form is positive semidefinite. Theorem 1 also holds when ∂Y is empty.

COROLLARY 3. *Let Y be as in Theorem* 1. *Then Y is a $K(\pi, 1)$ space.*

After the preparation of this paper, J. Eells and L. Lemaire published the report on harmonic maps [1], and we knew that our Theorem 1 was also obtained by R. Schoen [7][2]. But in [7], the proof of our Proposition 4 which is the essential step for Theorem 1 is not written in full detail. We are convinced that our Proposition 4 is not so trivial to verify. Moreover through a private correspondence, Professor R. Schoen told us that he has no intention to publish his proof of Theorem 1, and suggested us to publish this paper. We are very grateful for his generosity.

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§1. Heat equation

First we recall the method to prove the existence theorem in [3]. For given $f_0 \in M_\phi(X, Y)$, consider the following initial boundary value problem;

$$(1.1) \qquad f: X \to Y,\ \frac{\partial f}{\partial t} = \Delta f,\ f(\ ,0) = f_0,$$
$$f(\ ,t) \in M_\phi(X, Y) \quad \text{for all } t \in [0, \infty).$$

Next form the double $\tilde{Y}$ of Y and embed $\tilde{Y}$ into a suitable Euclidean space $\boldsymbol{R}^N$ as in [3], p. 108. With the metric of $\boldsymbol{R}^N$ chosen there, we can reduce (1.1) to the following equation;

$$(1.2) \qquad f: X \times [0, \infty) \to \boldsymbol{R}^N,\ \frac{\partial f}{\partial t} = \Delta_{\boldsymbol{R}^N} f,\ f(\ ,0) = f_0,$$
$$f|_{\partial X \times \{t\}} = \phi \quad \text{for all } t \in [0, \infty).$$

We can find the unique solution of (1.2), or equivalently of (1.1), in the following sense. The solution f is continuous and C^∞ except at the corner. At the corner f satisfies the equation in the L_2^p sense. That is, for any $0 < \omega < \infty$, $f|_{X \times [0,\omega]}$ belongs to $L_2^p(X \times [0, \omega], \boldsymbol{R}^N) = \overset{N}{\oplus} L_2^p(X \times [0, \omega])$ for arbitrary $p > \dim X + 2$. Here $L_k^p(X \times [0, \omega])$ is the Sobolev space, for its definition refer to Part II of [3]. In its definition we must note that the order of the differentiation with respect to t is counted twice as many as that with respect to x. So the first order time derivative $\frac{\partial f}{\partial t}$ of our solution f belongs to $L_0^p(X \times [0, \omega], \boldsymbol{R}^N)$ and is not continuous at the corner $\partial X \times \{0\}$ even though the first order space derivatives $\frac{\partial f}{\partial x^i}$ belong to $L_1^p(X \times [0, \omega], \boldsymbol{R}^N)$ and are continuous by the Sobolev em-

2) Through a private correspondence, J. Eells and L. Lemaire informed us that they also obtained their unpublished proof of Theorem 1.

bedding theorem (see [3], p. 49). Throughout the rest of this paper, $L_k^p(X\times[0, \omega], \boldsymbol{R}^N)$ will be denoted by $L_k^p(X\times[0, \omega])$ in short.

Finally it is proved in [3], pp. 158-161, that as t tends to ∞, $f(\ ,t)$ converge in the C^∞ topology to a harmonic map f_∞.

Now we are in position to state our main proposition.

As a main step for Theorem 1, we shall prove

PROPOSITION 4. *Let X, Y and $\phi:\partial X\to Y$ be as in Theorem 1. For a C^∞ map $f_0:[0, 1]\times X\to Y$ with $f_0(u,\ )\in M_\phi(X, Y)$ for each $u\in[0, 1]$, let $f:[0, 1]\times X\times[0, \infty)\to Y$ be the map with the following property. For fixed $u\in[0, 1]$, $f(u,\ ,\ ): X\times[0, \infty)\to Y$ is the solution of (1.1) with $f(u,\ , 0)=f_0(u,\ )$. Then f is C^∞ on $[0, 1]\times X\times(0, \infty)$.*

The proof will be carried out in the next three sections.

§ 2. The regularity with respect to the parameter variable

Take a small real number $\delta>0$, and extend f_0 smoothly to $f_0:(-\delta, 1+\delta)\times X\to Y$ with $f_0(u,\ )\in M_\phi(X, Y)$ for each $u\in(-\delta, 1+\delta)$. Next choose a C^∞ map $\eta:(-\delta, 1+\delta)\times X\times[0, \infty)\to\boldsymbol{R}^N$ such that $\eta(\ ,\ , 0)$ coincides with f_0 and that $\eta(u,\ , t)\in M_\phi(X, Y)$. We fix $a\in(-\delta, 1+\delta)$ for the time being, and define a map $\varphi:(-\delta, 1+\delta)\times X\times[0, \infty)\to\boldsymbol{R}^N$ as

$$\varphi(u, x, t)=\eta(u, x, t)-\eta(a, x, t)+f(a, x, t) \tag{2.1}$$

where $f(a, x, t)$ is the solution of (1.2) with the initial value $f_0(a,\ )$. Then $\varphi(\ ,\ , 0)$ coincides with f_0 and $\varphi(u,\ , t)$ has the boundary value ϕ. Since $f(a,\ ,\ )$ is continuous on $X\times[0, \infty)$ and C^∞ except at the corner $\partial X\times\{0\}$, φ is continuous on $(-\delta, 1+\delta)\times X\times[0, \infty)$ and C^∞ except at $(-\delta, 1+\delta)\times\partial X\times\{0\}$. But $\frac{\partial\varphi}{\partial u}$ is C^∞ on $(-\delta, 1+\delta)\times X\times[0, \infty)$ since $\frac{\partial\varphi}{\partial u}=\frac{\partial\eta}{\partial u}$.

Let $L_2^p(X\times[0, \omega]/0)$ be the completion by the L_2^p norm of the space of C^∞ maps $f: X\times[0, \omega]\to\boldsymbol{R}^N$ whose all derivatives vanish at $t=0$. And let $L_2^p(X\times[0, \omega]/0)_\#$ be the closed subspace of $L_2^p(X\times[0, \omega]/0)$ consisting of elements $f_\#$ with $f_\#|_{\partial X\times[0,\omega]}=0$.

LEMMA 1. *Let $f(u, x, t)$ be as in Proposition 4. Then the k-th derivative $\frac{\partial^k f}{\partial u^k}$ exists for every $k\geqq 0$ and is continuous on $(-\delta, 1+\delta)\times X\times[0, \infty)$.*

PROOF. Since $\varphi(u,\ ,\ )\in L_2^p(X\times[0, \omega])$, we can define the following infinitely Fréchet differentiable map;

$$\begin{aligned} P:(-\delta, 1+\delta)\times L_2^p(X\times[0, \omega]/0)_\# &\longrightarrow L_2^p(X\times[0, \omega]) \\ P(u, f_\#)(x, t)&=\varphi(u, x, t)+f_\#(x, t). \end{aligned} \tag{2.2}$$

And since $H=\frac{\partial}{\partial t}-\Delta: L_2^p(X\times[0, \omega])\to L^p(X\times[0, \omega])$ is also infinitely Fréchet differentiable, so is the composition map $Q=H\circ P: (-\delta, 1+\delta)\times L_2^p(X\times[0, \omega]/0)_\#$ $\to L^p(X\times[0, \omega])$. Then we have $Q(a, 0)=\left(\frac{\partial}{\partial t}-\Delta\right)f(a, \ , \ )=0$. Let D_2Q be the Fréchet derivative of Q with respect to the second factor. Then

$$D_2Q(a, 0)h^\alpha=\frac{\partial h^\alpha}{\partial t}-\Delta h^\alpha-g^{ij}\frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial y^\delta}(f_a)h^\delta\frac{\partial f_a^\beta}{\partial x^i}\frac{\partial f_a^\gamma}{\partial x^j}$$

$$-2g^{ij}\Gamma^\alpha_{\beta\gamma}(f_a)\frac{\partial h^\beta}{\partial x^i}\frac{\partial f_a^\gamma}{\partial x^j},$$

where $f_a(x, t)=f(a, x, t)$. As in the proof of Theorem in [3], p. 120, we can show that $D_2Q(a, 0): L_2^p(X\times[0, \omega]/0)_\#\to L^p(X\times[0, \omega])$ is an isomorphism. By the implicit function theorem there exist a sufficiently small positive number δ_a and the implicit function $G:(a-\delta_a, a+\delta_a)\to L_2^p(X\times[0, \omega]/0)_\#$ such that $G(a)=0$ and $Q(u, G(u))=0$. If we put

$$\tilde f(u, x, t)=\varphi(u, x, t)+G(u)(x, t),$$

then we have $\frac{\partial \tilde f}{\partial t}-\Delta\tilde f=0$, $\tilde f(\ , \ , 0)=f_0$ and $\tilde f(u, \ , t)|_{\partial X}=\psi$. Thus by the uniqueness of the heat equation we have $\tilde f=f$. Next let $F:(a-\delta_a, a+\delta_a)\to L_2^p(X\times[0, \omega])$ be $F(u)(x, t)=f(u, x, t)$, then F is C^∞ since G is C^∞. By the Sobolev embedding theorem,

$$\|f(u, \ , \ )-f(u', \ , \ )\|_{L^\infty(X\times[0,\omega])}\leqq C\|F(u)-F(u')\|_{L_2^p(X\times[0,\omega])}.$$

Thus the continuity of f with respect to u follows from that of F.

Let $D^kF(u)$ be the k-th Fréchet derivative of F at u. $D^kF(u)$ belongs to $L(\overset{k}{\otimes}\mathbf{R}, L_2^p(X\times[0, \omega]))$ and can be regarded canonically as an element of $L_2^p(X\times[0, \omega])$. If we prove $\frac{\partial^k f}{\partial u^k}=D^kF$ then we can prove the continuity of $\frac{\partial^k f}{\partial u^k}$ in the same way as above. When $k=1$, $DF(u)(x, t)=\frac{\partial f}{\partial u}(u, x, t)$ since

$$\begin{aligned}&\lim_{\theta\to 0}\left\|\frac{1}{\theta}\{f(u+\theta, \ , \ )-f(u, \ , \ )\}-DF(u)\right\|_{L^\infty(X\times[0,\omega])}\\ &\leqq\lim_{\theta\to 0}C\left\|\frac{1}{\theta}\{F(u+\theta)-F(u)\}-DF(u)\right\|_{L_2^p(X\times[0,\omega])}\\ &=0.\end{aligned}$$

In the same manner we can show $D^kF=\frac{\partial^k f}{\partial u^k}$. Since $a\in(-\delta, 1+\delta)$ and $\omega<\infty$ are arbitrary, $\frac{\partial^k f}{\partial u^k}$ exists and is continuous on $[0, 1]\times X\times[0, \infty)$, completing the proof.

§ 3. The continuity of $D_t^\alpha D_x^\beta f$ with respect to u

We know that $f(u, \ , \ )$ is C^∞ on $\{u\}\times X\times(0, \infty)$ for each $u\in(-\delta, 1+\delta)$. In this section we show

LEMMA 2. *Let $f(u, x, t)$ be as in Proposition 4. The time and space derivatives of f are continuous on $[0, 1]\times X\times(0, \infty)$.*

PROOF. Let $0<\alpha<\omega<\infty$, $p>\dim X+2$ and $n<\infty$. Define $F:(-\delta, 1+\delta)\to L_n^p(X\times[\alpha, \omega])$ to be $F(u)=f(u, \ , \ )$. If we can say that F is continuous, we will obtain Lemma 2 in the following way;

$$\|D_t^\alpha D_x^\beta f(u, \ , \ )-D_t^\alpha D_x^\beta f(v, \ , \ )\|_{L^\infty(X\times[\alpha,\omega])}$$
$$\leqq C\|f(u, \ , \ )-f(v, \ , \ )\|_{L^p_{2\alpha+\|\beta\|+1}(X\times[\alpha,\omega])}\to 0 \qquad \text{as } u\to v.$$

We prove the continuity of F by induction on n in increments 3/4. Note that it is known when $n=2$. The proof is a modification of the proof of Theorem in [3], p. 111. Hereafter we write f_u for $f(u, \ , \ )$ and $L_n^p[\alpha, \omega]$ for $L_n^p(X\times[\alpha, \omega])$ in short. A priori estimates (refer to [3], p. 96) shows that if $0<\pi<\alpha$ then we have

$$\|f_u-f_v\|_{L_n^p[\alpha,\omega]}\leqq C\left\|\left(\frac{\partial}{\partial t}-\Delta\right)(f_u-f_v)\right\|_{L^p_{n-2}[\pi,\omega]}+C\|f_u-f_v\|_{L^p[\pi,\alpha]}.$$

We know that the second term on the right hand side tends to 0 as u tends to v. Thus we have only to estimate

$$\left\|\left(\frac{\partial}{\partial t}-\Delta\right)(f_u^\alpha-f_v^\alpha)\right\|_{L^p_{n-2}[\pi,\omega]}$$
$$=\left\|g^{ij}\Gamma^\alpha_{\beta\gamma}(f_u)\frac{\partial f_u^\beta}{\partial x^i}\frac{\partial f_u^\gamma}{\partial x^j}-g^{ij}\Gamma^\alpha_{\beta\gamma}(f_v)\frac{\partial f_v^\beta}{\partial x^i}\frac{\partial f_v^\gamma}{\partial x^j}\right\|_{L^p_{n-2}[\pi,\omega]}.$$

Choose a partition of unity $\{\varphi_i\}$ subordinate to a finite covering by coordinate neighborhood of $X\times[\pi, \omega]$. We estimate $\varphi_i\{P(f_u)-P(f_v)\}$ where $P(f)=g^{ij}\Gamma^\alpha_{\beta\gamma}(f)\dfrac{\partial f^\beta}{\partial x^i}\dfrac{\partial f^\gamma}{\partial x^j}$.

First we consider the case when the support of φ_i does not meet the boundary of $X\times[\pi, \omega]$. We write φ for φ_i. Let $\rho\in\boldsymbol{R}\backslash\boldsymbol{Z}$ be $n-2<\rho<n-3/4$. Let Λ_β^p be the Besov space (refer to [3], p. 25). There exists an inclusion

$\Lambda^p_\rho \to L^p_{n-2}$. Thus we have only to estimate $\|\varphi P(f_u)-\varphi P(f_v)\|_{\Lambda^p_\rho}$. We write x^{n+1} for t and put $T^w_j g(x^1, \cdots, x^{n+1})=g(x^1, \cdots, x^j+w, \cdots, x^{n+1})$ and $\Delta^w_j=T^w_j-I$. By the definition of the Besov norm it is sufficient to show

$$\|D^\gamma(\varphi P(f_u)-\varphi P(f_v))\|_{L^p} \to 0 \quad \text{as } u \to v$$

and

$$\|\Delta^w_j D^\gamma(\varphi P(f_u)-\varphi P(f_v))\|_{L^p} \to 0 \quad \text{as } u \to v$$

for $\rho-1<\|\gamma\|<\rho$ when $j<n+1$ and for $\rho-2<\|\gamma\|<\rho$ when $j=n+1$. Since we can write $D^\gamma(\varphi P(f))=\sum c_{\beta_1\cdots\beta_\mu}(f)D^{\beta_1}f \cdots D^{\beta_\mu}f$ with $\max\|\beta_i\|\leqq\|\gamma\|+1$, $\sum\|\beta_i\|\leqq \|\gamma\|+2$, and $c_{\beta_1\cdots\beta_\mu}$ has a compact support, we can write $\Delta^w_j D^\gamma(\varphi P(f))$ as a sum of terms of the form $\Delta^w_j c(f)\cdot T^w_j D^{\beta_1}f \cdots T^w_j D^{\beta_\mu}f$ or $c(f)\cdot D^{\beta_1}f \cdots D^{\beta_{i-1}}f\cdot \Delta^w_j D^{\beta_i}f \times T^w_j D^{\beta_{i+1}}f \cdots T^j_w D^{\beta_\mu}f$. Let q be $p(\rho+2)<q\left(n-\frac{3}{4}\right)$ and p_i be slightly less than $q\left(n-\frac{3}{4}\right)/\|\beta_i\|$ for $1\leqq i\leqq\mu$. We can take $p_i>1$ since $q\left(n-\frac{3}{4}\right)/\|\beta_i\|>q\left(n-\frac{3}{4}\right)/\rho+2>p>1$. We can also make $\sum\frac{1}{p_i}\leqq\frac{1}{p}$ since $\sum\frac{1}{p_i}>\sum\left\{\|\beta_i\|/q\left(n-\frac{3}{4}\right)\right\}$ and $\sum\left\{\|\beta_i\|/q\left(n-\frac{3}{4}\right)\right\}<\frac{1}{p}$. Thus we can use Hölder's inequality since φ has a fixed compact support. Thus we have

$$\begin{aligned}
&\|c(f_u)\cdot D^{\beta_1}f_u \cdots D^{\beta_\mu}f_u-c(f_v)\cdot D^{\beta_1}f_v \cdots D^{\beta_\mu}f_v\|_{L^p} \\
&\leqq C\|c(f_u)-c(f_v)\|_{L^\infty}\cdot\|D^{\beta_1}f_u\|_{L^{p_1}} \cdots \|D^{\beta_\mu}f_u\|_{L^{p_\mu}} \\
&\quad +C\sum_i\|c(f_u)\|_{L^\infty}\cdot\|D^{\beta_1}f_v\|_{L^{p_1}} \cdots \|D^{\beta_{i-1}}f_v\|_{L^{p_{i-1}}} \\
&\quad \times\|D^{\beta_i}f_u-D^{\beta_i}f_v\|_{L^{p_i}}\cdot\|D^{\beta_{i+1}}f_u\|_{L^{p_{i+1}}} \cdots \|D^{\beta_\mu}f_u\|_{L^{p_\mu}}.
\end{aligned}$$

Since f is continuous, $\|c(f_u)-c(f_v)\|_{L^\infty}\to 0$ as $u\to v$. Using Lemma in [3], p. 110, we have

$$\|D^{\beta_i}f_u-D^{\beta_i}f_v\|_{L^{p_i}}\leqq C\|f_u-f_v\|_{L^q_{n-\frac{3}{4}}}^{\|\beta_i\|/n-\frac{3}{4}} \longrightarrow 0 \qquad \text{as } u\to v$$

by the induction hypothesis.

Using the same argument and the fact that the integration is invariant under translation T^w_j, we can show

$$\begin{aligned}
&\|\Delta^w_j c(f_u)\cdot T^w_j D^{\beta_1}f_u \cdots T^w_j D^{\beta_\mu}f_u-\Delta^w_j c(f_v)\cdot T^w_j D^{\beta_1}f_v \cdots T^w_j D^{\beta_\mu}f_v\|_{L^p} \\
&\qquad \longrightarrow 0 \qquad \text{as } u\to v, \\
&\|c(f_u)\cdot D^{\beta_1}f_u \cdots D^{\beta_{i-1}}f_u\cdot\Delta^w_j D^{\beta_i}f_u\cdot T^w_j D^{\beta_{i+1}}f_u \cdots T^w_j D^{\beta_\mu}f_u \\
&\quad -c(f_v)\cdot D^{\beta_1}f_v \cdots D^{\beta_{i-1}}f_v\cdot\Delta^w_j D^{\beta_i}f_v\cdot T^w_j D^{\beta_{i+1}}f_v \cdots T^w_j D^{\beta_\mu}f_v\|_{L^p} \\
&\qquad \longrightarrow 0 \qquad \text{as } u\to v.
\end{aligned}$$

When the support of φ_i meets the boundary of $M\times[\alpha, \omega]$, we have only to use the extension operator in [3], p. 32. This completes the proof.

§4. Proof of Proposition 4

In this section we show that $\dfrac{\partial^k f}{\partial u^k}$ is C^∞ on $\{u\}\times X\times(0, \infty)$ for each $u\in[0, 1]$ and that the time and space derivatives $D_t^\alpha D_x^\beta \dfrac{\partial^k f}{\partial u^k}$ are continuous in u.

In the notation of section 2, $H\circ F$ is C^∞ and $H\circ F(u)=0$ for all $u\in[0, 1]$. Thus taking Fréchet derivatives of $H\circ F$ we obtain the following lemma.

LEMMA 3. *In $L^p(X\times[0, \omega])$ the following equality holds for every $k\geqq 1$.*

$$0=\frac{\partial}{\partial t}\frac{\partial^k f}{\partial u^k}-\Delta\frac{\partial^k f}{\partial u^k}-g^{ij}\frac{\partial\Gamma_{\beta\gamma}^{\alpha}(f)}{\partial y^\delta}\frac{\partial^k f^\delta}{\partial u^k}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}$$

$$-2g^{ij}\Gamma_{\beta\gamma}^{\alpha}(f)\frac{\partial}{\partial x^i}\Big(\frac{\partial^k f^\beta}{\partial u^k}\Big)\cdot\frac{\partial f^\gamma}{\partial x^j}$$

$$-G_k\Big(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}, \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial u}\Big), \cdots, \frac{\partial^{k-1} f}{\partial u^{k-1}}, \frac{\partial}{\partial x}\Big(\frac{\partial^{k-1} f}{\partial u^{k-1}}\Big)\Big),$$

where G_k is a polynomial with coefficients in derivatives of g and Γ. For example, $G_1=0$.

LEMMA 4. *On $\{u\}\times X\times(0, \infty)$, $\dfrac{\partial f}{\partial u}$ is C^∞ for every $u\in[0, 1]$, and $D_t^\alpha D_x^\beta\dfrac{\partial f}{\partial u}$ is continuous in u for every α and β.*

PROOF. Let α, ω, p and n be $0<\alpha<\omega<\infty$, $p<\infty$, $n<\infty$. We shall show $\dfrac{\partial f}{\partial u}\in L_n^p(\{u\}\times X\times[\alpha, \omega])$ for every $u\in[0, 1]$ by induction on n. We know that it is true for $n=2$. We write $X\times[\alpha, \omega]$ for $\{u\}\times X\times[\alpha, \omega]$ in short. Suppose $\dfrac{\partial f}{\partial u}\in L_{n-1}^p(X\times[\alpha, \omega])$ for every p, α and ω. Let π be $0<\pi<\alpha$. Then we have $\dfrac{\partial f}{\partial u}=0$ on $\partial X\times[\pi, \omega]$, $\dfrac{\partial f}{\partial u}\in L^p(X\times[\pi, \omega])$. Since f is C^∞ on $X\times(0, \infty)$, $\dfrac{\partial f}{\partial u}\in L_{n-1}^p(X\times[\pi, \omega])$ and $G_1=0$ in Lemma 3,

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{\partial f^\alpha}{\partial u}=g^{ij}\frac{\partial\Gamma_{\beta\gamma}^{\alpha}(f)}{\partial y^\delta}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\frac{\partial f^\delta}{\partial u}+g^{ij}\Gamma_{\beta\gamma}^{\alpha}(f)\frac{\partial f^\beta}{\partial x}\frac{\partial}{\partial x^j}\Big(\frac{\partial f^\gamma}{\partial u}\Big)$$

belongs to $L_{n-2}^p(X\times[\pi, \omega])$. Hence by a priori estimate we have $\dfrac{\partial f}{\partial u}\in L_n^p(X\times[\alpha, \omega])$. Thus $\dfrac{\partial f}{\partial u}$ is C^∞ on $\{u\}\times X\times(0, \infty)$ for every $u\in[0, 1]$.

Next we shall show that $D_t^\alpha D_x^\beta \dfrac{\partial f}{\partial u}$ are continuous in u. We have only to show that $\left\|\dfrac{\partial f_u}{\partial u}-\dfrac{\partial f_v}{\partial u}\right\|_{L_n^p[\alpha,\omega]}\to 0$ as $u\to v$. We prove it by induction on n and the case $n=2$ is known. By a priori estimate we have

$$\begin{aligned}&\left\|\frac{\partial f_u}{\partial u}-\frac{\partial f_v}{\partial u}\right\|_{L_n^p[\alpha,\omega]}\\ &\leqq C\left\|\left(\frac{\partial}{\partial t}-\Delta\right)\left(\frac{\partial f_u}{\partial u}-\frac{\partial f_v}{\partial u}\right)\right\|_{L_{n-2}^p[\pi,\omega]}+C\left\|\frac{\partial f_u}{\partial u}-\frac{\partial f_v}{\partial u}\right\|_{L^p[\pi,\alpha]}.\end{aligned}$$

It suffices to estimate the first term on the right hand side. By Lemma 3 we obtain

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial t}-\Delta\right)\left(\frac{\partial f_u^\alpha}{\partial u}-\frac{\partial f_v^\alpha}{\partial u}\right)\right\|_{L_{n-2}^p[\pi,\omega]}\\ &=\left\|2g^{ij}\Gamma_{\beta\gamma}^\alpha(f_u)\frac{\partial f_u^\beta}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial f_u^\gamma}{\partial u}-\frac{\partial f_v^\gamma}{\partial u}\right)\right.\\ &\quad+2g^{ij}\left(\Gamma_{\beta\gamma}^\alpha(f_u)\frac{\partial f_u^\beta}{\partial x^i}-\Gamma_{\beta\gamma}^\alpha(f_v)\frac{\partial f_v^\beta}{\partial x^i}\right)\frac{\partial}{\partial x^j}\left(\frac{\partial f_v^\gamma}{\partial u}\right)\\ &\quad+g^{ij}\frac{\partial\Gamma_{\beta\gamma}^\alpha(f_u)}{\partial y^\delta}\frac{\partial f_u^\beta}{\partial x^i}\frac{\partial f_u^\gamma}{\partial x^j}\left(\frac{\partial f_u^\delta}{\partial u}-\frac{\partial f_v^\delta}{\partial u}\right)\\ &\quad\left.+g^{ij}\left(\frac{\partial\Gamma_{\beta\gamma}^\alpha(f_u)}{\partial y^\delta}\frac{\partial f_u^\beta}{\partial x^i}\frac{\partial f_u^\gamma}{\partial x^j}-\frac{\partial\Gamma_{\beta\gamma}^\alpha(f_v)}{\partial y^\delta}\frac{\partial f_v^\beta}{\partial x^i}\frac{\partial f_v^\gamma}{\partial x^j}\right)\frac{\partial f_v^\delta}{\partial u}\right\|_{L_{n-2}^p[\pi,\omega]}.\end{aligned}$$

For $\varphi\in C^\infty$, $f\in L_k^p$, we have $\|\varphi f\|_{L_k^p}\leqq\|\varphi\|_{C^k}\|f\|_{L_k^p}$ if k is even. Using the interpolation theorem we can show that $\|\varphi f\|_{L_k^p}\leqq\|\varphi\|_{C^{k+1}}\|f\|_{L_k^p}$ if k is odd. Thus the right hand side tends to 0 as u tends to v by Lemma 2 and the induction hypothesis, completing the proof.

LEMMA 5. *On* $\{u\}\times X\times(0,\infty)$, $\dfrac{\partial^k f}{\partial u^k}$ *is* C^∞ *for every* $u\in[0,1]$ *and* $k\geqq 0$. *And* $D_t^\alpha D_x^\beta\dfrac{\partial^k f}{\partial u^k}$ *are continuous in* u.

PROOF. We prove by induction on k. It is true when $k=1$ by the previous lemma. Suppose the statement is true up to $k-1$, then $G_k\left(f,\cdots,\dfrac{\partial}{\partial x}\left(\dfrac{\partial^{k-1}f}{\partial u^{k-1}}\right)\right)$ is C^∞ on $\{u\}\times X\times(0,\infty)$. We can prove that $\dfrac{\partial^k f_u}{\partial u^k}\in L_n^p[\alpha,\omega]$ for every $u\in[0,1]$, $p<\infty$, $n<\infty$ and $0<\alpha<\omega<\infty$ and that $\left\|\dfrac{\partial^k f_u}{\partial u^k}-\dfrac{\partial^k f_v}{\partial u^k}\right\|_{L_n^p[\alpha,\omega]}\to 0$ as $u\to v$ just an in Lemma 4. Thus Lemma 5 follows.

§6. Proof of Theorem 1

LEMMA 6. *Let Y be a compact Riemannian manifold with convex boundary. Then there exists a positive number δ such that for every $y\in Y$ any two points in $B(y, \delta)=\{z\in Y \mid d(y, z)<\delta\}$ can be joined by the unique minimal geodesic.*

PROOF. Let $\tilde{Y}$ be the double of Y whose metric is the one extended from Y. Then there exist a positive number δ_1 such that for any $y\in\tilde{Y}$ any two points of the open ball $V(y, \delta_1)$ of radius δ_1 centered at y can be joined by the unique minimal geodesic in $\tilde{Y}$. Let $(y^1, \cdots, y^n)$ be a coordinate near the boundary such that $y^n=0$ on ∂Y and that for $y\in\tilde{Y}\backslash Y$, y^n is the geodesic distance to ∂Y. Then the second fundamental form of ∂Y is (Γ_{ab}^n) where a and b run from 1 through $n-1$. Let $c:(-\varepsilon, \varepsilon)\to\tilde{Y}$ be a geodesic with $c(0)=y\in\partial Y$ and $\frac{dc}{dt}(0)\in T_y\partial Y$. Then $\frac{d^2c^n}{dt^2}(0)=-\Gamma_{ab}^n\frac{dc^a}{dt}(0)\frac{dc^b}{dt}(0)\leqq 0$. Without loss of generality we may assume that $\frac{dc}{dt}(0)=\frac{\partial}{\partial y^1}$. When $y\in \mathrm{Int}\{y\in\partial Y \mid \Gamma_{11}^n=0\}$, c does not get out from ∂Y in a neighborhood of 0. When $y\in\{y\in\partial Y \mid \Gamma_{11}^n>0$, c does not enter into the interior of Y. Since ∂Y is compact there exists a positive number δ_2 such that $c^n(t)$ is non increasing on $\{t \mid |t|<\delta_2\}$. Let δ be $0<\delta<\frac{1}{2}\min(\delta_1, \delta_2)$. We see that δ is a number that we want. If $V(y, \delta)$ is in Y then nothing need be done since $\delta<\delta_1$. When $V(y, \delta)\cap\tilde{Y}\backslash Y\neq\emptyset$ suppose that the minimal geodesic between p and q in $V(y, \delta)$ got out from Y at $r\in\partial Y$. Let H be the hyperplane $\exp_r(T_r\partial Y)\cap V(y, \delta)$. Then H does not meet Int Y since $\tilde{d}(s, r)\leqq\tilde{d}(s, y)+\tilde{d}(y, r)<\delta_2$ for $s\in H\cap V(y, \delta)$. If p or q is in Int Y then c is not tangent to H, otherwise c is wholly in H and neither p nor q can be in Int Y. Thus c intersects H transversely and c must intersect H at some point $t\in H$ again. This is a contradiction since r and t can be jointed by two geodesics in $V(y, \delta)$. In the same way we can lead a contradiction when p and q is in ∂Y. This completes the proof.

LEMMA 7. *Let f and g be C^∞ maps from X to Y and δ be as in Lemma 6. We suppose $d(f, g)=\sup_{x\in X}d(f(x), g(x))$. Let $F:[0, 1]\times X\to Y$ be a homotopy between f and g such that for each $x\in X$, $F(\ , x):[0, 1]\to Y$ is the unique minimal geodesic between $f(x)$ and $g(x)$, then F is C^∞.*

PROOF. Let $\tilde{Y}$ be the double of Y and U be the neighborhood $\{(x, v_x)\in |v_x|<\delta\}$ of the zero section in $f^{-1}T\tilde{Y}$. We define $\varphi: U\to X\times\tilde{Y}$ by $\varphi(x, v_x)=(x, \exp_{f(x)}v_x)$, then maps U diffeomorphically onto $\varphi(U)$. On the other hand the map $g': X\to X\times Y\hookrightarrow X\times\tilde{Y}$ defined by $g'(x)=(x, g(x))$ is C^∞ and its image is in $\varphi(U)$. Thus $\varphi^{-1}\circ g'$ is also C^∞. Since $\frac{\partial F}{\partial s}\Big|_{s=0}(x)=\varphi^{-1}(x, g(x))$, $F(0, x)$ and

$\frac{\partial F}{\partial s}(0, x)$ is C^∞. Hence F is C^∞.

LEMMA 8. *In the notation of Proposition 4, we put* $\pi(u, x, t)=\frac{1}{2}h_{\alpha\beta}\nabla_u f^\alpha \times \nabla_u f^\beta$. *Then*

$$(1) \qquad \frac{\partial \pi}{\partial t}=\Delta\pi-g^{ij}h_{\alpha\beta}\nabla_i\nabla_u f^\alpha\cdot\nabla_j\nabla_u f^\beta+g^{ij}R_{\alpha\beta\gamma\delta}\nabla_i f^\alpha\cdot\nabla_u f^\beta\cdot\nabla_j f^\gamma\cdot\nabla_u f^\delta$$

$$\text{on } [0, 1]\times X\times(0, \infty),$$

$$(2) \qquad \frac{\partial \pi}{\partial \nu}=0 \qquad \text{on } [0, 1]\times\partial X\times(0, \infty)$$

where $\nabla_u f^\alpha=\frac{\partial f^\alpha}{\partial u}$, $\nabla_i f^\alpha=\frac{\partial f^\alpha}{\partial x^i}$, $\nabla_i\nabla_u f^\alpha=\frac{\partial^2 f^\alpha}{\partial x^i\partial u}+\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial u}$ *and* ν *is the unit outer normal on* ∂X.

The proof is done by direct calculation, or refer to [4] for (1). By the curvature condition and the maximal principle for parabolic inequalities, we obtain the following lemma;

LEMMA 9. *In the notation of Lemma 8, if we set* $D(t)=\sup_{x\in X, u\in[0,1]}\pi(u, x, t)$, *then* $D(t)$ *is non increasing in* t.

Since $\sup_{x\in X}d(f(u, x, t), g(v, x, t))\leqq D(t)|u-v|\leqq D(0)|u-v|$, we obtain the following lemma;

LEMMA 10. *If we set* $f(u, x)=\lim_{t\to\infty}f(u, x, t)$, *then* f *is uniformly Lipschitz continuous in* u.

Now we are in position to prove Theorem 1. Let $f_{\infty,0}$ and $f_{\infty,1}$ be two harmonic maps in the same connected component of $M_\phi(X, Y)$, and $f_0:[0, 1]\times X\to Y$ be a C^∞ homotopy between $f_{\infty,0}$ and $f_{\infty,1}$. Let $f(u, x, t)$ be as in Proposition 4 for f_0 taken above, then $f(0, x, t)=f_{\infty,0}(x)$ and $f(1, x, t)=f_{\infty,1}(x)$. We put $f_\infty(u, x)=\lim_{t\to\infty}f(u, x, t)$, then f_∞ is a Lipschitz continuous homotopy through harmonic maps between $f_{\infty,0}$ and $f_{\infty,1}$. Take a subdivision $0=u_0<u_1<\cdots<u_k=1$ of $[0, 1]$ so that $\sup_{x\in X}d(f_\infty(u_j, x), f_\infty(u_{j+1}, x))<\delta$ for $j=0, \cdots, k-1$. Let $F:[0, 1]\times X\to Y$ be the homotopy between $f_\infty(u_j, \ )$ and $f_\infty(u_{j+1}, \ )$ obtained as in Lemma 7. Then F is C^∞ and $F|_{\{s\}\times X}=\phi$ for all $s\in[0, 1]$. Since $F(0, \ )$ and $F(1, \ )$ are harmonic, $\frac{\partial}{\partial s}E(F(s, \ ))$ is 0 at $s=0$ and $s=1$. On the other hand by a direct computation,

$$\frac{\partial^2}{\partial s^2}E(F(s, \ ))=\int_X g^{ij}h_{\alpha\beta}\nabla_i\nabla_s\nabla_s F\cdot\nabla_j F$$

$$+\int_X g^{ij} h_{\alpha\beta} \nabla_i \nabla_s F \cdot \nabla_j \nabla_s F$$

$$-\int_X g^{ij} R_{\alpha\beta\gamma\delta} \nabla_i F^\alpha \cdot \nabla_s F^\beta \cdot \nabla_j F^\gamma \cdot \nabla_s F^\delta$$

where $\nabla : \Gamma(\overset{p}{\otimes} T^*M \otimes F^{-1}TY) \to \Gamma(\overset{p+1}{\otimes} T^*M \otimes F^{-1}TY)$ is the connection defined by $\nabla = \nabla^M \otimes 1 + 1 \otimes F^{-1}\nabla^Y$, $M = [0, 1] \times X$. Since $F(\ ,x) : [0, 1] \to Y$ is a geodesic, $\nabla_s \nabla_s F = 0$. From this and the curvature condition of Y we obtain

$$\frac{\partial^2}{\partial s^2} E(F(s,\ )) \geqq \int_X g^{ij} h_{\alpha\beta} \nabla_i \nabla_s F \cdot \nabla_j \nabla_s F \geqq 0 .$$

But $\frac{\partial^2}{\partial s^2} E(F(s,\ ))$ must be identically 0 since the first variation is 0 at $s=0$ and $s=1$. Thus $\nabla_i \nabla_s F = 0$. Thus $\nabla_s F$ is parallel on $\{s\} \times X$. Since $\nabla_s F = 0$ on $\{s\} \times \partial X$ we obtain $\nabla_s F = 0$. Thus $f(u_j,\ ) \equiv f(u_{j+1},\ )$. Hence we have $f_{\infty,0} \equiv f_{\infty,1}$.

§7. Proof of Corollaries 1 and 2

Let $M_\phi^0(X, Y)$ be a connected component of $M_\phi(X, Y)$. For each $f_0 \in M^0(X, Y)$ let f be the solution of (1.1) with the initial value f_0. We define $\Phi : M_\phi^0(X, Y) \times [0, 1] \to M_\phi^0(X, Y)$ as $\Phi(f_0,\ ) = f\left(\ ,\ \tan\frac{\pi}{2}\theta\right)$. Clearly $\Phi(f_0, 0) = f_0$ and $\Phi(f_0, 1) = f_\infty$ where f is the unique harmonic map in $M_\phi^0(X, Y)$. To prove that $M_\phi^0(X, Y)$ is contractible, we show that Φ is continuous. Let $U_\varepsilon(f_0)$ be the set of all elements g in $M_\phi^0(X, Y)$ with $d(f_0, g) < \varepsilon$. For arbitrary $(h_0, \theta) \in \Phi^{-1}(U_\varepsilon(f_0))$ we put $\zeta = \Phi(h_0, \theta)$ and take ε_1 such that $0 < \varepsilon_1 < \max\left\{\frac{1}{2}[\varepsilon - d(f_0, \zeta)], \delta\right\}$. We set

$$U_{\varepsilon_1}(h_0) = \{\eta_0 \in M^0(X, Y) \mid d(h_0, \eta_0) < \varepsilon_1\}$$

$$V_{\varepsilon_1}(\theta) = \left\{\tau \in [0, 1] \,\middle|\, d\left(h\left(\ ,\ \tan\frac{\pi}{2}\theta\right), h\left(\ ,\ \tan\frac{\pi}{2}\tau\right)\right) < \varepsilon_1\right\}.$$

Then for $\eta_0 \in U_{\varepsilon_1}(h_0)$ and $\tau \in V_{\varepsilon_1}(\theta)$, we have

$$d\left(f_0, \eta\left(\ ,\ \tan\frac{\pi}{2}\tau\right)\right)$$

$$\leqq d\left(f_0, h\left(\ ,\ \tan\frac{\pi}{2}\theta\right)\right) + d\left(h\left(\ ,\ \tan\frac{\pi}{2}\theta\right), h\left(\ ,\ \tan\frac{\pi}{2}\tau\right)\right)$$

$$+ d\left(h\left(\ ,\ \tan\frac{\pi}{2}\tau\right), \eta\left(\ ,\ \tan\frac{\pi}{2}\tau\right)\right)$$

$$\leqq d(f_0, \zeta)+\varepsilon_1+d(h_0, \eta_0)$$

$$\leqq d(f_0, \zeta)+2\varepsilon_1<\varepsilon$$

where in the second inequality Lemma 9 was used. Thus $\Phi^{-1}(U_\varepsilon(f_0))$ is open and Φ is continuous.

Now we shall prove Corollary 2. Let Y be a topological space with the base point y_0 and ΩY be the space of continuous loops with the base point $[y_0]$ where $[y_0]$ is the constant map and ΩY be endowed with the compact open topology. Then $\pi_{i+1}(Y, y_0)\cong\pi_i(\Omega Y, [y_0])$ for $i\geqq 1$. This isomorphism is given as follows. For $[\varphi]\in\pi_i(\Omega Y, [y_0])$ can be regarded as a map from the reduced suspension $(S^i, p)\times(S^1, o)/S^i\times\{o\}\cup S^1\times\{p\}$ to (Y, y_0) which is an element in $\pi_{i+1}(Y, y_0)$. Note that when Y is a metric space the compact open topology coincides with the C^0 topology. In Theorem 1, we take $X=[0, 1]$ and $\phi(0)=\phi(1)=y_0\in Y$. By the C^∞ approximation theorem there exists a C^∞ map f_0 from $(S^i, p)\times(S^1, o)$ to Y homotopic to φ fixing $S^i\times\{o\}\cup S^1\times\{p\}$. Then f can be regarded as a map from (S^i, p) to $M_\phi^0(X, Y)$ which is a subspace of ΩY. By Corollary 1 f can be deformed to the constant map $[y_0]$. Thus $\pi_i(\Omega Y, [y_0])=0$ for $i\geqq 1$.

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