On a construction of the fundamental solution for Schrödinger equations[*)]

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(Communicated by H. Fujita)

§ 1. Introduction.

In [4] Feynman introduced the notion of path integrals and gave a reformulation of the quantum mechanics. Mathematically, the notion of the path integral was first formulated by K. Ito [7] as a certain kind of improper integrals. Albeverio and Høegh-Krohn [1] also gave a mathematical formulation equivalent to Ito's. Following the formulation by Albeverio and Høegh-Krohn, we may state Feynman's idea as follows permitting some abuse of terminology: first to assign a measure-like quantity to each classical path γ and then to represent a solution of Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial\phi}{\partial t}(x, t)+\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\phi(x, t)+V(t, x)\phi(x, t)=0 \tag{1.1}$$

by "integrating" $e^{i\hbar^{-1}S(\gamma)}\varphi(\gamma(s))$ with respect to that quantity. Here $S(\gamma)$ is the classical action along the path γ starting from the position $\gamma(s)$ at the time s; $\hbar$ is the Planck constant, $\hbar=h/2\pi$, $0<\hbar\leqq 1$; and φ is the initial state.

In arriving at his idea, Feynman considered a successive integral

$$\begin{aligned} &E_F(\hbar, t, t_{l-1})E_F(\hbar, t_{l-1}, t_{l-2})\cdots E_F(\hbar, t_1, s)\varphi(x) \\ &\quad=\prod_{j=1}^{l-1}\left(\frac{-i\hbar^{-1}m}{2\pi(t_j-t_{j-1})}\right)^{n/2}\int_{R^n}\cdots\int_{R^n} e^{i\hbar^{-1}S(\gamma(t, t_{l-1}, x, x_{l-1}))}\cdots \\ &\qquad\cdots e^{i\hbar^{-1}S(\gamma(t_1, s, x_1, x_0))}\varphi(x_0)dx_0\cdots dx_{l-1}, \end{aligned} \tag{1.2}$$

where

$$E_F(\hbar, t, s)\varphi(x)\equiv\left(\frac{-i\hbar^{-1}m}{2\pi(t-s)}\right)^{n/2}\int_{R^n} e^{i\hbar^{-1}S(\gamma(t, s, x, y))}\varphi(y)dy, \tag{1.3}$$

and then took a limit when $\delta(\Delta)\equiv\max_{1\leq j\leq l}|t_j-t_{j-1}|\to 0$. Here Δ is a subdivision of the interval $[s, t]$ or $[t, s]$: $t_0=s\gtrless t_1\gtrless\cdots\gtrless t_{l-1}\gtrless t_l=t$, and $\gamma(t, s, x, y)$ denotes the classical path passing through the position x and y at the time t and s, respectively.

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*) This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

Recently, instead of considering improper integrals, D. Fujiwara [5] directly proved the convergence of the integral (1.2) when $\delta(\Delta)\to 0$ in the *operator norm* in $L^2(R^n)$ and proved that the limit is actually the solution of Schrödinger equation (1.1). The purpose of the present note is to show that the construction procedure of Fujiwara [5] works well even when we replace $E_F(\hbar, t, s)$ by

$$E(\hbar, t, s)u(x)\equiv\int_{R^n} e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}(\mathcal{F}u)(\xi)d\xi\,, \tag{1.4}$$

which has the same form as the operator used by Kumano-go, Taniguchi and Tozaki [12] in constructing the fundamental solution for hyperbolic operators. Here $\phi(t, s, x, \xi)$ is the generating function of a certain canonical diffeomorphism of R^{2n} (see Propositions 3.2 and 3.5) and is related with the classical action along the classical path $\gamma(t, s, x, \xi)$ passing through the position x at time t and having the *momentum* ξ at time s (see the remark just before Proposition 3.5). And $\mathcal{F}$ denotes the usual Fourier transformation

$$(\mathcal{F}\varphi)(\xi)=\hat{\varphi}(\xi)\equiv(2\pi)^{-n}\int_{R^n} e^{-iy\cdot\xi}\varphi(y)dy\,. \tag{1.5}$$

Thus the main problem we shall consider is the convergence of the successive integral

$$\begin{aligned} E_\Delta(\hbar, t, s)\varphi(x)&\equiv E(\hbar, t, t_{l-1})E(\hbar, t_{l-1}, t_{l-2})\cdots E(\hbar, t_1, s)\varphi(x)\\ &=(2\pi)^{-n(l-1)}\iint_{R^n\times R^n}\cdots\iint_{R^n\times R^n} e^{i[\hbar^{-1}\phi(t,t_{l-1},x,\hbar\xi_{l-1})-\xi_{l-1}\cdot x_{l-1}]}\cdots\\ &\qquad\cdots e^{i[\hbar^{-1}\phi(t_1,s,x_1,\hbar\xi_0)-\xi_0\cdot x_0]}\varphi(x_0)dx_0d\xi_0\cdots dx_{l-1}d\xi_{l-1}\,. \end{aligned} \tag{1.6}$$

Our choice of $E(\hbar, t, s)$ makes the treatment of the problem somewhat simpler and more straightforward than that in Fujiwara [5]. Moreover it allows us to deal with the pseudo differential equation of Schrödinger's type

$$\begin{cases} \dfrac{\hbar}{i}\dfrac{\partial\psi}{\partial t}(x, t)+H(t, x, \hbar D_x)\psi(x, t)=0\,, & D_x=\dfrac{1}{i}\dfrac{\partial}{\partial x}\,,\\ \psi(x, s)=\varphi(x)\,, \end{cases} \tag{1.7}$$

which is somewhat more general than (1.1). The precise conditions imposed on the Hamiltonian $H(t, x, \xi)$ will be stated below. As a corollary we can also prove the essential self-adjointness of $H(t, x, \hbar D_x)$ in $L^2(R^n)$.

To take into account the dependency on $\hbar$ of the convergence of successive integrals, we shall also consider $E^{(N)}(\hbar, t, s)$ $(N\geqq 1)$ defined by

$$E^{(N)}(\hbar, t, s)u(x)\equiv\int_{R^n} e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}a^{(N)}(\hbar, t, s, x, \hbar\xi)\hat{u}(\xi)d\xi\,, \tag{1.8}$$

where the function $a^{(N)}$ has the form

$$a^{(N)}(\hbar, t, s, x, \xi)=\sum_{j=1}^{N}(i\hbar^{-1})^{-j+1}a_j(t, s, x, \xi), \tag{1.9}$$

$a_j(t, s, x, \xi)$ being a solution of the transport equation defined in section 3. For the sake of convenience, we shall write $E(\hbar, t, s)$ in (1.4) as $E^{(0)}(\hbar, t, s)$.

In [4] Feynman called $\left(\frac{-i\hbar^{-1}m}{2\pi(t-s)}\right)^{n/2} e^{i\hbar^{-1}S(\gamma(t,s,x,y))}$ the probability amplitude for the path $\gamma(t, s, x, y)$. Corresponding to this our $e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}$ may be called the probability amplitude for the path $\gamma(t, s, x, \xi)$. In this sense our formulation might give an alternate point of view to the theory of Feynman path integrals (see section 7).

Our assumption on the Hamiltonian $H(t, x, \xi)$ is as follows.

ASSUMPTION.

i) For each $t\in R^1$, $H(t, x, \xi)$ is a real-valued C^∞-function of $(x, \xi)\in R^n\times R^n$.

ii) The derivative $\partial_x^\alpha\partial_\xi^\beta H(t, x, \xi)$ is continuous in $(t, x, \xi)\in R^1\times R^n\times R^n$ for any α, β.

iii) For any $T>0$ and any multi-indices α, β satisfying $|\alpha|+|\beta|\geqq 2$,

$$\sup_{\substack{x,\xi\in R^n\\ |t|\leqq T}} |\partial_x^\alpha\partial_\xi^\beta H(t, x, \xi)|<\infty . \tag{1.10}$$

Here $\partial_x=(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ and $\partial_\xi=(\partial/\partial\xi_1, \cdots, \partial/\partial\xi_n)$.

iv) For any $t\in R^1$ and $\varphi, \psi\in\mathcal{S}$,

$$\langle H(t, x, \xi), e^{ix\cdot\xi}\overline{\psi(x)}\hat{\varphi}(\xi)\rangle=\langle H(t, x, \eta), e^{-ix\cdot\eta}\varphi(x)\overline{\hat{\psi}(\eta)}\rangle . \tag{1.11}$$

Here $H(t, x, \xi)$ is regarded as a tempered distribution in (x, ξ); $\mathcal{S}$ is the Schwartz space; $\bar{\psi}$ denotes the complex conjugate; and $\hat{\varphi}$ denotes the usual Fourier transform defined by (1.5).

The last condition iv) can be rewritten at least formally as

$$(\mathcal{F}H(t, \cdot, \xi))(\eta-\xi)=(\mathcal{F}H(t, \cdot, \eta))(\eta-\xi) \quad \text{in } \mathcal{S}'(R_\xi^n\times R_\eta^n). \tag{1.12}$$

Here $\mathcal{F}H(t, \cdot, \xi)$ denotes the Fourier transform of $H(t, x, \xi)$ with respect to x with t, ξ fixed as the tempered distribution in x.

We shall assume the above Assumption throughout this note.

REMARK 1.1. Under the above Assumption, $H(t, x, \xi)$ satisfies

$$\sup_{|t|\leqq T}|\partial_x^\alpha\partial_\xi^\beta H(t, x, \xi)|\leqq C_{T,\alpha,\beta}(|x|+|\xi|+1)^{2-|\alpha|-|\beta|} \tag{1.13}$$

for $T>0$ and $|\alpha|+|\beta|\leqq 1$, where $C_{T,\alpha,\beta}$ is independent of $x, \xi\in R^n$. This is easily proved by using the mean-value theorem.

Example 1.2. The Hamiltonian operator with a constant vector potential

$$H(t, x, \hbar D_x)=-\sum_{k=1}^{n}(\hbar\partial_k-ib_k(t))^2+V(t, x), \qquad \partial_k=\frac{\partial}{\partial x_k}, \tag{1.14}$$

or more generally

$$\begin{aligned}&H(t, x, \hbar D_x)\\ &=\sum_{j,k=1}^{n}\hbar^2 a_{jk}(t)D_jD_k+\sum_{\substack{j,k=1\\ j\neq k}}^{n}\hbar b_{jk}(t)x_jD_k+\sum_{k=1}^{n}\hbar b_k(t)D_k+V(t, x),\\ &\qquad\qquad D_k=-i\partial_k,\end{aligned} \tag{1.15}$$

satisfy our assumption, when $b_k(t)$, $a_{jk}(t)$ and $b_{jk}(t)$ are real-valued continuous functions of $t\in R^1$, and $V(t, x)$ is continuous in (t, x); C^∞ in x with t fixed; and satisfies for any $T>0$ and $|\alpha|\geqq 2$,

$$\sup_{\substack{x\in R^n\\ |t|\leq T}}|\partial_x^\alpha V(t, x)|<\infty. \tag{1.16}$$

This example covers the essential part of the Hamiltonians considered by Fuiiwara [5].

Before stating our main theorem, we make some preparation.

DEFINITION 1.3. For any $\varphi\in\mathcal{S}$, $t\in R^1$, $0<\hbar\leqq 1$, and $x\in R^n$, define

$$(H_0(\hbar, t)\varphi)(x)\equiv\int_{R^n}e^{ix\cdot\xi}H(t, x, \hbar\xi)\hat{\varphi}(\xi)d\xi. \tag{1.17}$$

PROPOSITION 1.4. *$H_0(\hbar, t)$ is a symmetric operator in $\mathfrak{H}=L^2(R^n)$ with the domain $\mathcal{S}$.*

Proof is immediate from iv) of Assumption.

DEFINITION 1.5. We denote by $H(\hbar, t)$ the minimal closed extension of $H_0(\hbar, t)$ in $\mathfrak{H}=L^2(R^n)$.

DEFINITION 1.6. For any integer $k\geqq 0$, we put

$$Y_k=\{f\in L^2(R^n)\mid \|f\|_k<\infty\}, \tag{1.18}$$

where

$$\|f\|_k=\Big[\sum_{|\alpha|+l\leq k}\|(1+|x|^2)^{l/2}\partial_x^\alpha f(x)\|_{L^2(R^n)}^2\Big]^{1/2}. \tag{1.19}$$

Y_k is a Hilbert space with respect to the norm $\|\ \|_k$. Moreover we have

$$\begin{cases} Y_{k+1}\subset Y_k \quad \text{(densely embedded for } k\geqq 0\text{)},\\ \|f\|_k\leqq\|f\|_{k+1},\\ Y_0=L^2(R^n).\end{cases} \tag{1.20}$$

Now we can state our main theorem.

THEOREM. *Let Assumption above be satisfied. Let* $0<\hbar\leqq 1$ *and let* $N\geqq 0$ *be an integer. Then for any* $T>0$ *and* $t, s\in[-T, T]$, $E_{\Delta}^{(N)}(\hbar, t, s)$ *defined by* (1.6) *with* $E(\hbar, t, s)$ *replaced by* $E^{(N)}(\hbar, t, s)$ *converges to a unitary operator* $U(\hbar, t, s)$ *(which does not depend on* N*) when* $\delta(\Delta)\to 0$ *in the uniform operator topology in* $\mathfrak{H}=L^{2}(R^{n})$. *More precisely we have*

$$\|U(\hbar, t, s)-E_{\Delta}^{(N)}(\hbar, t, s)\|\leqq\gamma b\hbar^{N}|t-s|e^{b\hbar^{N}|t-s|/2}\delta(\Delta) \tag{1.21}$$

for any subdivision Δ *of the interval* $[s, t]$ *or* $[t, s]$ *such that* $\delta(\Delta)$ *is sufficiently small. Here* $\gamma=4(e+2\log_{2}e)$; b *is some positive constant independent of* Δ, $\hbar\in(0, 1]$, *and* $t, s\in[-T, T]$ *(*b *is taken the same constant as* b *in* (5.30) *(see Theorem* 2.1)*); and in* (1.21), $\|\ \|$ *denotes the operator norm in* $\mathfrak{H}$.

The family $\{U(\hbar, t, s)\mid t, s\in R^{1}\}$ *of unitary operators thus constructed and* $H(\hbar, t)$ *satisfy the following properties* i)$\sim$vi):

i) $U(\hbar, t, t)=I$ *(identity operator) for any* $t\in R^{1}$.

ii) *For any* $u\in\mathfrak{H}$, *the mapping*

$$R^{2}\ni(t, s)\longmapsto U(\hbar, t, s)u\in\mathfrak{H} \tag{1.22}$$

is continuous.

iii) *For any* $t, r, s\in R^{1}$,

$$U(\hbar, t, r)U(\hbar, r, s)=U(\hbar, t, s). \tag{1.23}$$

iv) *For any* $t, s\in R^{1}$,

$$Y_{2}\subset\mathcal{D}(H(\hbar, t)) \tag{1.24}$$

and

$$U(\hbar, t, s)Y_{2}=Y_{2}. \tag{1.25}$$

v) *For any* $t, s\in R^{1}$ *and* $f\in Y_{2}$, *there exist the derivatives* $\frac{d}{dt}U(\hbar, t, s)f$ *and* $\frac{d}{ds}U(\hbar, t, s)f$ *in* $\mathfrak{H}=L^{2}(R^{n})$ *and they satisfy*

$$\frac{d}{dt}U(\hbar, t, s)f+iH(\hbar, t)U(\hbar, t, s)f=0, \tag{1.26}$$

$$\frac{d}{ds}U(\hbar, t, s)f-iU(\hbar, t, s)H(\hbar, s)f=0. \tag{1.27}$$

vi) *For any* $t\in R^{1}$, $H(\hbar, t)$ *is self-adjoint in* $\mathfrak{H}=L^{2}(R^{n})$, *in other words,* $H_{0}(\hbar, t)$ *is essentially self-adjoint in* $\mathfrak{H}$.

The equation (1.26) corresponds to the pseudo differential equation (1.7) with initial condition $\phi(\cdot, s)=f$.

REMARK 1.7. The family $\{U(\hbar, t, s) \mid t, s\in[-T, T]\}$ of bounded operators satisfying i)$\sim$v) above is usually called the evolution operator. There are various ways of construction of the evolution operator satisfying i)$\sim$v). Among them, we refer to the abstract approach by T. Kato [8], [9] and others through the so-called product integral of resolvents, which are applicable to the present problem under certain circumstances.

REMARK 1.8. It is well-known that the Schrödinger operator $H_0(\hbar, t)=-\hbar^2\Delta+V(t, x)$ is essentially self-adjoint under our Assumption (see e.g. T. Kato [10]).

REMARK 1.9. The convergence property (1.21) for $N\geqq 1$ may be useful in considering the quasi-classical limit as $\hbar\to 0$ for (1.7) (see K. Yajima [14], [15]).

The following sections are devoted to proving the above theorem. In section 2, we shall state two fundamental theorems which summarize a construction procedure of the solution for (1.26), in a rather general context. In section 3, we shall construct an operator $E^{(N)}(\hbar, t, s)$ which exactly corresponds to (1.8) and gives an approximate solution for (1.26) when $|t-s|$ is sufficiently small (see (4.25), (4.26), and ii) of Theorem 5.1). Some of the elementary properties of $E^{(N)}(\hbar, t, s)$ will then be considered in section 4. In sections 5 and 6, we shall investigate most important properties of $E^{(N)}(\hbar, t, s)$ and other operators for the construction of the solution for (1.26). In these sections, the L^2-boundedness theorem for a certain kind of integral transformations proved in the Appendix, will play a crucial role. In the last section 7, we shall make some remarks on our formulation.

§ 2. Fundamental theorems.

To construct the family $\{U(t, s) \mid t, s\in R^1\}$ of unitary operators in $\mathfrak{H}=L^2(R^n)$ satisfying i)$\sim$iii) of our Theorem, we shall use the following theorem which summarizes the Fujiwara's construction.

THEOREM 2.1. (due to Fujiwara [5]). *Let $\mathfrak{H}$ be a Hilbert space and let $T>0$, $a>0$, $b>0$, $1>\delta>0$, and $d\geqq 0$ be fixed as $b\delta^{2+d}<1/2$. Let $\{E(t, s) \mid |t|, |s|\leqq T, |t-s|<\delta\}$ be a family of bounded operators in $\mathfrak{H}$ which satisfies the following conditions* a)$\sim$d):

a) *$E(s, s)=I$ for $|s|\leqq T$.*

b) *$\|E(t, s)\|\leqq a$ for $|t|, |s|\leqq T, |t-s|<\delta$.*

c) *There exists a dense subset $\mathfrak{D}$ of $\mathfrak{H}$ such that the mapping*

(2.1) $$[-T,\ T]^2 \ni (t,\ s) \longmapsto E(t,\ s)u \in \mathfrak{H}$$

is continuous when $|t-s|<\delta$ *and* $u \in \mathcal{D}$.

d) *For any* t, r, s *satisfying* $|t|,\ |r|,\ |s| \leqq T$, $|t-s|<\delta$, $|r-s|<\delta$, *and* $|t-r|<\delta$,

(2.2) $$\begin{cases} \|E(t,\ r)^* E(t,\ s)-E(r,\ s)\| \leqq b(|t-r|^{2+d}+|r-s|^{2+d}), \\ \|E(t,\ r)E(r,\ s)-E(t,\ s)\| \leqq b(|t-r|^{2+d}+|r-s|^{2+d}). \end{cases}$$

(*In the above* $\|\ \|$ *denotes the operator norm in* $\mathfrak{H}$. *We shall also use the notation* $B(\mathfrak{H})$ *as the Banach space of bounded operators in* $\mathfrak{H}$ *equipped with this norm.*)

Let $[s,\ t] \subset [-T,\ T]$ $(s<t)$ *and let* Δ *be a subdivision of* $[s,\ t]$:

$$\Delta:\ t_0=s<t_1<\cdots<t_l=t\,.$$

Put $\delta(\Delta) \equiv \max_{1 \leq j \leq l} |t_j-t_{j-1}|$ *and define* E_Δ *for* Δ *satisfying* $\delta(\Delta)<\delta$ *as follows*:

(2.3) $$\begin{cases} E_\Delta(s,\ t) \equiv E(t_0,\ t_1) \cdots E(t_{l-1},\ t_l), \\ E_\Delta(t,\ s) \equiv E(t_l,\ t_{l-1}) \cdots E(t_1,\ t_0). \end{cases}$$

Then the following assertions i)$\sim$iii) *hold*:

i) $\|E(t,\ s)\| \leqq e^{b|t-s|^{2+d}/2}$ *for* $|t|,\ |s| \leqq T$, $|t-s|<\delta$.

ii) *There exist the limits*

(2.4) $$\begin{cases} U(s,\ t) \equiv \lim_{\delta(\Delta)\to 0} E_\Delta(s,\ t), \\ U(t,\ s) \equiv \lim_{\delta(\Delta)\to 0} E_\Delta(t,\ s) \end{cases}$$

in $B(\mathfrak{H})$. *More precisely we have*

(2.5) $$\begin{cases} \|U(s,\ t)-E_\Delta(s,\ t)\| \leqq \gamma b|t-s| e^{b|t-s|/2}\delta(\Delta)^{1+d}, \\ \|U(t,\ s)-E_\Delta(t,\ s)\| \leqq \gamma b|t-s| e^{b|t-s|/2}\delta(\Delta)^{1+d}, \end{cases}$$

where $\gamma \equiv 4(e+2\log_2 e)$.

iii) *If we put* $U(s,\ s)=I$ *for* $|s| \leqq T$, *then*:

1) *the mapping*: $[-T,\ T]^2 \ni (t,\ s) \mapsto U(t,\ s)u \in \mathfrak{H}$ *is continuous for any* $u \in \mathfrak{H}$;

2) $U(t,\ s)$ *is an unitary operator in* $\mathfrak{H}$ *for* $|t|,\ |s| \leqq T$; *and*

3) $U(t,\ r)U(r,\ s)=U(t,\ s)$ *for any* $|t|,\ |r|,\ |s| \leqq T$.

REMARK 2.2. If the assumptions of Theorem 2.1 hold with T replaced by another T' as well as for the original T, then $U(t,\ s)$ obviously remains the same as long as $|t|,\ |s| \leqq \min(T,\ T')$.

REMARK 2.3. $E_\Delta(t,\ s)$ and $E_\Delta(s,\ t)$ are abstract successive integrals corresponding to (1.2) or (1.6) in section 1.

To prove the assertions iv)$\sim$vi) of our Theorem we shall use the following theorem.

THEOREM 2.4. *Let all assumptions of Theorem* 2.1 *be satisfied and let* $U(t, s)$ *be the unitary operator constructed in Theorem* 2.1. *Let* $\{H_0(t) \mid |t| \leqq T\}$ *be a family of symmetric operators in* $\mathfrak{H}$ *and denote by* $H(t)$ *the minimal closed extension of* $H_0(t)$ *in* $\mathfrak{H}$. *Suppose further that there exists a subset* Y *of* $\mathfrak{H}$ *satisfying the following properties* a)$\sim$c):

a) $Y \subset \mathcal{D}(H(t))$ *for* $|t| \leqq T$.

b) $U(t, s)Y \subset Y$ *for* $|t|, |s| \leqq T$.

c) *For any* $f \in Y$ *and* $|t| \leqq T$, *there exists the derivative* $\frac{d}{dt}E(t, s)f|_{s=t}$ *in* $\mathfrak{H}$ *and satisfies*

$$\frac{d}{dt}E(t, s)f|_{s=t} = -iH(t)f. \tag{2.6}$$

Then the following assertions i) *and* ii) *hold*:

i) *For any* $f \in Y$ *and* $t, s \in [-T, T]$, *there exist the derivatives* $\frac{d}{dt}U(t, s)f$ *and* $\frac{d}{ds}U(t, s)f$ *in* $\mathfrak{H}$ *and we have*

$$\begin{cases} \dfrac{d}{dt}U(t, s)f = -iH(t)U(t, s)f, \\[2ex] \dfrac{d}{ds}U(t, s)f = iU(t, s)H(s)f. \end{cases} \tag{2.7}$$

ii) *When* $H_0(t)$ *is independent of* $t \in [-T, T]$ *and* Y *is dense in* $\mathfrak{H}$, $H_0 \equiv H_0(t)$ *is essentially self-adjoint in* $\mathfrak{H}$.

PROOF. i) Let $f \in Y$ and $t, s \in [-T, T]$. Then by iii), 3) of Theorem 2.1, we have

$$\begin{aligned} &\left\| \frac{1}{h}(U(t+h, s)-U(t, s))f + iH(t)U(t, s)f \right\| \\ &\leqq \left\| \frac{1}{h}(U(t+h, t)-E(t+h, t))U(t, s)f \right\| \\ &\quad + \left\| \frac{1}{h}(E(t+h, t)-I)U(t, s)f + iH(t)U(t, s)f \right\|. \end{aligned} \tag{2.8}$$

The first term on the right-hand side is bounded by $|h|^{-1}\gamma b|h||h|^{1+d}e^{b|h|/2}$ from (2.5), which converges to zero as $h \to 0$. The second term also converges to zero by b) and c).

The second part of (2.7) is easily proved by the following inequality, if we use the first part and iii), 1) of Theorem 2.1:

$$\left\| \frac{1}{h}(U(t, s+h)-U(t, s))f - iU(t, s)H(s)f \right\| \tag{2.9}$$

$$\leqq\left\|\frac{1}{h}(U(s+h,\ s)-I)f+iH(s)f\right\|+\|i(I-U(s+h,\ s))H(s)f\|\,.$$

ii) We have only to show $\mathrm{Ker}(H^*\pm i)=\{0\}$. For example let $\varphi\in\mathscr{D}(H^*)$ satisfy $H^*\varphi=i\varphi$. Following page 267 of Reed and Simon [13], consider $F_f(t,\ s)=(U(t,\ s)f,\ \varphi)$ for $f\in Y$ and $t,\ s\in[-T,\ T]$. Then using i), we obtain $\frac{d}{dt}F_f(t,s)=-F_f(t,\ s)$. Thus $(U(t,\ s)f,\ \varphi)=(f,\ \varphi)e^{-(t-s)}$. Because Y is dense in $\mathfrak{H}$ we get from this $(U(t,\ s)\varphi,\ \varphi)=\|\varphi\|^2e^{-(t-s)}$ and hence $e^{-(t-s)}\|\varphi\|^2\leqq\|\varphi\|^2$ for $t,\ s\in[-T,\ T]$. Taking $t<s$, we have proved $\varphi=0$. Q.E.D.

Thus in order to prove our main Theorem we have to prove that all assumptions of Theorems 2.1 and 2.4 are satisfied for some $T>0$ which we shall fix throughout the rest of this note and for a suitably defined $E^{(N)}(\hbar,\ t,\ s)$ in accordance with $H_0(\hbar,\ t)$ in Definition 1.3. In the next section, we shall define $E^{(N)}(\hbar,\ t,\ s)$ for $t,\ s\in[-T,\ T]$ which yields $E_\Delta^{(N)}(\hbar,\ t,\ s)$ that corresponds to the successive integral (1.6).

§ 3. Definition of $E^{(N)}(\hbar,\ t,\ s)$.

In this section we shall construct $E^{(N)}(\hbar,\ t,\ s)$ for $t,\ s\in[-T,\ T]$ under the Assumption stated in section 1. To do this we shall first investigate some properties of the classical orbit corresponding to the Hamiltonian $H(t,\ x,\ \xi)$. Hereafter the time parameters $t,\ s,\ r$, etc. are confined to the interval $[-T,\ T]$.

Let $q(t,\ s,\ x,\ \xi)$ and $p(t,\ s,\ x,\ \xi)$ be the solution of the Hamilton equation

$$\left\{\begin{aligned}&\frac{dq}{dt}(t)=\partial_\xi H(t,\ q(t),\ p(t))\,,\\&\frac{dp}{dt}(t)=-\partial_x H(t,\ q(t),\ p(t))\end{aligned}\right.\tag{3.1}$$

with initial condition

$$q(s)=x\,,\qquad p(s)=\xi\,.\tag{3.2}$$

$(q(t,\ s,\ x,\ \xi),\ p(t,\ s,\ x,\ \xi))$ is the classical orbit in the phase space.

PROPOSITION 3.1. *The solution of* (3.1) *and* (3.2) *exists and* C^1 *in* $(t,\ s,\ x,\ \xi)$ *and* C^∞ *in* $(x,\ \xi)$ *with* $t,\ s$ *fixed, and satisfies the following estimates* i)$\sim$iii) *for* $|t-s|<1$:

i)

$$\left\{\begin{aligned}&|q(t,\ s,\ x,\ \xi)-x|\leqq C_0(|x|+|\xi|+1)|t-s|\,,\\&|p(t,\ s,\ x,\ \xi)-\xi|\leqq C_0(|x|+|\xi|+1)|t-s|\,.\end{aligned}\right.\tag{3.3}$$

ii)

$$(3.4)\qquad \begin{cases} |(\partial_s q)(t, s, x, \xi)+\partial_\xi H(s, x, \xi)| \leqq C_0(|x|+|\xi|+1)|t-s|, \\ |(\partial_s p)(t, s, x, \xi)-\partial_x H(s, x, \xi)| \leqq C_0(|x|+|\xi|+1)|t-s|. \end{cases}$$

The constant C_0 *in* (3.3) *and* (3.4) *is independent of* x, ξ, t, s.

iii) *For* $|\alpha|+|\beta|\geqq 1$,

$$(3.5)\qquad \begin{cases} |\partial_x^\alpha \partial_\xi^\beta (q(t, s, x, \xi)-x)| \leqq C_{|\alpha|,|\beta|}|t-s|, \\ |\partial_x^\alpha \partial_\xi^\beta (p(t, s, x, \xi)-\xi)| \leqq C_{|\alpha|,|\beta|}|t-s|. \end{cases}$$

The constant $C_{|\alpha|,|\beta|}$ *is independent of* x, ξ, t, s.

Proof is easily done by successive approximation as in [5] hence is omitted.

PROPOSITION 3.2. *Let* δ_1 *be fixed as* $0<\delta_1\leqq 1$ *and* $n\cdot\max(C_{1,0}, C_{0,1})\delta_1<1/2$, *where* $C_{1,0}$ *and* $C_{0,1}$ *are the constants in* (3.5). *Let* $|t-s|<\delta_1$. *Then*:

i) *For any fixed* t, s, ξ, *the mapping*

$$(3.6)\qquad R^n \ni y \longmapsto x=q(t, s, y, \xi)\in R^n$$

is a C^∞ *diffeomorphism. We write the inverse* C^∞ *diffeomorphism as*

$$(3.7)\qquad R^n \ni x \longmapsto y(t, s, x, \xi)\in R^n .$$

This mapping is C^1 *in* (t, s, x, ξ) *and* C^∞ *in* (x, ξ) *with* t, s *fixed.*

ii) *For any fixed* t, s, x, *the mapping*

$$(3.8)\qquad R^n \ni \eta \longmapsto \xi=p(t, s, x, \eta)\in R^n$$

is a C^∞ *diffeomorphism. We write the inverse* C^∞ *diffeomorphism as*

$$(3.9)\qquad R^n \ni \xi \longmapsto \eta(t, s, x, \xi)\in R^n .$$

This mapping is C^1 *in* (t, s, x, ξ) *and* C^∞ *in* (x, ξ) *with* t, s *fixed.*

PROOF. i) Because of (3.5) with $|\alpha|=1$ and $|\beta|=0$, we have only to prove the bijectiveness of the mapping (3.6). Following Kumano-go [11], put $T_x(y)=x+y-q(t, s, y, \xi)$ for fixed t, s, x, ξ. Then T_x is a contraction mapping from R^n into R^n by (3.5). Thus T_x has a unique fixed point, from which the bijectiveness of (3.6) follows. ii) is similarly proved. Q. E. D.

The mappings y and η defined above satisfy the following properties.

PROPOSITION 3.3. *Let* $|t-s|<\delta_1$. *Then*:

i)

$$(3.10)\qquad \begin{cases} q(t, s, y(t, s, x, \xi), \xi)=x , \\ p(t, s, x, \eta(t, s, x, \xi))=\xi . \end{cases}$$

$$(3.11)\qquad \begin{cases} q(t, s, x, \eta(t, s, x, \xi))=y(s, t, x, \xi) , \\ p(t, s, y(t, s, x, \xi), \xi)=\eta(s, t, x, \xi) . \end{cases}$$

ii) *For* $|\alpha|+|\beta|\geqq 1$,

$$\begin{cases} |\partial_x^\alpha \partial_\xi^\beta (y(t, s, x, \xi)-x)| \leqq C_{|\alpha|, |\beta|} |t-s|, \\ |\partial_x^\alpha \partial_\xi^\beta (\eta(t, s, x, \xi)-\xi)| \leqq C_{|\alpha|, |\beta|} |t-s|. \end{cases} \tag{3.12}$$

The constant $C_{|\alpha|, |\beta|}$ *is independent of* x, ξ, t, s.

iii)

$$\begin{cases} |y(t, s, x, \xi)-x| \leqq C_1(|x|+|\xi|+1)|t-s|, \\ |\eta(t, s, x, \xi)-\xi| \leqq C_1(|x|+|\xi|+1)|t-s|. \end{cases} \tag{3.13}$$

The constant C_1 *is independent of* x, ξ, t, s.

PROOF. i) is obvious by definition. ii) is proved by differentiating the relation (3.10) and then by using induction. We shall prove iii) using ii). We can write

$$y(t, s, x, \xi)-x=\xi\cdot\int_0^1 \partial_\xi y(t, s, x, r\xi)dr$$
$$+x\cdot\int_0^1 (\partial_x y(t, s, rx, 0)-I)dr+y(t, s, 0, 0),$$

where I denotes the $n\times n$ identity matrix. Hence by ii)

$$|y(t, s, x, \xi)-x| \leqq C(|\xi|+|x|)|t-s|+|y(t, s, 0, 0)|$$

for some constant C. By (3.11) and (3.3) we get

$$\begin{aligned} |y(t, s, 0, 0)| &= |q(s, t, 0, \eta(s, t, 0, 0))-0| \\ &\leqq C_0(|0|+|\eta(s, t, 0, 0)|+1)|t-s| \\ &\leqq C|t-s|, \end{aligned}$$

since $\eta(s, t, 0, 0)$ is continuous in t, s. This proves the first inequality of (3.13). The second is proved similarly. Q. E. D.

Next we shall construct the phase function $\phi(t, s, x, \xi)$ for our approximate solutions (1.4) and (1.8). The following definition is due to Kumano-go [11].

DEFINITION 3.4. Let $|t-s|<\delta_1$. Put

$$u(t, s, y, \eta)\equiv y\cdot\eta+\int_s^t (\xi\cdot\partial_\xi H-H)(\tau, q(\tau, s, y, \eta), p(\tau, s, y, \eta))d\tau \tag{3.14}$$

and

$$\phi(t, s, x, \xi)\equiv u(t, s, y(t, s, x, \xi), \xi). \tag{3.15}$$

The second term on the right-hand side of (3.14) is the classical action along the classical path starting from the position x at time s with the momentum η. Thus the corresponding term of ϕ is exactly the classical action along the path $\gamma(t, s, x, \xi)$ passing through the position x at time t and having the momentum ξ at time s. For later argument, however, the fact that ϕ is the solution of the Hamilton-Jacobi equation with initial condition $\phi(s, s, x, \xi) = x\cdot\xi$ (which we shall state in the next proposition) will play the crucial role in constructing the solution of (1.7) or (1.26).

PROPOSITION 3.5. *Let* $|t-s|<\delta_1$. *Then*:

i) $\phi(s, s, x, \xi)=x\cdot\xi$.

ii) $\partial_t\phi(t, s, x, \xi)+H(t, x, \partial_x\phi(t, s, x, \xi))=0$.

iii)

$$\begin{cases} \partial_x\phi(t, s, x, \xi)=\eta(s, t, x, \xi), \\ \partial_\xi\phi(t, s, x, \xi)=y(t, s, x, \xi). \end{cases} \tag{3.16}$$

iv)

$$|\phi(t', s', x, \xi)-\phi(t, s, x, \xi)|\leqq C_2(|x|+|\xi|+1)^2(|t'-t|+|s'-s|) \tag{3.17}$$

for $|t'-s|, |s'-s|, |t'-s'|<\delta_1$, *where the constant* C_2 *is independent of* x, ξ, t', s', t, s.

PROOF. i) is obvious by definition. ii) and iii) are easily proved by direct calculation (or see Kumano-go [11]). iv) is directly proved by using the expression (3.15) and (3.4), (3.10) and Remark 1.1, we omit the details. Q.E.D.

Next we shall construct the amplitude function $a^{(N)}(\hbar, t, s, x, \xi)$. For this purpose we consider the following transport equation for $j\geqq 1$:

$$\begin{aligned} &\partial_t a_j(t, s, x, \xi)+\sum_{k=1}^{n}(\partial_{\xi_k}H)(t, x, \eta(s, t, x, \xi))(\partial_{x_k}a_j)(t, s, x, \xi) \\ &+\frac{1}{2}\sum_{k,l=1}^{n}(\partial_{\xi_k}\partial_{\xi_l}H)(t, x, \eta(s, t, x, \xi))(\partial_{x_k}\partial_{x_l}\phi)(t, s, x, \xi)a_j(t, s, x, \xi) \\ &+B_j(t, s, x, \xi)=0, \end{aligned} \tag{3.18}$$

with the initial condition $a_1(s, s, x, \xi)=1$ and $a_j(s, s, x, \xi)=0$ for $j\geqq 2$. Here $B_1\equiv 0$, and for $j\geqq 2$ $B_j(t, s, x, \xi)$ is defined inductively as follows:

$$\begin{aligned} &B_j(t, s, x, \xi) \\ &\equiv\sum_{2\leq|\alpha|\leq j}\frac{1}{\alpha!}\Big\{(\partial_\xi^\alpha H)(t, x, \partial_x\phi(t, s, x, \xi))(\partial_x^\alpha a_{j-|\alpha|+1})(t, s, x, \xi) \\ &+\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\geq 1}}\sum_{l=1}^{|\beta|}\sum_{\substack{(k_1+\cdots+k_l=|\beta| \\ k_i\geq 1)}}(\partial_\xi^{\gamma+l}H)(t, x, \partial_x\phi(t, s, x, \xi)) \\ &\times\prod_{i=1}^{l}\frac{1}{k_i+1}(\partial_x^{k_i+1}\phi)(t, s, x, \xi)(\partial_x^\gamma a_{j-|\alpha|+1})(t, s, x, \xi)\Big\}, \end{aligned} \tag{3.19}$$

where $\partial_\xi^{\alpha+l}H$ and $\partial_x^{k_i+1}\phi$ denote one of the derivatives of H and ϕ of order $|\alpha|+l$ and k_i+1. The solution of the transport equation (3.18) is given by the classical theory of a first order differential equation as follows:

$$a_1(t, s, x, \xi)=\exp\left\{-\frac{1}{2}\int_s^t \sum_{k,l=1}^{n}(\partial_{\xi_k}\partial_{\xi_l}H)(\tau, X(\tau), \eta(s, t, X(\tau), \xi))\right. \\ \left.\times(\partial_{x_k}\partial_{x_l}\phi)(\tau, s, X(\tau), \xi)d\tau\right\} \tag{3.20}$$

and for $j\geqq 2$

$$a_j(t, s, x, \xi)=-a_1(t, s, x, \xi)\int_s^t \frac{B_j(\tau, s, X(\tau), \xi)}{a_1(\tau, s, X(\tau), \xi)}d\tau, \tag{3.21}$$

where

$$X(\tau)\equiv q(\tau, s, y(t, s, x, \xi), \xi). \tag{3.22}$$

These solutions satisfy the following estimates.

PROPOSITION 3.6. *Let* $|t-s|<\delta_1$.

i) *For any* α, β,

$$|\partial_x^\alpha\partial_\xi^\beta(a_1(t, s, x, \xi)-1)|\leqq C_{\alpha\beta}|t-s|^2 \tag{3.23}$$

and for $j\geqq 2$

$$|\partial_x^\alpha\partial_\xi^\beta a_j(t, s, x, \xi)|\leqq C_{\alpha\beta}|t-s|^2. \tag{3.24}$$

where the constant $C_{\alpha\beta}$ *is independent of* t, s, x, ξ.

ii) *For any* α, β *and* $j\geqq 1$,

$$|(\partial_x^\alpha\partial_\xi^\beta\partial_t a_j)(t, s, x, \xi)|\leqq C_{\alpha\beta}, \tag{3.25}$$

where $C_{\alpha\beta}$ *is independent of* t, s, x, ξ.

Proof is clear.

Put $a^{(0)}(\hbar, t, s, x, \xi)\equiv 1$ and for $N\geqq 1$,

$$a^{(N)}(\hbar, t, s, x, \xi)\equiv \sum_{j=1}^{N}(i\hbar^{-1})^{-j+1}a_j(t, s, x, \xi). \tag{3.26}$$

Now we can define the approximate solution for (1.7).

DEFINITION 3.7. Let $|t-s|<\delta_1$ and $0<\hbar\leqq 1$. Put $\mathcal{D}\equiv\mathcal{F}^{-1}(C_0^\infty(R^n))$ $(\subset\mathcal{S})$. For $u\in\mathcal{D}$ and an integer $N\geqq 0$, we define

$$E^{(N)}(\hbar, t, s)u(x)\equiv\int_{R^n} e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}a^{(N)}(\hbar, t, s, x, \hbar\xi)\hat{u}(\xi)d\xi. \tag{3.27}$$

§ 4. Elementary properties of $E^{(N)}(\hbar, t, s)$.

In this section we shall investigate some of the properties of $E^{(N)}(\hbar, t, s)$ which does not concern its L^2-property.

PROPOSITION 4.1. *Let $|t-s|<\delta_1$, $0<\hbar\leqq 1$, and $u\in\mathcal{D}$. Then $E^{(N)}(\hbar, t, s)u(x)$ is C^1 in (t, s, x), and C^∞ in x with t, s fixed, and satisfies the following properties:*

i) $E^{(N)}(\hbar, s, s)u(x)=u(x)$.

ii)
$$\frac{\hbar}{i}\frac{\partial}{\partial t}E^{(N)}(\hbar, t, s)u(x)$$
$$=\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}\left\{\frac{\hbar}{i}(\partial_t a^{(N)})(\hbar, t, s, x, \hbar\xi)\right.$$
$$\left.-H(t, x, \eta(s, t, x, \hbar\xi))a^{(N)}(\hbar, t, s, x, \hbar\xi)\right\}\hat{u}(\xi)d\xi\,.$$

(Here we omitted the integration region R^n for no confusion arises. In what follows we also follow this convention.)

Proof is immediate by Propositions 3.5 and 3.6.

Let δ_2 be fixed in the following as $0<\delta_2\leqq\delta_1$ and $C_1\delta_2<1/2$, where C_1 is the constant in (3.13) of Proposition 3.3. The next lemma will be useful in studying regularity and the decaying property of $E^{(N)}(\hbar, t, s)u(x)$.

LEMMA 4.2. *Let real numbers $A>0$ and $K\geqq 0$ be fixed. Let a C^∞ function $F(x, \xi)$ satisfy*

$$\sup_{\substack{|\xi|\leqq A\\ |\alpha|\leqq m}}|\partial_\xi^\alpha F(x, \xi)|\leqq a_m(1+|x|)^K \tag{4.1}$$

for any $x\in R^n$ and any integer $m\geqq 0$. Here a_m is a constant independent of $x\in R^n$. For $|t-s|<\delta_2$, $0<\hbar\leqq 1$, and $u\in\mathcal{D}$ such that $\operatorname{supp}\hat{u}\subset\{\xi\mid|\xi|\leqq A\}$, *define*

$$T(\hbar, t, s)u(x)\equiv\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}F(x, \hbar\xi)\hat{u}(\xi)d\xi\,. \tag{4.2}$$

Then $T(\hbar, t, s)u(x)$ satisfies the following properties:

i) *For any integer $m\geqq 0$, there exists a constant $C_m>0$ such that for any $x\in R^n$, $0<\hbar\leqq 1$, and t, s satisfying $|t-s|<\delta_2$,*

$$|T(\hbar, t, s)u(x)|\leqq C_m a_m(1+|x|)^{K-m}|||u|||_m\,. \tag{4.3}$$

Here

$$|||u|||_m=\sum_{|\alpha|\leqq m}\int_{R^n}|\partial_\xi^\alpha\hat{u}(\xi)|\,d\xi\,. \tag{4.4}$$

ii) *We have*

(4.5) $$\lim_{t'\to t,\, s'\to s}\|T(\hbar,\, t',\, s')u-T(\hbar,\, t,\, s)u\|=0\,.$$

PROOF. i) From the first estimate of (3.13) in Proposition 3.3, we have

(4.6) $$|\partial_\xi\phi(t,\, s,\, x,\, \xi)|=|y(t,\, s,\, x,\, \xi)|\geqq\frac{1}{2}(|x|-(A+1))$$

when $|\xi|\leqq A$ and $|t-s|<\delta_2$. Thus for x such that $|x|\geqq 2(A+1)$, we get

(4.7) $$|\partial_\xi\phi(t,\, s,\, x,\, \xi)|\geqq\frac{1}{4}|x|\geqq\frac{1}{2}(A+1)>0\,.$$

In the following we always assume that $|x|\geqq 2(A+1)$, $|\xi|\leqq A$, $0<\hbar\leqq 1$, and $|t-s|<\delta_2$. Then

(4.8) $$L=\sum_{j=1}^n\frac{1}{i}\frac{\partial_{\xi_j}\phi(t,\, s,\, x,\, \hbar\xi)}{|\partial_\xi\phi(t,\, s,\, x,\, \hbar\xi)|^2}\frac{\partial}{\partial\xi_j}=\frac{1}{i|\partial_\xi\phi|^2}\partial_\xi\phi\cdot\partial_\xi$$

is well-defined and satisfies

(4.9) $$L(e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)})=e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}\,.$$

The formal adjoint L^* of L is given by

(4.10) $$L^*=i[a(t,\, s,\, x,\, \hbar\xi)\cdot\partial_\xi+\hbar\, b(t,\, s,\, x,\, \hbar\xi)]$$

where a and b satisfy

(4.11) $$\begin{cases}|\partial_\xi^\gamma(a(t,\, s,\, x,\, \hbar\xi))|\leqq C_\gamma|x|^{-1}\,, \\ |\partial_\xi^\gamma(b(t,\, s,\, x,\, \hbar\xi))|\leqq C_\gamma|x|^{-1}\end{cases}$$

for any γ by Proposition 3.3 and (4.6). Integrating by parts we get

(4.12) $$T(\hbar,\, t,\, s)u(x)=\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}(L^*)^m(F(x,\, \hbar\xi)\hat{u}(\xi))d\xi\,.$$

The integrand is majorized by

(4.13) $$Ca_m|x|^{K-m}\sum_{|\alpha|\leqq m}|\partial_\xi^\alpha\hat{u}(\xi)|$$

uniformly in $|x|\geqq 2(A+1)$, $|\xi|\leqq A$, $0<\hbar\leqq 1$, and $|t-s|<\delta_2$, by using (4.11). Hence we obtain

(4.14) $$|T(\hbar,\, t,\, s)u(x)|\leqq Ca_m|x|^{K-m}|\!|\!|u|\!|\!|_m$$

for $|x|\geqq 2(A+1)$, $0<\hbar\leqq 1$, and $|t-s|<\delta_2$.

ii) By iv) of Proposition 3.5, we have

$$|T(\hbar,\, t',\, s')u(x)-T(\hbar,\, t,\, s)u(x)|$$

(4.15) $$\leqq C_2\hbar^{-1}\int(|x|+|\xi|+1)^2(|t'-t|+|s'-s|)|F(x,\, \hbar\xi)\hat{u}(\xi)|\, d\xi$$

$$\leqq C_2\hbar^{-1}a_0(|x|+1)^{2+K}(|t'-t|+|s'-s|)\int(|\xi|+1)^2|\hat{u}(\xi)|d\xi,$$

which converges to zero as $t'\to t$, $s'\to s$ for each x. On the other hand from i) we obtain

$$|T(\hbar, t', s')u(x)-T(\hbar, t, s)u(x)|\leqq C(|x|+1)^{-n-1}. \tag{4.16}$$

Therefore Lebesgue's dominated convergence theorem proves ii). Q. E. D.

PROPOSITION 4.3. *Let* $|t-s|<\delta_2$, $0<\hbar\leqq 1$, $N\geqq 0$, *and* $u\in\mathcal{D}$. *Then*:

i) $E^{(N)}(\hbar, t, s)u\in\mathcal{S}\subset L^2$. *Furthermore for any* α, β,

$$\sup_{\substack{x\in R^n\\ |t-s|<\delta_2}}|x^\alpha\partial_x^\beta E^{(N)}(\hbar, t, s)u(x)|<\infty, \tag{4.17}$$

$$\sup_{\substack{\xi\in R^n\\ |t-s|<\delta_2}}|\xi^\alpha\partial_\xi^\beta(\mathcal{F}E^{(N)}(\hbar, t, s)u(\xi)|<\infty. \tag{4.18}$$

ii) *For any* α, β,

$$\sup_{\substack{x\in R^n\\ |t-s|<\delta_2}}|x^\alpha\partial_x^\beta(H_0(\hbar, t)E^{(N)}(\hbar, t, s)u)(x)|<\infty. \tag{4.19}$$

In particular

$$\sup_{|t-s|<\delta_2}\|H_0(\hbar, t)E^{(N)}(\hbar, t, s)u\|<\infty. \tag{4.20}$$

iii) *The mappings*

$$[-T, T]^2\ni(t, s)\longmapsto E^{(N)}(\hbar, t, s)u,\ H_0(\hbar, t)E^{(N)}(\hbar, t, s)u\in\mathfrak{H} \tag{4.21}$$

are continuous.

PROOF. i) We have only to prove (4.17). When $|\beta|=0$ (4.17) follows from Lemma 4.2 and Proposition 3.6 if we take $F(x, \xi)=a^{(N)}(\hbar, t, s, x, \xi)$. For $|\beta|\geqq 1$, we have

$$\begin{aligned}&\partial_x^\beta E^{(N)}(\hbar, t, s)u(x)\\ &=\sum_{\gamma+\delta=\beta}\sum_{l=1}^{|\gamma|}\sum_{\substack{k_1+\cdots+k_l=|\gamma|\\ k_j\geqq 1}}(i\hbar^{-1})^l\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}\\ &\quad\times\prod_{j=1}^l\partial_x^{k_j}\phi(t, s, x, \hbar\xi)(\partial_x^\delta a^{(N)})(\hbar, t, s, x, \hbar\xi)\hat{u}(\xi)d\xi.\end{aligned} \tag{4.22}$$

By Propositions 3.5, 3.3 and 3.6, $F(x, \xi)=\prod_{j=1}^l\partial_x^{k_j}\phi(t, s, x, \xi)(\partial_x^\delta a^{(N)})(\hbar, t, s, x, \xi)$ satisfies (4.1). Thus by Lemma 4.2 we obtain (4.17).

ii) We have by direct calculation,

$$x^\alpha \partial_x^\beta (H_0(\hbar, t)E^{(N)}(\hbar, t, s)u)(x)$$

$$(4.23)\qquad = \sum_{\gamma+\delta=\alpha} c_{\gamma\delta} \int e^{ix\cdot\xi} (\partial_\xi^\gamma H)(t, x, \hbar\xi) \partial_\xi^\delta (\xi^\beta \mathcal{F} E^{(N)}(\hbar, t, s)u(\xi)) d\xi .$$

Thus by (4.18) we obtain (4.19).

iii) For $E^{(N)}(\hbar, t, s)u$, the result follows from iii) of Lemma 4.2 if we take $F=a^{(N)}$ there. For $H_0(\hbar, t)E^{(N)}(\hbar, t, s)u$, by (4.19) and Lebesgue's theorem, we have only to prove that $H_0(\hbar, t)E^{(N)}(\hbar, t, s)u(x)$ is continuous in t, s for each fixed $x \in R^n$. We have

$$(4.24)\qquad \begin{aligned} &H_0(\hbar, t')E^{(N)}(\hbar, t', s')u(x) - H_0(\hbar, t)E^{(N)}(\hbar, t, s)u(x) \\ &= \int e^{ix\cdot\xi} (H(t', x, \hbar\xi) - H(t, x, \hbar\xi)) \mathcal{F} E^{(N)}(\hbar, t', s')u(\xi) d\xi \\ &\quad + \int e^{ix\cdot\xi} H(t, x, \hbar\xi) \mathcal{F}(E^{(N)}(\hbar, t', s')u - E^{(N)}(\hbar, t, s)u)(\xi) d\xi . \end{aligned}$$

The first term on the right-hand side converges to zero as $t' \to t$, $s' \to s$ by (4.18) and the continuity of $H(t, x, \xi)$ in t. For the second term, the same thing is proved from (4.18), (4.17), and (4.15) with $F=a^{(N)}$, by using Lebesgue's theorem twice. Q. E. D.

DEFINITION 4.4. For $|t-s|<\delta_2$, $0<\hbar\leqq 1$, $N\geqq 0$, and $u\in\mathcal{D}$, define

$$(4.25)\qquad F^{(N)}(\hbar, t, s)u(x) \equiv -\frac{\hbar}{i}\frac{\partial}{\partial t} E^{(N)}(\hbar, t, s)u(x)$$

and

$$(4.26)\qquad G^{(N)}(\hbar, t, s)u(x) \equiv H_0(\hbar, t)E^{(N)}(\hbar, t, s)u(x) - F^{(N)}(\hbar, t, s)u(x) .$$

PROPOSITION 4.5. *Let $|t-s|<\delta_2$, $0<\hbar\leqq 1$, $N\geqq 0$, and $u\in\mathcal{D}$. Then the following assertions hold:*

i) $G^{(N)}(\hbar, s, s)u(x)=0$.

ii)

$$(4.27)\qquad \sup_{|t-s|<\delta_2} \|F^{(N)}(\hbar, t, s)u\| < \infty, \quad \sup_{|t-s|<\delta_2} \|G^{(N)}(\hbar, t, s)u\| < \infty ,$$

and the mappings

$$(4.28)\qquad [-T, T]^2 \ni (t, s) \longmapsto F^{(N)}(\hbar, t, s)u,\ G^{(N)}(\hbar, t, s)u \in \mathfrak{H}$$

are continuous.

iii) *For any real number r between t and s, we have*

$$(4.29)\qquad E^{(N)}(\hbar, t, s)u - E^{(N)}(\hbar, r, s)u = -i\int_r^t F^{(N)}(\hbar, \tau, s)u\, d\tau ,$$

where the integral on the right-hand side is the Bochner integral in $\mathfrak{H}=L^2(R^n)$. *In other words, there exists the derivative* $\frac{d}{dt}E^{(N)}(\hbar, t, s)u$ *in* $\mathfrak{H}$ *and we have*

$$\text{(4.30)} \qquad \hbar\frac{d}{dt}E^{(N)}(\hbar, t, s)u=-iF^{(N)}(\hbar, t, s)u$$
$$=i(G^{(N)}(\hbar, t, s)u-H_0(\hbar, t)E^{(N)}(\hbar, t, s)u).$$

PROOF. i) is obvious from $F^{(N)}(\hbar, s, s)u(x)=H_0(\hbar, s)u(x)$ which follows from ii) of Proposition 4.1 and the transport equation (3.18) and (3.19). ii) follows from Lemma 4.2, ii) of Proposition 3.6, and Proposition 4.3. We next prove iii). Take the inner product of the right-hand side of (4.29) with $\phi\in C_0^\infty(R^n)$. Then using the definition (4.25) and Fubini's theorem, we can easily see the inner product is equal to $(E^{(N)}(\hbar, t, s)u-E^{(N)}(\hbar, r, s)u, \phi)$. Since $C_0^\infty(R^n)$ is dense in $L^2(R^n)$, this proves iii). Q.E.D.

§5. Construction of the unitary operator $U(\hbar, t, s)$.

In this section we shall investigate the L^2-properties of the operators $E^{(N)}(\hbar, t, s)$, $G^{(N)}(\hbar, t, s)$, $F^{(N)}(\hbar, t, s)$ defined in the previous sections, using Theorem A.3 in the Appendix, and prove all assumptions of Theorem 2.1 and hence (1.21) and i)∼iii) of our main Theorem.

In the following we shall fix δ_3 as $0<\delta_3\leqq\delta_2$ and $nC_{1,0}\delta_3<1/2$, where $C_{1,0}$ is the constant in (3.12) of Proposition 3.3.

THEOREM 5.1. *Let* $|t-s|<\delta_3$, $0<\hbar\leqq 1$, $N\geqq 0$, *and* $u\in\mathcal{D}$. *Then*:

i) $\|E^{(N)}(\hbar, t, s)u\|\leqq C\|u\|$.

ii) $\|G^{(N)}(\hbar, t, s)u\|\leqq C\hbar^{N+1}|t-s|\,\|u\|$.

iii) $\|F^{(N)}(\hbar, t, s)u\|\leqq C\|u\|_2$.

iv) $\|H_0(\hbar, t)u\|\leqq C\|u\|_2$. *In particular we have* $Y_2\subset\mathcal{D}(H(\hbar, t))$ *for* $|t|\leqq T$. *Here the constant C is independent of t, s,* $\hbar$, *and u.* Y_2 *and the norm* $\|\ \|_2$ *are defined in Definition* 1.6.

PROOF. We use Theorem A.3 in the Appendix. First we note that $\phi(\lambda, x, \xi)=\phi(t, s, x, \xi)$ satisfies the condition (Aϕ) in the Appendix by Propositions 3.5 and 3.3, if we take $\lambda=(t, s)\in\Lambda\equiv\{(t, s)\mid t, s\in[-T, T], |t-s|<\delta_3\}$, and that the function $x(t, s)$ corresponding to $x(\lambda)$ of (A.5) satisfies the estimate

$$\text{(5.1)} \qquad |x(t, s)|\leqq C_0|t-s|<C_0\delta_3,$$

since from the definition and (3.10) we have $x(t, s)=q(t, s, 0, 0)$. Thus i) follows immediately from Theorem A.3 if we take $k=1$, $l=0$, $a_1(x, \xi)=a^{(N)}(\hbar, t, s, x, \hbar\xi)$, and $\nu=\hbar$ there, by using Proposition 3.6.

ii) Take $A>0$ and $\chi\in\mathcal{S}$ such that $\operatorname{supp}\hat{u}\subset\{\xi\mid|\xi|\leqq A\}$ and $\chi(0)=1$. Then we have using Fubini's Theorem

$$
\begin{aligned}
& H_0(\hbar, t)E^{(N)}(\hbar, t, s)u(x) \\
&= \lim_{\varepsilon\downarrow 0}\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}a_\varepsilon(\hbar, t, s, x, \xi)\hat{u}(\xi)d\xi\,,
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{aligned}
a_\varepsilon(\hbar, t, s, x, \xi) &= (2\pi)^{-n}\iint e^{i[(x-y)\cdot\eta-\hbar^{-1}(\phi(t,s,x,\hbar\xi)-\phi(t,s,y,\hbar\xi))]} \\
&\qquad\times H(t, x, \hbar\eta)a^{(N)}(\hbar, t, s, y, \hbar\xi)\chi(\varepsilon\eta)\chi(\varepsilon y)dy\,d\eta\,.
\end{aligned}
\tag{5.3}
$$

Putting

$$
\begin{aligned}
& \theta(\hbar, t, s, x, y, \xi)=\theta(\hbar, x, y, \xi) \\
&= \hbar^{-1}\int_0^1\partial_x\phi(t, s, y+r(x-y), \hbar\xi)dr=\hbar^{-1}\int_0^1\eta(s, t, y+r(x-y), \hbar\xi)dr\,,
\end{aligned}
\tag{5.4}
$$

and making a change of variables, we obtain

$$
a_\varepsilon(\hbar, t, s, x, \xi)=(2\pi)^{-n}\iint e^{-iz\cdot\zeta}p_\varepsilon(\hbar, x, \xi+\zeta, x+z, \xi)d\zeta dz\,,
\tag{5.5}
$$

where

$$
\begin{aligned}
& p_\varepsilon(\hbar, x, \eta, y, \xi)=p_\varepsilon(\hbar, t, s, x, \eta, y, \xi) \\
&= H(t, x, \hbar(\eta-\xi+\theta(\hbar, x, y, \xi)))a^{(N)}(\hbar, t, s, y, \hbar\xi)\chi(\varepsilon(\eta-\xi+\theta(\hbar, x, y, \xi)))\chi(\varepsilon y)\,.
\end{aligned}
\tag{5.6}
$$

By Taylor's formula we get

$$
\begin{aligned}
& p_\varepsilon(\hbar, x, \xi+\zeta, x+z, \xi) \\
&= \sum_{|\alpha|\leqq N}\frac{\zeta^\alpha}{\alpha!}(\partial_\eta^\alpha p_\varepsilon)(\hbar, x, \xi, x+z, \xi) \\
&\quad+\sum_{|\alpha|=N+1}\frac{|\alpha|}{\alpha!}\zeta^\alpha\int_0^1(1-\theta)^{|\alpha|-1}(\partial_\eta^\alpha p_\varepsilon)(\hbar, x, \xi+\theta\zeta, x+z, \xi)d\theta\,.
\end{aligned}
\tag{5.7}
$$

Thus by integration by parts we obtain

$$
\begin{aligned}
& a_\varepsilon(\hbar, t, s, x, \xi) \\
&= (2\pi)^{-n}\sum_{|\alpha|\leqq N}\frac{(-i)^{|\alpha|}}{\alpha!}\lim_{\varepsilon'\downarrow 0}\iint e^{-iz\cdot\zeta}(\partial_y^\alpha\partial_\eta^\alpha p_\varepsilon)(\hbar, x, \xi, x+z, \xi)\chi(\varepsilon'\zeta)\chi(\varepsilon' z)d\zeta dz \\
&\quad+(2\pi)^{-n}\sum_{|\alpha|=N+1}\frac{|\alpha|}{\alpha!}(-i)^{|\alpha|} \\
&\quad\times\int_0^1(1-\theta)^{|\alpha|-1}\lim_{\varepsilon'\downarrow 0}\iint e^{-iz\cdot\zeta}(\partial_y^\alpha\partial_\eta^\alpha p_\varepsilon)(\hbar, x, \xi+\theta\zeta, x+z, \xi)\chi(\varepsilon'\zeta)\chi(\varepsilon' z)d\zeta dz d\theta\,.
\end{aligned}
\tag{5.8}
$$

The first term on the right-hand side is equal to

$$\sum_{|\alpha|\leqq N}\frac{(-i)^{|\alpha|}}{\alpha!}(\partial_y^\alpha\partial_\eta^\alpha p_\varepsilon)(\hbar, x, \xi, x, \xi) \tag{5.9}$$

by Fourier's inversion formula. For each fixed $\hbar$, t, s, x, this is uniformly bounded in $|\xi|\leqq A$ and $\varepsilon>0$, and has the following limit when $\varepsilon\downarrow 0$:

$$\begin{aligned}
&a_1(\hbar, t, s, x, \xi)\\
&\equiv H(t, x, \partial_x\phi(t, s, x, \hbar\xi))a^{(N)}(\hbar, t, s, x, \hbar\xi)\\
&+(i\hbar^{-1})^{-1}\Big\{\sum_{k=1}^{n}(\partial_{\xi_k}H)(t, x, \partial_x\phi(t, s, x, \hbar\xi))(\partial_{x_k}a^{(N)})(\hbar, t, s, x, \hbar\xi)\\
&+\frac{1}{2}\sum_{k,l=1}^{n}(\partial_{\xi_l}\partial_{\xi_k}H)(t, x, \partial_x\phi(t, s, x, \hbar\xi))(\partial_{x_l}\partial_{x_k}\phi)(t, s, x, \hbar\xi)a^{(N)}(\hbar, t, s, x, \hbar\xi)\Big\}\\
&+\sum_{2\leqq|\alpha|\leqq N}\frac{(i\hbar^{-1})^{-|\alpha|}}{\alpha!}\Big[(\partial_\xi^\alpha H)(t, x, \partial_x\phi(t, s, x, \hbar\xi))(\partial_x^\alpha a^{(N)})(\hbar, t, s, x, \hbar\xi)\\
&+\sum_{\substack{\beta+\gamma=\alpha\\|\beta|\geqq 1}}\sum_{l=1}^{|\beta|}\sum_{\substack{k_1+\cdots+k_l=|\beta|\\k_j\geqq 1}}(\partial_\xi^{\alpha+l}H)(t, x, \partial_x\phi(t, s, x, \hbar\xi))\\
&\times\prod_{j=1}^{l}\frac{1}{k_j+1}(\partial_x^{k_j+1}\phi)(t, s, x, \hbar\xi)(\partial_x^\gamma a^{(N)})(\hbar, t, s, x, \hbar\xi)\Big].
\end{aligned} \tag{5.10}$$

Denoting the second summand of (5.8) by $b_\varepsilon(\hbar, t, s, x, \xi)$ and using the relation $\langle\zeta\rangle^{-2l}\langle D_z\rangle^{2l}e^{-iz\cdot\zeta}=\langle z\rangle^{-2l'}\langle D_\zeta\rangle^{2l'}e^{-iz\cdot\zeta}=e^{-iz\cdot\zeta}$, $\langle z\rangle=\sqrt{1+|z|^2}$, we obtain by integration by parts

$$\begin{aligned}
&b_\varepsilon(\hbar, t, s, x, \xi)\\
&=\sum_{|\alpha|=N+1}\frac{|\alpha|}{\alpha!}(-i)^{|\alpha|}\int_0^1(1-\theta)^{|\alpha|-1}\lim_{\varepsilon'\downarrow 0}\iint g_{\varepsilon,\varepsilon',l,l',\alpha}(\hbar, x, \xi, z, \zeta, \theta)dzd\zeta d\theta,
\end{aligned} \tag{5.11}$$

where $g_{\varepsilon,\varepsilon',l,l',\alpha}$ satisfies

$$|g_{\varepsilon,\varepsilon',l,l',\alpha}(\hbar, x, \xi, z, \zeta, \theta)|\leqq C\langle z\rangle^{-2(l'-1)}\langle\zeta\rangle^{-2(l-1)} \tag{5.12}$$

for some constant C independent of $0<\varepsilon<1$, $0<\varepsilon'<1$, $0<\theta<1$, $|\xi|\leqq A$, z, and ζ, but depends on α, l, l', t, s, x, and $\hbar$. Further the following limit exists and we have

$$\begin{aligned}
&g_{l,l',\alpha}(\hbar, x, \xi, z, \zeta, \theta)\equiv\lim_{\varepsilon\downarrow 0,\varepsilon'\downarrow 0}g_{\varepsilon,\varepsilon',l,l',\alpha}(\hbar, x, \xi, z, \zeta, \theta)\\
&=(2\pi)^{-n}e^{-iz\cdot\zeta}\langle z\rangle^{-2l'}\langle D_\zeta\rangle^{2l'}\langle\zeta\rangle^{-2l}\langle D_z\rangle^{2l}(\partial_y^\alpha\partial_\eta^\alpha p)(\hbar, x, \xi+\theta\zeta, x+z, \xi),
\end{aligned} \tag{5.13}$$

where $p(\hbar, x, \eta, y, \xi)\equiv p_0(\hbar, x, \eta, y, \xi)$. Thus for each fixed $\hbar$, t, s, and x,

$b_\varepsilon(\hbar, t, s, x, \xi)$ is uniformly bounded in $|\xi| \leqq A$, and the next limit exists and we have

$$b(\hbar, t, s, x, \xi) \equiv \lim_{\varepsilon \downarrow 0} b_\varepsilon(\hbar, t, s, x, \xi)$$

$$(5.14) \qquad = \sum_{|\alpha|=N+1} \frac{|\alpha|}{\alpha!}(-i)^{|\alpha|}\int_0^1 (1-\theta)^{|\alpha|-1}\iint g_{l,l',\alpha}(\hbar, x, \xi, z, \zeta, \theta)dzd\zeta d\theta,$$

when l and l' are sufficiently large.

Summing up, we have proved that for each fixed $\hbar$, t, s, x,

$$(5.15) \qquad \sup_{\substack{|\xi| \leqq A \\ 0<\varepsilon<1}} |a_\varepsilon(\hbar, t, s, x, \xi)| < \infty$$

and

$$(5.16) \qquad \lim_{\varepsilon \downarrow 0} a_\varepsilon(\hbar, t, s, x, \xi) = a_1(\hbar, t, s, x, \xi) + b(\hbar, t, s, x, \xi).$$

From this we get for each fixed $\hbar$, t, s, x that

$$(5.17) \qquad H_0(\hbar, t)E^{(N)}(\hbar, t, s)u(x)$$

$$= \int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)} a_1(\hbar, t, s, x, \xi)\hat{u}(\xi)d\xi$$

$$+ \int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)} b(\hbar, t, s, x, \xi)\hat{u}(\xi)d\xi.$$

Therefore we obtain from (4.26), (4.25), ii) of Proposition 4.1, and the transport equation (3.18) and (3.19),

$$G^{(N)}(\hbar, t, s)u(x)$$

$$(5.18) \qquad = \int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}(b(\hbar, t, s, x, \xi) + c(\hbar, t, s, x, \xi))\hat{u}(\xi)d\xi,$$

where c is the remainder term which does not cancel out with a_1 by the transport equation.

So we have now only to prove

$$(5.19) \qquad \sup_{\substack{x,\xi \in R^n \\ |t-s|<\delta_3}} |\partial_x^\alpha \partial_\xi^\beta (b+c)(\hbar, t, s, x, \xi)| \leqq C_{\alpha\beta}\hbar^{N+1}|t-s|,$$

since from this and (5.1), ii) follows by Theorem A.3 with $k=1$, $l=0$, and $\nu=\hbar$. But (5.19) is not difficult to prove by using (5.14), (5.13), and the expansion formula for $\partial_y^\alpha \partial_\eta^\beta p$ similar to (5.10), where we also use iii) of Assumption, (3.12), and i) of Proposition 3.6.

iii) By definition we have

$$
\begin{aligned}
(5.20)\qquad & F^{(N)}(\hbar, t, s)u(x) \\
&= \int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}(H(t, x, \eta(s, t, x, \hbar\xi))-H(t, x, \hbar\xi)) \\
&\qquad\qquad \times a^{(N)}(\hbar, t, s, x, \hbar\xi)\hat{u}(\xi)d\xi \\
&\quad + \int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}H(t, x, \hbar\xi)a^{(N)}(\hbar, t, s, x, \hbar\xi)\hat{u}(\xi)d\xi \\
&\quad - \frac{\hbar}{i}\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}(\partial_t a^{(N)})(\hbar, t, s, x, \hbar\xi)\hat{u}(\xi)d\xi \\
&\equiv F_1(\hbar, t, s)u(x)+F_2(\hbar, t, s)u(x)+F_3(\hbar, t, s)u(x).
\end{aligned}
$$

We have to prove

$$(5.21)\qquad \|F_j(\hbar, t, s)u\| \leqq C\|u\|_2 \qquad (j=1, 2, 3).$$

By ii) of Proposition 3.6 and Theorem A.3 with $k=1$, $l=0$ and $\nu=\hbar$, F_3 clearly satisfies (5.21).

We next treat F_1. The amplitude of F_1 is written as

$$
\begin{aligned}
& (H(t, x, \eta(s, t, x, \hbar\xi))-H(t, x, \hbar\xi))a^{(N)}(\hbar, t, s, x, \hbar\xi) \\
(5.22)\qquad &= (\eta(s, t, x, \hbar\xi)-\hbar\xi)\cdot\int_0^1(\partial_\xi H)(t, x, \hbar\xi+r(\eta(s, t, x, \hbar\xi)-\hbar\xi))dr \\
&\quad \times a^{(N)}(\hbar, t, s, x, \hbar\xi) \\
&\equiv a_1(x, \xi)a_2(x, \xi)a_3(x, \xi).
\end{aligned}
$$

We shall apply Theorem A.3 with $k=3$, $l=(1, 1, 0)$, and $\nu=\hbar$. We get by Proposition 3.3

$$(5.23)\qquad \begin{cases} |\partial_x^\alpha\partial_\xi^\beta a_1(x, \xi)| \leqq C_{\alpha\beta}|t-s| & \text{for } |\alpha|+|\beta| \geqq 1, \\ |a_1(0, 0)| \leqq C_1|t-s|, \end{cases}$$

where $C_{\alpha\beta}$ and C_1 are independent of $t, s, x, \xi, \hbar$. Thus A_1 corresponding to a_1 in (A.11) of Theorem A.3 is bounded as

$$(5.24)\qquad \sup_{0<\hbar\leqq 1} A_1 \leqq C|t-s|.$$

On the other hand a_2 is estimated by using Assumption and Remark 1.1 as follows:

$$(5.25)\qquad \begin{cases} |\partial_x^\alpha\partial_\xi^\beta a_2(x, \xi)| \leqq C_{\alpha\beta} & \text{for } |\alpha|+|\beta| \geqq 1, \\ |a_2(0, 0)| \leqq C. \end{cases}$$

Thus

$$(5.26)\qquad \sup_{\left(\substack{0<\hbar\leqq 1 \\ |t-s|<\delta_3}\right)} A_2 < \infty.$$

By Proposition 3.6 A_3 clearly satisfies (5.26) with A_2 replaced by A_3. Therefore noting (5.1) we obtain by Theorem A.3

$$\|F_1(\hbar, t, s)u\| \leqq C|t-s|\|u\|_2, \tag{5.27}$$

where C is independent of t, s, $\hbar$.

Finally we consider F_2. In Theorem A.3 let $k=2$, $l=(2, 0)$, and $\nu=\hbar$. Then by iii) of Assumption and Proposition 3.6, we have

$$\sup_{j=1,2} \sup_{\left(\substack{0<\hbar\leqq 1 \\ |t-s|<\delta_3}\right)} A_j < \infty, \tag{5.28}$$

hence we get for some constant C independent of t, s, $\hbar$,

$$\|F_2(\hbar, t, s)u\| \leqq C\|u\|_2, \tag{5.29}$$

which proves iii).

iv) is now obvious by iii) and $H_0(\hbar, t)=F^{(0)}(\hbar, t, t)$. Q. E. D.

PROPOSITION 5.2. *Let* $0<\hbar\leqq 1$, $N\geqq 0$, $|t-s|<\delta_3$, $|r-s|<\delta_3$ *and* $|t-r|<\delta_3$. *Then we have*

$$\begin{cases} \|E^{(N)}(\hbar, t, r)^* E^{(N)}(\hbar, t, s)-E^{(N)}(\hbar, r, s)\| \leqq b\hbar^N(|t-r|^2+|r-s|^2), \\ \|E^{(N)}(\hbar, t, r) E^{(N)}(\hbar, r, s)-E^{(N)}(\hbar, t, s)\| \leqq b\hbar^N(|t-r|^2+|r-s|^2), \end{cases} \tag{5.30}$$

where constant b *is independent of* t, r, s, $\hbar$.

Proof is done quite similarly to that of Propositions 5.1 and 5.2 of Fujiwara [5] by using ii) of Theorem 5.1 and (4.30) hence is omitted.

Now we have proved all assumptions of Theorem 2.1. In fact a) of Theorem 2.1 follows from i) of Proposition 4.1 and i) of Theorem 5.1. b) was proved in Theorem 5.1. c) holds by Proposition 4.3 with $\mathcal{D}=\mathcal{F}^{-1}(C_0^\infty(R^n))$. Finally d) holds with $d=0$ by Proposition 5.2 just above. Thus we have constructed a family $\{U^{(N)}(\hbar, t, s) \mid t, s\in[-T, T]\}$ of unitary operators satisfying i)$\sim$iii) of our Theorem.

From Proposition 3.6 and Theorem A.3, we can easily obtain

$$\|E^{(N)}(h, t, s)-E^{(M)}(h, t, s)\| \leqq C_\hbar |t-s|^2 \tag{5.31}$$

for $|t-s|<\delta_3$, N, $M\geqq 0$, $0<\hbar\leqq 1$. Thus we have proved the following proposition.

PROPOSITION 5.3. *The family* $\{U^{(N)}(\hbar, t, s) \mid t, s\in[-T, T]\}$ *of unitary operators thus constructed does not depend on the choice of the integer* $N\geqq 0$. *We shall denote it by* $\{U(\hbar, t, s) \mid t, s\in[-T, T]\}$.

This proposition completes the proof of (1.21), and moreover shows that for the discussion of the property of $U(\hbar, t, s)$, it suffices to study $E^{(N)}(\hbar, t, s)$

for a particular integer $N\geqq 0$. This we shall do in the next section with $N=0$ to prove the strong differentiability of $U(\hbar, t, s)$, though even for $N\geqq 1$, we can develop an argument similar to the one in the next section.

§6. Strong differentiability of $U(\hbar, t, s)$ and the self-adjointness of $H(\hbar, t)$.

In this section we shall study the L^2-properties of $E^{(0)}(\hbar, t, s)$ more closely and prove all assumptions of Theorem 2.4 and hence iv)~vi) of our main theorem. In the following we fix $\hbar$ as $0<\hbar\leqq 1$ and write $E^{(0)}(\hbar, t, s)$, $H(\hbar, t)$ and $F^{(0)}(\hbar, t, s)$ as $E(t, s)$, $H(t)$ and $F(t, s)$, respectively.

PROPOSITION 6.1. *Let $|t-s|<\delta_3$ and $u\in\mathcal{D}$. Then:*

i) *For any α, β and j satisfying $1\leqq j\leqq n$, we have*

$$(6.1)\qquad \begin{cases} \|x^\alpha\partial_x^\beta[\partial_{x_j}, E(t, s)]u\|\leqq C_{\alpha\beta}|t-s|\,\|u\|_{|\alpha|+|\beta|+1}, \\ \|x^\alpha[x_j, E(t, s)]u\|\leqq C_\alpha|t-s|\,\|u\|_{|\alpha|+1}, \end{cases}$$

where constants $C_{\alpha\beta}$ and C_α are independent of t, s, and u, and $[A, B]=AB-BA$.

ii) *For any α, β, we have*

$$(6.2)\qquad \begin{aligned} \|x^\alpha\partial_x^\beta E(t, s)u\| &\leqq \|E(t, s)x^\alpha\partial_x^\beta u\|+C_{\alpha\beta}|t-s|\,\|u\|_{|\alpha|+|\beta|} \\ &\leqq e^{b|t-s|/2}\|x^\alpha\partial_x^\beta u\|+C_{\alpha\beta}|t-s|\,\|u\|_{|\alpha|+|\beta|}, \end{aligned}$$

where b is the constant appeared in (5.30) *with $N=0$ and $C_{\alpha\beta}$ is a constant independent of t, s, u.*

PROOF. i) We have

$$(6.3)\qquad \begin{aligned} &[\partial_{x_j}, E(t, s)]u(x) \\ &=i\hbar^{-1}\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}(\eta_j(s, t, x, \hbar\xi)-\hbar\xi_j)\hat{u}(\xi)d\xi\,. \end{aligned}$$

By direct calculation we obtain

$$(6.4)\qquad \begin{aligned} &x^\alpha\partial_x^\beta[\partial_{x_j}, E(t, s)]u(x) \\ &=i\hbar^{-1}\sum_{\gamma+\delta=\beta}\sum_{l=1}^{|\gamma|}\sum_{\binom{k_1+\cdots+k_l=|\gamma|}{k_j\geqq 1}}(i\hbar^{-1})^l \\ &\quad\times\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)}\{x^\alpha\partial_x^{k_1}\phi\cdots\partial_x^{k_l}\phi\partial_x^\delta(\eta_j(s, t, x, \hbar\xi)-\hbar\xi_j)\}\,\hat{u}(\xi)d\xi\,. \end{aligned}$$

Let, in Theorem A.3, $k=|\alpha|+l+1$ $(\leqq|\alpha|+|\beta|+1)$, $l_j=1$, $\nu=\hbar$, and

$$(6.5)\qquad \begin{cases} a_1(x, \xi)\cdots a_{|\alpha|}(x, \xi)=x^\alpha \quad (a_j(x, \xi) \text{ is first order in } x), \\ a_{|\alpha|+j}(x, \xi)=\partial_x^{k_j}\phi(t, s, x, \hbar\xi)=\partial_x^{k_j-1}\eta(s, t, x, \hbar\xi) \quad (1\leqq j\leqq l), \\ a_{|\alpha|+l+1}(x, \xi)=\partial_x^\delta(\eta_j(s, t, x, \hbar\xi)-\hbar\xi_j). \end{cases}$$

Then we have from (3.12)

$$
\left\{\begin{array}{ll} A_j \leqq C & (1 \leqq j \leqq |\alpha|+l), \\ A_{|\alpha|+l+1} \leqq C|t-s|, & \end{array}\right. \tag{6.6}
$$

where C is a constant independent of t, s. Therefore by Theorem A.3 and (5.1) we get the first part of (6.1).

Next we consider the second part of (6.1). We have

$$
x^\alpha[x_j, E(t, s)]u(x)=\int e^{i\hbar^{-1}\phi(t,s,x,\hbar\xi)} x^\alpha(x_j-y_j(t, s, x, \hbar\xi))\hat{u}(\xi)d\xi . \tag{6.7}
$$

Thus in Theorem A.3, taking $k=|\alpha|+1$, $l_j=1$ and $\nu=\hbar$, and putting

$$
\left\{\begin{array}{l} a_1(x, \xi)\cdots a_{|\alpha|}(x, \xi)=x^\alpha \quad (a_j(x, \xi) \text{ is first order in } x), \\ a_{|\alpha|+1}(x, \xi)=x_j-y_j(t, s, x, \hbar\xi), \end{array}\right. \tag{6.8}
$$

we get

$$
\left\{\begin{array}{ll} A_j \leqq C & (1 \leqq j \leqq |\alpha|), \\ A_{|\alpha|+1} \leqq C|t-s|, & \end{array}\right. \tag{6.9}
$$

from which we obtain the second part of (6.1).

ii) When $k\equiv|\alpha|+|\beta|=0$, (6.2) is a consequence of the results obtained in section 5 by Theorem 2.1, i). When $k\geqq 1$ assume (6.2) holds for $|\alpha|+|\beta|\leqq k-1$. For the case $|\beta|=0$, using the second part of (6.1) we obtain

$$
\begin{aligned}
\|x^\alpha E(t, s)u\| &\leqq \|x^{\alpha-\varepsilon_j}E(t, s)x_j u\|+\|x^{\alpha-\varepsilon_j}[x_j, E(t, s)]u\| \\
&\leqq \|E(t, s)x^\alpha u\|+C|t-s|\|x_j u\|_{|\alpha|-1}+C|t-s|\|u\|_{|\alpha|} \\
&\leqq \|E(t, s)x^\alpha u\|+C_\alpha|t-s|\|u\|_{|\alpha|},
\end{aligned} \tag{6.10}
$$

where ε_j is the multi-index whose j-th component is 1 and others are zero. For $|\beta|\geqq 1$, similarly we have from the first part of (6.1)

$$
\begin{aligned}
\|x^\alpha\partial_x^\beta E(t, s)u\| &\leqq \|x^\alpha\partial_x^{\beta-\varepsilon_j}E(t, s)\partial_{x_j}u\|+\|x^\alpha\partial_x^{\beta-\varepsilon_j}[\partial_{x_j}, E(t, s)]u\| \\
&\leqq \|E(t, s)x^\alpha\partial_x^\beta u\|+C_{\alpha\beta}|t-s|\|u\|_{|\alpha|+|\beta|}.
\end{aligned} \tag{6.11}
$$

Thus by induction we have proved ii). Q.E.D.

COROLLARY 6.2. *Let $k\geqq 0$ be an integer and let $|t-s|<\delta_3$ and $u\in\mathcal{D}$. Then we have*

$$
\|E(t, s)u\|_k \leqq e^{b_k|t-s|}\|u\|_k \tag{6.12}
$$

for some positive constant b_k.

PROOF. First note that

$$\|f\|_k^2=\sum_{l+|\alpha|\leq k}\|(1+|x|^2)^{l/2}\partial_x^\alpha f(x)\|^2$$

$$(6.13)\qquad =\sum_{l+|\alpha|\leq k}\sum_{m=0}^{l}\binom{l}{m}\sum_{|\beta|=m}c_\beta\|x^\beta\partial_x^\alpha f(x)\|^2,$$

for some constants $c_\beta\geqq 0$. Thus by (6.2)

$$\|E(t, s)u\|_k^2=\sum_{l+|\alpha|\leq k}\sum_{m=0}^{l}\binom{l}{m}\sum_{|\beta|=m}c_\beta\|x^\beta\partial_x^\alpha E(t, s)u\|^2$$

$$\leqq\sum_{l+|\alpha|\leq k}\sum_{m=0}^{l}\binom{l}{m}\sum_{|\beta|=m}c_\beta\{e^{b|t-s|/2}\|x^\beta\partial_x^\alpha u\|+C_{\alpha\beta}|t-s|\|u\|_k\}^2$$

$$(6.14)\qquad \leqq e^{b|t-s|}\sum_{l+|\alpha|\leq k}\sum_{m=0}^{l}\binom{l}{m}\sum_{|\beta|=m}c_\beta\|x^\beta\partial_x^\alpha u\|^2$$

$$+C|t-s|(e^{b|t-s|/2}+|t-s|)\|u\|_k^2$$

$$\leqq e^{b|t-s|}\|u\|_k^2+C|t-s|e^{b'|t-s|}\|u\|_k^2$$

$$\leqq e^{b_k|t-s|}\|u\|_k^2.$$

Q. E. D.

THEOREM 6.3. *Let $k\geqq 0$ be an integer and let $|t|, |s|\leqq T$ and $u\in\mathcal{D}$. Then we have*

$$(6.15)\qquad \|U(t, s)u\|_k\leqq e^{b_k|t-s|}\|u\|_k,$$

for the same constant b_k as in (6.12).

Proof is quite similar to that of Theorem 4 of Fujiwara [5] hence is omitted.

COROLLARY 6.4. *Let $k\geqq 0$ be an integer. Then*

$$(6.16)\qquad U(t, s)Y_k=Y_k.$$

PROOF. From (6.15) we have $U(t, s)Y_k\subset Y_k$ and $U(t, s)^{-1}Y_k=U(s, t)Y_k\subset Y_k$.

PROPOSITION 6.5. *Let $|t-s|<\delta_3$ and $f\in Y_2$. Then for any r between t and s, we have*

$$(6.17)\qquad E(t, s)f-E(r, s)f=-i\int_r^t F(\tau, s)f\,d\tau \quad \text{in}\quad \mathfrak{H}=L^2(R^n).$$

In other words, there exists the derivative $\frac{d}{dt}E(t, s)f$ in $\mathfrak{H}$ and we have

$$(6.18)\qquad \frac{d}{dt}E(t, s)f=-iF(t, s)f.$$

In particular we have

$$\frac{d}{dt} E(t, s)f\Big|_{s=t} = -iH(t)f. \tag{6.19}$$

PROOF. Take a sequence $\{u_m\}_{m=1}^{\infty} \subset \mathcal{D}$ so that $u_m \to f$ in Y_2 as $m \to \infty$. Then by iii) of Theorem 5.1, $F(\tau, s)u_m \to F(\tau, s)f$ in $\mathfrak{H}$ as $m \to \infty$ for each τ. Since $\tau \mapsto F(\tau, s)u_m \in \mathfrak{H}$ is strongly continuous by ii) of Proposition 4.5, it follows from this and iii) of Theorem 5.1 that the mapping $\tau \mapsto F(\tau, s)f \in \mathfrak{H}$ is strongly measurable and uniformly bounded. Thus the right-hand side of (6.17) is well-defined as the Bochner integral in $\mathfrak{H}$.

Now (6.17) follows from (4.29) if we take $u = u_m$ and let $m \to \infty$ there. Q.E.D.

Now we have proved all assumptions of Theorem 2.4 with $Y = Y_2$. In fact, a) of Theorem 2.4 was proved as iv) of Theorem 5.1. b) was proved as Corollary 6.4 and c) as Proposition 6.5. Therefore we have completed the proof of our Theorem.

§7. Concluding remarks.

We first remark on the relation of our $E^{(N)}(\hbar, t, s)$ with Fujiwara's. For our $E^{(0)}(\hbar, t, s)$ for (1.1) and Fujiwara's $E_F(\hbar, t, s)$, we can prove the following estimate

$$\|E_F(\hbar, t, s) - E^{(0)}(\hbar, t, s)\| \leqq C|t-s|^2, \tag{7.1}$$

when $|t-s|$ is sufficiently small. Proof is done quite similarly to the proof of (5.11) of [5] by using Proposition 4.18 of [5] and our (4.26) and ii) of Theorem 5.1. From (7.1) we can easily prove that our $U(\hbar, t, s)$ coincides with $U(i\hbar^{-1}, t, s) \equiv \lim_{\delta(\Delta) \to 0} E_F(h, t, t_{l-1}) \cdots E_F(t_1, s)$ of Fujiwara [5], though this fact is also a corollary of Theorem 5 of [5] and our main Theorem.

Secondly we consider the meaning of our formulation from the physical point of view. As was noted by Feynman [4], in quantum mechanics, the law of superposition of probability amplitude is given by $\varphi_{ac} = \sum_b \varphi_{ab}\varphi_{bc}$ and $P_{ac}^q = |\varphi_{ac}|^2$, where P_{ac}^q denotes the quantum mechanical probability that a measurement A which follows the measurement C giving c results in a, and φ_{ab}, φ_{bc}, φ_{ac} are some complex numbers giving the probability amplitudes for such measurements. Under the above preparation, Feynman gave two postulates which gives the formula expressing the probability amplitude that a particle will be found in a certain region in space-time (see (12) of [4]), which leads us to (1.2). His postulates concern only with the measurement of the position of the particle, that is A, B, C above are the position measurements in [4], and it is in fact sufficient in itself. However, his postulates are not the unique one. We may think the measurement A as the position measurement while the B as the momentum measurement in considering φ_{ab}. What we have shown in the present note is that this is in fact possible and the treatment becomes somewhat simpler than that of Feynman's integral (1.2), e.g. in the analysis of the L^2-properties of $E^{(N)}(\hbar, t, s)$, etc., with the operator norm convergence of $E_{\Delta}^{(N)}(\hbar, t, s)$ in $L^2(R^n)$ being preserved, though the physical meaning is not so easy to get acquainted with as that of Feynman.

Finally we remark on our Assumption. Although we have assumed that $H(t, x, \xi)$ is C^∞ in (x, ξ) and its derivatives of order$\geqq 2$ are bounded, this assumption is redundant. It suffices to assume this up to a certain finite order, since in Theorem A.3 we only need the estimates for finite order derivatives of ϕ and a_j.

Appendix

L^2-boundedness of some integral transformations.

In this appendix we shall consider the integral transformations of the form

$$T(\lambda, \nu)u(x) \equiv \int_{R^n} e^{i\nu^{-1}\phi(\lambda, x, \nu\xi)} a_1(x, \xi) \cdots a_k(x, \xi)\hat{u}(\xi)d\xi, \tag{A.1}$$

and prove the L^2-boundedness of $T(\lambda, \nu)$ under certain assumptions, where λ and ν are parameters such that $0<\nu\leqq 1$ and $\lambda\in\Lambda$, Λ being some fixed set.

Let $k\geqq 1$ and let $l=(l_1, \cdots, l_k)$ be multi-index and put $M=2([n/2]+[5n/4]+2)+2|l|+\max_{1\leqq j\leqq k} l_j$. We impose on the functions ϕ and a_j $(1\leqq j\leqq k)$ the following conditions:

(Aϕ) i) $\phi(\lambda, x, \xi)$ is real-valued and $\phi(\lambda, \cdot, \cdot)\in C^{M+2}(R^n\times R^n)$ for any $\lambda\in\Lambda$.

ii) $\rho\equiv \sup\limits_{\substack{\lambda\in\Lambda \\ x,\xi\in R^n}} |\partial_x\partial_\xi\phi(\lambda, x, \xi)-I|<1$.

iii) $\sup\limits_{\substack{\lambda\in\Lambda, x,\xi\in R^n \\ 3\leqq|\alpha|+|\beta|\leqq M+2 \\ 1\leqq|\beta|}} |\partial_x^\alpha\partial_\xi^\beta\phi(\lambda, x, \xi)|<\infty$.

(Aa_j) i) $a_j(\cdot, \cdot)\in C^M(R^n\times R^n)$.

ii) $\sup\limits_{\substack{x,\xi\in R^n \\ l_j\leqq|\alpha|+|\beta|\leqq M}} |\partial_x^\alpha\partial_\xi^\beta a_j(x, \xi)|<\infty$.

Before stating our main theorem, we prepare the following two lemmas.

LEMMA A.1. *Let* i) *and* ii) *of* (Aϕ) *be satisfied. Put*

$$\theta(\lambda, \nu, x, \eta, \xi) \equiv \int_0^1 (\partial_\xi\phi)(\lambda, x, \nu(\eta+r(\xi-\eta)))dr \tag{A.2}$$

for $\lambda\in\Lambda$, $0<\nu\leqq 1$, *and* $x, \eta, \xi\in R^n$. *Then*

$$|\partial_x\theta(\lambda, \nu, x, \eta, \xi)-I|\leqq\rho(<1) \tag{A.3}$$

for any $\lambda\in\Lambda$, $0<\nu\leqq 1$, *and* $x, \eta, \xi\in R^n$. *Thus* $R^n\ni x\mapsto\theta(\lambda, \nu, x, \eta, \xi)\in R^n$ *is a* C^{M+1} *diffeomorphism for any fixed* λ, ν, η, ξ. *We write the inverse* C^{M+1} *diffeomorphism as*

$$R^n\ni\theta\longmapsto x(\lambda, \nu, \theta, \eta, \xi)\in R^n. \tag{A.4}$$

Then $x(\lambda, \nu, 0, 0, 0)$ *is independent of* $0<\nu\leqq 1$. *We write this as*

$$x(\lambda)\equiv x(\lambda, \nu, 0, 0, 0). \tag{A.5}$$

PROOF. We have by ii) of (Aϕ),

$$|\partial_x\theta(\lambda, \nu, x, \eta, \xi)-I|\leqq\int_0^1|\partial_x\partial_\xi\phi(\lambda, x, \nu(\eta+r(\xi-\eta)))-I|\,dr$$

$$\leqq\rho<1.$$

From this we can prove that $x\mapsto\theta(\lambda, \nu, x, \eta, \xi)$ is a C^{M+1} diffeomorphism in the same way as in the proof of Proposition 3.2.

$x(\lambda, \nu, 0, 0, 0)$ is uniquely determined by

$$\theta(\lambda, \nu, x(\lambda, \nu, 0, 0, 0), 0, 0)=0,$$

which is rewritten as

$$\partial_\xi\phi(\lambda, x(\lambda, \nu, 0, 0, 0), 0)=0$$

by (A.2). This shows that $x(\lambda, \nu, 0, 0, 0)$ is independent of ν. Q.E.D.

LEMMA A.2. *Let* (Aϕ) *be satisfied. Then*

$$\sup_{\substack{\lambda\in\Lambda, 0<\nu\leqq 1\\ \theta, \eta, \xi\in R^n}}|\partial_\theta^\alpha\partial_\eta^\beta\partial_\xi^\gamma x(\lambda, \nu, \theta, \eta, \xi)|<\infty \tag{A.6}$$

for any α, β, γ *satisfying* $1\leqq|\alpha|+|\beta|+|\gamma|\leqq M+1$.

PROOF. $x(\lambda, \nu, \theta, \eta, \xi)$ is defined by

$$\theta(\lambda, \nu, x(\lambda, \nu, \theta, \eta, \xi), \eta, \xi)=\theta. \tag{A.7}$$

Differentiating this with respect to θ we have

$$(\partial_x\theta)(\lambda, \nu, x(\lambda, \nu, \theta, \eta, \xi), \eta, \xi)\cdot(\partial_\theta x)(\lambda, \nu, \theta, \eta, \xi)=I,$$

from which and (A.3) we get (A.6) for $|\alpha|=1$ and $|\beta|=|\gamma|=0$. For other α, β, γ satisfying $1\leqq|\alpha|+|\beta|+|\gamma|\leqq M+1$, operating $\partial_\theta^\alpha\partial_\eta^\beta\partial_\xi^\gamma$ to (A.7), we obtain

$$\begin{aligned}0=&(\partial_x\theta)(\lambda, \nu, x(\lambda, \nu, \theta, \eta, \xi), \eta, \xi)\cdot(\partial_\theta^\alpha\partial_\eta^\beta\partial_\xi^\gamma x)(\lambda, \nu, \theta, \eta, \xi)\\ &+\sum c(\partial_\eta^{\alpha'}\partial_\xi^{\beta'}\partial_x^{\gamma'}\theta)(\lambda, \nu, x(\lambda, \nu, \theta, \eta, \xi), \eta, \xi)\\ &\times\prod_{j=1}^{|\gamma'|}(\partial_\eta^{\alpha'_j}\partial_\xi^{\beta'_j}\partial_\theta^{\gamma'_j}x)(\lambda, \nu, \theta, \eta, \xi),\end{aligned} \tag{A.8}$$

where the summation ranges over those systems of multi-indices α', β', γ', α'_j, β'_j, γ'_j such that $2\leqq|\alpha'|+|\beta'|+|\gamma'|\leqq M+1$ and $1\leqq|\alpha'_j|+|\beta'_j|+|\gamma'_j|<|\alpha|+|\beta|+|\gamma|$ $(1\leqq j\leqq|\gamma'|)$, and the constant c depends on them. By iii) of (Aϕ) and (A.2), $|(\partial_\eta^{\alpha'}\partial_\xi^{\beta'}\partial_x^{\gamma'}\theta)(\lambda, \nu, x, \eta, \xi)|$ is uniformly bounded in $\lambda, \nu, x, \eta, \xi$, since $0<\nu\leqq 1$. Thus from (A.8) we obtain (A.6) by induction. Q.E.D.

Now we can state and prove our main theorem.

THEOREM A.3[1]. *Let $k\geqq 1$, and $l=(l_1, \cdots, l_k)$ be a multi-index. Put* $M=2\left(\left[\frac{n}{2}\right]+\left[\frac{5n}{4}\right]+2\right)+2|l|+\max_{1\leq j\leq k} l_j$. *Let* (A$\phi$) *and* (A$a_j$) *for* $1\leqq j\leqq k$ *be satisfied. Then for any* $u\in\mathcal{D}\equiv\mathcal{F}^{-1}(C_0^\infty(R^n))$, $0<\nu\leqq 1$, *and* $\lambda\in\Lambda$, *we have*

$$\|T(\lambda, \nu)u\|_{L^2}\leqq K_\lambda\|u\|_{|l|}, \tag{A.9}$$

where $\|\ \|_m$ *is defined by* (1.19). *The constant* K_λ *is given by*

$$K_\lambda=C_{n,k,l}\prod_{j=1}^{k}A_j\cdot(1+|x(\lambda)|)^{|l|} \tag{A.10}$$

where $x(\lambda)$ *is defined by* (A.5); $C_{n,k,l}$ *is independent of* λ *and* ν; *and* A_j *is defined by*

$$A_j=\sup_{\substack{x,\xi\in R^n\\ l_j\leq|\alpha+\beta|\leq M}}|\partial_x^\alpha\partial_\xi^\beta a_j(x, \xi)|+\sup_{|\alpha+\beta|<l_j}|\partial_x^\alpha\partial_\xi^\beta a_j(0, 0)| \tag{A.11}$$

for $1\leqq j\leqq k$.

PROOF. For $\omega\in C_0^\infty(R^n)$ satisfying $\omega(x)\geqq 0$ on R^n, define

$$\begin{aligned}(K_\omega(\lambda, \nu)\hat{u})(\eta)\\ =\iint e^{i\nu^{-1}(\phi(\lambda,x,\nu\xi)-\phi(\lambda,x,\nu\eta))}\prod_{j=1}^{k}a_j(x, \xi)\overline{a_j(x, \eta)}\omega(x)\hat{u}(\xi)d\xi dx\,.\end{aligned} \tag{A.12}$$

Then we have

$$\int|T(\lambda, \nu)u(x)|^2\omega(x)dx=\int(K_\omega(\lambda, \nu)\hat{u})(\eta)\overline{\hat{u}(\eta)}d\eta\,. \tag{A.13}$$

Using Taylor's formula we write

$$a_j(x, \xi)=\sum_{|\alpha|\leq l_j}(x, \xi)^\alpha a_j^{(\alpha)},\quad \overline{a_j(x, \eta)}=\sum_{|\beta|\leq l_j}(x, \eta)^\beta b_j^{(\beta)}. \tag{A.14}$$

Here for $|\alpha|, |\beta|<l_j$,

$$a_k^{(\alpha)}=\frac{1}{\alpha!}(\partial_{(x,\xi)}^\alpha a_j)(0, 0),\qquad b_j^{(\beta)}=\frac{1}{\beta!}\overline{(\partial_{(x,\xi)}^\beta a_j)(0, 0)}, \tag{A.15}$$

and for $|\alpha|=|\beta|=l_j$,

1) When $l=0$ and $\nu=1$, this theorem is almost included in the results of Fujiwara [6] or Asada and Fujiwara [2], [3].

(A.16)
$$\begin{cases} a_j^{(\alpha)}=a_j^{(\alpha)}(x,\xi)=\dfrac{|\alpha|}{\alpha!}\displaystyle\int_0^1(1-r)^{|\alpha|-1}(\partial_{(x,\xi)}^{\alpha}a_j)(rx,r\xi)dr\,, \\ b_j^{(\beta)}=b_j^{(\beta)}(x,\eta)=\dfrac{|\beta|}{\beta!}\displaystyle\int_0^1(1-r)^{|\beta|-1}\overline{(\partial_{(x,\xi)}^{\beta}a_j)(rx,r\eta)}dr\,. \end{cases}$$

Thus we get

(A.17)
$$(K_\omega(\lambda,\nu)\hat{u})(\eta)=\sum\nolimits_1\iint e^{i\nu^{-1}(\phi(\lambda,x,\nu\xi)-\phi(\lambda,x,\nu\eta))}\prod_{j=1}^{k}(x,\xi)^{\alpha_j}(x,\eta)^{\beta_j}\times a_j^{(\alpha_j)}b_j^{(\beta_j)}\omega(x)\hat{u}(\xi)d\xi dx\,,$$

where the summation $\sum_1$ ranges over the indices α_j, β_j such that $|\alpha_j|$, $|\beta_j|\leqq l_j$ $(1\leqq j\leqq k)$.

Writing $\nu^{-1}(\phi(\lambda,x,\nu\xi)-\phi(\lambda,x,\nu\eta))=(\xi-\eta)\cdot\theta(\lambda,\nu,\eta,x,\xi)$ for $\theta(\lambda,\nu,\eta,x,\xi)=\int_0^1\partial_\xi\phi(\lambda,x,\nu(\eta+r(\xi-\eta)))dr$ of (A.2), and making a change of variable $\theta=\theta(\lambda,\nu,\eta,x,\xi)$, we obtain

(A.18)
$$\begin{aligned}&(K_\omega(\lambda,\nu)\hat{u})(\eta)\\&=\sum\nolimits_1\iint e^{i(\xi-\eta)\cdot\theta}\prod_{j=1}^{k}A_j(\theta,\eta,\xi)|\det\partial_\theta x(\theta,\eta,\xi)|\omega(x(\theta,\eta,\xi))\hat{u}(\xi)d\xi d\theta\,,\end{aligned}$$

where $x(\theta,\eta,\xi)=x(\lambda,\nu,\theta,\eta,\xi)$ of (A.4). Here

(A.19)
$$A_j(\theta,\eta,\xi)=(x(\theta,\eta,\xi),\xi)^{\alpha_j}(x(\theta,\eta,\xi),\eta)^{\beta_j}a_j^{(\alpha_j)}b_j^{(\beta_j)}\,,$$

and for $|\alpha|=|\beta|=l_j$

(A.20)
$$a_j^{(\alpha)}=a_j^{(\alpha)}(x(\theta,\eta,\xi),\xi)\,,\qquad b_j^{(\beta)}=b_j^{(\beta)}(x(\theta,\eta,\xi),\eta)\,.$$

Using Taylor's formula again we have

(A.21)
$$x(\theta,\eta,\xi)=x(0,0,0)+\sum_{|\gamma|=1}(\theta,\eta,\xi)^{\gamma}x^{(\gamma)}(\theta,\eta,\xi)\,,$$

where

(A.22)
$$x^{(\gamma)}(\theta,\eta,\xi)=\int_0^1(\partial_{(\theta,\eta,\xi)}^{\gamma}x)(r\theta,r\eta,r\xi)dr$$

and $x(0,0,0)=x(\lambda)$ by (A.5). Then substituting (A.21) into (A.19) we obtain

(A.23)
$$A_j(\theta,\eta,\xi)=\sum_{|\mu|\leq 2l_j}(\theta,\eta,\xi)^{\mu}d_j^{(\mu)}(\theta,\eta,\xi)a_j^{(\alpha_j)}b_j^{(\beta_j)}$$

for some $d_j^{(\mu)}(\theta,\eta,\xi)=d_j^{(\mu)}(\lambda,\nu,\theta,\eta,\xi)\in C^M$ satisfying $|\partial_\theta^{\alpha}\partial_\eta^{\beta}\partial_\xi^{\gamma}d_j^{(\mu)}(\theta,\eta,\xi)|\leqq C(1+|x(\lambda)|)^{|\mu|}$ for $|\alpha+\beta+\gamma|\leqq M$. Then we have

$$
\begin{aligned}
&(K_\omega(\lambda,\nu)\hat{u})(\eta) \\
&=\Sigma_1\Sigma_2\iint e^{i(\xi-\eta)\cdot\theta}(\theta,\eta,\xi)^{\sum_{j=1}^{k}\mu_j}\prod_{j=1}^{k}\{d_j^{(\mu_j)}(\theta,\eta,\xi)a_j^{(\alpha_j)}b_j^{(\beta_j)}\} \\
&\qquad\times|\det\partial_\theta x(\theta,\eta,\xi)|\omega(x(\theta,\eta,\xi))\hat{u}(\xi)d\xi d\theta,
\end{aligned}
\tag{A.24}
$$

where the summation Σ_2 ranges over the indices μ_j such that $|\mu_j|\leqq 2l_j$ $(1\leqq j\leqq k)$. Thus using the identities

$$
\left\{
\begin{aligned}
&i\xi e^{i(\xi-\eta)\cdot\theta}=\partial_\theta(e^{i\xi\cdot\theta})e^{-i\eta\cdot\theta}, \\
&i\eta e^{i(\xi-\eta)\cdot\theta}=-\partial_\theta(e^{-i\eta\cdot\theta})e^{i\xi\cdot\theta}, \\
&i\theta e^{i(\xi-\eta)\cdot\theta}=\partial_\xi(e^{i(\xi-\eta)\cdot\theta})=-\partial_\eta(e^{i(\xi-\eta)\cdot\theta}),
\end{aligned}
\right.
\tag{A.25}
$$

we obtain by an appropriate series of integration by parts

$$
\begin{aligned}
&(K_\omega(\lambda,\nu)\hat{u})(\eta) \\
&=\Sigma_1\Sigma_2\Sigma_3\partial_\eta^\alpha\eta^\beta\iint e^{i(\xi-\eta)\cdot\theta}\partial_\theta^{\rho_1}\partial_\eta^{\rho_2}\partial_\xi^{\rho_3}\{\prod_{j=1}^{k}(d_j^{(\mu_j)}(\theta,\eta,\xi)a_j^{(\alpha_j)}b_j^{(\beta_j)}) \\
&\qquad\times|\det\partial_\theta x(\theta,\eta,\xi)|\omega(x(\theta,\eta,\xi))\}\xi^\gamma\partial_\xi^\delta\hat{u}(\xi)d\xi d\theta,
\end{aligned}
\tag{A.26}
$$

where the summation Σ_3 ranges over those systems of multi-indices $\alpha,\beta,\gamma,\rho_1,\rho_2,\rho_3$ such that $|\alpha+\beta|, |\gamma+\delta|\leqq|l|$ and $|\rho_1+\rho_2+\rho_3|\leqq 2|l|$.

Now by Lemma A.2 we obtain for α,β,γ satisfying $|\alpha+\beta+\gamma|\leqq 2\left(\left[\frac{n}{2}\right]+\left[\frac{5n}{4}\right]+2\right)$,

$$
\begin{aligned}
&|\partial_\theta^\alpha\partial_\eta^\beta\partial_\xi^\gamma\{\partial_\theta^{\rho_1}\partial_\eta^{\rho_2}\partial_\xi^{\rho_3}(\prod_{j=1}^{k}(d_j^{(\mu_j)}(\theta,\eta,\xi)a_j^{(\alpha_j)}b_j^{(\beta_j)}) \\
&\qquad\times|\det\partial_\theta x(\theta,\eta,\xi)|\omega(x(\theta,\eta,\xi)))\}| \\
&\leqq C_{k,l}(\prod_{j=1}^{k}A_j)^2(1+|x(\lambda)|)^{2|l|}\sup_{\substack{x\in R^n\\|m|\leqq M}}|\partial_x^m\omega(x)|,
\end{aligned}
\tag{A.27}
$$

where $|\rho_1+\rho_2+\rho_3|\leqq 2|l|$, $|\alpha_j|, |\beta_j|\leqq l_j$ $(1\leqq j\leqq k)$. Thus by Calderón-Vaillancourt theorem we obtain

$$
\begin{aligned}
&\int|T(\lambda,\nu)u(x)|^2\omega(x)dx=\int(K_\omega(\lambda,\nu)\hat{u})(\eta)\overline{\hat{u}(\eta)}d\eta \\
&=\Sigma_1\Sigma_2\Sigma_3\int\overline{\eta^\beta\partial_\eta^\alpha\hat{u}(\eta)}\iint e^{i(\xi-\eta)\cdot\theta}\{\partial_\theta^{\rho_1}\partial_\eta^{\rho_2}\partial_\xi^{\rho_3}(\prod_{j=1}^{k}(d_j^{(\mu_j)}(\theta,\eta,\xi) \\
&\qquad\times a_j^{(\alpha_j)}b_j^{(\beta_j)})|\det\partial_\theta x(\theta,\eta,\xi)|\omega(x(\theta,\eta,\xi)))\}\xi^\gamma\partial_\xi^\delta\hat{u}(\xi)d\xi d\theta d\eta \\
&\leqq\Sigma_1\Sigma_2\Sigma_3 C_{k,l}(\prod_{j=1}^{k}A_j)^2(1+|x(\lambda)|)^{2|l|}\sup_{\substack{x\in R^n\\|m|\leqq M}}|\partial_x^m\omega(x)|\cdot\|u\|_{|l|}^2.
\end{aligned}
\tag{A.28}
$$

Take a sequence $\{\omega_N\}_{N=1}^{\infty}\subset C_0^{\infty}(R^n)$ so that $0\leqq\omega_N(x)\leqq 1$; $\omega_N(x)=1$ $(|x|\leqq N)$; $\omega_N(x)\leqq\omega_{N+1}(x)$; and $\sup_{x,N}|\partial_x^{\alpha}\omega_N(x)|<\infty$ for any α, and replace ω by ω_N in (A.28). Then we get (A.9) by letting $N\to\infty$. Q. E. D.

Acknowledgement

The author wishes to express his sincere appreciation to Professor Atsushi Inoue for encouraging and stimulating discussions which turned the author's interest to the Feynman path integral. Also he wishes to express his hearty thanks to Professor Daisuke Fujiwara for making a manuscript of [5] available to him and for valuable comments on our original manuscript, especially for suggesting to the author that he had better take into account the Planck constant.

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(Received June 14, 1979)

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