Spectral and scattering theory for Schrödinger operators with Stark-effect

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1. Introduction, Assumption and Theorem

The purpose of the present paper is to study the spectral and scattering theory for Schrödinger operators related to the Stark-effect.

The Hamiltonian which governs the motion of a quantum mechanical particle (with mass m and charge e) moving in a homogeneous electrostatic field $\boldsymbol{E}$ can be written as

$$H_0=-(1/2m)\Delta+e\boldsymbol{E}\cdot x\,. \tag{1.1}$$

Here Δ is the n-dimensional Laplacian and H_0 is considered to be an operator acting on the Hilbert space $\mathfrak{H}=L^2(\boldsymbol{R}^n)$. Let us suppose that the motion of such a particle is perturbed by some external force given by potential $V(x)$. Then the Hamiltonian is changed to

$$H=H_0+V\,. \tag{1.2}$$

We shall study the spectral properties of H and the scattering theory between H and H_0.

We take the coordinate such that $\boldsymbol{E}=(1, 0, \cdots, 0)$ and the units $m=\frac{1}{2}$, $e=1$ and we assume that the potential V satisfies the following condition.

Assumption (A). $V(x)$ is a real-valued function and $V(x)$ can be decomposed as

$$V(x)=\{\tilde{\chi}(x_1)(1+x_1^2)^{-\sigma/2}+\chi(x_1)(1+x_1^2)^{1/2}\}(V_1(x)+V_2(x_1))\,. \tag{1.3}$$

Here σ is a positive constant such that

$$\sigma>1/2 \tag{1.4}$$

and functions χ, $\tilde{\chi}$, V_1 and V_2 satisfy:

$$\chi\in C^\infty(\boldsymbol{R}^1) \quad \text{such that} \tag{1.5}$$

$$\chi(x_1)=\begin{cases} 1, & x_1>-1\,, \\ 0, & x_1<-2\,; \end{cases}$$

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(1.6) $$\tilde{\chi}(x_1)=\chi(-x_1);$$

(1.7) $$V_1\in L^\infty(\boldsymbol{R}^n) \quad \text{and} \quad \lim_{|x|\to\infty} V_1(x)=0;$$

(1.8) $V_2\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ and there exists a constant $4>\mu>0$ such that

$$\lim_{|x|\to\infty}(1+|x_1|^2)\int_{|x-y|\le 1}|V_2(y)|^2|x-y|^{-n+\mu}dy=0.$$

We write as $\rho(x_1)=\tilde{\chi}(x_1)(1+x_1^2)^{-\sigma/2}+\chi(x_1)(1+x_1^2)^{1/2}$.

H_0 defined on $\mathcal{S}(\boldsymbol{R}^n)$ is essentially selfadjoint and under Assumption (A), V is H_0-compact (Theorems 2.4 and 2.5). Hence $H=H_0+V$ with $D(H)=D(H_0)$ is selfadjoint. The main theorem of this paper is the following theorem.

THEOREM 1.1. *Let $V(x)$ satisfy Assumption* (A). *Then the following statements hold.*

1) $$\sigma(H)=\sigma_p(H)\cup\sigma_{\mathrm{ac}}(H)=\boldsymbol{R}^1 \quad \textit{and} \quad \sigma_{\mathrm{sc}}(H)=\phi.$$

2) *$\sigma_p(H)$ is discrete, that is, each $\lambda\in\sigma_p(H)$ is of finite multiplicity and $\sigma_p(H)$ has no accummulation points except possibly $\pm\infty$.*

3) *The limits of the following formulas*

$$W_\pm=\text{s-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$$

exist and are isometries. Moreover

$$R(W_\pm)=\mathfrak{H}_{\mathrm{ac}}(H),$$

where $\mathfrak{H}_{\mathrm{ac}}(H)$ is the spectrally absolutely continuous subspace of $\mathfrak{H}$ with respect to H.

4) *$H_{\mathrm{ac}}=H|_{\mathfrak{H}_{\mathrm{ac}}(H)}$ is unitarily equivalent to H_0 via the operators $W_\pm$: $W_\pm^* H_{\mathrm{ac}} W_\pm=H_0$.*

We shall prove the theorem in the subsequent sections, using the abstract stationary method of Kato-Kuroda [7]. In the course of the proof, an essential role will be played by an integral operator of Fourier type with a highly oscillating kernel.

The study of the spectral and scattering theory for the operators of type H has a long history and there are many references. We mention here, among others, the work of Titchmarsh [12], Kato [6], Conley-Rejto [4], Avron-Herbst [2], Veselič-Weidmann [13] and Herbst [5]. Titchmarsch [12] first pointed out that $\sigma(H)=\boldsymbol{R}^1$, suggesting that H should be considered to be a perturbed operator of H_0 by V rather than the one of $-\Delta+V(x)$ by $\boldsymbol{E}\cdot x$, while the latter point of view had been taken by most physicists (see, for instance, Landau-Lifschitz [9]). Kato [6] and Conley-Rejto [4] studied the asymptotic behavior of the spectral projection of H in the case where $\boldsymbol{E}$ tends to zero. The scattering theory for H and H_0 was begun by Avron-Herbst [2] and Veselič-Weidmann [13]. They proved the existence of the wave operator under a condition similar to Assumption (A). Herbst [5] prove the completeness of the wave operators $R(W_\pm)=\mathfrak{H}_{ac}(H)$ under the condition that, roughly speaking, $|V(x)|\leqq C\tilde{\chi}(x_1)(1+x_1^2)^{-\sigma/2}+C_1\chi(x_1)(1+x_1^2)^{1/4}$.

Our results and method are new in the following two respects: (1) We prove the completeness of the wave operators under the assumption that $V(x)$ may growth in the direction of $\boldsymbol{E}$ with order $o(\boldsymbol{E}\cdot x)$, weakening the assumption of Herbst [5]; (2) we use the integral operator with highly oscillating kernel for studying the properties of the resolvent $R_0(z)=(H_0-z)^{-1}$. By virtue of this method (2) our results can be readily extended to more general cases: Replacing $-\Delta$ and $V(x)$ by a constant coefficient, elliptic, formally selfadjoint, differential operator $P_0(D)$ of degree $2m$ and a perturbation by a differential operator $V_D=\sum_\alpha V_\alpha(x)D^\alpha$, with coefficients $|V_\alpha(x)|\leqq C(1+|x_1|)^{-\sigma/2}$, $\sigma>1/m$ (roughly speaking), we can prove the results similar to Theorem 1.1. However, we shall not go into details in this direction here.

We list here the notation and conventions used in the following sections.

$x=(x_1, x_2, \cdots, x_n)=(x_1, x')$ stands for a generic point of $\boldsymbol{R}^n$, $x'\in\boldsymbol{R}^{n-1}$. $p=(p_1, p')=(p_1, p_2, \cdots, p_n)$ denotes the conjugate variable of x.

$$|x|=(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}.$$

For multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)\in\boldsymbol{N}^n$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$,

$$D_j=\frac{\partial}{\partial x_j}.\quad |\alpha|=\sum_{j=1}^n|\alpha_j|.$$

$\mathfrak{F}$ is the Fourier transform:

$$(\mathfrak{F}f)(p)=\hat{f}(p)=(2\pi)^{-n/2}\int_{R^n}e^{-ix\cdot p}f(x)dx\,.$$

$\mathfrak{F}^{-1}f=\check{f}$. $L^2(\boldsymbol{R}^n)$ is the Hilbert space of all square integrable functions on $\boldsymbol{R}^n$ with the inner product $(f, g)=\int_{R^n}f(x)\overline{g(x)}dx$ and norm $\|f\|=(f, f)^{1/2}$.

$L^2_{loc}(\boldsymbol{R}^n)$ is the space of all locally square integrable functions. $\mathcal{S}(\boldsymbol{R}^n)$ is the space of all rapidly decreasing functions and $\mathcal{S}'(\boldsymbol{R}^n)$ is the space of all tempered distributions. $H^s(\boldsymbol{R}^n)$ is the Sobolev space:

$$H^s(\boldsymbol{R}^n)=\{f\in\mathcal{S}'(\boldsymbol{R}^n):\ \|f\|_s=\|(1+p^2)^{s/2}\hat{f}\|<\infty\}$$

$H^0(\boldsymbol{R}^n)=L^2(\boldsymbol{R}^n)$. For γ, $\delta\in\boldsymbol{R}^1$, we set

$$\mathfrak{X}_{\gamma,\delta}=\{f\in L^2_{\text{loc}}(\boldsymbol{R}^n):$$
$$\|f\|_{\mathfrak{X}_{\gamma,\delta}}=\|\{\tilde{\chi}(x_1)(1+x_1^2)^{\gamma/2}+\chi(x_1)(1+x_1^2)^{\delta/2}\}f\|<\infty\},$$
$$\mathfrak{X}_\delta=\mathfrak{X}_{\delta,\delta}.$$

For these space we have from the interpolation theorem that for $0\leqq t\leqq 1$

$$[\mathfrak{X}_{\gamma,\delta},\ \mathfrak{X}_{\gamma',\delta'}]_t=\mathfrak{X}_{\gamma' t+(1-t)\gamma,\,\delta' t+(1-t)\delta},$$

where $[\ ,\ ]_t$ is the intermediate space (Lions [10]). For any pair of Banach spaces X and Y, $B(X, Y)$ stands for the set of all linear bounded operators from X to Y, $B_\infty(X, Y)$ the set of all compact operators. $B(X)=B(X, X)$ and $B_\infty(X)=B_\infty(X, X)$.

For the operator T in a Hilbert space X, $D(T)$ and $R(T)$ stand for the domain of T and the range of T. For a selfadjoint T, $\sigma(T)$, $\sigma_{\text{ess}}(T)$, $\sigma_{\text{sc}}(T)$, $\sigma_p(T)$ and $\rho(T)$ are the spectrum, the essential spectrum, the singular continuous spectrum, the point spectrum and the resolvent set of T. For $z\in\rho(T)$, $R(z)=(T-z)^{-1}$. $\Pi^\pm=\{z\in \boldsymbol{C}^1: \text{Im}\, z\gtrless 0\}$. For any pair of intervals I_1 and I_2 we write as $I_1\Subset I_2$ if the closure $\bar{I}_1$ of I_1 is compact and $\bar{I}_1$ is contained in the interior of I_2.

The composition of the paper is as follows. In section 2, the essential selfadjointness of H_0 and H on $\mathcal{S}(\boldsymbol{R}^n)$ is studied. In section 3, we analyze the property of the resolvents $R_0(z)=(H_0-z)^{-1}$ and $R(z)=(H-z)^{-1}$ when z is near the real line, and the so called limiting absorption principle is proved for H_0 and H. In section 4, the main theorem is proved by use of the materials developed in the previous sections. Section 5 is an appendix and the proof of Lemma 2.1 is given.

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2. The essential selfadjointness

Here we shall discuss the construction of the Hamiltonians, that is, we shall prove the essential selfadjointness of H and H_0 defined on $\mathcal{S}(\boldsymbol{R}^n)$. However, since the integral operators U and U^* defined below play essential role in the subsequent sections, we first record some important properties of U and U^*, postponing the proof till the end of the paper.

Let $G(p)=(1/3)p_1^3+p_1(p_2^2+\cdots+p_n^2)$. We define as

$$(2.1)\qquad (Uf)(x)=(2\pi)^{-n/2}\int_{R^n}e^{ix\cdot p-iG(p)}\hat{f}(p)dp,$$

$$(U^*f)(x)=(2\pi)^{-n/2}\int_{R^n} e^{ix\cdot p+iG(p)}\hat{f}(p)dp, \qquad f\in \mathcal{S}(\boldsymbol{R}^n). \tag{2.2}$$

LEMMA 2.1. *The operators U and U^* satisfy the following properties.*

1) *U and U^* are isomorphisms on $\mathcal{S}(\boldsymbol{R}^n)$ and $U^*=U^{-1}$.*

2) *For any $s\in\boldsymbol{R}^1$, $\|Uf\|_s=\|U^*f\|_s=\|f\|_s$. In particular, U and U^* can be extended to $\mathfrak{H}$ by continuity and are unitary operators: $U^*=U^{-1}$.*

3) *If $f\in\mathcal{S}(\boldsymbol{R}^n)$ has support in the half space $\{x\in\boldsymbol{R}^n: x_1>a\}$* (resp. $\{x\in\boldsymbol{R}^n: x_1<a\}$), *then $(Uf)(x)$* (resp. $(U^*f)(x)$) *is real analytic. Moreover for any l, $s\in\boldsymbol{R}^1$, $q_1<0$* (resp. $q_1>0$) *and integer $m\geqq 0$, there exists a constant c independent of f such that*

$$\sum_{|\alpha|\leqq m}\|e^{q_1\cdot x_1}D^\alpha Uf\|\leqq c\|(1+x_1^2)^{l/2}f\|_s \tag{2.3}$$

$$(\textit{resp.} \sum_{|\alpha|\leqq m}\|e^{q_1\cdot x_1}D^\alpha U^*f\|\leqq c\|(1+x_1^2)^{l/2}f\|_s).$$

4) *For any integer $m\geqq 0$, any s, $l\in\boldsymbol{R}^1$ with $0\leqq s\leqq m$, there exists a constant c independent of f such that*

$$\sum_{|\alpha|\leqq 2m-2s-l}\|(1+x_1^2)^{s/2}D^\alpha Uf\|_{L^2(x_1<-1)}\leqq c\|(1+x_1^2)^{m/2}f\|_{-l}, \tag{2.4}$$

$$\sum_{|\alpha|\leqq 2m-2s-l}\|(1+x_1^2)^{s/2}D^\alpha U^*f\|_{L^2(x_1>1)}\leqq c\|(1+x_1^2)^{m/2}f\|_{-l}.$$

5) *Let m be non-negative integer and $l\in\boldsymbol{R}^1$. Then there exists a constant $c>0$ independent of f such that*

$$\sum_{k=0}^{m}\sum_{|\alpha|\leqq 2k}\|(1+x_1^2)^{-k/2}D^\alpha Uf\|_l\leqq c\|(1+x_1^2)^{m/2}f\|_l \tag{2.5}$$

$$\sum_{k=0}^{m}\sum_{|\alpha|\leqq 2k}\|(1+x_1^2)^{-k/2}D^\alpha U^*f\|_l\leqq c\|(1+x_1^2)^{m/2}f\|_l.$$

COROLLARY 2.2. 1) $\rho(x_1)U^*\in B(\mathfrak{X}_{1,1}, \mathfrak{X}_{\sigma,0})$.

2) $(\chi(x_1)+(1+x_1^2)^{-1/2}\tilde{\chi}(x_1))U^*\in B(\mathfrak{X}_{1,1}, H^2(\boldsymbol{R}^n))$.

3) *For $\phi\in C^\infty(\boldsymbol{R}^1)$, let Φ be the multiplication operator by $\phi(x_1)$.*

i) *If ϕ has support in the left half line, then*

$$\Phi\cdot U\in B(\mathfrak{X}_{\gamma,\delta}, H^{2\gamma}(\boldsymbol{R}^n)), \tag{2.6}$$

$$U^*\cdot\Phi\in B(H^{-2\gamma}(\boldsymbol{R}^n), \mathfrak{X}_{-\gamma,-\delta}), \tag{2.7}$$

for any $\gamma\geqq 0$, $\delta\in\boldsymbol{R}^1$.

ii) *If ϕ has compact support, then*

$$\{\chi(x_1)e^{sx_1}+(1+x_1^2)^{-(1+\gamma)/2}\}U^*\cdot\Phi\in B(H^{-2\gamma}(\boldsymbol{R}^n), H^2(\boldsymbol{R}^n)) \tag{2.8}$$

for any $\gamma\geqq 0$, $s\geqq 0$.

PROOF. 1) Setting $m=s=1$ and $l=0$ in (2.4), we have

$$\|\rho(x_1)U^*f\|_{L^2(x_1>1)}\leqq c\|(1+x_1^2)^{1/2}f\|.$$

By Lemma 2.1,1),

$$\|(1+x_1^2)^{\sigma/2}\rho(x_1)U^*f\|_{L^2(x_1<1)}\leqq c\|U^*f\|=c\|f\|.$$

Combining these two inequalities, we get 1).

2) Set $m=1$ and $l=0$ in (2.4) and (2.5). Combine them.

3.i) First we note that (2.4) and a simple interpolation theorem ([10]) imply that

$$\Phi\cdot U\in B(\mathfrak{X}_{\gamma,\gamma},\ H^{2\gamma}(\boldsymbol{R}^n)),\qquad \gamma\geqq 0. \tag{2.9}$$

Hence by (2.3) and (2.9), we get for any $\delta\in\boldsymbol{R}^1$,

$$\begin{aligned}\|\Phi\cdot Uf\|_{2\gamma}&\leqq\|\Phi\cdot U(\chi(x_1)f)\|_{2\gamma}+\|\Phi\cdot U(1-\chi(x_1))f\|_{2\gamma}\\&\leqq c\|f\|_{\mathfrak{X}_{\gamma,\delta}}.\end{aligned}$$

This proves (2.6). (2.7) is the adjoint of (2.6).

3.ii) To prove (2.8) it suffices to prove that

$$U^*\cdot\Phi\in B(H^{-2\gamma}(\boldsymbol{R}^n),\ H^{2,-(1+\gamma)}(\boldsymbol{R}^n)),$$

where for $\delta\in\boldsymbol{R}^1$, $H^{2,\delta}(\boldsymbol{R}^n)$ is the Hilbert space

$$H^{2,\delta}(\boldsymbol{R}^n)=\{f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n):\ \sum_{|\alpha|\leqq 2}\|(1+x_1^2)^{\delta/2}D^\alpha f\|^2=\|f\|^2_{2,\delta}<\infty\}.$$

By an interpolation theorem ([10]),

$$[H^{2,\delta}(\boldsymbol{R}^n),\ H^{2,\rho}(\boldsymbol{R}^n)]_\theta=H^{2,(1-\theta)\delta+\theta\rho},\ 0\leqq\theta\leqq 1.$$

Hence it suffices to prove (2.8) for γ natural numbers. Setting $l=-2\gamma$ and $m=\gamma+1$ in (2.5), we get

$$\sum_{|\beta|\leqq 2\gamma+2}\|(1+x_1^2)^{-(1+\gamma)/2}D^\beta U^*\Phi f\|_{-2\gamma}\leqq c\|f\|_{-2\gamma}.$$

However the left side is greater than a constant times

$$\sum_{|\beta|\leqq 2\gamma}\sum_{|\alpha|\leqq 2}\|D^\beta(1+x_1^2)^{-(1+\gamma)/2}D^\alpha U^*\Phi f\|_{-2\gamma}$$

which is again greater than a constant times

$$\sum_{|\alpha|\leqq 2}\|(1+x_1^2)^{-(1+\gamma)/2}D^\alpha U^*\Phi f\|.$$

This proves (2.8). (Q. E. D.)

LEMMA 2.3. *Let V satisfy Assumption* (A). *Then* $VU^*\in B_\infty(\mathfrak{X}_{1,1},\ \mathfrak{X}_{\sigma/2,0})$.

PROOF. $VU^* = V_1\rho(x_1)U^* + V_2\rho(x_1)U^*$. By Corollary 2.2,1) and 2), and Assumption (A), (1.7), $V_1\rho(x_1)U^* \in B_\infty(\mathfrak{X}_{1,1}, \mathfrak{X}_{\sigma/2,0})$ (Rellich's compactness theorem). By Corollary 2.2,2) and Assumption (A), (1.8), for proving

$$\rho(x_1)V_2U^* = \rho(x_1)V_2(x)(\chi(x_1)+(1+x_1^2)^{-1/2}\tilde{\chi}(x_1))^{-1}\cdot(\chi(x_1)+(1+x_1^2)\tilde{\chi}(x_1))U^*$$
$$\in \boldsymbol{B}_\infty(\mathfrak{X}_{1,1}, \mathfrak{X}_{\sigma/2,0}),$$

it suffices to prove that if $W \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ and

$$\lim_{|x|\to\infty}\int_{|x-y|\leq 1} |x-y|^{-n+\mu}|W(y)|^2 dy = 0$$

for some $0<\mu<4$, the multiplication operator by $W(x)$ is a compact operator from $H^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^n)$. However this fact is proved by Schechter [11].

(Q. E. D.)

THEOREM 2.4. *H_0 with $D(H_0)=\mathcal{S}(\boldsymbol{R}^n)$ is essentially selfadjoint on $\mathfrak{H}$ (we write the closure by the same symbol H_0). If T_0 is the maximal operator defined by the multiplication by x_1*

$$H_0 = U^*T_0U\,. \tag{2.10}$$

PROOF. If $f \in \mathcal{S}(\boldsymbol{R}^n)$, a simple calculation shows that

$$(U^*T_0Uf)(x) = (-\Delta + x_1)f(x) = H_0f(x)\,.$$

Since T_0 is selfadjoint and $\mathcal{S}(\boldsymbol{R}^n)$ is a core of T_0, statements 1) and 2) of Lemma 2.1 imply that $H_0|_{\mathcal{S}(\boldsymbol{R}^n)}$ is essentially selfadjoint. (2.10) is obvious.

(Q. E. D.)

THEOREM 2.5. *Let V satisfy Assumption* (A). *Then $V(x)$ is H_0-compact. $H = H_0 + V$ with $D(H)=D(H_0)$ is selfadjoint.*

PROOF. The selfadjointness of $H=H_0+V$ with $D(H)=D(H_0)$ is a consequence of H_0-compactness of V (see Combes [3]). For proving the H_0-compactness of V, it is sufficient to prove $VR_0(i) \in B_\infty(\mathfrak{H})$. Since $R_0(i) = U^*(T_0-i)^{-1}U$ and $(T_0-i)^{-1} \in B(\mathfrak{H}, \mathfrak{X}_{1,1})$, $VR_0(i) \in B_\infty(\mathfrak{H})$ is an immediate consequence of Lemma 2.3.

(Q. E. D.)

3. Limiting absorption principles

In this section we shall prove the so called limiting absorption principle for H and H_0, which we shall use for proving the main theorem.

LEMMA 3.1. *Let $\gamma > 1/2$. Then the $B(H^\gamma(\boldsymbol{R}^n), H^{-\gamma}(\boldsymbol{R}^n))$-valued analytic function $(T_0-z)^{-1}$ on $\Pi^\pm$ can be extended to $\Pi^\pm \cup \boldsymbol{R}^1$ as a Hölder continuous function.*

Lemma 3.1 is well-known and the proof is omitted here.

PROPOSITION 3.2. i) *Let* $\gamma>1/4$. *Then the* $B(\mathfrak{X}_{\gamma,0}, \mathfrak{X}_{-\gamma,0})$-*valued analytic function* $R_0(z)$ *on* $\Pi^{\pm}$ *can be extended to* $\Pi^{\pm}\cup \boldsymbol{R}^1$ *as a locally Hölder continuous function.*

ii) *Let* $V(x)$ *satisfy Assumption* (A). *Then* $VR_0(z)$ *is a* $B(\mathfrak{X}_{\sigma/2,0})$-*valued analytic function on* $\Pi^{\pm}$ *and can be extended to* $\Pi^{\pm}\cup \boldsymbol{R}^1$ *as a locally Hölder continuous function.*

PROOF. Let us take a compact interval $I\subset \boldsymbol{R}^1$. Then it suffices to prove that $R_0(z)$ (or $VR_0(z)$) can be extended to $\Pi^{\pm}\cup I$ as a Hölder-continuous function. We take $\varphi_1(x_1)\in C_0^\infty(\boldsymbol{R}^1)$ and $\varphi_2(x_1)\in C^\infty(\boldsymbol{R}^1)$ such that $\varphi_1(x_1)^2+\varphi_2(x_1)^2\equiv 1$ and $\varphi_1(x_1)\equiv 1$ on some I', $I\Subset I'$. We write the multiplication operator by φ_1 and φ_2 as Φ_1 and Φ_2. Then

$$
\begin{aligned}
R_0(z) &= U^*\varphi_1(x_1)(T_0-z)^{-1}\varphi_1(x_1)U \\
&\quad + U^*\varphi_2(x_1)(T_0-z)^{-1}\varphi_2(x_1)U\,.
\end{aligned}
\tag{3.1}
$$

1) Setting $\delta=0$, $\gamma=\sigma/2$ in (2.6) and (2.7), we have $\Phi_1 U\in B(\mathfrak{X}_{\sigma/2,0}, H^{\sigma}(\boldsymbol{R}^n))$ and $U^*\Phi_1\in B(H^{-\sigma}(\boldsymbol{R}^n), \mathfrak{X}_{-\sigma/2,0})$. Hence Lemma 3.1 implies that the first summand of (3.1) satisfies the statement i). The second of (3.1) obviously satisfies i), since $\Phi_2(T_0-z)^{-1}\Phi_2$ is a $B(L^2(\boldsymbol{R}^n))$-valued analytic function on $(\boldsymbol{C}^1\backslash \boldsymbol{R}^1)\cup I'$.

2) $$
\begin{aligned}
VR_0(z) &= V_1\rho(x_1)U^*\Phi_1(T_0-z)^{-1}\Phi_1 U + V_2\rho(x_1)U^*\Phi_1(T_0-z)^{-1}\Phi_1 U \\
&\quad + VU^*\Phi_2(T_0-z)^{-1}\Phi_2 U\,.
\end{aligned}
$$

By (2.7) ($\gamma=\sigma/2$, $\delta=-1$) and (1.7), $V_1\rho(x_1)U^*\Phi_1\in B(H^{-\sigma}(\boldsymbol{R}^n), \mathfrak{X}_{\sigma/2,0})$; by (2.8) ($\gamma=\sigma/2$, $s>0$) and (1.8), $V_2\rho(x_1)U^*\Phi_1\in B(H^{-\sigma}(\boldsymbol{R}^n), \mathfrak{X}_{\sigma/2,0})$. Hence by Lemma 3.1, the first two members satisfy ii). Since $(T_0-z)^{-1}\Phi_2 U$ is a $B(\mathfrak{X}_{\sigma/2,0}, \mathfrak{X}_{1,1})$-valued analytic function on $(\boldsymbol{C}^1\backslash \boldsymbol{R}^1)\cup I$, Lemma 2.3 implies that the last member also satisfies ii). (Q. E. D.)

PROPOSITION 3.3. *Let* V *satisfy Assumption* (A). *Then there exists closed null set* $e_{\pm}$ *such that* $B(\mathfrak{X}_{\sigma/2,0}, \mathfrak{X}_{-\sigma/2,0})$-*valued analytic function* $R(z)$ *on* $\Pi^{\pm}$ *can be extended to* $\Pi^{\pm}\cup(\boldsymbol{R}^1\backslash e_{\pm})$ *as a locally Hölder continuous function.*

PROOF. Since V is H_0-compact, the resolvent equation gives $R(z)=R_0(z)\times(1+VR_0(z))^{-1}$ for $\operatorname{Im} z\neq 0$. Since $R_0(z)$ and $VR_0(z)$ satisfy Proposition 3.2 and $VR_0(z)\in B_\infty(\mathfrak{X}_{\sigma/2,0})$ for $\operatorname{Im} z\neq 0$ by Lemma 2.3, the proposition follows from the theorem of Kato and Kuroda [7]. (Q. E. D.)

4. Proof of the theorem

4.1. Proof of statement 3) and 4)

Since statement 4) is an obvious consequence of 3), it suffices to prove 3).

For this purpose we use a following simple version of the theorem of Kato-Kuroda [7].

THEOREM (Kato-Kuroda). *Let H_1 and H_2 be selfadjoint operators in a Hilbert space $\mathfrak{H}$. Suppose that there exists a dense linear manifold $\mathfrak{X}$ of $\mathfrak{H}$ such that*

1) *$\mathfrak{X}$ is a Hilbert space with its own structure and is continuously embedd in H;*

2) *there exist closed null sets $e_\pm$ of $\boldsymbol{R}^1$ such that $R_j(z)=(H_j-z)^{-1}$* (resp. $VR_1(z)$) *($j=1, 2$) can be extended to $\Pi^\pm\cup(\boldsymbol{R}^1\backslash e_\pm)$ as a $B(\mathfrak{X}, \mathfrak{X}^*)$-valued* (resp. $B(\mathfrak{X})$-*valued*) *continuous function, where $\mathfrak{X}^*$ is the dual space of $\mathfrak{X}$ with respect to the inner product of $\mathfrak{H}$. Then the limits in the following formulas*

$$W_\pm = \text{s-}\lim_{t\to\pm\infty} e^{itH_2}e^{-itH_1}P_{\text{ac}}(H_1) \tag{4.1}$$

exist and are isometries. Moreover

$$R(W_\pm)=\mathfrak{H}_{\text{ac}}(H_2)\,. \tag{4.2}$$

Continuation of the proof of 3).

We apply the theorem to our problem. We take as $\mathfrak{X}=\mathfrak{X}_{\sigma/2,0}$. Then condition 1) is obvious. Since $\mathfrak{X}^*=\mathfrak{X}_{-\sigma/2,0}$, condition 2) is an immediate consequence of Proposition 3.2 and Proposition 3.3. Hence statement 3) follows.

4.2. Proof of statement 1)

In this subsection we assume that $V_2=0$ for simplicity. The case that $V_2\neq 0$ can be treated by a modification of the following argument. By a standard argument we have $\sigma_p(H)\subset e_\pm$ and $\boldsymbol{R}^1\backslash e_\pm\subset\sigma_{\text{ac}}(H)$. We prove $e_+\subset\sigma_p(H)$. We prove only $e_+\subset\sigma_p(H)$. The other case can be proved similarly.

For proving this, we need the following lemma, the proof of which can be found in Agmon [1] and Yajima [14].

LEMMA 4.1. *Let $\mathfrak{R}$ be separable Hilbert space and $H^s(\boldsymbol{R}; \mathfrak{R})$ be the Sobolev space of $\mathfrak{R}$-valued functions on $\boldsymbol{R}^1$. Suppose that $f\in H^s(\boldsymbol{R}^1; \mathfrak{R})$ ($s>1/2$) and $f(\mu)=0$. Then $(\lambda-\mu)^{-1}f(\lambda)\in H^{s-1}(\boldsymbol{R}^1; \mathfrak{R})$, and there exists a constant C independent of such $f\in H^s(\boldsymbol{R}^n; \mathfrak{R})$ and μ such that*

$$\|(\lambda-\mu)^{-1}f(\lambda)\|_{s-1}\leq C\|f\|_s\,.$$

Continuation of the proof.

Let $\lambda\in e_+$. Then there exists a function $f\in\mathfrak{X}_{\sigma/2,0}$ such that $f+VR_0(\lambda+i0)f=0$. We put $g=R_0(\lambda+i0)f$. It is obvious that $(-\Delta+x_1+V(x))g=0$ in the generalized sense. Hence it is sufficient to prove that $g\in D(H)=D(H_0)$. Since f, $VR_0(\lambda+i0)f\in\mathfrak{X}_{\sigma/2,0}$ by Proposition 3.2,

$$\langle f, R_0(\lambda+i0)f\rangle+\langle VR_0(\lambda+i0)f, R_0(\lambda+i0)f\rangle=0\,. \tag{4.3}$$

Here and hereafter $\langle\ ,\ \rangle$ is the natural coupling between $\mathfrak{X}_{\sigma/2,0}$ and $\mathfrak{X}_{-\sigma/2,0}$. Taking the imaginary parts of (4.3), remembering that $V(x)$ is real-valued, we have

$$\operatorname{Im}\langle f, R_0(\lambda+i0)f\rangle=0\,. \tag{4.4}$$

On the other hand

$$\begin{aligned} 0&=\operatorname{Im}\langle f, R_0(\lambda+i0)f\rangle=\lim_{\varepsilon\downarrow 0}\operatorname{Im}\langle f, R_0(\lambda+i\varepsilon)f\rangle \\ &=\lim_{\varepsilon\downarrow 0}\operatorname{Im}(Uf, (T_0-\lambda-i\varepsilon)^{-1}Uf) \\ &=-(\pi/2)\lim_{\varepsilon\downarrow 0}(\varepsilon/\pi)\int_{R^n}\frac{|Uf(x)|^2}{(x_1-\lambda)^2+\varepsilon^2}dx \\ &=-(\pi/2)\int_{R^{n-1}}|Uf(\lambda, x')|^2dx'\,. \end{aligned} \tag{4.5}$$

Here we used the fact that $Uf\in H^{\sigma}_{\text{loc}}(\boldsymbol{R}^n)\cap L^2(\boldsymbol{R}^n)$ $(\sigma>1/2)$ and the embedding theorem for the Sobolev spaces. Therefore using Lemma 4.1 for

$$\begin{aligned} (T_0-(\lambda+i0))^{-1}Uf=\varphi_1(x_1)^2(T_0-\lambda-i0)^{-1}Uf \\ +\varphi_2(x_1)^2(T_0-\lambda-i0)^{-1}Uf\,, \end{aligned}$$

we have $\qquad \varphi_1(x_1)^2(T_0-\lambda-i0)^{-1}Uf\in H^{\sigma-1}(\boldsymbol{R}^1)\otimes L^2(\boldsymbol{R}^{n-1})$

and

$$\varphi_2(x_1)^2(T_0-\lambda-i0)^{-1}Uf\in\mathfrak{X}_{1,1} \tag{4.6}$$

with corresponding estimates which are uniform as long as λ runs over a compact subset of the reals. Then by Corollary 2.2.1) and (2.7), we get

$$\begin{aligned} f=-VR_0(\lambda+i0)f&=-VU^*\phi_1(x_1)(T_0-\lambda-i0)^{-1}\phi_1(x_1)Uf \\ &-VU^*\phi_2(x_1)(T_0-\lambda-i0)^{-1}\phi_2(x_1)Uf \\ &\in\mathfrak{X}_{\min(\sigma,\,\sigma/2+(\sigma-1/2)),\,0}\,. \end{aligned}$$

Repeating the foregoing arguement n-times, we have

$$f\in\mathfrak{X}_{\min(\sigma,\,\sigma/2+n(\sigma-1/2)),\,0}\,.$$

Thus if we take n sufficiently large such that $\sigma/2+n(\sigma-1/2)>\sigma$, we get $f\in\mathfrak{X}_{\sigma,0}$. Then we have by Lemma 4.1 that

$$(T_0-\lambda-i0)^{-1}\phi_1(x_1)Uf\in H^{2\sigma-1}(\boldsymbol{R}^1)\otimes L^2(\boldsymbol{R}^{n-1})\subset L^2(\boldsymbol{R}^n)\,.$$

Therefore $(T_0-\lambda-i0)^{-1}Uf\in\mathfrak{X}_{1,1}=D(T_0)$, hence $g=R_0(\lambda+i0)f=U^*(T_0-\lambda-i0)^{-1}Uf\in U^*D(T_0)=D(H)$. This proves the statement.

4.3. Proof of statement 2)

Let $\lambda \in \boldsymbol{R}^1$ be the eigenvalue and let g be the corresponding eigenfunction of H. Let us put $f=(H_0-\lambda)g$. Then it can be easily seen that $R_0(\lambda+i0)f=g$ and $f+VR_0(\lambda+i0)f=0$. Since $g=-(i-\lambda)R_0(i)g-R_0(i)Vg$, $-Vg=f\in\mathfrak{X}_{\sigma/2,0}$ and $\|f\|_{\mathfrak{X}_{\sigma/2,0}}\leqq C_\lambda\|g\|$. Here C_λ can be taken uniformly as long as λ runs over a compact subset of the real line. Now let us assume that the eigenvalues $\{\lambda_j\}$ with corresponding normalized eigenfunctions $\{g_j\}$ converge to λ. It suffices to prove that $\{g_j\}$ forms a precompact subset of $\mathfrak{H}$. By virtue of the foregoing argument and the remark following (4.6) it suffices to prove that the functions $\{f_j\}$ corresponding to $\{g_j\}$ by $f_j=-Vg_j$ forms a precompact subset of $\mathfrak{X}_{\sigma/2,0}$. Since $VR_0(\lambda+i0)\in B_\infty(\mathfrak{X}_{\sigma/2,0})$ by Proposition 3.3, we can find a subsequence of $\{f_j\}$ (which we write as $\{f_j\}$ again) such that $\{VR_0(\lambda+i0)f_j\}$ is convergent in $\mathfrak{X}_{\sigma/2,0}$. Since

$$f_j-f_k=(VR_0(\lambda_j+i0)f_j-VR_0(\lambda+i0)f_j)+(VR_0(\lambda_k+i0)f_k-VR_0(\lambda+i0)f_k)$$
$$+VR_0(\lambda+i0)(f_k-f_j)$$

and $VR_0(\mu+i0)$ is $B(\mathfrak{X}_{\sigma/2,0})$-valued locally Hölder continuous function, $\{f_j\}$ obviously forms a convergent sequence in $\mathfrak{X}_{\sigma/2,0}$. This is the desired result.

(Q. E. D)

5. Appendix

Here we give a proof of Lemma 2.1.

5.1. Proof of statement 1) and 2)

Since the Fourier transform and the multiplication by $e^{-iG(p)}$ (resp. $e^{iG(p)}$) are isomorphisms on $\mathcal{S}(\boldsymbol{R}^n)$. U (resp. U^*) is isomorphism. Trivial relation $e^{iG(p)}e^{-iG(p)}=1$ implies $U^*=U^{-1}$. Statement 2) is an obvious consequence of the relation

$$(1+p^2)^s|\hat{U}f(p)|^2=(1+p^2)^s|e^{-iG(p)}\hat{f}(p)|^2=(1+p^2)^s|\hat{f}(p)|^2 .$$

5.2. Proof of statement 3)

We give the proof for U only. The other case can be proved similarly. Since $\operatorname{supp} f\subset\{x\in\boldsymbol{R}^n,\ x_1>a\}$, $\hat{f}(z_1, p_2, \cdots, p_n)$ is analytic in $z_1=p_1+iq_1$ on the region $q_1<0$. Hence by the change of the integral region $\boldsymbol{R}^n$ to $\boldsymbol{R}^n+i(q_1, 0, \cdots, 0)$, we get

$$(5.1)\qquad (Uf)(x)=e^{-q_1^3/3-x_1q_1}(2\pi)^{-n/2}\int e^{ixp-(i/3)(p_1^3-3p_1q_1^2+3p_1p'^2)+q_1p^2}\hat{f}(p_1+iq_1\cdot p')dp .$$

Note here that $\hat{f}(p_1+iq_1,\ p')=(e^{q_1x_1}f)\hat{\ }(p)$ and $e^{q_1x_1}(1+x_1^2)^{l/2}$ is uniformly bounded on $\{x_1>a\}$, since $q_1<0$. Thus from the expression (5.1), we can easily see that $Uf(x)$ is real analytic and satisfies the inequality (2.3).

5.3. Proof of Statement 4)

By differentiating (2.1), we have

$$D^\alpha Uf(x)=(2\pi)^{-n/2}\int_{R^n}e^{ix\cdot p-iG(p)}(ip)^\alpha \hat{f}(p)dp\,.$$

Using the relation $(\partial/i\partial p_1)e^{ix\cdot p-iG(p)}=(x_1-p^2)e^{ix\cdot p-iG(p)}$, we get, for $x_1<-1$, by partial integration that

$$\begin{aligned}D^\alpha Uf(x)&=(2\pi)^{-n/2}\int_{R^n}\left\{\left(\frac{1}{x_1-p^2}\frac{\partial}{i\partial p_1}\right)^m e^{ix\cdot p-iG(p)}\right\}(ip)^\alpha\hat{f}(p)dp\\ &=(2\pi)^{-n/2}\int_{R^n}e^{ix\cdot p-iG(p)}\left(\frac{i\partial}{\partial p_1}\circ\frac{1}{x_1-p^2}\right)^m\{(ip)^\alpha\hat{f}(p)\}\,dp\\ &=\sum_{k=0}^m(2\pi)^{-n/2}\int_{R^n}e^{ix\cdot p}a_{m,k,\alpha,l}(x_1,\ p)\\ &\quad\times\left\{e^{-iG(p)}(1+p^2)^{-l/2}\left(\frac{\partial}{\partial p_1}\right)^k\hat{f}(p)\right\}dp\,.\end{aligned}$$

Here we can easily see that for $0\leqq s\leqq m$, $|\alpha|\leqq 2m-2s-l$,

$$\sup_{x,p}\left|\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial p}\right)^\beta[\tilde{\chi}(x_1+2)(1+x_1^2)^{s/2}a_{m,k,\alpha,l}(x_1,\ p)]\right|<\infty$$

for any multi-index γ and β. Hence by the L^2-boundedness theorem for pseudo-differential operators (Kumano-go [8]), we see that

$$\begin{aligned}&\sum_{|\alpha|\leqq 2m-2s-l}\left\|(1+x_1^2)^{s/2}\left(\frac{\partial}{\partial x}\right)^\alpha\tilde{\chi}(x_1+2)Uf(x)\right\|\\ &\leqq c\sum_{k=0}^m\left\|e^{-iG(p)}(1+p^2)^{-l/2}\left(\frac{\partial}{\partial p_1}\right)^k f(p)\right\|\leqq c\|(1+x_1^2)^{m/2}f\|_{-l}\,,\end{aligned}$$

which obviously implies the desired estimate.

5.4. Proof of statement 5)

We first observe that the norm $\|(1+x_1^2)^{s/2}(1-\Delta)^{t/2}f\|$ is equivalent to the norm $\|(1-\Delta)^{t/2}(1+x_1^2)^{s/2}f\|$ for any $s,\ t\in \boldsymbol{R}^1$. This can be seen exactly the same way as the equivalence of the norms $\|(1+x^2)^{s/2}(1-\Delta)^{t/2}f\|$ and $\|(1-\Delta)^{t/2}\times(1+x^2)^{s/2}f\|$; $(1-\Delta)^{t/2}$ commutes with $D^\alpha U$ for any multi-index α. Hence it suffices to prove only the case $l=0$. We only prove the case $m=1$, since the other cases can be proved similarly. Let us take functions $\varphi_1(p)$ and $\varphi_2(p)$ such that $\varphi_1(p)+\varphi_2(p)=1$, $\varphi_1(p)\equiv 1$ near $p=0$ and $\varphi_1,\ \varphi_2\in C^\infty(\boldsymbol{R}^n)$. Then

$$(Uf)(x)=(2\pi)^{-n/2}\int e^{ix\cdot p-iG(p)}\varphi_1(p)\hat{f}(p)dp$$

$$-ix_1(2\pi)^{-n/2}\int e^{ixp-iG(p)}p^{-2}\varphi_2(p)\hat{f}(p)dp$$

$$+2i(2\pi)^{-n/2}\int e^{ixp-iG(p)}(p_1\cdot p^{-4}\varphi_2(p))\hat{f}(p)dp$$

$$-i(2\pi)^{-n/2}\int e^{ixp-iG(p)}p^{-2}\frac{\partial}{\partial p_1}(\varphi_2(p)\hat{f}(p))dp\,.$$

Hence the statement is an obvious consequence of the Parseval relation.

(Q. E. D.)

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