# *Singular hyperbolic systems, II*

## *Pseudo-differential operators with a parameter and their applications to singular hyperbolic systems*

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In our previous paper [6], we have established the existence, the uniqueness and the differentiability of solutions for a certain class of singular hyperbolic systems of type $(\sigma, \rho)$ with respect to $t$. However, these theorems are stated in a rather abstract form and no explanations of their assumptions are given there. For the application of these results to concrete partial differential equations, we must prepare some new classes of pseudo-differential operators so that we can deal with operators with some degeneracy at $t=0$. Once such a new class is well introduced, the application is straightforward. Hence, in this paper we want to discuss the following two problems:

(1) Introduce a good class of pseudo-differential operators.

(2) Apply the results to concrete partial differential equations.

In Section 1, we define a class of pseudo-differential operators such that their symbols are estimated by a quadratic form with a parameter $t$ and investigate basic properties of pseudo-differential operators in this class. In Section 2, we introduce a certain class of quadratic forms with a parameter $t$, which we call *basic quadratic forms*, to measure the degree of the degeneracy at $t=0$ of pseudo-differential operators. Using the results in Sections 1 and 2, we establish in Section 3 the symmetrizability in the sense of [6] for first order systems of pseudo-differential operators in our class. These three sections answer the problem (1). The problem (2) is treated in Sections 4, 5 and 6, that is, we treat symmetric singular hyperbolic systems in Section 4, non-symmetric singular hyperbolic systems in Section 5 and single singular hyperbolic equations in Section 6. These are easy applications of the problem (1) and the results in [6]. Thus, our program of this paper is as follows:

1. Pseudo-differential operators with a parameter $t$,
2. Basic quadratic form,
3. Symmetrization,
4. Symmetric singular hyperbolic systems,
5. Singular hyperbolic systems,

6. Singular hyperbolic equations.

This paper is the second part of a series of my papers and the results in this paper will be applied to the study of the Cauchy problem for a certain class of weakly hyperbolic equations with variable multiplicity and with non-smooth characteristic roots. A part of its study is announced in Tahara [5]. The systematic discussion will be done in Part III.

## Part I. PSEUDO-DIFFERENTIAL OPERATORS

### 1. Pseudo-differential operators with a parameter $t$

Let $Q(t, \xi)=\sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j$ be a quadratic form of $\xi$ satisfying the following:

(G-1) $a_{ij}(t)\in C^0([0, T])$, $a_{ij}(t)=a_{ji}(t)$ and $a_{ij}(t)$ is a real valued function.

(G-2) $Q(t, \xi)\geqq 0$ holds for any $(t, \xi)\in[0, T]\times \boldsymbol{R}^n$.

Then we have

PROPOSITION 1.1. *Put* $\lambda_Q(t, \xi)=(1+Q(t, \xi))^{1/2}$. *Then for any* $\alpha$ *the estimate*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\lambda_Q(t, \xi)\right|\leqq A_{\alpha}\lambda_Q(t, \xi)^{1-|\alpha|}, \ (t, \xi)\in[0, T]\times \boldsymbol{R}^n \tag{1.1}$$

*is valid for some constant* $A_{\alpha}$.

PROOF. Take any $t\in[0, T]$ and fix it. Then by an orthogonal transformation $\eta=T\xi$, we can transform $Q(t, \xi)$ into a diagonal form $\tilde{Q}(t, \eta)=\lambda_1(t)\eta_1^2+\cdots+\lambda_n(t)\eta_n^2$, where $\lambda_i(t)\geqq 0$ for $1\leqq i\leqq n$. Since the elements of $T$ are bounded, we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\lambda_Q(t, \xi)\right|\leqq M\sum_{|\beta|=|\alpha|}\left|\left(\frac{\partial}{\partial\eta}\right)^{\beta}(1+\lambda_1(t)\eta_1^2+\cdots+\lambda_n(t)\eta_n^2)^{1/2}\right|$$

for some constant $M$, which is independant of $t$. Since it is well known that $\lambda(\xi)=(1+|\xi|^2)^{1/2}$ satisfies (1.1) for some constant $C_{\alpha}$, we obtain

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\lambda_Q(t, \xi)\right|\leqq\{M\sum_{|\beta|=|\alpha|}\lambda_1(t)^{\beta_1/2}\cdots\lambda_n(t)^{\beta_n/2}C_{\beta}\}\lambda_Q(t, \xi)^{1-|\alpha|}.$$

This immediately leads us to (1.1). Q.E.D.

Since $1\leqq\lambda_Q(t, \xi)\leqq A(1+|\xi|^2)^{1/2}$ is valid for some constant $A$, (1.1) means that $\lambda_Q(t, \xi)$ is a basic weight function of $\xi$ in the sense of Kumano-go [3]. Therefore we can follow the argument in [3] to define a class of pseudo-differential operators with a parameter $t$ corresponding to this basic weight function. Thus we obtain

DEFINITION 1.2. Let $m$ be a real number and let

$$p(t, x, \xi)\in C^0([0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n).$$

We say that $p(t, x, \xi)$ belongs to a class $S_Q^m([0, T])$, if $p(t, x, \xi)$ satisfies the following conditions:

(i) $p(t, x, \xi)$ is of $C^\infty$ class with respect to $(x, \xi)$,

(ii) for any $\alpha$ and $\beta$, $\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial \xi}\right)^\beta p(t, x, \xi)\in C^0([0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n)$,

(iii) for any $\alpha$ and $\beta$, the estimate

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial \xi}\right)^\beta p(t, x, \xi)\right|\leqq C_{\alpha,\beta}\lambda_Q(t, \xi)^{m-|\beta|} \tag{1.2}$$

is valid on $[0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n$ for some constant $C_{\alpha,\beta}$.

For such a function $p(t, x, \xi)$, we define the corresponding operator by

$$p(t, x, D_x)u(t, x)=\int_{\boldsymbol{R}^n} e^{i(x,\xi)}p(t, x, \xi)\hat{u}(t, \xi)d\xi\,, \tag{1.3}$$

$$u(t, x)\in C^0([0, T], \mathcal{S}(\boldsymbol{R}^n)),$$

where $\mathcal{S}(\boldsymbol{R}^n)$ is the Schwartz space of all rapidly decreasing functions. If $Q(t, \xi)=\xi_1^2+\cdots+\xi_n^2$, we write $S^m([0, T])$ instead of $S_Q^m([0, T])$.

Clearly, the linear map (1.3) is a continuous mapping from $C^0([0, T], \mathcal{S}(\boldsymbol{R}^n))$ into $C^0([0, T], \mathcal{S}(\boldsymbol{R}^n))$. The operator $P(t)=p(t, x, D_x)$ defined by (1.3) is called *a pseudo-differential operator with a parameter t* in the class $S_Q^m([0, T])$ and the function $p(t, x, \xi)$ is called the symbol of $P(t)$ and denoted by $\sigma(P(t))$. The convenience to introduce this symbol class lies in the next proposition.

PROPOSITION 1.3. (1) *Assume that $Q_1(t, \xi)\leqq AQ_2(t, \xi)$ for some constant $A$. If $m\leqq 0$, we have $S_{Q_2}^m([0, T])\subset S_{Q_1}^m([0, T])$.* (2) *Let $A(t)$ be a $n\times n$ matrix with coefficients in $C^0([0, T])$. If $p(t, x, \xi)\in S_Q^m([0, T])$, we have $p(t, x, A(t)\xi)\in S_R^m([0, T])$ with $R(t, \xi)=Q(t, A(t)\xi)$.* (3) *Let $\rho\geqq 0$ and let $m\geqq 0$. If $p(t, x, \xi)\in S_Q^m([0, T])$, we have $t^{\rho m}p(t, x, \xi)\in S_R^m([0, T])$ with $R(t, \xi)=Q(t, t^\rho\xi)$.*

PROOF. Easy from the definition. Q. E. D.

The following theorem is the most fundamental results in the theory of pseudo-differential operators.

THEOREM 1.4. (1) (*Product*). *Let $P_j(t)\in S_Q^{m_j}([0, T])$ $(j=1, 2)$. Then the product $P(t)=P_1(t)P_2(t)$ belongs to $S_Q^{m_1+m_2}([0, T])$ and satisfies $\sigma(P(t))-\sigma(P_1(t))\times\sigma(P_2(t))\in S_Q^{m_1+m_2-1}([0, T])$.*

(2) (*Adjoint*). *Let $P(t)\in S_Q^m([0, T])$. Then the operator $P(t)^*$ defined by $(P(t)u, v)=(u, P(t)^*v)$ for $u, v\in\mathcal{S}(\boldsymbol{R}^n)$ belongs to $S_Q^m([0, T])$ and satisfies $\sigma(P(t)^*)-{}^t\overline{\sigma(P(t))}\in S_Q^{m-1}([0, T])$.*

(3) (*$L^2$-boundedness*). *Let $P(t)\in S_Q^0([0, T])$. Then there exists a positive con-*

*stant $C$ such that*

$$\|P(t)u\|_{L^2}\leqq C\|u\|_{L^2}$$

*is valid for $0\leqq t\leqq T$ and any $u\in\mathcal{S}(\boldsymbol{R}^n)$, hence for any $u\in L^2(\boldsymbol{R}^n)$.*

PROOF. See Kumano-go [3]. Q.E.D.

Now, we will study some more properties in connection with the assumptions supposed in Tahara [6]. Let $A(t)\in S_Q^1([0, T])$ and let $\wedge_Q$ and $\wedge$ be pseudo-differential operators defined by the symbols $\lambda_Q(t, \xi)$ and $\lambda(\xi)=(1+|\xi|^2)^{1/2}$ respectively. Then the operator $A_0(t)=A(t)\wedge^{-1}$ belongs to $S_Q^0([0, T])$ and the following

$$A_j(t)=[\wedge_Q, A_{j-1}(t)]=\wedge_Q A_{j-1}(t)-A_{j-1}(t)\wedge_Q,\ j\geqq 1$$

can be defined inductively on $j$. Clearly, $A_j(t)\in S_Q^0([0, T])$ holds. However, since $\wedge$ does not belong to $S_Q^1([0, T])$ in general, we can not apply Theorem 1.4 to define the commutator $[\wedge, A_0(t)]$ in $S_Q^0([0, T])$. This commutator is justified in the following way.

PROPOSITION 1.5 (*Commutator with* $\wedge$). *Let $P(t)\in S_Q^0([0, T])$. Then the commutator $P_1(t)=\wedge P(t)-P(t)\wedge$ also belongs to $S_Q^0([0, T])$.*

PROOF. Clearly, the mappings

$$\wedge P(t),\ P(t)\wedge : C^0([0, T], \mathcal{S}(\boldsymbol{R}^n)) \longrightarrow C^0([0, T], \mathcal{S}(\boldsymbol{R}^n))$$

are well defined. Therefore, combining the Taylor expansion of $\sigma(P(t))$ with Fourier's inversion formula, we have

$$\sigma(P_1(t))=\sum_{|\gamma|=1}\int_0^1 r_{\gamma,\theta}(t, x, \xi)d\theta, \tag{1.4}$$

where

$$r_{\gamma,\theta}(t, x, \xi)=O_s\text{-}\iint\left(\frac{1}{i}\right)^{|\gamma|}\left(\frac{\partial}{\partial\xi}\right)^{\gamma}\lambda(\xi+\theta\eta)\left(\frac{\partial}{\partial x}\right)^{\gamma}p(t, x+y, \xi)dy\, d\eta,$$

$p(t, x, \xi)=\sigma(P(t))$ and $\lambda(\xi)=(1+|\xi|^2)^{1/2}$.

Since $|\gamma|=1$, we have $\left(\frac{\partial}{\partial\xi}\right)^{\gamma}\lambda(\xi)\in S^0\subset S_Q^0([0, T])$. Therefore (1.4) means that $\sigma(P_1(t))\in S_Q^0([0, T])$, that is, $P_1(t)$ belongs to $S_Q^0([0, T])$. Q.E.D.

Thus, we can define the commutators

$$A_j(t)=[\wedge, A_{j-1}(t)]=\wedge A_{j-1}(t)-A_{j-1}(t)\wedge,\ j\geqq 1 \tag{1.5}$$

for any $A_0(t)\in S_Q^0([0, T])$ inductively on $j$. From Proposition 1.5, we have $A_j(t)\in S_Q^0([0, T])$ for any $j$. Next, we will show the continuity in the parameter $t$.

PROPOSITION 1.6 (*Continuity in* $t$). *Let* $P(t)\in S_Q^0([0, T])$. *Then* $P(t)u\in C^0([0, T], L^2(\boldsymbol{R}^n))$ *holds for any* $u\in L^2(\boldsymbol{R}^n)$. *Further, the linear map* $P(t)$: $C^0([0, T], L^2(\boldsymbol{R}^n))\to C^0([0, T], L^2(\boldsymbol{R}^n))$ *is a continuous mapping.*

PROOF. From Theorem 1.4, we have $P(t)u\in L^2(\boldsymbol{R}^n)$ for $0\leqq t\leqq T$ and $u\in L^2(\boldsymbol{R}^n)$. For $0\leqq t$, $t'\leqq T$ and $u\in L^2(\boldsymbol{R}^n)$, we have

$$\|P(t)u-P(t')u\|_{L^2}\leqq 2(\sup_{0\leq t\leq T}\|P(t)\|)\|u-w\|_{L^2}+\|P(t)w-P(t')w\|_{L^2}$$

for any $w\in\mathcal{S}(\boldsymbol{R}^n)$. Since $\mathcal{S}(\boldsymbol{R}^n)$ is dense in $L^2(\boldsymbol{R}^n)$, we can choose $w$ so that the first term of the right hand side is sufficiently small. For a fixed $w\in\mathcal{S}(\boldsymbol{R}^n)$, $P(t)w\in C^0([0, T], L^2(\boldsymbol{R}^n))$ is obtained by Lebesgue's convergence theorem. Therefore we have $\|P(t)u-P(t')u\|\to 0$ as $t'\to t$. Thus $P(t)u\in C^0([0, T], L^2(\boldsymbol{R}^n))$ is proved. On the other hand, from Theorem 1.4 we have

$$\sup_{0\leq t\leq T}\|P(t)u(t)\|_{L^2}\leqq(\sup_{0\leq t\leq T}\|P(t)\|)(\sup_{0\leq t\leq T}\|u(t)\|_{L^2})$$

for any $u(t)\in C^0([0, T], L^2(\boldsymbol{R}^n))$. Hence, $P(t)$ is a continuous mapping from $C^0([0, T], L^2(\boldsymbol{R}^n))$ into $C^0([0, T], L^2(\boldsymbol{R}^n))$. Q.E.D.

Thus, $A_j(t)u\in C^0([0, T], L^2(\boldsymbol{R}^n))$ holds for any $u\in L^2(\boldsymbol{R}^n)$ for the operator $A_j(t)$ defined by (1.5). Further, from Theorem 1.4 and the definition (1.5) we have

$$A_j(t)u=\wedge A_{j-1}(t)u-A_{j-1}(t)\wedge u$$

for any $u\in H^1(\boldsymbol{R}^n)$, where $H^1(\boldsymbol{R}^n)$ is Sobolev's space. Since the domain of $\wedge$ in $L^2(\boldsymbol{R}^n)$ coincides with $H^1(\boldsymbol{R}^n)$, we can conclude that our pseudo-differential operator with a parameter $t$ satisfies the condition (C-1) in Section 3 of [6]. As for the differentiability in $t$, the next proposition is prepared.

PROPOSITION 1.7 (*Differentiability in* $t$). *Let* $p(t, x, \xi)\in C^1([0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ *and assume that* $p(t, x, \xi)$ *and* $p_t'(t, x, \xi)$ *belong to* $S_Q^0([0, T])$. *Then the corresponding operators* $P(t)$ *and* $P_t'(t)$ *satisfy the following*: $P(t)u\in C^1([0, T], L^2(\boldsymbol{R}^n))$ *holds for any* $u\in L^2(\boldsymbol{R}^n)$ *and its derivative* $\dfrac{d}{dt}(P(t)u)$ *in* $C^1([0, T], L^2(\boldsymbol{R}^n))$ *coincides with* $P_t'(t)u$.

PROOF. If $u\in\mathcal{S}(\boldsymbol{R}^n)$, by use of Lebesgue's convergence theorem we have $P(t)u\in C^1([0, T], L^2(\boldsymbol{R}^n))$ and $\dfrac{d}{dt}(P(t)u)=P_t'(t)u$. Therefore the proof of this proposition is reduced to the following lemma.

LEMMA 1.8. *Let* $X$ *be a Hilbert space and let* $P(t)$ *be a bounded operator in* $X$ *for* $0\leqq t\leqq T$. *We assume the following conditions on* $P(t)$: (i) $\|P(t)\|$ *is uniformly bounded for* $0\leqq t\leqq T$, (ii) *there exists a dense subspace* $D$ *in* $X$ *such that*

*$P(t)x\in C^1([0, T], X)$ for any $x\in D$,* (iii) *the operator $\frac{d}{dt}P(t)$ defined by $\left(\frac{d}{dt}P(t)\right)x=\frac{d}{dt}(P(t)x)$ for $x\in D$ can be extended to a bounded operator $P_t'(t)$ in $X$, and* (iv) *$\|P_t'(t)\|$ is also uniformly bounded for $0\leqq t\leqq T$. Then we have $P(t)x\in C^1([0, T], X)$ and $\frac{d}{dt}(P(t)x)=P_t'(t)x$ for any $x\in X$.*

PROOF. Take any $x\in X$ and fix it. By the same argument as in the proof of Proposition 1.6, we have $P(t)x$, $P_t'(t)x\in C^0([0, T], X)$. Since $D$ is dense in $X$, we can choose a sequence $\{x_n\}$ in $D$ such that $x_n$ converges to $x$ in $X$ as $n\to\infty$. Then we have

$$P_t'(t)x=\lim_{n\to\infty}P_t'(t)x_n=\lim_{n\to\infty}\frac{d}{dt}(P(t)x_n) \tag{1.6}$$

for $0\leqq t\leqq T$. Take any $y\in X$ and put $h_n(t)=(P_t'(t)x_n, y)=\frac{d}{dt}(P(t)x_n, y)$ and $h(t)=(P_t'(t)x, y)$. These functions are complex valued continuous functions on $[0, T]$ and (1.6) means that $\lim_{n\to\infty}h_n(t)=h(t)$ for a fixed $t$. Therefore using Lebesgue's convergence theorem we have

$$\lim_{n\to\infty}\int_0^t h_n(\tau)d\tau=\int_0^t h(\tau)\,d\tau. \tag{1.7}$$

Since $\frac{d}{dt}(P(t)x_n)$, $P_t'(t)x\in C^0([0, T], X)$ holds, (1.7) implies

$$\lim_{n\to\infty}\left(\int_0^t\frac{d}{d\tau}(P(\tau)x_n)d\tau,\ y\right)=\left(\int_0^t P_\tau'(\tau)x\,d\tau,\ y\right).$$

Since $P(t)x_n\in C^1([0, T], X)$ and $y$ is an arbitrary element in $X$, we obtain

$$P(t)x-P(0)x=\lim_{n\to\infty}(P(t)x_n-P(0)x_n)=\int_0^t P_\tau'(\tau)x\,d\tau.$$

This means that $P(t)x\in C^1([0, T], X)$ and $\frac{d}{dt}(P(t)x)=P_t'(t)x$. Q.E.D.

Thus, the differentiability in $t$ is established. The assumption in Theorem 3.5 of [6] can be translated into the following.

COROLLARY 1.9. *Let $a(t, x, \xi)\in C^0([0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)\cap C^m((0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ and assume that $t^{\sigma k}\left(\frac{\partial}{\partial t}\right)^k a(t, x, \xi)\in S_Q^0([0, T])$ for $k=0, 1, \cdots, m$, where $\sigma\geqq 1$. Then the corresponding operator $A(t)$ satisfies the following conditions:* (i) *$t^{\sigma k}A(t)u\in C^k([0, T], L^2(\boldsymbol{R}^n))$ for any $u\in L^2(\boldsymbol{R}^n)$ and $k=0, 1, \cdots, m$, and* (ii) *$\left(\frac{d}{dt}\right)^k(t^{\sigma k}A(t))$ is a bounded operator in $X$ for $k=0, 1, \cdots, m$.*

PROOF. Clear. Q.E.D.

## 2. Basic quadratic form

Let $\sigma$ be a real number $\geqq 1$ and let $Q_\sigma(t, \xi)=\sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j$ be a quadratic form of $\xi$ satisfying the following:

(H-1) $a_{ij}(t)$ satisfies (G-1) and $a_{ij}(t)\in C^1([0, T])$ for $1\leqq i, j\leqq n$.

(H-2) $Q_\sigma(t, \xi)>0$ holds for any $(t, \xi)\in(0, T]\times(\boldsymbol{R}^n\backslash 0)$.

(H-3) $\max_{|\xi|=1}\left|\left(\frac{\partial}{\partial t}\right)\log Q_\sigma(t, \xi)\right|=O\left(\frac{1}{t^\sigma}\right)$ as $t\to+0$.

Then we say that $Q_\sigma(t, \xi)$ is *a basic quadratic form of class* $\sigma$. In the case with $\sigma=1$, this is introduced in Tahara [5] and is applied to the Cauchy problem. From the definition, we immediately have

PROPOSITION 2.1. *Let $Q_\sigma(t, \xi)$ and $R_\sigma(t, \xi)$ be basic quadratic forms of class $\sigma$. Then we have the following.* (1) *$Q_\sigma(t, \xi)$ is of class $\sigma'$ for any $\sigma'\geqq\sigma$.* (2) *$Q_\sigma(t, \xi)+R_\sigma(t, \xi)$ is of class $\sigma$.* (3) *Let $a(t)\in C^1([0, T])$ be a real valued function such that $a(t)>0$ for $t>0$ and $\left|\left(\frac{d}{dt}\right)\log a(t)\right|=O\left(\frac{1}{t^\sigma}\right)$ as $t\to+0$. Then $a(t)Q_\sigma(t, \xi)$ is of class $\sigma$.* (4) *Let $A(\xi)$ be a quadratic form satisfying $A(\xi)\geqq 0$ for any $\xi\in\boldsymbol{R}^n$. Then $A(\xi)+Q_\sigma(t, \xi)$ is of class $\sigma$.*

PROOF. Clear. Q. E. D.

First, we will give typical examples to illustrate the conditions (H-1), (H-2) and (H-3).

EXAMPLE 2.2. (1) Let $\kappa_i$ be an integer $\geqq 0$ for $1\leqq i\leqq n$. Then $Q(t, \xi)=t^{\kappa_1}\xi_1^2+\cdots+t^{\kappa_n}\xi_n^2$ is of class 1. (2) Let $\sigma_i$ and $m_i$ be real numbers such that $\sigma_i>1$ and $m_i>0$ for $1\leqq i\leqq n$. Then $Q(t, \xi)=e_{m_1,\sigma_1}(t)\xi_1^2+\cdots+e_{m_n,\sigma_n}(t)\xi_n^2$ (where $e_{m,\sigma}(t)=e^{-m/t^{\sigma-1}}$) is of class $\sigma=\max\{\sigma_i ; 1\leqq i\leqq n\}$. (3) Let $m$ be an integer $\geqq 0$. Then $Q(t, \xi)=\sum_{i,j=1}^{n}\xi_i\xi_j+t^m\sum_{i=1}^{n}\xi_i^2$ is of class 1. (Apply Proposition 2.1.) (4) Let $\sigma$ be a real number such that $\sigma>1$. Then $Q(t, \xi)=t^m\sum_{i,j=1}^{n}\xi_i\xi_j+e^{-1/t^{\sigma-1}}\sum_{i=1}^{n}\xi_i^2$ is of class $\sigma$. (5) $Q(t, \xi)=e^{-1/t}(\sin(1/t)+2)\xi^2$ is of class 2. (6) $Q(t, \xi)=e^{-1/t}(\sin(1/t^2)+2)\xi^2$ is of class 3. (7) Let $\sigma$ and $\kappa$ be real numbers such that $\sigma>1$ and $\kappa>0$. Then $Q(t, \xi)=e^{-1/t^{\sigma-1}}(\sin(1/t^{\sigma-1})+e^{t^{2\kappa}})\xi^2$ is of class $\sigma+\kappa$. And so on$\cdots$.

From the above examples, we can understand that (H-3) is closely related to the degree of the degeneracy of $Q(t, \xi)$ at $t=0$. Roughly speaking, we may say that $Q(t, \xi)$ has the degeneracy at most of finite order in the case $\sigma=1$ and that of infinite order in the case $\sigma>1$.

Now, we will study some properties of basic quadratic forms. Let $Q_\sigma(t, \xi)$ be as above and let $S^m_{Q_\sigma}([0, T])$ be the corresponding class of pseudo-differential operators.

PROPOSITION 2.3. *There exists a positive constant $C$ such that the estimates*

$$\left|\left(\frac{\partial}{\partial \xi_j}\right)Q_\sigma(t, \xi)\right| \leqq C\, Q_\sigma(t, \xi)^{1/2} \tag{2.1}$$

$$\left|t^\sigma\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \xi_j}\right)Q_\sigma(t, \xi)\right| \leqq C\, Q_\sigma(t, \xi)^{1/2} \tag{2.2}$$

*hold for any $(t, \xi) \in [0, T] \times \boldsymbol{R}^n$ and $1 \leqq j \leqq n$. Hence, $Q_\sigma(t, \xi)$ and $t^\sigma\left(\frac{\partial}{\partial t}\right)Q_\sigma(t, \xi)$ belong to $S^2_{Q_\sigma}([0, T])$.*

PROOF. First, we will show (2.1). Put $e_j = (0, \cdots, \overset{j}{1}, \cdots, 0)$. Then we have

$$\begin{aligned} 0 &\leqq Q_\sigma(t, \xi + s e_j) \\ &= Q_\sigma(t, \xi) + \left(\frac{\partial}{\partial \xi_j}\right)Q_\sigma(t, \xi)\cdot s + \frac{1}{2}\left(\frac{\partial}{\partial \xi_j}\right)^2 Q_\sigma(t, \xi)\cdot s^2 \end{aligned}$$

for any $s \in \boldsymbol{R}$. Since $\left(\frac{\partial}{\partial \xi_j}\right)^2 Q_\sigma(t, \xi) = 2Q_\sigma(t, e_j) \geqq 0$ holds, we have

$$\left[\left(\frac{\partial}{\partial \xi_j}\right)Q_\sigma(t, \xi)\right]^2 - 4Q_\sigma(t, \xi)Q_\sigma(t, e_j) \leqq 0\,. \tag{2.3}$$

Hence, (2.3) immediately leads us to (2.1). Thus (2.1) is proved. Next we will show (2.2). From the condition (H-3), we have

$$\left|t^\sigma\left(\frac{\partial}{\partial t}\right)Q_\sigma(t, \xi)\right| \leqq M\, Q_\sigma(t, \xi) \tag{2.4}$$

for some constant $M$. Put $R(t, \xi) = M\, Q_\sigma(t, \xi) - t^\sigma\left(\frac{\partial}{\partial t}\right)Q_\sigma(t, \xi)$. Clearly $R(t, \xi) \geqq 0$ holds for any $(t, \xi) \in [0, T] \times \boldsymbol{R}^n$. Therefore, by the same argument as in the proof of (2.1) we have

$$\left|\left(\frac{\partial}{\partial \xi_j}\right)R(t, \xi)\right| \leqq N\, R(t, \xi)^{1/2} \tag{2.5}$$

for some constant $N$. From (2.1), (2.4) and (2.5), we have

$$\begin{aligned} \left|t^\sigma\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \xi_j}\right)Q_\sigma(t, \xi)\right|^2 &\leqq 2\left(M^2\left|\left(\frac{\partial}{\partial \xi_j}\right)Q_\sigma(t, \xi)\right|^2 + \left|\left(\frac{\partial}{\partial \xi_j}\right)R(t, \xi)\right|^2\right) \\ &\leqq 2(M^2C^2Q_\sigma(t, \xi) + N^2R(t, \xi)) \\ &\leqq 2\left\{M^2C^2Q_\sigma(t, \xi) + N^2\left(MQ_\sigma(t, \xi) + \left|t^\sigma\left(\frac{\partial}{\partial t}\right)Q_\sigma(t, \xi)\right|\right)\right\} \\ &\leqq 2(M^2C^2 + N^2M + N^2M)Q_\sigma(t, \xi)\,. \end{aligned}$$

This is (2.2). Q. E. D.

PROPOSITION 2.4. *Put* $\lambda_{Q_\sigma}(t, \xi)=(1+Q_\sigma(t, \xi))^{1/2}$. *Then* $\lambda_{Q_\sigma}(t, \xi)$ *and* $t^\sigma\left(\frac{\partial}{\partial t}\right)\lambda_{Q_\sigma}(t, \xi)$ *belong to* $S^1_{Q_\sigma}([0, T])$.

PROOF. Clear from Proposition 1.1 and Proposition 2.3. Q. E. D.

PROPOSITION 2.5. *Let* $\rho(t)\in C^\infty(\boldsymbol{R})$ *such that* $\rho(t)=0$ *for* $t\leqq 1$, $0\leqq\rho(t)\leqq 1$ *for* $1\leqq t\leqq 2$ *and* $\rho(t)=1$ *for* $t\geqq 2$. *Put*

$$\rho_s(t, \xi)=\rho(1-s+Q_\sigma(t, \xi)^{1/2})$$

*for* $s\geqq 1$. *Then* $\rho_s(t, \xi)$ *satisfies the following*: (i) *if* $Q_\sigma(t, \xi)\leqq s^2$, $\rho_s(t, \xi)=0$ *holds*, (ii) *if* $Q_\sigma(t, \xi)\geqq(s+1)^2$, $\rho_s(t, \xi)=1$ *holds, and* (iii) $\rho_s(t, \xi)$ *and* $t^\sigma\left(\frac{\partial}{\partial t}\right)\rho_s(t, \xi)$ *belong to* $S^0_{Q_\sigma}([0, T])$.

PROOF. (i) and (ii) are clear. (iii) is obtained by an easy but rather tedious calculation. We omit the details. Q. E. D.

PROPOSITION 2.6. *Let* $\rho(t)$ *be as above and put*

$$\Theta_\sigma(t, \xi)=\rho(4-Q_\sigma(t, \xi)^{1/2})+Q_\sigma(t, \xi)^{1/2}\cdot\rho(Q_\sigma(t, \xi)^{1/2})\,.$$

*Then* $\Theta_\sigma(t, \xi)$ *satisfies the following*: (i) $\Theta_\sigma(t, \xi)\geqq 1$ *for any* $(t, \xi)\in[0, T]\times\boldsymbol{R}^n$, (ii) *if* $Q_\sigma(t, \xi)\geqq 9$, $\Theta_\sigma(t, \xi)=Q_\sigma(t, \xi)^{1/2}$ *holds*, (iii) $\Theta_\sigma(t, \xi)$ *and* $t^\sigma\left(\frac{\partial}{\partial t}\right)\Theta_\sigma(t, \xi)$ *belong to* $S^1_{Q_\sigma}([0, T])$, *and* (iv) $\Theta_\sigma(t, \xi)^{-1}$ *and* $t^\sigma\left(\frac{\partial}{\partial t}\right)(\Theta_\sigma(t, \xi)^{-1})$ *belong to* $S^{-1}_{Q_\sigma}([0, T])$.

PROOF. (i) and (ii) are clear. (iii) and (iv) are obtained by easy but rather tedious calculations. We omit the details. Q. E. D.

The operators defined by the symbols $\lambda_{Q_\sigma}(t, \xi)$, $\rho_s(t, \xi)$ and $\Theta_\sigma(t, \xi)$ will play important roles in the later sections.

## 3. Symmetrization

In this section, we establish the symmetrizability for the first order system of pseudo-differential operators with a parameter $t$.

Let $Q_\sigma(t, \xi)$ be a basic quadratic form of class $\sigma(\geqq 1)$ and let $H(t, x, \xi)=(h_{ij}(t, x, \xi))_{1\leqq i, j\leqq m}$ be an $m\times m$ matrix with coefficients in $C^1([0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that it satisfies the following:

(I-1) $h_{ij}(t, x, \xi)$ is a real valued function for $1\leqq i, j\leqq m$.

(I-2) $h_{ij}(t, x, \xi)$ and $t^\sigma\left(\frac{\partial}{\partial t}\right)h_{ij}(t, x, \xi)$ belong to $S^0_{Q_\sigma}([0, T])$ for $1\leqq i, j\leqq m$.

(I-3) Let $\lambda_i(t, x, \xi)$ $(1\leqq i\leqq m)$ be eigen-values of $H(t, x, \xi)$. Then $\lambda_i(t, x, \xi)$ is a real valued function on $[0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ for $1\leqq i\leqq m$.

(I-4) There exists a positive constant $c$ such that the estimate

$$|\lambda_i(t, x, \xi)-\lambda_j(t, x, \xi)|\geqq c$$

holds for any $(t, x, \xi)\in\tilde{\Omega}=\{(t, x, \xi)\in[0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n;\ Q_\sigma(t, \xi)\geqq 9\}$ and $1\leqq i\neq j\leqq m$.

Then we have the following theorem.

THEOREM 3.1 (*Symmetrizability*). *For the above $H(t, x, \xi)$, we can find $m\times m$ matrices $N(t, x, \xi)=(n_{ij}(t, x, \xi))_{1\leqq i,j\leqq m}$, $M(t, x, \xi)=(m_{ij}(t, x, \xi))_{1\leqq i,j\leqq m}$ and $D(t, x, \xi)=(d_{ij}(t, x, \xi))_{1\leqq i,j\leqq m}$ such that they satisfy the following*: (i) *$n_{ij}(t, x, \xi)$, $m_{ij}(t, x, \xi)$ and $d_{ij}(t, x, \xi)$ belong to $C^1([0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ for $1\leqq i, j\leqq m$,* (ii) *$n_{ij}(t, x, \xi)$, $m_{ij}(t, x, \xi)$, $d_{ij}(t, x, \xi)$, $t^\sigma\left(\frac{\partial}{\partial t}\right)n_{ij}(t, x, \xi)$, $t^\sigma\left(\frac{\partial}{\partial t}\right)m_{ij}(t, x, \xi)$ and $t^\sigma\left(\frac{\partial}{\partial t}\right)d_{ij}(t, x, \xi)$ belong to $S^0_{Q_\sigma}([0, T])$ for $1\leqq i, j\leqq m$,* (iii) *${}^t\overline{D(t, x, \xi)}=D(t, x, \xi)$,* (iv) *$N(t, x, \xi)H(t, x, \xi)=D(t, x, \xi)N(t, x, \xi)$, and* (v) *$N(t, x, \xi)M(t, x, \xi)=M(t, x, \xi)N(t, x, \xi)=\rho_4(t, \xi)^2$, where $\rho_4(t, \xi)$ is defined in Proposition* 2.5.

Before the proof of this theorem, we prepare some lemmas.

LEMMA 3.2. *$\lambda_i(t, x, \xi)$ in* (I-3) *satisfies the following conditions on $\tilde{\Omega}$*: (i) *$\lambda_i(t, x, \xi)$ is of $C^1$ class with respect to $t$ and of $C^\infty$ class with respect to $(x, \xi)$, and* (ii) *for any $\alpha$ and $\beta$ there exists a positive constant $C_{\alpha,\beta}$ such that the estimates*

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial \xi}\right)^\beta\lambda_i(t, x, \xi)\right|\leqq C_{\alpha,\beta}\lambda_{Q_\sigma}(t, \xi)^{-|\beta|},$$

$$\left|t^\sigma\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial \xi}\right)^\beta\lambda_i(t, x, \xi)\right|\leqq C_{\alpha,\beta}\lambda_{Q_\sigma}(t, \xi)^{-|\beta|}$$

*hold for any $(t, x, \xi)\in\tilde{\Omega}$. Here $\tilde{\Omega}$ is the same as in* (I-4).

PROOF. Put $P(\lambda: t, x, \xi)=\det(\lambda I-H(t, x, \xi))$. Then $\lambda_i(t, x, \xi)$ is a root of the equation $P(\lambda: t, x, \xi)=0$ with respect to $\lambda$. Since the coefficients of $P(\lambda: t, x, \xi)$ are bounded continuous functions on $[0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n$, the root $\lambda_i(t, x, \xi)$ is also a bounded continuous function on $[0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n$. Further, (i) is easily obtained from the condition (I-4). Next we will show (ii). By use of the theorem of implicit functions, we have

$$t^\sigma\left(\frac{\partial}{\partial t}\right)\lambda_i=-\left(t^\sigma\left(\frac{\partial}{\partial t}\right)P\Big/\left(\frac{\partial}{\partial\lambda}\right)P\right)_{\lambda=\lambda_i},$$

$$\left(\frac{\partial}{\partial x_k}\right)\lambda_i=-\left(\left(\frac{\partial}{\partial x_k}\right)P\Big/\left(\frac{\partial}{\partial\lambda}\right)P\right)_{\lambda=\lambda_i},$$

$$\left(\frac{\partial}{\partial\xi_k}\right)\lambda_i=-\left(\left(\frac{\partial}{\partial\xi_k}\right)P/\left(\frac{\partial}{\partial\lambda}\right)P\right)_{\lambda=\lambda_i}.$$

Since $\left|\left(\left(\frac{\partial}{\partial\lambda}\right)P\right)_{\lambda=\lambda_i}\right|\geqq c^{m-1}$ holds on $\tilde{\Omega}$, we have

$$\left|t^\sigma\left(\frac{\partial}{\partial t}\right)\lambda_i(t, x, \xi)\right|\leqq C,\ \left|\left(\frac{\partial}{\partial x_k}\right)\lambda_i(t, x, \xi)\right|\leqq C$$

and

$$\left|\left(\frac{\partial}{\partial\xi_k}\right)\lambda_i(t, x, \xi)\right|\leqq C\lambda_{Q_\sigma}(t, \xi)^{-1}$$

on $\tilde{\Omega}$ for some constant $C$. Considering successive derivative with respect to $(x, \xi)$, we can easily see the condition (ii). Q. E. D.

LEMMA 3.3. *There exists an* $m\times m$ *matrix* $\tilde{N}(t, x, \xi)=(\tilde{n}_{ij}(t, x, \xi))_{1\leq i, j\leq m}$ *defined on* $\tilde{\Omega}$ *such that it satisfies the following*: (i) $\tilde{N}(t, x, \xi)H(t, x, \xi)=\tilde{D}(t, x, \xi)\tilde{N}(t, x, \xi)$ *holds on* $\tilde{\Omega}$, *where*

$$\tilde{D}(t, x, \xi)=\begin{pmatrix}\lambda_1(t, x, \xi) & & \\ & \ddots & \\ & & \lambda_m(t, x, \xi)\end{pmatrix},$$

(ii) $\tilde{n}_{ij}(t, x, \xi)$ *satisfies the same conditions as in Lemma* 3.2, *and* (iii) *there exists a positive constant* $\delta$ *such that* $|\det N(t, x, \xi)|\geqq\delta$ *holds for any* $(t, x, \xi)\in\tilde{\Omega}$.

PROOF. Note that $\tilde{\Omega}$ has the same homotopy structure as $[0, T]\times\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0)$. Therefore, the argument in the proof of Proposition 6.4 of Mizohata [4] is also valid in this case. Hence the proof is easy. Q. E. D.

PROOF OF THEOREM 3.1. By the above lemmas, we can define $N(t, x, \xi)$, $M(t, x, \xi)$ and $D(t, x, \xi)$ by

$$N(t, x, \xi)=\rho_4(t, \xi)\tilde{N}(t, x, \xi),$$

$$M(t, x, \xi)=\rho_4(t, \xi)(\tilde{N}(t, x, \xi)^{-1}),$$

$$D(t, x, \xi)=\rho_3(t, \xi)\tilde{D}(t, x, \xi).$$

Then, it is clear that they satisfy the conditions in Theorem 3.1. Q. E. D.

COROLLARY 3.4. *Let* $B(t)$ *be a pseudo-differential operator defined by the symbol* $B(t, x, \xi)=\sqrt{-1}\,H(t, x, \xi)\Theta_\sigma(t, \xi)$. *Then* $B(t)$ *satisfies the symmetrizability condition* (D-3) *in Tahara* [6].

PROOF. Let $N(t)$, $M(t)$ and $D(t)$ be pseudo-differential operators defined by $N(t, x, \xi)$, $M(t, x, \xi)$ and $\sqrt{-1}\,D(t, x, \xi)\Theta_\sigma(t, \xi)$. Then we have $N(t)$, $t^\sigma N_t'(t)$, $M(t)$, $t^\sigma M_t'(t)$, $D(t)^*+D(t)$, $N(t)B(t)-D(t)N(t)\in S^0_{Q_\sigma}([0, T])$. Therefore we can put $S(t)=D(t)^*+D(t)$ and $T(t)=N(t)B(t)-D(t)N(t)$. Let $\Delta(t)$ be a pseudo-differential operator defined by $\lambda_{Q_\sigma}(t, \xi)^{-1}$. Then we have $\Delta(t)\in S^{-1}_{Q_\sigma}([0, T])$. Since $B(t)\in S^1_{Q_\sigma}([0, T])$, we have $\Delta(t)B(t)\in S^0_{Q_\sigma}([0, T])$. Therefore we can conclude that $N(t)$, $D(t)$, $S(t)$, $T(t)$ and $\Delta(t)$ satisfy the conditions (i), (ii) and (iii) in (D-3). The condition (iv) in (D-3) is verified as follows. Since $I-M(t)N(t)=(I-\rho_4(t)^2)+(\rho_4(t)^2-M(t)N(t))\in S^{-1}_{Q_\sigma}([0, T])$ holds, we have

$$\begin{aligned}\|u\| &\leqq \|M(t)N(t)u\|+\|(I-M(t)N(t))u\| \\ &\leqq \|M(t)\|\cdot\|N(t)u\|+\|(I-M(t)N(t))\Delta(t)^{-1}\|\cdot\|\Delta(t)u\| \\ &\leqq (\|M(t)\|\cdot\|N(t)\|+\|(I-M(t)N(t))\Delta(t)^{-1}\|\cdot\|\Delta(t)\|)\|u\|\,.\end{aligned}$$

Therefore we can find positive constants $\beta$, $c_1$ and $c_2$ such that $c_1\|u\|\leqq\|N(t)u\|+\beta\|\Delta(t)u\|\leqq c_2\|u\|$ holds for any $u\in L^2(\boldsymbol{R}^n)$ and $0\leqq t\leqq T$. This is (iv).

Q. E. D.

REMARK 3.5. Since the conditions (I-1)$\sim$(I-4) are valid for the matrix $-{}^t\overline{H(t, x, \xi)}$, we can also obtain the symmetrizability of $B(t)^*$ in the sense of (D-4) in Tahara [6].

## Part II. APPLICATIONS

In the next three sections, we will apply the results in Part I to concrete singular hyperbolic partial differential equations and establish the existence and uniqueness theorem.

### 4. Symmetric singular hyperbolic systems

Let us consider a symmetric singular hyperbolic system of type $(\sigma, \rho)$ of the form

$$\text{(J)}\qquad t^\sigma\frac{\partial u}{\partial t}+A(t, x)u-t^\rho\sum_{j=1}^n B_j(t, x)\frac{\partial u}{\partial x_j}=f(t, x)$$

on $\Omega=[0, T]\times\boldsymbol{R}^n$, where $A(t, x)=(a_{ik}(t, x))_{1\leq i, k\leq m}$ and $B_j(t, x)=(b_{ik}^{(j)}(t, x))_{1\leq i, k\leq m}$ $(1\leqq j\leqq n)$ are $m\times m$ matrices on $\Omega$. We assume the following conditions on (J):

(J-1) $\sigma\geqq 1$ and $\rho-\sigma+1>0$.

(J-2) (Coefficients). $a_{ik}(t, x)$ and $b_{ik}^{(j)}(t, x)$ are of $C^0$ class with respect to $t$ and of $C^\infty$ class with respect to $x$. Further, $\left(\frac{\partial}{\partial x}\right)^\alpha a_{ik}(t, x)$ and $\left(\frac{\partial}{\partial x}\right)^\alpha b_{ik}^{(j)}(t, x)$

belong to $\mathscr{B}^0(\Omega)$ for any $\alpha$, $i$, $j$ and $k$, where $\mathscr{B}^0(\Omega)$ is the space of all bounded continuous functions on $\Omega$.

(J-3) (Positivity). Let $\mu_i(t, x)$ $(1\leqq i\leqq m)$ be eigen-values of the matrix $(A(t, x)+{}^t\overline{A(t, x)})$. Then there exists a positive constant $a$ such that $\mu_i(t, x)\geqq a$ holds for any $(t, x)\in\Omega$ and $1\leqq i\leqq m$.

(J-4) (Symmetric hyperbolicity). $B_j(t, x)$ $(1\leqq j\leqq n)$ are Hermitian matrices, that is, $\overline{b_{ik}^{(j)}(t, x)}=b_{ki}^{(j)}(t, x)$ holds for any $i$, $j$ and $k$.

Under these assumptions, we have the following theorem.

THEOREM 4.1. *Let $l$ be an integer$\geqq 1$. Then, for an arbitrary $f(t)\in C^0([0, T], H^l(\boldsymbol{R}^n))$, there exists a unique solution $u(t)\in C^0([0, T], H^l(\boldsymbol{R}^n))\cap C^1((0, T], H^{l-1}(\boldsymbol{R}^n))$ of the equation* (J) *such that it satisfies the following conditions*: (i) $t^\sigma\left(\dfrac{\partial}{\partial t}\right)u(t)\in C^0([0, T], H^{l-1}(\boldsymbol{R}^n))$ *holds, and* (ii) *the energy inequality*

$$\|u(t)\|_k\leqq C_k\int_0^\infty e^{-a_k s}\|f(\phi_\sigma(t, s))\|_k ds$$

*holds for $0\leqq t\leqq T$ and $k=0, 1, \cdots, l$, where $C_k$ and $a_k$ are positive constants and $\phi_\sigma(t, s)$ is defined by*

$$\phi_\sigma(t, s)=\begin{cases} te^{-s}, & \text{when } \sigma=1\,, \\ t\left(\dfrac{1}{(\sigma-1)st^{\sigma-1}+1}\right)^{1/(\sigma-1)}, & \text{when } \sigma>1\,. \end{cases}$$

*The norm $\|\cdot\|_k$ means the Sobolev norm in the usual sense.*

PROOF. Let $A(t)$ be a bounded operator in $L^2(\boldsymbol{R}^n)$ defined by $A(t)u=A(t, x)u$ and let $B(t)$ be a closed operator in $L^2(\boldsymbol{R}^n)$ defined by $B(t)u=\sum_{j=1}^n B_j(t, x)\dfrac{\partial u}{\partial x_j}$. Then (J) is rewritten into the form

$$t^\sigma\frac{du}{dt}+A(t)u-t^\rho B(t)u=f(t)\,, \qquad 0<t\leqq T\,.$$

Therefore, from Theorem 3.1 and Theorem 3.8 in Tahara [6] we can easily obtain this theorem. Q. E. D.

To have the differentiability in $t$, we assume the following condition (J-5) in addition to (J-1)$\sim$(J-4).

(J-5) (Coefficients (2)). $a_{ik}(t, x)$ and $b_{ik}^{(j)}(t, x)$ are of $C^{l-1}$ class with respect to $t$ (for $0<t\leqq T$) and of $C^\infty$ class with respect to $x$. Further,

$t^{\sigma h}\left(\frac{\partial}{\partial t}\right)^h\left(\frac{\partial}{\partial x}\right)^\alpha a_{ik}(t, x)$ and $t^{\sigma h}\left(\frac{\partial}{\partial t}\right)^h\left(\frac{\partial}{\partial x}\right)^\alpha b_{ik}^{(j)}(t, x)$ belong to $\mathscr{B}^0(\Omega)$ for $0\leqq h\leqq l-1$ and any $\alpha$.

Then we have

COROLLARY 4.2. *If $t^{\sigma k}f(t)\in C^k([0, T], H^{l-1-k}(\boldsymbol{R}^n))$ holds for $k=0, 1, \cdots, l-1$, then the solution $u(t)$ in Theorem 4.1 satisfies $t^{\sigma k}u(t)\in C^k([0, T], H^{l-k}(\boldsymbol{R}^n))$ for $k=0, 1, \cdots, l$.*

PROOF. Clear from Theorem 3.5 in [6]. Q. E. D.

Thus, we have established the existence, uniqueness and differentiability theorem.

REMARK 4.3. Note that the equation (J) is nothing but a symmetric positive system of differential equations in the sense of Friedrichs [1]. Therefore, we can treat this equation from the standpoint of Friedrichs. We will sketch this treatment in brief. Note that the argument below is local.

(1) Let $\Omega$ be a bounded open subset of $(0, T)\times\boldsymbol{R}^n$ and let

$$P\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)=t^\sigma\frac{\partial}{\partial t}+A(t, x)-t^\sigma\sum_{j=1}^n B_j(t, x)\frac{\partial}{\partial x_j}$$

be a system of differential operators such that it satisfies the following:

(i) $\sigma\geqq 1$ and $\rho\geqq 0$.

(ii) $A(t, x)$ and $B_j(t, x)$ $(1\leqq j\leqq n)$ are $m\times m$ matrices with coefficients in $C^1(\bar{\Omega})$.

(iii) $B_j(t, x)$ $(1\leqq j\leqq n)$ are Hermitian matrices.

(iv) Let $\mu_i(t, x)$ $(1\leqq i\leqq m)$ be eigen-values of the matrix $J(t, x)$:

$$J(t, x)=-\sigma t^{\sigma-1}+t^\rho\sum_{j=1}^n\frac{\partial B_j}{\partial x_j}(t, x)+A(t, x)+{}^t\overline{A(t, x)}.$$

Then there exists a positive constant $a$ such that $\mu_i(t, x)\geqq a$ holds for any $(t, x)\in\Omega$ and $1\leqq i\leqq m$.

(v) $\Omega$ is a lens domain with $\partial\bar{\Omega}=S_0\cup S_+$, where $S_0=\partial\bar{\Omega}\cap\{t=0\}$ and $S_+=\partial\bar{\Omega}\backslash S_0$.

Under these assumptions, we can follow the argument in Friedrichs [1].

(2) (A priori estimate). There exists a positive constant $\gamma$ such that the inequality

$$\|u\|_{L^2(\bar{\Omega})}^2+\int_{S_+}Hu\cdot\bar{u}\,d\sigma\leqq\gamma^2\|Pu\|_{L^2(\bar{\Omega})}^2$$

holds for any $u\in C^1(\bar{\Omega})$, where $d\sigma$ means Lebesgue's measure on $S_+$ and $H$ is a matrix function on $S_+$ defined by

$$H(t, x)=t^{\sigma} n_0 I - t^{\rho} \sum_{j=1}^{n} n_j B_j(t, x), \ (t, x) \in S_+ .$$

Here $(n_0, n_1, \cdots, n_n)$ is the outer normal unit vector of $S_+$.

(3) (Existence). Using the same a priori estimate for $P^*$, we can easily obtain the existence of a weak solution in a suitable sense.

(4) (Uniqueness). Assume that $H(t, x)\xi\cdot\bar{\xi}\geqq 0$ holds for any $(t, x)\in S_+$ and $\xi\in \boldsymbol{C}^n$. Then, if $u\in C^1(\bar{\Omega})$ and $Pu=0$, we have $u=0$ on $\Omega$. Further, combining this result with the argument of John [2] we can discuss the dependence domain.

## 5. Singular hyperbolic systems

Let us consider a singular hyperbolic system of type $(\sigma, \rho)$ of the form

$$(\mathrm{K})_s \qquad t^{\sigma}\frac{\partial u}{\partial t}+(s+A(t, x))u-t^{\rho}\sum_{j=1}^{n} B_j(t, x)\frac{\partial u}{\partial x_j}=f(t, x)$$

on $\Omega=[0, T]\times \boldsymbol{R}^n$, where $s$ is a real parameter and $A(t, x)=(a_{ik}(t, x))_{1\leq i, k\leq m}$, $B_j(t, x)=(b_{ik}^{(j)}(t, x))_{1\leq i, k\leq m}(1\leqq j\leqq n)$ are $m\times m$ matrices on $\Omega$. We assume the following conditions on $(\mathrm{K})_s$:

(K-1) $\sigma\geqq 1$ and $\rho-\sigma+1>0$.

(K-2) (Coefficients of lower order part). $a_{ik}(t, x)$ is of $C^0$ class with respect to $t$ and of $C^\infty$ class with respect to $x$. Further, $\left(\frac{\partial}{\partial x}\right)^{\alpha} a_{ik}(t, x)$ belongs to $\mathscr{B}^0(\Omega)$ for any $\alpha$, $i$ and $k$.

(K-3) (Coefficients of principal part). $b_{ik}^{(j)}(t, x)$ is of $C^1$ class with respect to $t$ and of $C^\infty$ class with respect to $x$. Further, $\left(\frac{\partial}{\partial x}\right)^{\alpha} b_{ik}^{(j)}(t, x)$ and $t^{\sigma}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}\right)^{\alpha} b_{ik}^{(j)}(t, x)$ belong to $\mathscr{B}^0(\Omega)$ for any $\alpha$, $i$, $j$ and $k$.

(K-4) (Hyperbolicity). The following two conditions are satisfied.

(i) $b_{ik}^{(j)}(t, x)$ is a real valued function on $\Omega$ for any $i$, $j$ and $k$.

(ii) Let $\lambda_i(t, x, \xi)$ $(1\leqq i\leqq m)$ be eigen-values of the matrix $\sum_{j=1}^{n} B_j(t, x)\xi_j$. Then $\lambda_i(t, x, \xi)$ is a real valued function on $\Omega\times(\boldsymbol{R}^n\backslash 0)$ for $1\leqq i\leqq m$.

(K-5) (Distinctness). There exist a positive constant $c$ and a basic quadratic form $Q_{\sigma}(t, \xi)$ of class $\sigma$ such that the estimate

$$|\lambda_i(t, x, \xi)-\lambda_j(t, x, \xi)|\geqq c\, Q_{\sigma}(t, \xi)^{1/2}$$

holds for any $(t, x, \xi)\in\Omega\times(\boldsymbol{R}^n\backslash 0)$ and $1\leqq i\neq j\leqq m$.

(K-6) (Estimates). For any $\alpha$ there exists a positive constant $C_{\alpha}$ such that the estimates

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\sum_{j=1}^{n}B_j(t,\ x)\xi_j\right|\leqq C_\alpha Q_\sigma(t,\ \xi)^{1/2},$$

$$\left|t^{\sigma}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}\right)^{\alpha}\sum_{j=1}^{n}B_j(t,\ x)\xi_j\right|\leqq C_\alpha Q_\sigma(t,\ \xi)^{1/2}$$

hold for any $(t,\ x,\ \xi)\in\Omega\times(\boldsymbol{R}^n\backslash 0)$.

Under these assumptions (K-1)$\sim$(K-6), we have the following theorem.

THEOREM 5.1. *There exists a positive constant $s_0$ which satisfies the following. Let $l$ be an integer $\geqq 1$ and let $s$ be a real number $> s_0$. Then, for an arbitrary $f(t)\in C^0([0,\ T],\ H^l(\boldsymbol{R}^n))$, there exists a unique solution $u(t)\in C^0([0,\ T],\ H^l(\boldsymbol{R}^n))\cap C^1((0,\ T],\ H^{l-1}(\boldsymbol{R}^n))$ of the equation $(K)_s$ such that it satisfies the following conditions:*

(i) $t^{\sigma}\left(\frac{\partial}{\partial t}\right)u(t)\in C^0([0,\ T],\ H^{l-1}(\boldsymbol{R}^n))$ *holds, and* (ii) *the energy inequality*

$$\|u(t)\|_k\leqq C_k\int_0^{\infty}e^{-a_k s}\|f(\phi_\sigma(t,\ s))\|_k ds$$

*holds for $0\leqq t\leqq T$ and $k=0,\ 1,\ \cdots,\ l$, where $C_k$ and $a_k$ are positive constants.*

PROOF. To prove this, we have only to apply Theorem 4.4 in Tahara [6]. Let $A(t)$ be a bounded operator in $L^2(\boldsymbol{R}^n)$ defined by $A(t)u=A(t,\ x)u$ and let $B(t)$ be a closed operator in $L^2(\boldsymbol{R}^n)$ defined by $B(t)u=\sum_{j=1}^{n}B_j(t,\ x)\frac{\partial u}{\partial x_j}$. Then, $(\mathrm{K})_s$ is rewritten into the form

$$t^{\sigma}\frac{du}{dt}+(s+A(t))u-t^{\rho}B(t)u=f(t)\,,\ 0<t\leqq T\,.$$

From the conditions (K-1)$\sim$(K-6), we have the following:

(1) Put $B(t,x,\xi)=\sqrt{-1}\sum_{j=1}^{n}B_j(t,x)\xi_j$. Then $B(t,x,\xi)$ and $t^{\sigma}\left(\frac{\partial}{\partial t}\right)B(t,x,\xi)$ belong to $S^1_{Q_\sigma}([0,\ T])$.

(2) Put $H(t,\ x,\ \xi)=-\sqrt{-1}\,B(t,\ x,\ \xi)\Theta_\sigma(t,\ \xi)^{-1}$. Then $H(t,\ x,\ \xi)$ satisfies the conditions (I-1)$\sim$(I-4) in Section 3.

Therefore, from Corollary 3.4 and Remark 3.5 we have the symmetrizabilities of $B(t)$ and $B(t)^*$. Thus, we can apply Theorem 4.4 in [6]. Q.E.D.

COROLLARY 5.2. *Assume the same condition as* (J-5). *If $t^{\sigma k}f(t)\in C^k([0,\ T],\ H^{l-1-k}(\boldsymbol{R}^n))$ holds for $k=0,\ 1,\ \cdots,\ l-1$, then the solution $u(t)$ in Theorem* 5.1. *satisfies $t^{\sigma k}u(t)\in C^k([0,\ T],\ H^{l-k}(\boldsymbol{R}^n))$ for $k=0,\ 1,\ \cdots,\ l$.*

PROOF. Clear from Theorem 4.3 in [6]. Q.E.D.

REMARK 5.3. The advantage of the above formulation will be illustrated as follows. Assume that the equation

$$t^{\sigma}\frac{\partial u}{\partial t}+(s+A(t,\ x))u-t^{\rho}\sum_{j=1}^{n}B_{j}(t,\ x)\frac{\partial u}{\partial x_{j}}=f(t,\ x)$$

satisfies our conditions with a basic quadratic form $Q_{\sigma}(t, \xi)$ of class $\sigma$. Then, for any integers $\kappa_{i}\geqq 0$ $(1\leqq j\leqq n)$ the following equation

$$t^{\sigma}\frac{\partial u}{\partial t}+(s+A(t,\ x))u-t^{\rho}\sum_{j=1}^{n}t^{\kappa_{j}}B_{j}(t,\ x)\frac{\partial u}{\partial x_{j}}=f(t,\ x)$$

also satisfies our conditions with a basic quadratic form

$$R(t,\ \xi)=Q_{\sigma}(t,\ t^{\kappa_{1}}\xi_{1},\ \cdots,\ t^{\kappa_{n}}\xi_{n}).$$

When $\sigma>1$, for any integers $\kappa_{j}\geqq 0$ $(1\leqq j\leqq l)$ and for any real numbers $\sigma_{j}$ such that $1<\sigma_{j}\leqq\sigma$ $(l+1\leqq j\leqq n)$, the following equation

$$t^{\sigma}\frac{\partial u}{\partial t}+(s+A(t,\ x))u-t^{\rho}\Big\{\sum_{j=1}^{l}t^{\kappa_{j}}B_{j}(t,\ x)\frac{\partial u}{\partial x_{j}}$$

$$+\sum_{j=l+1}^{n}e_{\sigma_{j}}(t)B_{j}(t,\ x)\frac{\partial u}{\partial x_{j}}\Big\}=f(t,\ x)$$

$$(\text{where } e_{\sigma}(t)=e^{-1/t^{\sigma-1}})$$

also satisfies our conditions with a basic quadratic form

$$R(t,\ \xi)=Q_{\sigma}(t,\ t^{\kappa_{1}}\xi_{1},\ \cdots,\ t^{\kappa_{l}}\xi_{1},\ e_{\sigma_{l+1}}(t)\xi_{l+1},\ \cdots,\ e_{\sigma_{n}}(t)\xi_{n}).$$

See also Proposition 1.3 and Proposition 2.1.

## 6. Singular hyperbolic equations

Let us consider a singular hyperbolic equation of type $(\sigma, \rho)$ of the form

$$(\mathrm{L})_{s}\qquad \Big(t^{\sigma}\frac{\partial}{\partial t}+s\Big)^{m}u+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}a_{j,\alpha}(t,\ x)\Big(t^{\rho}\frac{\partial}{\partial x}\Big)^{\alpha}\Big(t^{\sigma}\frac{\partial}{\partial t}+s\Big)^{j}u=f(t,\ x)$$

on $\Omega=[0,\ T]\times\boldsymbol{R}^{n}$, where $s$ is a real parameter and

$$\Big(t^{\rho}\frac{\partial}{\partial x}\Big)^{\alpha}=t^{\rho|\alpha|}\Big(\frac{\partial}{\partial x}\Big)^{\alpha}=\Big(t^{\rho}\frac{\partial}{\partial x_{1}}\Big)^{\alpha_{1}}\cdots\Big(t^{\rho}\frac{\partial}{\partial x_{n}}\Big)^{\alpha_{n}}.$$

We assume the following conditions on $(\mathrm{L})_{s}$:

(L-1) $\sigma\geqq 1$ and $\rho-\sigma+1>0$.

(L-2) (Coefficients of principal part). $a_{j,\alpha}(t,\ x)$ $(j+|\alpha|=m)$ is of $C^{1}$ class with respect to $t$ and of $C^{\infty}$ class with respect to $x$. Further, $\Big(\frac{\partial}{\partial x}\Big)^{\beta}a_{j,\alpha}(t,\ x)$ and $t^{\sigma}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{\partial}{\partial x}\Big)^{\beta}a_{j,\alpha}(t,\ x)$ belong to $\mathscr{B}^{0}(\Omega)$ for any $\beta$.

(L-3) (Coefficients of lower order part). $a_{j,\alpha}(t, x)$ $(j+|\alpha|<m)$ is of $C^0$ class with respect to $t$ and of $C^\infty$ class with respect to $x$. Further, $\left(\frac{\partial}{\partial x}\right)^\beta a_{j,\alpha}(t, x)$ belongs to $\mathcal{B}^0(\Omega)$ for any $\beta$.

(L-4) (Hyperbolicity). Let $\lambda_i(t, x, \xi)$ $(1\leqq i\leqq m)$ be roots of the equation

$$\lambda^m+\sum_{j=0}^{m-1}\Big(\sum_{|\alpha|=m-j} a_{j,\alpha}(t, x)\xi^\alpha\Big)\lambda^j=0.$$

Then $\lambda_i(t, x, \xi)$ is a real valued function on $\Omega\times(\boldsymbol{R}^n\backslash 0)$ for $1\leqq i\leqq m$.

(L-5) (Distinctness). There exist a positive constant $c$ and a basic quadratic form $Q_\sigma(t, \xi)$ of class $\sigma$ such that the estimate

$$|\lambda_i(t, x, \xi)-\lambda_j(t, x, \xi)|\geqq c\, Q_\sigma(t, \xi)^{1/2}$$

holds for any $(t, x, \xi)\in\Omega\times(\boldsymbol{R}^n\backslash 0)$ and $1\leqq i\neq j\leqq m$.

(L-6) (Estimates of principal part). For any $\beta$ and $\gamma$ $(0\leqq|\gamma|\leqq m-j)$, there exists a positive constant $C_{\beta,\gamma}$ such that the estimates

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial \xi}\right)^\gamma\sum_{|\alpha|=m-j} a_{j,\alpha}(t, x)\xi^\alpha\right|\leqq C_{\beta,\gamma}Q_\sigma(t, \xi)^{(m-j-|\gamma|)/2},$$

$$\left|t^\sigma\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial \xi}\right)^\gamma\sum_{|\alpha|=m-j} a_{j,\alpha}(t, x)\xi^\alpha\right|\leqq C_{\beta,\gamma}Q_\sigma(t, \xi)^{(m-j-|\gamma|)/2}$$

hold for any $(t, x, \xi)\in\Omega\times(\boldsymbol{R}^n\backslash 0)$ and $0\leqq j\leqq m-1$.

(L-7) (Estimates of lower order part). For any $\beta$ and $\gamma$ $(0\leqq|\gamma|<m-j)$, there exists a positive constant $C_{\beta,\gamma}$ such that the estimate

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial \xi}\right)^\gamma\sum_{|\alpha|<m-j} a_{j,\alpha}(t, x)(\sqrt{-1}\,\xi)^\alpha\right|\leqq C_{\beta,\gamma}(1+Q_\sigma(t, \xi))^{(m-j-1-|\gamma|)/2}$$

holds for any $(t, x, \xi)\in\Omega\times(\boldsymbol{R}^n\backslash 0)$ and $0\leqq j\leqq m-1$.

Under these assumptions (L-1)~(L-7), we have the following theorem.

THEOREM 6.1. *There exists a positive constant $s_0$ which satisfies the following. Let $l$ be an integer $\geqq 1$ and let $s$ be a real number $>s_0$. Then, for an arbitrary $f(t)\in C^0([0, T], H^l(\boldsymbol{R}^n))$, there exists a unique solution $u(t)$ of the equation $(L)_s$ such that it satisfies the following conditions:*

(i) $$\tilde{u}(t)\equiv\left(\wedge_{Q_\sigma,\rho}(t)^{m-1}u(t),\ \wedge_{Q_\sigma,\rho}(t)^{m-2}\left(t^\sigma\frac{\partial}{\partial t}\right)u(t),\ \cdots,\ \left(t^\sigma\frac{\partial}{\partial t}\right)^{m-1}u(t)\right)$$
$$\in C^0([0, T], H^l(\boldsymbol{R}^n))\cap C^1((0, T], H^{l-1}(\boldsymbol{R}^n))$$

*holds,* (ii) $\left(t^\sigma\frac{\partial}{\partial t}\right)\tilde{u}(t)\in C^0([0, T], H^{l-1}(\boldsymbol{R}^n))$ *holds, and* (iii) *the energy inequality*

$$\|u(t)\|_k'\leqq C_k\int_0^\infty e^{-a_k s}\|f(\phi_\sigma(t, s))\|_k ds$$

*holds for* $0 \leqq t \leqq T$ *and* $k=0, 1, \cdots l$, *where* $C_k$ *and* $a_k$ *are positive constants and*

$$\|u(t)\|'_k = \sum_{j=1}^{m} \|\wedge_{Q_\sigma,\rho}(t)^{m-j}\Big(t^\sigma \frac{\partial}{\partial t}\Big)^{j-1} u(t)\|_k .$$

*Here, we denote by* $\wedge_{Q_\sigma,\rho}(t)$ *a pseudo-differential operator defined by* $(1+t^{2\rho}Q_\sigma(t, \xi))^{1/2}$.

PROOF. To prove this, first we transform the equation $(L)_s$ into a singular hyperbolic system of the first order and next we apply the results of singular hyperbolic systems. To do so, we introduce unknown functions

$$u_j(t) = (\sqrt{-1})^{m-j}(1+t^\rho \Theta_\sigma(t))^{m-j}\Big(t^\sigma \frac{\partial}{\partial t}+s\Big)^{j-1} u(t),\ j=1, \cdots, m,$$

where $\Theta_\sigma(t)$ is a pseudo-differential operator defined by $\Theta_\sigma(t, \xi)$ in Proposition 2.6. Since the relation

$$\Big(t^\sigma \frac{\partial}{\partial t}+s\Big)u_j(t) = (m-j)\Big\{\Big(t^\sigma \frac{\partial}{\partial t}\Big)(t^\rho \Theta_\sigma(t))\Big\}(1+t^\rho \Theta_\sigma(t))^{-1}u_j(t)$$
$$+\sqrt{-1}\,(1+t^\rho \Theta_\sigma(t))u_{j+1}(t)$$

holds for $1 \leqq j \leqq m-1$, we can rewrite the equation $(L)_s$ into the form

$$\Big(t^\sigma \frac{\partial}{\partial t}+s\Big)u_m = -\sqrt{-1}\sum_{j=0}^{m-1} k_j(t)(1+t^\rho \Theta_\sigma(t))u_{j+1}+f(t),$$

where $f(t)=f(t, x)$ and $k_j(t)$ is a pseudo-differential operator defined by

$$\sigma(k_j(t)) = (\sqrt{-1})^{-m+j}\sum_{|\alpha|\leqq m-j} a_{j,\alpha}(t, x)(\sqrt{-1}\,t^\rho \xi)^\alpha (1+t^\rho \Theta_\sigma(t, \xi))^{-m+j}.$$

Therefore, $(L)_s$ is equivalent to the following first order system

$$\Big(t^\sigma \frac{\partial}{\partial t}+s\Big)\vec{u} = M(t)\vec{u}+\sqrt{-1}\,K(t)(1+t^\rho \Theta_\sigma(t))\vec{u}+\vec{f}(t), \tag{6.1}$$

where

$$M(t) = \begin{pmatrix} m-1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}\Big\{\Big(t^\sigma \frac{\partial}{\partial t}\Big)(t^\rho \Theta_\sigma(t))\Big\}(1+t^\rho \Theta_\sigma(t))^{-1},$$

$$K(t) = \begin{pmatrix} 0, & 1, & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -k_0(t), & -k_1(t), & \cdots, & & -k_{m-1}(t) \end{pmatrix},\ \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix},\ \vec{f} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

Here, we divide the term $t^{\sigma}k_j(t)\Theta_{\sigma}(t)$ into the following three parts:

$$t^{\sigma}k_j(t)\Theta_{\sigma}(t)=t^{\sigma}h_j(t)\Theta_{\sigma}(t)+s_j(t)+r_j(t),$$

where

$$\sigma(h_j(t))=\sum_{|\alpha|=m-j}a_{j,\alpha}(t, x)\xi^{\alpha}\Theta_{\sigma}(t, \xi)^{-m+j},$$

$$\sigma(s_j(t))=-\Big(\sum_{|\alpha|=m-j}a_{j,\alpha}(t, x)\xi^{\alpha}\Theta_{\sigma}(t, \xi)^{-m+j}\Big)\Big(\sum_{l=1}^{m-j}\Big(\frac{t^{\sigma}\Theta_{\sigma}(t, \xi)}{1+t^{\rho}\Theta_{\sigma}(t, \xi)}\Big)^{l}\Big),$$

$$\sigma(r_j(t))=(\sqrt{-1})^{-m+j}\sum_{|\alpha|<m-j}a_{j,\alpha}(t, x)(\sqrt{-1}\,t^{\rho}\xi)^{\alpha}(t^{\rho}\Theta_{\sigma}(t, \xi))(1+t^{\rho}\Theta_{\sigma}(t, \xi))^{-m+j}.$$

Then, we can rewrite the equation (6.1) into the form

$$\Big(t^{\sigma}\frac{\partial}{\partial t}+s\Big)\vec{u}=(M(t)+\sqrt{-1}\,K(t)+\sqrt{-1}\,R(t)+\sqrt{-1}\,S(t))\vec{u} \qquad$$
$$+\sqrt{-1}\,t^{\rho}H(t)\Theta_{\sigma}(t)\vec{u}+\vec{f}(t), \tag{6.2}$$

where

$$R(t)=-\begin{pmatrix} & 0 & \\ & & \\ r_0(t), & r_1(t), \cdots, & r_{m-1}(t)\end{pmatrix}, \quad S(t)=-\begin{pmatrix} & 0 & \\ & & \\ s_0(t), & s_1(t), \cdots, & s_{m-1}(t)\end{pmatrix}.$$

$$H(t)=\begin{pmatrix} 0, & 1, & & \\ & & & 1 \\ -h_0(t), & -h_1(t), & \cdots, & -h_{m-1}(t)\end{pmatrix}.$$

From the conditions (L-1)$\sim$(L-7), we have the following:

(1) $M(t)$, $K(t)$, $R(t)$ and $S(t)$ belong to $S_R^0([0, T])$ with $R(t, \xi)=Q_{\sigma}(t, t^{\rho}\xi)$.

(2) $H(t)$ satisfies the conditions (I-1)$\sim$(I-4) in Section 3.

Therefore, we can apply Theorem 4.4 in [6] to the equation (6.2). Thus, the existence of the solution is proved. Since the estimate

$$c_1(1+Q_{\sigma}(t, t^{\rho}\xi))^{1/2}\leqq(1+t^{\rho}\Theta_{\sigma}(t, \xi))\leqq c_2(1+Q_{\sigma}(t, t^{\rho}\xi))^{1/2}$$

holds for some constants $c_1$ and $c_2$, the rest part of the theorem is easy.

Q. E. D.

COROLLARY 6.2. *If* $l\geqq m$, *the solution* $u(t)$ *in Theorem* 6.1 *satisfies* $t^{\sigma k}u(t)\in C^k([0, T], H^{l-k}(\boldsymbol{R}^n))$ *for* $k=0, 1, \cdots, m$.

PROOF. From the conditions (i) and (ii) in Theorem 6.1, we have

$$\wedge_{Q_{\sigma},\rho}(t)^{m-k}\Big(t^{\sigma}\frac{\partial}{\partial t}\Big)^{k-1}u(t)\in C^0([0, T], H^{l-k+1}(\boldsymbol{R}^n)),$$

$$\wedge_{Q_{\sigma,\rho}}(t)^{m-k}\Big(t^{\sigma}\frac{\partial}{\partial t}\Big)^{k}u(t)\in C^{0}([0,\ T],\ H^{l-k}(\boldsymbol{R}^{n}))$$

for $k=0, 1, \cdots, m$. Therefore, we have $\Big(t^{\sigma}\frac{\partial}{\partial t}\Big)^{k}u(t)\in C^{0}([0,\ T],\ H^{l-k}(\boldsymbol{R}^{n}))$ for $k=0, 1, \cdots, m$. This means the above corollary. Q. E. D.

To have the differentiability in $t$ more precisely, we assume the following condition in addition to (L-1)$\sim$(L-7). Let $l$ be an integer $\geqq m$.

(L-8) (Coefficients (2)). $a_{j,\alpha}(t,\ x)$ is of $C^{l-m}$ class with respect to $t$ (such that $0<t\leqq T$) and of $C^{\infty}$ class with respect to $x$. Further, $\Big(t^{\sigma}\frac{\partial}{\partial t}\Big)^{k}\Big(\frac{\partial}{\partial x}\Big)^{\beta}a_{j,\alpha}(t,\ x)$ belongs to $\mathscr{B}^{0}(\Omega)$ for any $\beta$ and $0\leqq k\leqq l-m$.

Then we have

COROLLARY 6.3. *If* $t^{\sigma k}f(t)\in C^{k}([0,\ T],\ H^{l-m-k}(\boldsymbol{R}^{n}))$ *holds for* $k=0, 1, \cdots, l-m$, *then the solution* $u(t)$ *in Theorem* 6.1 *satisfies* $t^{\sigma k}u(t)\in C^{k}([0,\ T],\ H^{l-k}(\boldsymbol{R}^{n}))$ *for* $k=0, 1, \cdots, l$.

PROOF. Let $\vartheta=t^{\sigma}\frac{\partial}{\partial t}+s$ and let $\wedge$ and $h_{m-j}(t)$ be pseudo-differential operators defined by

$$\sigma(\wedge)=\lambda(\xi)=(1+|\xi|^{2})^{1/2},$$

$$\sigma(h_{m-j}(t))=-\sum_{|\alpha|\leqq m-j}a_{j,\alpha}(t,\ x)(\sqrt{-1}\,t^{\rho}\xi)^{\alpha}\lambda(\xi)^{-m+j}.$$

We define the commutator $h_{m-j}^{(p)}(t)$ by

$$h_{m-j}^{(p)}(t)=[\wedge,\ h_{m-j}^{(p-1)}(t)],\ h_{m-j}^{(0)}(t)=h_{m-j}(t)$$

inductively on $p$. Then we have the following conditions: (i) $h_{m-j}^{(p)}(t)\in S^{0}([0,\ T])$, (ii) $t^{\sigma k}h_{m-j}^{(p)}(t)u\in C^{k}([0,\ T],\ L^{2}(\boldsymbol{R}^{n}))$ for any $u\in L^{2}(\boldsymbol{R}^{n})$ and $k=0, 1, \cdots, l-m$, and (iii) $\Big(\frac{d}{dt}\Big)^{k}(t^{\sigma k}h_{m-j}^{(p)}(t))$ is a bounded operator in $L^{2}(\boldsymbol{R}^{n})$ for $k=0, 1, \cdots, l-m$. (See Theorem 3.5 and Remark 3.6 in Tahara [6]). Since the condition $t^{\sigma k}u(t)\in C^{k}([0,\ T],\ H^{l-k}(\boldsymbol{R}^{n}))$ is proved for $k=0, 1, \cdots, m$ in Corollary 6.2, we have the following differential equation

$$\vartheta^{m}\wedge^{r}u=\sum_{j=0}^{m-1}\Big\{\sum_{i=0}^{r}\binom{r}{i}h_{m-j}^{(i)}(t)\wedge^{m-j+r-i}\Big\}\vartheta^{j}u+\wedge^{r}f(t)$$

for $0\leqq t\leqq T$ and $r=0, 1, \cdots, l-m$. Therefore, to prove the condition $t^{\sigma k}u(t)\in C^{k}([0,\ T],\ H^{l-k}(\boldsymbol{R}^{n}))$ for $k=m+1, \cdots, l$, we have only to apply the same argument as in the proof of Theorem 3.5 in [6]. Therefore, the proof is easy.

Q. E. D.

## References

[1] Friedrichs, K. O., Symmetric positive linear differential equations, Comm. Pure. Appl. Math. **11** (1958), 333-418.
[2] John, F., On linear partial differential equations with analytic coefficients. Unique continuation of data, Comm. Pure. Appl. Math. **2** (1949), 209-253.
[3] Kumano-go, H., Pseudo-differential operators, Iwanami, Tokyo, 1974 (in Japanese).
[4] Mizohata, S., The theory of partial differential equations, Cambridge Univ. Press, 1973.
[5] Tahara, H., Cauchy problems for Fuchsian hyperbolic partial differential equations, Proc. Japan Acad., **54** (1978), 92-96.
[6] Tahara, H., Singular hyperbolic systems, I. Existence, uniqueness and differentiability, J. Fac. Sci. Univ. Tokyo Sec. IA **26** (1979), 213-238.

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