# *Multi-dimensional wave equation and an associated group of operators on the boundary*

Dedicated to the memory of the late Professor William Feller

By Tadashi UENO

Let $D$ be a bounded open domain in $R^N$ with smooth boundary. The first problem in this paper is to solve the wave equation

$$(0.1)\qquad \frac{\partial^2 u}{\partial t^2}(t, x) = Au(t, x)$$
$$= \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i}(t, x) + c(x)u(t, x)$$

on $D$ with boundary condition:

$$Lu(x)=0, \qquad x \in \partial D$$

$$(0.2)\qquad Lu(x) = \sum_{i,j=1}^{N-1} \alpha_{ij}(x)\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}(x) + \sum_{i=1}^{N-1} \beta_i(x)\frac{\partial u}{\partial \xi_i}(x) + \gamma(x)u(x)$$
$$+\delta(x)Au(x) + \mu(x)\frac{\partial u}{\partial n}(x)$$
$$+\int_{\bar{D}}\Big\{u(y) - u(x) - \sum_{i=1}^{N-1}\frac{\partial u}{\partial \xi_i}(x)\xi_i^x(y)\Big\}\nu(x, dy),$$

where $\{\alpha_{ij}(x)\}$ is symmetric, non-negative definite, $\gamma(x)$, $\delta(x) \leqq 0$, $\mu(x) \geqq 0$ and $\{\xi_i^x(y), 1 \leqq i \leqq N\}$ is a local coordinate near $x \in \partial D$ and defined on $\bar{D}$ as a set of functions in $y$. $\nu(x, \cdot)$ is a measure on $\bar{D}$ such that $\nu(x, \bar{D} - U_x)$ and $\int_{U_x \cap \bar{D}}\Big\{\sum_{i=1}^{N-1}(\xi_i^x(y))^2 + |(\xi_N^x(y)|\Big\}\nu(x, dy)$ are finite for each neighbourhood $U_x$ of $x$.

This boundary condition was obtained by A. D. Wentzell [14] as a partial extension of the well known result of Feller [1, 3] in one dimension, and is the most general boundary condition for the diffusion equation

$$(0.3)\qquad \frac{\partial u}{\partial t}(t, x) = Au(t, x)$$

in the following sense. If $\{T_t, t \geqq 0\}$ is a strongly continuous semigroup of non-negative linear operators on $C(\bar{D})$[1] with a contraction of $\bar{A}$ as the generator[2], then each smooth function $u$ in the domain of the generator necessarily satisfies (0.2). For the *diffusion equation*, there are some works to prove the existence of solution which satisfies this boundary condition.

The motivation of this paper is to know whether this type of boundary condition is also possible for the *wave equation*, or not. The answer is in the affirmative under some auxiliary conditions.

As to the boundary condition of this kind, Feller [4] already solved the one dimensional wave equation of type.

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(t, x)=\Omega u(t, x), \qquad r_1<x<r_2, \\ (0.4) \qquad & \\ &\gamma_i u(t, r_i)+\delta_i \Omega u(t, r_i)+\frac{\partial}{\partial n} u(t, r_i)=0, \qquad i=1, 2, \end{aligned}$$

by using Fourier series and discussed the intuitive meanings of the generator and the boundary condition, where $\Omega f(x)=\frac{d}{dm}\frac{d^+}{dx}f(x)$ is the generalized derivative of the second order discoverd by Feller [2][3].

Here, we first assume some auxiliary conditions on $A$ and $L$, and prepare certain Hilbert spaces depending on $A$ and $L$ in § 1. Then, we define a closed extension $A_L$ of $A$ relative to $L$, and prove that there is a semigroup $\{T_t, t\geqq 0\}$ with generator $A_L$ in § 3. Thus, the equation, an operator-theoretical version of (0.3)-(0.2),

$$\frac{d}{dt}u_t=A_L u_t, \qquad u_t\to f \quad \text{as} \quad t>0,$$

is solved by $u_t=T_t f$ for given initial data $f$. The main tool for the proof is a bilinear form, an extension of

$$B_\lambda(f, g)=((\lambda-A)f, g)_\partial-\langle Lf, g\rangle$$

for smooth $f$ and $g$ in § 2. Here,

1) $C(\bar{D})$ and $C(\partial D)$ are the space of continuous functions on $\bar{D}$ and $\partial D$ with norms $\max_{x\in D}|f(x)|$ and $\max_{x\in\partial D}|f(x)|$, respectively.

2) $\bar{A}$ is the closure in $C(\bar{D})$ of $A$ which is defined for twice continuously differentiable functions on $\bar{D}$.

3) This is the intrinsic form of the one-dimensional diffusion generator discovered by Feller [2]. Here, $\frac{d^+}{dx}f$ is the right-derivative of $f$ and $\frac{d}{dm}$ is the Radon-Nikodym derivative with respect to the measure $m$.

$$(f, g)_\delta=\int_D f(x)g(x)dx+\int_{\partial D} f(x)g(x)|\delta(x)|dx, \ \langle f, g\rangle=\int_{\partial D} f(x)g(x)dx,$$

where $\delta(x)$ is a coefficient of $L$, and $dx$ denotes the volume element for the integral on $D$, and denotes the surface element for the integral on $\partial D$.

In §4, we first prove a fundamental lemma (Lemma 4.1), and then prove that there is a group of linear operators $\{U_t, -\infty<t<\infty\}$ on a vector valued function space with generator $\mathfrak{G}$ such that $\mathfrak{G}\binom{f}{g}=\binom{g}{A_L f}$. Hence, the equation

$$\frac{d^2}{dt^2}u_t=A_L u_t, \quad u_t\to f, \quad v_t=\frac{d}{dt}u_t\to g \quad \text{as} \quad t\to 0,$$

a version of (0.1)-(0.2), is solved by $\binom{u_t}{v_t}=U_t\binom{f}{g}$. This method is an abstract extension of K. Yosida [15], where the wave equation with free boundary in $R^N$ is solved. But, Lemma 4.1 applies also for a wider class of operators than that of differential operators of type $A$, as we see in §5.

As to the differentiability, there are versions of $u_t$ in $\binom{u_t}{v_t}=U_t\binom{f}{g}$ and $w_t=T_t f$ which satisfy (0.1) and (0.3) in the strict sense, under the condition that the mass of $\nu(x, \cdot)$ is concentrated on $D$, the coefficients of $A$ are in $C^n(\bar{D})$, and $f$ and $g$ belong to $C_0^n(D)$ for sufficiently large $n$[4]. Moreover, if the coefficients of $A$, $f$ and $g$ are infinitely differentiable, so are the solutions on $(-\infty, \infty)\times D$ and $(0, \infty)\times D$, respectively.

The second problem in this paper is to construct a group of linear operators $\{\tilde{U}_t, -\infty<t<\infty\}$ on a space of vector valued functions on the boundary in §5, which is supposed to describe the wave propagation on the boundary.

Let $u=H\varphi$ be the solution of the Dirichlet problem

$$Au(x)=0, \qquad x\in D,$$

$$u(x)=\varphi(x), \qquad x\in\partial D,$$

for $\varphi\in C(\partial D)$. We define $LH$ for smooth functions $\varphi$ on $\partial D$ by

$$\varphi\to(LH)\varphi(x)=L(H\varphi)(x), \qquad x\in\partial D.$$

Then, a closed extension $\overline{LH}$ of $LH$ is the generator of a semigroup $\{\tilde{T}_t, t\geqq 0\}$ of linear operators on a space of functions on $\partial D$. The group of operators $\{\tilde{U}_t, -\infty<t<\infty\}$ on the boundary is obtained out of $\{\tilde{T}_t\}$ by Lemma 4.1, just

4) $C^n(\partial D)$ and $C(\bar{D})$ are the space of $n$-times continuously differentiable functions on $\partial D$ and some neighbourhood of $\bar{D}$, respectively. $C_0^n(D)$ is the space of $n$-times continuously differentiable functions with supports in $D$. Clearly, a function in $C_0^n(D)$ can be considered as an element in $C^n(\bar{D})$.

as $\{U_t\}$ is obtained from $\{T_t\}$. Hence,

$$(0.5)\qquad \frac{d^2}{dt^2}\varphi_t=LH\varphi_t\,,\quad \varphi_t\to\varphi\,,\quad \psi_t=\frac{d}{dt}\varphi_t\to\psi\,,\quad \text{as}\quad t\to 0\,,$$

is solved by $\binom{\varphi_t}{\psi_t}=\tilde{U}_t\binom{\varphi}{\psi}$. As for the nature of $\overline{LH}$, it can be proved that $\overline{LH}$ is an integro-differential operator of elliptic character for smooth $\varphi$ on $\partial D$.

Finally, intuitive interpretations of $\{\tilde{U}_t\}$ and the terms of $Lu(x)$ are discussed in § 6. For the case of diffusion equation (0.3)-(0.2), the semigroup with generator $\overline{LH}$, in the setup of [12], corresponds to the *Markov process on the boundary of the diffusion*, which is the trace on $\partial D$ of the diffusion, described on a random time scale on $\partial D$. Here, $\{\tilde{U}_t\}$ is conjectured by analogy to describe the wave propagation on $\partial D$, which is coupled with the wave propagation through the domain $D$. This is also suggested by (0.1) and (0.5).

As for the terms in $Lu(x)$, it seems natural that the wave can propagate through $\partial D$ according to the term $\sum_{i,j}\alpha_{ij}\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}+\sum_i\beta_i\frac{\partial u}{\partial\xi_i}+\gamma\cdot u$ (with absorption by the last term), and the boundary has the mass distribution $|\delta(x)|dx$ by the term $\delta(x)Au(x)$ like a vessel.

But, these interpretations are not yet justified mathematically. It is desirable that exact intuitive explanations will be formulated and proved rigorously on some mathematical setup, as in the case of the diffusion equation.

The author thanks very much to Daisuke Fujiwara for his kind help during the research of this problem, and also to Kazuhiko Aomoto and Akio Yamada who answered his questions very kindly.

## § 1. Preliminaries

First, we assume auxiliary conditions on $D$, $A$ and $L$. Let $D$ be a bounded domain in $R^N$ with boundary $\partial D$ of class $C^3$. The coefficients of $A$ in (0.1) are in $C^3(\bar{D})$, $c(x)\leqq 0$ and $\{a_{ij}(x)\}$ is symmetric and uniformly elliptic, that is, there are positive constants $\underline{a}$ and $\bar{a}$ such that, for real numbers $\eta_i$, $1\leqq i\leqq N$,

$$(1.1)\qquad \underline{a}\sum_{i=1}^N\eta_i^2\leqq\sum_{i,j=1}^N a_{ij}(x)\eta_i\eta_j\leqq\bar{a}\sum_{i=1}^N\eta_i^2\,,\qquad x\in\bar{D}\,.$$

Let $\phi(x)$ be a function on $R^N$ such that $\phi(x)=0$ characterizes $\partial D$, and $\phi(x)>0$ on $D$. The normal derivative $\frac{\partial u}{\partial n}(x)$ in $Lu(x)$ of (0.2) is defined relative to $\{a_{ij}(x)\}$ by $\sum_{i=1}^N\Big(\sum_{j=1}^N a_{ij}(x)\phi_j(x)\Big)\frac{\partial u}{\partial x_i}(x)$, where $\phi_i(x)=\frac{\partial\phi}{\partial x_i}(x)\Big(\sum_{i=1}^N\Big(\frac{\partial\phi}{\partial x_i}(x)\Big)^2\Big)^{-1/2}$. As the local coordinate in $Lu(x)$, we take a set of functions $\{\xi_i^r(y),\ 1\leqq i\leqq N\}$

defined on a neighbourhood of $\bar{D}$ for each $x\in\partial D$, such that it is a local coordinate of class $C^3$ in a neighbourhood $U_x$ of $x$, and $\{\xi_i^x(y),\ 1\leqq i\leqq N-1\}$ is also a local coordinate on $\partial D$ in $U_x\cap\partial D$, $\xi_N^x(y)=0$ characterizes $\partial D$ in $U_x$, $\xi_N^x(y)\geqq 0$ on $\bar{D}$, and $\xi_i^x(x)=0$ for $1\leqq i\leqq N$.

We assume, for $\mu(x)$ and $\nu(x,\cdot)$ in $Lu(x)$

$$\mu(x)>0, \qquad x\in\partial D \tag{1.2}$$

$$\nu(x,\bar{D}-U_x)+\int_{U_x}\sum_{i=1}^{N}|\xi_i^x(y)|\nu(x,dy)<\infty. \tag{1.3}$$

Then, we can rewrite the boundary condition (0.2) to a simpler form

$$Lu(x)=0,$$

$$Lu(x)=\sum_{i,j=1}^{N-1}\alpha_{ij}(x)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}(x)+\sum_{i=1}^{N-1}\beta_i(x)\frac{\partial u}{\partial\xi_i}(x)+\gamma(x)u(x)+\delta(x)Au(x) \tag{1.4}$$
$$+\frac{\partial u}{\partial n}(x)+\int_{\bar{D}}(u(y)-u(x))\nu(x,dy),$$

by replacing $\sum_{i=1}^{N-1}\beta_i(x)\frac{\partial u}{\partial\xi_i}(x)$ in (0.2) by $\sum_{i=1}^{N-1}\left\{\beta_i(x)-\int_D\xi_i^x(y)\nu(x,dy))\frac{\partial u}{\partial\xi_i}(x)\right\}$, and then multiplying the both sides of (0.2) by $\mu^{-1}(x)$ and changing the notations. Thus, we use this simpler form (1.4) hereafter. We assume

$$\alpha_{ij},\ \beta_i,\ \gamma,\ \delta\in C^2(\partial D), \qquad 1\leqq i,\ j\leqq N-1, \tag{1.5}$$

$$f\to\nu[f]\ \text{ maps }\ C^2(\bar{D})\ \text{ into }\ C(\partial D), \tag{1.6}$$

$$\text{where}\quad \nu[f](x)=\int_{\bar{D}}(f(y)-f(x))\nu(x,dy).$$

If $f$ is smooth near $x\in\partial D$, $\nu[f]$ clearly exists by (1.3). By (1.5)-(1.6), it is clear that

$$f\to Lf\ \textit{ maps }\ C^2(\bar{D})\ \textit{ into }\ C(\partial D). \tag{1.7}$$

We also assume that the measure $\nu(x,\cdot)$ is decomposed as

$$\nu(x,\cdot)=\nu_\partial(x,\cdot)+\nu'(x,\cdot),\quad \nu'(x,\cdot)=\nu'_\partial(x,\cdot)+\nu_D(x,\cdot), \tag{1.8}$$

where $\nu_\partial(x,\cdot)$ and $\nu'_\partial(x,\cdot)$ have the masses only on $\partial D$ and have the density functions $\nu_\partial(x,y)$ and $\nu'_\partial(x,y)$ with respect to the surface element on $\partial D$, and $\nu_D(x,\cdot)$ has the mass only on $D$ and has the density function $\nu_D(x,y)$ with respect to the volume element on $D$. Moreover, these density functions satisfy

$$\nu_\partial(x,y)=\nu_\partial(y,x), \qquad x,\ y\in\partial D \tag{1.9}$$

$$\bar{\nu}^2=\int_{\partial D}\left\{\int_D \nu_D(x,\ y)^2 dy+\int_{\partial D}\nu'_\partial(x,\ y)^2 dy\right\}dx<\infty. \tag{1.10}$$

Finally, we assume the condition

(L) $\{\alpha_{ij}(x)\}$ *is uniformly elliptic, or* $L$ *is formally self-adjoint*, that is, at least one of the following (1.11) and (1.12) holds:

$$\underline{\alpha}\sum_{i=1}^{N-1}\eta_i^2\leqq\sum_{i,j=1}^{N-1}\alpha_{ij}(x)\eta_i\eta_j\leqq\bar{\alpha}\sum_{i=1}^{N-1}\eta_i^2,\quad x\in\partial D,\quad \text{for constants } \underline{\alpha},\ \bar{\alpha}>0, \tag{1.11}$$

$$\begin{aligned}&\sum_{j=1}^{N-1}\frac{\partial}{\partial\xi_i}\alpha_{ij}(x)-\beta_i(x)=0, && 1\leqq i\leqq N-1,\quad x\in\partial D\\ &\nu(x,\ \cdot)\equiv\nu_\partial(x,\ \cdot) && x\in\partial D.\end{aligned} \tag{1.12}$$

Now, we define, for measurable functions $f$ and $g$ on $\bar{D}$,

$$(f,\ g)=\int_D f(x)g(x)dx,\ \langle f,\ g\rangle=\int_{\partial D}f(x)g(x)dx,$$

$$\|f\|=(f,\ f)^{1/2},\quad \|f\|_\partial=\langle f,\ f\rangle^{1/2},$$

$$\nu(f,\ g)=\frac{1}{2}\int_{\partial D}dx\int_{\partial D}(f(y)-f(x))(g(y)-g(x))\nu_\partial(x,\ y)dy$$

$$+\int_{\partial D}f(x)g(x)\nu'(x,\ \bar{D})dx,$$

$$\nu'(f)(x)=\int_{\bar{D}}f(y)\nu'(x,\ dy),$$

when the integrals converge.

PROPOSITION 1.1. i) *$\nu(f,\ g)$ and $\nu'(f)$ can be defined for $f,\ g\in C^2(\bar{D})$, and*

$$\langle\nu[f],\ g\rangle=-\nu(f,\ g)+\langle\nu'(f),\ g\rangle \tag{1.13}$$

$$|\langle\nu'(f),\ g\rangle|<\bar{\nu}(\|f\|+\|f\|_\partial)\|g\|_\partial. \tag{1.14}$$

ii) *If a sequence $\{f_n\}$ in $C^2(\bar{D})$ satisfies $\lim_{n\to\infty}\|f_n\|_\partial=0$ and $\lim_{m,\ n\to\infty}\nu(f_m-f_n,\ f_m-f_n)=0$, then $\lim_{n\to\infty}\nu(f_n,\ f_n)=0$.*

PROOF. i) By (1.8) and (1.10), $\nu'(f)(x)$ exists and (1.14) holds for bounded $f$ and $g$:

$$\left|\int_D f(y)\nu'(x,\ dy)\right|\leqq\left|\int_D f(y)\nu_D(x,\ y)dy\right|+\left|\int_{\partial D}f(y)\nu'_\partial(x,\ y)dy\right|$$

$$\leqq \|f\| \Big(\int_D \nu_D(x, y)^2 dy\Big)^{1/2} + \|f\|_\partial \Big(\int_{\partial D} \nu_{\partial D}(x, y)^2 dy\Big)^{1/2}$$

$$\leqq (\|f\| + \|f\|_\partial)\Big(\int_D \nu_D(x, y)^2 dy + \int_{\partial D} \nu'_\partial(x, y)^2 dy\Big)^{1/2}.$$

$$|\langle \nu'(f), g\rangle| = \Big|\int_{\partial D} \nu'(f)(x) g(x) dx\Big| \leqq \|g\|_\partial \cdot \Big(\int_{\partial D} \nu'(f)(x)^2 dx\Big)^{1/2}$$

$$\leqq \bar{\nu} \cdot (\|f\| + \|f\|_\partial) \|g\|_\partial .$$

Since $\nu'(x, \bar{D}) = \nu'(1)(x)$, $\nu'(x, \cdot)$ is a bounded measure for almost all $x \in \partial D$.

For $f$ and $g$ in $C^2(\bar{D})$, $\nu[f](x)$ is continuous by (1.6), and hence $\langle \nu[f], g\rangle$ exists. Thus, by

$$\Big|\int_{\partial D} f(x) g(x) \nu'(x, \bar{D}) dx\Big| \leqq \max_{x \in \partial D} |f(x)| \int_{\partial D} |g(x)| \nu'(x, \bar{D}) dx$$

$$= \max_{x \in \partial D} |f(x)| \cdot \langle \nu'(1), |g|\rangle < \infty ,$$

and

$$\langle \nu[f], g\rangle = \int_{\partial D} \Big\{\int_{\bar{D}} (f(y) - f(x)) \nu(x, dy)\Big\} g(x) dx \tag{1.15}$$

$$= \int_{\partial D} \Big\{\int_{\partial D} (f(y) - f(x)) \nu_\partial(x, y) dy\Big\} g(x) dx$$

$$+ \langle \nu'(f), g\rangle - \int_{\partial D} f(x) g(x) \nu'(x, \bar{D}) dx ,$$

the first term on the right side of (1.15) is finite for $f, g \in C^2(\bar{D})$. By the symmetry of $\nu_\partial(x, y)$, this term is equal to

$$\int_{\partial D} \Big\{\int_{\partial D} (f(y) - f(x)) \nu_\partial(y, x) dy\Big\} g(x) dx$$

$$= \int_{\partial D} \Big\{\int_{\partial D} (f(x) - f(y)) \nu_\partial(x, y) dx\Big\} g(y) dy$$

$$= -\int_{\partial D} dx \int_{\partial D} (f(y) - f(x)) g(y) \nu_\partial(x, y) dy .$$

Hence, it is equal to the mean of this and the original expression in (1.15), that is,

$$-\frac{1}{2} \int_{\partial D} dx \int_{\partial D} (f(y) - f(x))(g(y) - g(x)) \nu_\partial(x, y) dy .$$

Thus, $\nu(f, g)$ exists and (1.13) is clear.

ii) Since $\|f_n\|_\partial \to 0$, there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to 0$ on $\partial D$

almost everywhere. For each $\varepsilon>0$, there is an $N_\varepsilon$ such that $\nu(f_m-f_n, f_m-f_n)<\varepsilon$ for $m, n\geqq N_\varepsilon$. Thus, the assertion is true by

$$\nu(f_m, f_m)=\int_{\partial D}dx\left\{\frac{1}{2}\int_{\partial D}(f_m(y)-f_m(x))^2\nu_\partial(x, y)dy+f_m(x)^2\nu'(x, \bar{D})\right\}$$

$$=\int_{\partial D}dx\left\{\frac{1}{2}\int_{\partial D}\lim_{k\to\infty}(f_m(y)-f_{n_k}(y)-f_m(x)+f_{n_k}(x))^2\nu_\partial(x, y)dy\right.$$

$$\left.+\lim_{k\to\infty}(f_m(x)-f_{n_k}(x))^2\nu'(x, \bar{D})\right\}$$

$$\leqq\varliminf_{k\to\infty}\int_{\partial D}dx\left\{\frac{1}{2}\int_{\partial D}(f_m(y)-f_{n_k}(y)-f_m(x)+f_{n_k}(x))^2\nu_\partial(x, y)dy\right.$$

$$\left.+(f_m(x)-f_{n_k}(x))^2\nu'(x, \bar{D})\right\}$$

$$=\varliminf_{k\to\infty}\nu(f_m-f_{n_k}, f_m-f_{n_k})\leqq\varepsilon, \quad \text{for } m\geqq N_\varepsilon,$$

completing the proof.

We define, for $f$ and $g$ in $C^2(\bar{D})$,

$$\tilde{D}\langle f, g\rangle=\int_{\partial D}\sum_{i,j=1}^{N-1}\alpha_{ij}(x)\frac{\partial f}{\partial\xi_i}(x)\frac{\partial g}{\partial\xi_j}(x)dx=\sum_{i,j=1}^{N-1}\langle\alpha_{ij}f_{\xi_i}, g_{\xi_j}\rangle.$$

PROPOSITION 1.2[5]. *If a sequence* $\{f_n\}$ *in* $C^2(\bar{D})$ *satisfies* $\lim_{n\to\infty}\|f_n\|_\partial=0$ *and* $\lim_{m,n\to\infty}\tilde{D}\langle f_m-f_n, f_m-f_n\rangle=0$, *then* $\lim_{n\to\infty}\tilde{D}\langle f_n, f_n\rangle=0$.

PROOF. Since $\alpha_{ij}(x)$ are in $C^2(\partial D)$, it can be proved that there are Lipschitz continuous functions $\beta_{ij}(x)$ such that $\beta_{ij}(x)=\beta_{ji}(x)$ and

$$\alpha_{ij}(x)=\sum_{k=1}^{N-1}\beta_{ik}(x)\beta_{kj}(x),$$

by modifying the result of Freidlin [5] or Phillips and Sarason [8]. Thus, for $g_{n,i}(x)=\sum_{j=1}^{N-1}\beta_{ij}(x)\frac{\partial f_n}{\partial\xi_i}(x)$, $1\leqq i\leqq N-1$, $x\in\partial D$, we have

$$\tilde{D}\langle f_m-f_n, f_m-f_n\rangle=\int_{\partial D}\sum_{i,j=1}^{N-1}\alpha_{ij}(x)\frac{\partial(f_m-f_n)}{\partial\xi_i}(x)\frac{\partial(f_m-f_n)}{\partial\xi_j}(x)dx$$

$$=\int_{\partial D}\sum_{i=1}^{N-1}(g_{m,i}(x)-g_{n,i}(x))^2dx=\sum_{i=1}^{N-1}\|g_{m,i}\|_\partial^2\to 0, \quad \text{as } m, n\to\infty,$$

by the assumption, and hence $\{g_{n,i}\}$ converge to the limits $g_i$, $1\leqq i\leqq N-1$, in the Hilbert space $H_\partial$ of the square integrable functions on $\partial D$ with inner product $\langle f, g\rangle$. Thus, $\{g_{n,i}\}$ also converge to $g_i$ in the Sobolev space $H^{-1}(\partial D)$.

5) The proof of this proposition owes to Daisuke Fujiwara.

On the other hand, $\frac{\partial f_n}{\partial \xi_i}(x)\to 0$ in $H^{-1}(\partial D)$ by the assumption $\|f_n\|_\delta \to 0$, and hence

$$g_{n,i}(x)=\sum_{j=1}^{N-1}\beta_{ij}(x)\frac{\partial f_n}{\partial \xi_i}(x)\to 0, \quad \text{in} \quad H^{-1}(\partial D),$$

since the multiplication by a Lipschitz continuous function is a continuous mapping in $H^{-1}(\partial D)$. Thus, $g_i=0$ in $H^{-1}(\partial D)$, and $g_i$ also vanishes in $H_\delta$, implying

$$\tilde{D}\langle f_n, f_n\rangle=\sum_{i=1}^{N-1}\|g_{n,i}\|_\delta^2\to\sum_{i=1}^{N-1}\|g\|_\delta^2=0.$$

We define, for $f$ and $g$ in $C^2(\bar{D})$,

$$D(f, g)=\int_D \sum_{i=1}^{N}\frac{\partial f}{\partial x_i}(x)\frac{\partial g}{\partial x_i}(x)dx, \quad D\langle f, g\rangle=\int_{\partial D}\sum_{i=1}^{N-1}\frac{\partial f}{\partial \xi_i}(x)\frac{\partial g}{\partial \xi_i}(x)dx$$

$$(f, g)_\delta=(f, g)+\langle f, g\cdot|\delta|\rangle, \quad (f, g)_1=(f, g)+D(f, g)$$

$$((f, g))=(f, g)_\delta+D(f, g)+\tilde{D}\langle f, g\rangle+\nu(f, g)$$

$$\|f\|_\delta=(f, f)_\delta^{1/2}, \quad \|f\|_1=(f, f)_1^{1/2}, \quad |||f|||=((f, f))^{1/2}.$$

Let $H_\delta$ and $K$ be the completions of $C^2(\bar{D})$ with respect to the norms $\|\ \|_\delta$ and $|||\ |||$, respectively. $H_\delta$ is clearly a space of functions on $D_\delta \doteq D\cup\partial D_\delta$, where $\partial D_\delta$ is the support of $\delta(x)$. The Sobolev space $H^1(D)$ is obtained as the completion of $C^2(\bar{D})$ with respect to $\|\ \|_1$, and each $f$ in $H^1(D)$ has the square integrable boundary value on $\partial D$ almost everywhere. It is also known that, for each $\varepsilon>0$, there is a constant $C(\varepsilon)>0$ such that

$$\|f\|_\delta^2 \leqq \varepsilon D(f, f)+C(\varepsilon)\|f\|^2, \qquad \text{for} \quad f\in H^1(D). \tag{1.16}$$

Here, $D(f, f)$ is defined by continuity and coincides with $\int_D \sum_{i=1}^{N}(D_{x_i}f(x))^2 dx$, where $D_{x_i}f$, $1\leqq i\leqq N$, are the derivatives in the Schwartz's sense.

Since $H^1(D)$ is a space of functions on $\bar{D}$ as above, so is $K$ by

PROPOSITION 1.3. i) *If a sequence* $\{f_n\}$ *in* $C^2(\bar{D})$ *satisfies* $\lim_{n\to\infty}\|f_n\|_\delta=0$ *and* $\lim_{m,n\to\infty}\|f_m-f_n\|_1=0$, *then* $\lim_{n\to\infty}\|f_n\|_1=0$.

ii) *If a sequence* $\{f_n\}$ *in* $C^2(\bar{D})$ *satisfies* $\lim_{n\to\infty}\|f_n\|_1=0$ *and* $\lim_{m,n\to\infty}|||f_m-f_n|||=0$, *then* $\lim_{n\to\infty}|||f_n|||=0$.

*Hence, $K$ is imbedded in $H^1(D)$ and $H^1(D)$ is imbedded in $H_\delta$ uniquely.*

PROOF. i) The assertion is clear, since $\|f_n\|\leqq\|f_n\|_\delta$, and it is known that $\lim_{n\to\infty}\|f_n\|=0$ and $\lim_{m,n\to\infty}\|f_m-f_n\|_1=0$ imply $\|f_n\|_1\to 0$.

ii) By the assumption $\|f_n\|_1^2=\|f_n\|^2+D(f_n, f_n)\to 0$ and (1.16), we have $\|f_n\|_\delta\to 0$. By $|||f_m-f_n|||\to 0$, we have $\nu(f_m-f_n, f_m-f_n)\to 0$, $\tilde{D}\langle f_m-f_n, f_m-f_n\rangle \to 0$, and hence $\nu(f_n, f_n)\to 0$ and $\tilde{D}\langle f_n, f_n\rangle\to 0$ by Propositions 1.1 and 1.2 and $\|f_n\|_\delta\to 0$. Moreover, $\|f_n\|_\delta^2=(f_n, f_n)+\langle f_n, f_n|\delta|\rangle\leqq\|f_n\|^2+\max_{x\in\partial D}|\delta(x)|\,\|f_n\|_\delta^2\to 0$, and hence $|||f_n|||\to 0$.

## §2. A bilinear form $B_\lambda(f, g)$

For $f$ and $g$ in $C^2(\bar{D})$ and $\lambda\geqq 0$, we define

$$B_\lambda(f, g)=((\lambda-A)f, g)_\delta-\langle Lf, g\rangle \tag{2.1}$$

PROPOSITION 2.1. *For $f$ and $g$ in $C^2(\bar{D})$ and $\lambda\geqq 0$,*

$$\begin{aligned} B_\lambda(f, g)=&\sum_{i,j=1}^{N}(a_{ij}f_{x_i}, g_{x_j})+(f, (\lambda-c)g)+\tilde{D}\langle f, g\rangle \\ &+\langle f, |\lambda\delta+\gamma|g\rangle+\nu(f, g) \\ &+\sum_{i=1}^{N}(b_i'f_{x_i}, g)+\sum_{i=1}^{N-1}\langle\beta_i'f_{\xi_i}, g\rangle-\langle\nu'(f), g\rangle, \end{aligned} \tag{2.2}$$

*where* $b_i'=\sum_{j=1}^{N}(a_{ij})_{x_j}-b_i$ *and* $\beta_i'=\sum_{i=1}^{N-1}(\alpha_{ij})_{\xi_j}-\beta_j$.

$$\begin{aligned} B_\lambda(f, g)-B_\lambda(g, f)=&2\sum_{i=1}^{N}(b_i'f_{x_i}, g)+2\sum_{i=1}^{N-1}\langle\beta_i'f_{\xi_i}, g\rangle \\ &+(c'f, g)+\langle\gamma'f, g\rangle-\langle\nu'(f), g\rangle+\langle f, \nu'(g)\rangle \end{aligned} \tag{2.3}$$

*where* $c'=\sum_{i=1}^{N}(b_i')_{x_i}$ *and* $\gamma'=\sum_{i=1}^{N-1}(\beta_i')_{\xi_i}-\sum_{i=1}^{N}b_i'\phi_i$.

PROOF. By Green-Stokes Theorem, we have

$$\Big(\sum_{i,j=1}^{N}a_{ij}f_{x_ix_j}, g\Big)=-\sum_{i,j=1}^{N}(a_{ij}f_{x_i}, g_{x_j})-\sum_{i=1}^{N}\Big(\sum_{j=1}^{N}(a_{ij})_{x_j}f_{x_i}, g\Big)-\Big\langle\frac{\partial f}{\partial n}, g\Big\rangle,$$

$$\Big\langle\sum_{i,j=1}^{N-1}\alpha_{ij}f_{\xi_i\xi_j}, g\Big\rangle=-\sum_{i,j=1}^{N-1}\langle\alpha_{ij}f_{\xi_i}, g_{\xi_j}\rangle-\sum_{i=1}^{N-1}\Big\langle\sum_{j=1}^{N-1}(\alpha_{ij})_{\xi_j}f_{\xi_i}, g\Big\rangle.$$

Then, by (1.13),

$$\begin{aligned} B_\lambda(f, g)=&((\lambda-A)f, g)+\langle(\lambda-A)f, g|\delta|\rangle-\langle Lf, g\rangle \\ =&\Big\{\sum_{i,j=1}^{N}(a_{ij}f_{x_i}, g_{x_j})+\sum_{i=1}^{N}\Big(\Big(\sum_{j=1}^{N}(a_{ij})_{x_i}-b_i\Big)f_{x_i}, g\Big)+((\lambda-c)f, g)+\Big\langle\frac{\partial f}{\partial n}, g\Big\rangle\Big\} \\ &+\{\langle\lambda|\delta|f, g\rangle+\langle\delta Af, g\rangle\}+\Big\{\tilde{D}\langle f, g\rangle+\sum_{i=1}^{N-1}\Big\langle\Big(\sum_{j=1}^{N-1}(\alpha_{ij})_{\xi_j}-\beta_i\Big)f_{\xi_i}, g\Big\rangle \end{aligned}$$

$$-\langle\gamma\cdot f, g\rangle-\langle\delta Af, g\rangle-\left\langle\frac{\partial f}{\partial n}, g\right\rangle+\nu(f, g)-\langle\nu'(f), g\rangle\Big\}.$$

Hence, (2.2) is clear. Thus, (2.3) is obtained by

$$B_\lambda(f, g)-B_\lambda(g, f)=\sum_{i=1}^{N}\{(b_i' f_{x_i}, g)-(b_i' g_{x_i}, f)\}$$

$$+\sum_{i=1}^{N-1}\{\langle\beta_i' f_{\xi_i}, g\rangle-\langle\beta_i' g_{\xi_i}, f\rangle\}-\langle\nu'(f), g\rangle+\langle\nu'(g), f\rangle,$$

$$(b_i' g_{x_i}, f)=-(b_i' f_{x_i}, g)-((b_i')_{x_i} f, g)+\langle b_i'\phi_i f, g\rangle,$$

$$\langle\beta_i' g_{\xi_i}, f\rangle=-\langle\beta_i' f_{\xi_i}, g\rangle-\langle(\beta_i')_{\xi_i} f, g\rangle.$$

PROPOSITION 2.2. i) *$B_\lambda(f, g)$ can be extended to a bilinear form on $K$ uniquely. The extension, written by the same notation, satisfies the following*

$$|B_\lambda(f, g)|\leqq k_\lambda |||f|||\cdot|||g|||, \quad \lambda\geqq 0, \quad f, g\in K, \tag{2.4}$$

$$B_\lambda(f, f)\geqq \underline{k}|||f|||^2, \quad \lambda\geqq\lambda_0, \quad f\in K, \tag{2.5}$$

$$B_\lambda(f, f)\geqq\|f\|_\partial^2, \quad \lambda\geqq\lambda_0, \quad f\in K, \tag{2.6}$$

*where $\lambda_0\geqq 0$, $\underline{k}\geqq 0$ and $k_\lambda>0$ are constants, independent of $f$ and $g$.*

ii) *If $\delta$ is strictly negative, then there is a constant $\bar{k}>0$ such that*

$$|B_\lambda(f, g)-B_\lambda(g, f)|\leqq\bar{k}|||f|||\cdot\|g\|_\partial, \quad \lambda\geqq 0, \quad f, g\in K. \tag{2.7}$$

*If $A$ and $L$ are formally self-adjoint, then $B_\lambda(f, g)$ is symmetric:*

$$B_\lambda(f, g)=B_\lambda(g, f), \quad \lambda\geqq 0, \quad f, g\in K. \tag{2.8}$$

PROOF. i) By the definition of $K$, it is sufficient to prove (2.4)-(2.6) for $f, g$ in $C^2(\bar{D})$. By (2.2), (1.1) and (1.14), with $\bar{b}=\max_{1\leq i\leq N}\sup_{x\in\bar{D}}|b_i'(x)|$ and $\bar{\beta}=\max_{1\leq i\leq N-1}\sup_{x\in\partial D}|\beta_i'(x)|$,

$$B_\lambda(f, f)\geqq\lambda\|f\|_\partial^2+\underline{a}D(f, f)+\tilde{D}\langle f, f\rangle+\nu(f, f)-\sqrt{N}\bar{b}D(f, f)^{1/2}\|f\|$$

$$-\sqrt{N}\bar{\beta}D\langle f, f\rangle^{1/2}\|f\|_\partial-\bar{\nu}(\|f\|+\|f\|_\partial)\|f\|_\partial.$$

There are two cases of the condition (L). *When $\{\alpha_{ij}(x)\}$ is uniformly elliptic,* $\underline{\alpha}D\langle f, f\rangle\leqq\tilde{D}\langle f, f\rangle$ by (1.11). Noting (1.16), we have

$$D(f, f)^{1/2}\cdot\|f\|=\sqrt{\eta}D(f, f)^{1/2}\cdot 1/\sqrt{\eta}\cdot\|f\|\leqq\eta D(f, f)+1/\eta\cdot\|f\|^2, \quad \eta>0,$$

$$D\langle f, f\rangle^{1/2}\|f\|_\partial\leqq 1/\underline{\alpha}\cdot\tilde{D}\langle f, f\rangle\|f\|_\partial\leqq\eta/\underline{\alpha}\cdot\tilde{D}\langle f, f\rangle+1/(\eta\underline{\alpha})\cdot\|f\|_\partial^2$$

$$\leqq\eta/\underline{\alpha}\cdot\tilde{D}\langle f, f\rangle+\varepsilon/(\underline{\alpha}\eta)\cdot D(f, f)+C(\varepsilon)/(\underline{\alpha}\eta)\cdot\|f\|^2, \quad \eta>0, \quad \varepsilon>0,$$

$$(\|f\|+\|f\|_\partial)\|f\|_\partial \leqq \|f\|^2+2\|f\|_\partial^2 \leqq (1+2C(\varepsilon))\|f\|^2+2\varepsilon D(f, f), \qquad \varepsilon>0.$$

Hence, by taking $\eta$ and $\varepsilon$, and then $\lambda_0$ so that

$$\underline{k}_1=\lambda_0-\{\sqrt{N}\,\bar{b}/\eta+\sqrt{N}\,\bar{\beta}C(\varepsilon)/(\underline{\alpha}\eta)+\bar{\nu}(1+2C(\varepsilon))\}\geqq 1$$

$$\underline{k}_2=\underline{a}-(\sqrt{N}\,\bar{b}/\eta+\sqrt{N}\,\bar{\beta}\varepsilon/(\underline{\alpha}\eta)+2\bar{\nu}\varepsilon)>0, \qquad \underline{k}_3=1-\sqrt{N}\,\bar{\beta}\eta/\underline{\alpha}>0,$$

we have

$$B_\lambda(f, f)\geqq B_{\lambda_0}(f, f)=\underline{k}_1\|f\|_\delta^2+\underline{k}_2 D(f, f)+\underline{k}_3\tilde{D}\langle f, f\rangle+\nu(f, f), \qquad \lambda\geqq\lambda_0.$$

Thus, (2.5) and (2.6) hold for $\lambda\geqq\lambda_0$ and $\underline{k}=\min(\underline{k}_1, \underline{k}_2, \underline{k}_3, 1)$.

*When $L$ is formally self-adjoint*, then $\bar{\beta}=\bar{\nu}=0$. Hence, we take $\eta$ and then $\lambda_0$ so that $\underline{k}_1=\lambda_0-\sqrt{N}\,\bar{b}/\eta\geqq 1$ and $\underline{k}_2=\underline{a}-\sqrt{N}\,\bar{b}\eta>0$, and put $\underline{k}=\min(\underline{k}_1, \underline{k}_2)$.

To prove (2.4), let $B_\lambda^{(1)}(f, g)$ be the sum of the first five terms of (2.2) and $B_\lambda^{(2)}(f, g)$ be the sum of the remaining terms. Since $B_\lambda^{(1)}(f, g)$ is symmetric and

$$B_\lambda^{(1)}(f, f)\leqq \bar{a}D(f, f)+\tilde{D}\langle f, f\rangle+(\lambda+\bar{c}+\bar{\gamma})\|f\|_\delta^2+\nu(f, f), \qquad f\in C^2(\bar{D})$$

with $\bar{c}=\sup_{x\in\bar{D}}|c(x)|$ and $\bar{\gamma}=\sup_{x\in\partial D}|\gamma(x)|$, we have

$$|B_\lambda^{(1)}(f, g)|\leqq B_\lambda^{(1)}(f, f)^{1/2}\cdot B_\lambda^{(1)}(g, g)^{1/2}\leqq\max(\bar{a}, \lambda+\bar{c}+\bar{\gamma}, 1)|||f|||\cdot|||g|||.$$

On the other hand, $|B_\lambda^{(2)}(f, g)|$ is also bounded by a constant multiple of $|||f|||\cdot|||g|||$ by

$$|B_\lambda^{(2)}(f, g)|<\sqrt{N}\,\bar{b}D(f, f)^{1/2}\|g\|+\sqrt{N}\,\bar{\beta}D\langle f, f\rangle^{1/2}\|g\|_\partial+\bar{\nu}(\|f\|+\|f\|_\partial)\|g\|_\partial.$$

ii) When $A$ and $L$ are formally self-adjoint, (2.8) is clear. Otherwise, we have

$$\begin{aligned}(2.9)\qquad |B_\lambda(f, g)-B_\lambda(g, f)|\leqq\ & 2\sqrt{N}\,\bar{b}D(f, f)^{1/2}\cdot\|g\|\\ &+2\sqrt{N}\,\bar{\beta}D\langle f, f\rangle^{1/2}\|g\|_\partial+\bar{c}'\|f\|\cdot\|g\|+\bar{\gamma}'\|f\|_\partial\cdot\|g\|_\partial\\ &+2\bar{\nu}(\|f\|+\|f\|_\partial)(\|g\|+\|g\|_\partial),\end{aligned}$$

by (2.3), where $\bar{c}'=\max_{x\in\bar{D}}|c'(x)|$ and $\bar{\gamma}'=\max_{x\in\partial D}|\gamma'(x)|$. Since $\delta(x)$ is strictly negative, $\|g\|_\partial$ is bounded by a constant multiple of $\|g\|_\delta$ by

$$\|g\|_\delta^2=(g, g)+\langle g, g|\delta|\rangle\geqq\|g\|^2+\inf_{x\in\partial D}|\delta(x)|\cdot\|g\|_\partial^2.$$

When $L$ is formally self-adjoint, $\bar{\beta}=\bar{\nu}=0$, and when $\{\alpha_{ij}(x)\}$ is uniformly elliptic, $\underline{\alpha}D\langle f, f\rangle\leqq\tilde{D}\langle f, f\rangle$. Hence, (2.9) is clearly bounded by a constant multiple of $|||f|||\cdot\|g\|_\delta$.

REMARK. *For $f$ in $C^2(\bar{D})$, $g$ in $K$ and $\lambda\geqq 0$,*

$$(2.10)\qquad B_\lambda(f, g)=((\lambda-A)f, g)_\delta-\langle Lf, g\rangle.$$

This is clear by the definition of $B_\lambda(f, g)$ and the continuity in $g$.

## § 3. The semigroup for the diffusion equation

PROPOSITION 3.1. *If* $\{f_n\}$ *in* $C^2(\bar{D})$ *and* $g$ *in* $H_\delta$ *satisfy* $\lim_{n\to\infty}|||f_n|||=0$ *and* $\lim_{n\to\infty}\{(Af_n-g, h)_\delta+\langle Lf_n, h\rangle\}=0$ *for each* $h$ *in* $C^2(\bar{D})$, *then* $g=0$.

PROOF. Since $|||f_n|||\to 0$, we have $B_\lambda(f_n, h)\to 0$ and $(f_n, h)_\delta\to 0$ and hence

$$0=\lim_{n\to\infty}\{(Af_n-g, h)_\delta+\langle Lf_n, h\rangle\}=\lim_{n\to\infty}\{(\lambda f_n-g, h)_\delta-B_\lambda(f_n, h)\}=-(g, h)_\delta$$

for each $h\in C^2(\bar{D})$. Since $C^2(\bar{D})$ is dense in $H_\delta$, we have $g=0$.

DEFINITION 3.1. *If, for* $f$ *in* $K$, *there are a sequence* $\{f_n\}$ *in* $C^2(\bar{D})$ *and* $g$ *in* $H_\delta$ *such that* $\lim_{n\to\infty}|||f_n-f|||=0$ *and*

$$\lim_{n\to\infty}\{(Af_n-g, h)_\delta+\langle Lf_n, h\rangle\}=0, \qquad \text{for each} \quad h\in C^2(\bar{D}),$$

*then we define* $A_L$ *by* $A_Lf=g$. *We write* $\mathcal{D}(A_L)$ *for the domain of* $A_L$.

PROPOSITION 3.2. *Let* $\lambda\geqq\lambda_0$. *For* $u$ *in* $K$ *and* $v$ *in* $H_\delta$,

$$B_\lambda(u, f)=(v, f)_\delta, \tag{3.1}$$

*for each* $f$ *in* $K$, *if and only if* $u$ *is in* $\mathcal{D}(A_L)$ *and*

$$(\lambda-A_L)u=v. \tag{3.2}$$

ii) *The solution of* (3.1) *or* (3.2) *is unique for* $v$ *in* $H_\delta$, *and satisfies*

$$\underline{k}|||u|||<\|v\|_\delta. \tag{3.3}$$

PROOF. i) For $u$ in $K$, there is $\{u_n\}$ in $C^2(\bar{D})$ such that $|||u_n-u|||\to 0$, and hence $B_\lambda(u_n, f)\to B_\lambda(u, f)$ for $f\in K$. Thus, for $v$ in $H_\delta$

$$\begin{aligned}(v, f)_\delta-B_\lambda(u, f)&=(v, f)_\delta-\lim_{n\to\infty}B_\lambda(u_n, f)\\&=(v, f)_\delta-\lim_{n\to\infty}\{((\lambda-A)u_n, f)_\delta-\langle Lu_n, f\rangle\}\\&=\lim_{n\to\infty}\{(Au_n-\lambda u+v, f)_\delta+\langle Lu_n, f\rangle\}, \qquad f\in C^2(\bar{D}).\end{aligned} \tag{3.4}$$

If $u$ and $v$ satisfy (3.1), we have (3.2) by (3.4). Conversely, if (3.2) holds, then there is $\{u_n\}$ such that the right hand side of (3.4) vanishes. Thus, we have (3.1) for $f$ in $C^2(\bar{D})$, and hence for $f$ in $K$ by continuity

ii) (3.3) is clear by (2.5) and by taking $f=u$ in (3.1):

$$\underline{k}|||u|||^2\leqq B_\lambda(u, u)=(v, u)_\delta\leqq\|v\|_\delta\|u\|_\delta\leqq\|v\|_\delta|||u|||.$$

Since $v=0$ implies $u=0$ by (3.3), the uniqueness of the solution of (3.1) is clear.

PROPOSITION 3.3. *For $v$ in $H_\delta$ and $\lambda\geqq\lambda_0$, there is a unique solution of* (3.1), *or* (3.2). *Hence, a linear operator $(\lambda-A_L)^{-1}$ is defined on $H_\delta$ which takes values in $K$ and satisfies*

$$\|\|(\lambda-A_L)^{-1}v\|\|\leqq \underline{k}^{-1}\|v\|_\delta . \tag{3.5}$$

PROOF. First, we take $v$ in $K$, and define $F(f)=(v, f)_\delta$ for $f\in K$. Since $|F(f)|=|(v, f)_\delta|\leqq\|v\|_\delta\cdot\|f\|_\delta\leqq\|v\|_\delta\cdot\|\|f\|\|$, $F(f)$ is continuous in $f$. Hence, there is $w$ in $K$ such that $F(f)=((w, f))$ by Riesz Theorem. Then, by (2.4)-(2.5) and Milgram-Lax Theorem, there is $u$ in $K$ such that $((w, f))=B_\lambda(u, f)$. Hence, $(v, f)_\delta=B_\lambda(u, f)$ for each $f\in K$. Since $u$ satisfies $\underline{k}\|\|u\|\|\leqq\|v\|_\delta$ and $K$ is dense in $H_\delta$, the mapping $v\to u$ can be extended over $H_\delta$ so that $B_\lambda(u, f)=(v, f)_\delta$ holds, completing the proof.

DEFINITION 3.2. We define a norm $\|f\|_L$ for $f\in K$ by

$$\|f\|_L=B_{\lambda_0}(f, f)^{1/2}.$$

PROPOSITION 3.4.

$$\underline{k}^{1/2}\|\|f\|\|\leqq\|f\|_L\leqq k_{\lambda_0}^{1/2}\|\|f\|\|, \qquad f\in K. \tag{3.6}$$

$$\|f\|_\delta\leqq\|f\|_L, \qquad f\in K. \tag{3.7}$$

$$\|f\|_L^2=(\lambda_0 f-A_L f, f)_\delta, \qquad f\in\mathfrak{D}(A_L). \tag{3.8}$$

$$(\lambda-\lambda_0)\|f\|_\delta^2+\|f\|_L^2=B_\lambda(f, f), \quad f\in K. \tag{3.9}$$

$$|(A_L f, g)_\delta-(A_L g, f)_\delta|<\bar{k}_1(\|f\|_L^2+\|g\|_\delta^2), \qquad f, g\in\mathfrak{D}(A_L), \tag{3.10}$$

*when $\delta$ is strictly negative, or $A$ and $L$ are formally self-adjoint.*

PROOF. (3.6) and (3.7) are clear by (2.4)-(2.5). We have (3.8) by taking $\lambda=\lambda_0$, $u=f$, $v=(\lambda_0-A_L)f$ in (3.1). (3.9) is obtained by taking $f=g$ in

$$(\lambda-\lambda_0)(f, g)_\delta+B_{\lambda_0}(f, g)=B_\lambda(f, g), \qquad f, g\in K. \tag{3.11}$$

(3.10) is obtained by (3.6), (3.8) and (2.7):

$$\begin{aligned}|(A_L f, g)_\delta-(A_L g, f)_\delta|&=|(\lambda_0 f-A_L f, g)_\delta-(\lambda_0 g-A_L g, f)_\delta|\\&=|B_{\lambda_0}(f, g)-B_{\lambda_0}(g, f)|\leqq k\|\|f\|\|\cdot\|g\|_\delta\\&\leqq\bar{k}\cdot\underline{k}^{-1/2}\|f\|_L\cdot\|g\|_\delta\leqq\bar{k}\cdot\underline{k}^{-1/2}(\|f\|_L^2+\|g\|_\delta^2)\end{aligned}$$

PROPOSITION 3.5. *For $u=(\lambda-A_L)^{-1}v$ and $\lambda\geqq\lambda_0$,*

$$(\lambda-\lambda_0)\|u\|_\delta^2+\|u\|_L^2=(u, v)_\delta, \qquad v\in H_\delta \tag{3.12}$$

(3.13) $$\|\lambda u-v\|_\delta^2+\lambda(\|u\|_L^2-\lambda_0\|u\|_\delta^2)=B_{\lambda_0}(u, v)-\lambda_0(u, v)_\delta, \qquad v\in K$$

(3.14) $$\|\lambda u-v\|_L^2+(\lambda-\lambda_0)\|\lambda u-v\|_\delta^2=(Av, \lambda u-v)_\delta+\langle Lv, \lambda u-v\rangle, \qquad v\in C^2(\bar{D}).$$

PROOF. By taking $f=u$ in (3.1), and by (3.9), we have (3.12):

$$(v, u)_\delta=B_\lambda(u, u)=(\lambda-\lambda_0)\|u\|_\delta^2+\|u\|_L^2.$$

(3.13) is obtained by (3.1), (3.9) and (3.11):

$$\begin{aligned}\|\lambda u-v\|_\delta^2&=\lambda(u, \lambda u-v)_\delta-(v, \lambda u-v)_\delta=\lambda(u, \lambda u-v)_\delta-B_\lambda(u, \lambda u-v)\\&=\lambda(u, \lambda u-v)_\delta-(\lambda-\lambda_0)(u, \lambda u-v)_\delta-B_{\lambda_0}(u, \lambda u-v)\\&=\lambda_0(u, \lambda u-v)_\delta-\lambda B_{\lambda_0}(u, u)+B_{\lambda_0}(u, v)\\&=\lambda_0\lambda\|u\|_\delta^2-\lambda_0(u, v)_\delta-\lambda\|u\|_L^2+B_{\lambda_0}(u, v).\end{aligned}$$

To prove (3.14), we use (3.1) and apply (2.10) for $f=v\in C^2(\bar{D})$ and $g=\lambda u-v$:

$$\|\lambda u-v\|_L^2=B_{\lambda_0}(\lambda u-v, \lambda u-v)=(\lambda_0-\lambda)\|\lambda u-v\|_\delta^2+B_\lambda(\lambda u-v, \lambda u-v),$$

where the last term is

$$\begin{aligned}&\lambda B_\lambda(u, \lambda u-v)-B_\lambda(v, \lambda u-v)\\&=\lambda(v, \lambda u-v)_\delta-((\lambda-A)v, \lambda u-v)_\delta+\langle Lv, \lambda u-v\rangle\\&=(Av, \lambda u-v)_\delta+\langle Lv, \lambda u-v\rangle.\end{aligned}$$

PROPOSITION 3.6. *Let* $\lambda\geqq\lambda_0$.

(3.15) $$(\lambda-\lambda_0)\|(\lambda-A_L)^{-1}v\|_\delta\leqq\|v\|_\delta, \qquad v\in H_\delta.$$

(3.16) $$(\lambda-\lambda_0^2)\|(\lambda-A_L)^{-1}v\|_L\leqq k'\|v\|_L, \qquad v\in K, \quad \textit{for a constant } k', \quad \lambda\geqq\lambda_0^2.$$

(3.17) $$\lim_{\lambda\to\infty}\|\lambda(\lambda-A_L)^{-1}v-v\|_\delta=0, \qquad v\in H_\delta.$$

(3.18) $$\lim_{\lambda\to\infty}\|\lambda(\lambda-A_L)^{-1}v-v\|_L=0, \qquad v\in K.$$

PROOF. We write $u=(\lambda-A_L)^{-1}$ as in Proposition 3.5. We have (3.15) by (3.12):

$$(\lambda-\lambda_0)\|u\|_\delta^2\leqq(u, v)_\delta\leqq\|u\|_\delta\cdot\|v\|_\delta.$$

We multiply the both sides of (3.12) by $\lambda_0$, and add to (3.13) to obtain

(3.19) $$\|\lambda u-v\|_\delta^2+(\lambda+\lambda_0)\|u\|_L^2-\lambda_0^2\|u\|_\delta^2=B_{\lambda_0}(u, v),$$

$$(\lambda-\lambda_0^2)\|u\|_L^2\leqq B_{\lambda_0}(u, v)\leqq k_{\lambda_0}|||u|||\cdot|||v|||\leqq k_{\lambda_0}\cdot\underline{k}^{-1}\|u\|_L\cdot\|v\|_L,$$

by (2.4)-(2.6) and (3.6), and hence we have (3.16). By (3.19) and (3.16), we have, for $\lambda\geqq\lambda_0^2$ and $v\in K$,

$$\|\lambda u-v\|_\delta^2\leqq B_{\lambda_0}(u, v)\leqq k_{\lambda_0}\underline{k}^{-1}\|u\|_L\cdot\|v\|_L$$

$$\leqq k_{\lambda_0}\underline{k}^{-1}k'\|v\|_L^2/(\lambda-\lambda_0^2)\to 0\,,\qquad \text{as}\quad \lambda\to\infty\,.$$

Since $K$ is dense in $H_\delta$, this holds for all $v$ in $H_\delta$. By (3.14),

$$\|\lambda u-v\|_L^2\leqq(Av,\ \lambda u-v)_\delta+\langle Lv,\ \lambda u-v\rangle$$
$$\leqq\|Av\|_\delta\|\lambda u-v\|_\delta+\|Lv\|_\partial\|\lambda u-v\|_\partial\,.$$

But, by (1.16) and $D(f,\ f)<|||f|||^2\leqq\underline{k}^{-1}\|f\|_L^2$, we have

$$\|\lambda u-v\|_\partial\leqq(\varepsilon D(\lambda u-v,\ \lambda u-v)+C(\varepsilon)\|\lambda u-v\|^2)^{1/2}$$
$$\leqq(\varepsilon/\underline{k})^{1/2}\|\lambda u-v\|_L+C(\varepsilon)^{1/2}\|\lambda u-v\|_\delta$$
$$\|\lambda u-v\|_L^2\leqq(\varepsilon/\underline{k})^{1/2}\|\lambda u-v\|_L+(\|Av\|_\delta+C(\varepsilon)^{1/2})\|\lambda u-v\|_\partial\,.$$

Thus, by (3.17), $\overline{\lim\limits_{\lambda\to\infty}}\|\lambda u-v\|_L$ is finite and

$$(\overline{\lim_{\lambda\to\infty}}\|\lambda u-v\|_L)^2\leqq(\varepsilon/\underline{k})^{1/2}\,\overline{\lim_{\lambda\to\infty}}\|\lambda u-v\|_L+0\,.$$

But, since $\varepsilon>0$ is arbitrary, $\lim\limits_{\lambda\to\infty}\|\lambda u-v\|_L=0$, completing the proof.

THEOREM 3.1. *$A_L$ is the generator of a semigroup of linear operators $\{T_t,\ t\geqq 0\}$ on $H_\delta$, which is strongly continuous in $t\geqq 0$ and satisfies $\|T_tf\|_\delta\leqq e^{\lambda_0 t}\|f\|_\delta$ for each $f\in H_\delta$. The domain $\mathcal{D}(A_L)$ is a dense subset of the space $K$.*

PROOF. By Proposition 3.3, $(\lambda-A_L)^{-1}$ is defined on $H_\delta$ for $\lambda\geqq\lambda_0$, and the norm is bounded by $(\lambda-\lambda_0)^{-1}$ by (3.15). The domain $\mathcal{D}(A_L)$ is dense in $H_\delta$ by (3.17). Hence, by Hille-Yosida Theorem, $A_L$ is the generator of a semigroup $\{T_t,\ t\geqq 0\}$ on $H_\delta$ as in the theorem. $\mathcal{D}(A_L)$ is dense $K$ by (3.18).

PROPOSITION 3.7. i) *If $f\in C^2(\bar{D})$ satisfies $Lf(x)=0$ on $\partial D$, then $f$ belongs to $\mathcal{D}(A_L)$ and $A_Lf(x)=Af(x)$ on $D_\delta$. Especially, when the mass of $\nu(x,\ \cdot)$ is concentrated on $\partial D$, that is, $\nu_D(x,\ \cdot)\equiv 0$, then $C_0^2(D)$ is contained in $\mathcal{D}(A_L)$ and $f\in C_0^2(D)$ satisfies $A_Lf(x)=Af(x)$ on $D_\delta$.*

ii) *If $f\in\mathcal{D}(A_L)$ is twice continuously differentiable on $D$, then $A_Lf(x)=Af(x)$ on $D$.*

iii) *If $f$ belongs to $\mathcal{D}(A_L)\cap C^2(\bar{D})$ and $A_Lf(x)$ is continuous on $D_\delta$, then $f$ satisfies $Lf(x)=0$ on $\partial D$ almost everywhere.*

PROOF. i) In this case, we can take $f_n=f$, $n=1, 2, \cdots$, and $g=Af$ in Definition 3.1, and hence $A_Lf=Af$ in $H_\delta$. If $\nu_D(x,\ \cdot)\equiv 0$, then each $f\in C_0^2(D)$ trivially satisfies $Lf(x)=0$, $x\in\partial D$.

ii) Let $\{f_n\}$ be a sequence as in Definition 3.1. Then, for $h$ in $C_0^2(D)$,

$$0=\{\lim_{n\to\infty}(Af_n-A_Lf,\ h)_\delta+\langle Lf_n,\ h\rangle\}=\lim_{n\to\infty}(Af_n,\ h)-(A_Lf,\ h)$$

$$=\lim_{n\to\infty}(f_n,\ A^*h)-(A_Lf,\ h)=(f,\ A^*h)-(A_Lf,\ h)=(Af-A_Lf,\ h)$$

and hence, $Af(x)=A_Lf(x)$ on $D$, where $A^*$ is the formal adjoint of $A$.

iii) Since $A_Lf(x)$ is continuous on $D_\delta$, $Af(x)=A_Lf(x)$ on $D_\delta$ by ii). Hence, for a sequence $\{f_n\}$ as in Definition 3.1 and $h\in C^2(\bar{D})$,

$$0=\lim_{n\to\infty}B_\lambda(f_n-f,\ h)=\lim_{n\to\infty}\lambda(f_n-f,\ h)_\delta-\lim_{n\to\infty}\{(Af_n-Af,\ h)_\delta+\langle Lf_n-Lf,\ h\rangle\}$$
$$=-\lim_{n\to\infty}\{(Af_n-A_Lf,\ h)_\delta+\langle Lf_n,\ h\rangle-(Af-A_Lf,\ h)_\delta-\langle Lf,\ h\rangle\}=\langle Lf,\ h\rangle .$$

Hence, $Lf(x)=0$ on $\partial D$ almost everywhere.

## §4. The group of operators for the wave equation

First, we prepare an abstract lemma which is independent of the preceding results.

LEMMA 4.1. *Let $H$ be a Hilbert space with inner product $(f, g)_s$ and norm $\|f\|_s=(f, f)^{1/2}$, and let the following conditions be satisfied.*

1) *There is a dense subspace $K$ of $H$, which is a Banach space with respect to a norm $\|f\|_l$ such that $\|f\|_s\leqq\|f\|_l$, for $f\in K$.*

2) *There is a semigroup of linear operators $\{T_t, f\geqq 0\}$ on $H$, which is strongly continuous in $t$ and satisfies $\|T_tf\|_s\leqq e^{\beta t}$, for $f\in H$.*

3) *The domain $\mathfrak{D}(\mathfrak{A})$ of the generator $\mathfrak{A}$ of $\{T_t\}$ is contained in $K$ and is dense as a subset of $K$.*

4) *There are positive constants $\lambda_0$ and $k$ such that*

$$(\lambda_0 f-\mathfrak{A}f,\ f)_s=\|f\|_l^2,\qquad f\in\mathfrak{D}(\mathfrak{A}), \tag{4.1}$$

$$|(\mathfrak{A}f,\ g)_s-(\mathfrak{A}g,\ f)_s|\leqq k(\|f\|_l^2+\|g\|_s^2),\qquad f,\ g\in\mathfrak{D}(\mathfrak{A}). \tag{4.2}$$

*Let $M$ be the product Banach space $\begin{pmatrix}K\\H\end{pmatrix}$ of $K$ and $H$ with norm*

$$\left\|\begin{pmatrix}f\\g\end{pmatrix}\right\|=(\|f\|_l^2+\|g\|_s^2)^{1/2}$$

*and let*

$$\mathfrak{G}\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}g\\\mathfrak{A}f\end{pmatrix},\quad \textit{for}\quad \begin{pmatrix}f\\g\end{pmatrix}\in\mathfrak{D}(\mathfrak{G})=\begin{pmatrix}\mathfrak{D}(\mathfrak{A})\\K\end{pmatrix}.$$

*Then, $\mathfrak{G}$ is the generator of a group of linear operators $\{U_t, -\infty<t<\infty\}$ on $M$, which is strongly continuous in $t$, and $\left\|U_t\begin{pmatrix}f\\g\end{pmatrix}\right\|\leqq e^{\beta'|t|}\left\|\begin{pmatrix}f\\g\end{pmatrix}\right\|$, for $\begin{pmatrix}f\\g\end{pmatrix}\in M$.*

PROOF. $\mathfrak{D}(\mathfrak{G})$ is dense in $M$ by 1) and 3). $\mathfrak{G}$ is closed, since $\mathfrak{A}$ is closed by 2). For $\lambda^2>\beta$, (4.3) $$(\lambda-\mathfrak{G})\binom{u}{v}=\binom{\lambda u-v}{\lambda v-\mathfrak{A}u}=\binom{f}{g}, \qquad \text{for } \binom{f}{g}\in M,$$

has a solution by 2) and

$$u=(\lambda^2-\mathfrak{A})^{-1}(\lambda f+g), \quad v=\lambda(\lambda^2-\mathfrak{A})^{-1}(\lambda f+g)-f.$$

The solution is unique, since $\lambda u-v=0$ and $\lambda v-\mathfrak{A}u=0$ imply $(\lambda^2-\mathfrak{A})u=0$, and hence $u=0$ by 2), and then $v=0$. Thus, $(\lambda-\mathfrak{G})^{-1}$ is defined on $M$ for $\lambda^2>\beta$. Hence, it is sufficient to prove the following for $\binom{f}{g}\in M$.

(4.4) $$\left\|\binom{u}{v}\right\|\leqq\left\|\binom{f}{g}\right\|/(|\lambda|-\beta'), \qquad \text{for a constant } \beta'.$$

In case $\binom{f}{g}$ belongs to $\mathfrak{D}(\mathfrak{G})$, this is obtained by (4.1), (4.2) and (4.3).

$$\begin{aligned}\left\|\binom{f}{g}\right\|^2&=\|f\|_l^2+\|g\|_s^2=(\lambda_0 f-\mathfrak{A}f, f)_s+(g, g)_s\\&=(\lambda_0(\lambda u-v)-\mathfrak{A}(\lambda u, u), \lambda u-v)_s+(\lambda v-\mathfrak{A}u, \lambda v-\mathfrak{A}u)_s\\&=\lambda^2\{(\lambda_0 u-\mathfrak{A}u, u)_s+(v, v)_s\}+\lambda\{(\mathfrak{A}v, u)_s-(\mathfrak{A}u, v)_s-2_{\lambda_0}(u, v)_s\}\\&\quad+(\lambda_0 v-\mathfrak{A}v, v)_s+(\mathfrak{A}u, \mathfrak{A}u)_s\\&=\lambda^2\left\|\binom{u}{v}\right\|^2+\lambda\{(\mathfrak{A}v, u)_s-(\mathfrak{A}u, v)_s-2\lambda_0(u, v)_s\}+\|v\|_l^2+\|\mathfrak{A}u\|_s^2\\&\geqq\lambda^2\left\|\binom{u}{v}\right\|^2-|\lambda|(k+2\lambda_0)\left\|\binom{u}{v}\right\|^2\\&\geqq\{|\lambda|-(k+2\lambda_0)\}^2\left\|\binom{u}{v}\right\|^2, \qquad \text{for } |\lambda|>k+2\lambda_0=\beta'.\end{aligned}$$

In case $\binom{f}{g}$ is in $M$, there is a sequence $\left\{\binom{f_n}{g_n}\right\}$ in $\mathfrak{D}(\mathfrak{G})$ such that $\binom{f_n}{g_n}\to\binom{f}{g}$. Then $\left\{\binom{u_n}{v_n}=(\lambda-\mathfrak{G})^{-1}\binom{f_n}{g_n}, n=1, 2, \cdots\right\}$ is a Cauchy sequence, since (4.4) applies for $\binom{f_n}{g_n}$, and hence has a limit $\binom{u}{v}$. Since $\mathfrak{G}$ is closed, this is the solution of (4.3), and hence,

$$\left\|\binom{u}{v}\right\|=\lim_{n\to\infty}\left\|\binom{u_n}{v_n}\right\|\leqq\lim_{n\to\infty}\left\|\binom{f_n}{g_n}\right\|/(|\lambda|-\beta')=\left\|\binom{f}{g}\right\|/(|\lambda|-\beta').$$

THEOREM 4.1. *Let $M_\delta$ be the product space $\binom{K}{H_\delta}$ of $K$ and $H_\delta$ with norm $\left\|\binom{f}{g}\right\|=(\|f\|_L^2+\|g\|_\delta^2)^{1/2}$, and let*

$$\mathfrak{G}_\delta\binom{f}{g}=\binom{g}{A_L f}, \quad for \quad \binom{f}{g}\in\mathcal{D}(\mathfrak{G}_\delta)=\binom{\mathcal{D}(A_L)}{K}.$$

*If $\delta(x)$ is strictly negative, or if $A$ and $L$ are formally self-adjoint, then $\mathfrak{G}_\delta$ is the generator of a group of linear operators $\{U_t, -\infty<t<\infty\}$ on $M_\delta$, which is strongly continuous in $t$, and $\left\|U_t\binom{t}{g}\right\|<e^{\beta'|t|}\left\|\binom{f}{g}\right\|$ for a constant $\beta'$. Thus for $f\in\mathcal{D}(A_L)$ and $g\in K$, there is a solution $u_t$ of*

$$(4.5)\qquad \begin{aligned} &\frac{d^2}{dt^2}u_t=A_L u_t, \qquad -\infty<t<\infty, \\ &U_t\to f \quad in \quad K, \qquad \frac{d}{dt}u_t\to g \quad in \quad H_\delta, \qquad as \quad t\to 0. \end{aligned}$$

PROOF. If we take $H_\delta$, $\|f\|_\delta$, $\|f\|_L$, $A_L$ and $\lambda_0$ for $H$, $\|f\|_s$, $\|f\|_t$, $\mathfrak{A}$ and $\beta$ in Lemma 4.1, and if $\delta$ is strictly negative or $A$ and $L$ and formally self-adjoint, then all conditions 1)–4) in Lemma 4.1 are satisfied. In fact, 1) is clear by the definitions of $H_\delta$ and $K$ and (3.7). 2) and 3) are satisfied by Theorem 3.1, and 4) holds by Proposition 3.4. For each $\binom{f}{g}\in\mathcal{D}(\mathfrak{G})$, $\binom{u_t}{v_t}=U_t\binom{f}{g}$ is the solution of (4.5) by

$$\frac{d}{dt}\binom{u_t}{v_t}=\mathfrak{G}U_t\binom{f}{g}=\mathfrak{G}\binom{u_t}{v_t}=\binom{v_t}{A_L u_t}, \quad \frac{d^2}{dt^2}u_t=\frac{d}{dt}v_t=A_L u_t.$$

THEOREM 4.2. *Let the coefficients of $A$ be in $C^{[N/2]+4}(\bar{D})$ and the mass of $\nu(x,\cdot)$ be concentrated on $\partial D$, that is, $\nu_D(x,\cdot)\equiv 0$. If $f$ and $g$ are in $C_0^{[N/2]+4}(D)$, then a version $u(t,x)$ of $u_t$ in $\binom{u_t}{v_t}=U_t\binom{f}{g}$ satisfies*

$$(4.6)\qquad \frac{\partial^2}{\partial t^2}u(t,x)=Au(t,x), \qquad (t,x)\in(-\infty,\infty)\times D.$$

*Moreover, if the coefficients of $A$ and $f$, $g$ are infinitely differentiable, then there is an infinitely differentiable version of $u_t$.*

REMARK 4.1. *If $f$ and $g$ belong to $\mathcal{D}(A_L{}^{[N/2]+4})$, then a version of $u_t$ in $\binom{u_t}{v_t}=U_t\binom{f}{g}$ satisfies (4.6). If $f$ and $g$ belong to $\mathcal{D}(A_L^n)$ for all $n\geqq 1$, then there is an infinitely differentiable version of $u_t$.*

PROOF. Let the coefficients of $A$ be in $C^{2m}(\bar{D})$, $f$ and $g$ in $C_0^{2m}(D)$ and let $\nu_D(x,\cdot)\equiv 0$. Then, $A^k f$ and $A^k g$ belong to $C_0^2(D)$ for $0\leqq k\leqq m-1$, and hence $f$ and $g$ belong to $\mathcal{D}(A_L^m)$ and $A_L^k f=A^k f$, $A_L^k g=A^k g$ on $D_\delta$ by i) of Proposition 3.7

for $1\leqq k\leqq m$. Since $\begin{pmatrix} f \\ g \end{pmatrix}\in\mathcal{D}(\mathfrak{G})$,

$$\frac{d}{dt}\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\mathfrak{G}U_t\begin{pmatrix} f \\ g \end{pmatrix}=U_t\mathfrak{G}\begin{pmatrix} f \\ g \end{pmatrix}=U_t\begin{pmatrix} g \\ A_L f \end{pmatrix} \quad\text{and}\quad \frac{d^2}{dt^2}\begin{pmatrix} u_t \\ v_t \end{pmatrix}=U_t\begin{pmatrix} A_L g \\ A_L f \end{pmatrix}.$$

Similarly, $\begin{pmatrix} u_t \\ v_t \end{pmatrix}=U_t\begin{pmatrix} f \\ g \end{pmatrix}$ belong to $\mathcal{D}(\mathfrak{G}^m)$ and $\frac{d^{2k}}{dt^{2k}}\begin{pmatrix} u_t \\ v_t \end{pmatrix}=U_t\begin{pmatrix} A_L^k f \\ A_L^k g \end{pmatrix}$, $1\leqq k\leqq m$, and

$$\left(\frac{d^2}{dt^2}+A_L\right)^k\begin{pmatrix} u_t \\ v_t \end{pmatrix}=2^k U_t\begin{pmatrix} A_L^k f \\ A_L^k g \end{pmatrix}, \qquad 1\leqq k\leqq m.$$

The strong derivative $\frac{d}{dt}u_t$ of $u_t$ in $H_\delta$ can be considered as the derivative $D_t u_t$ in the sense of Schwartz's distribution on $(-\infty, \infty)\times D$. $A_L f$ for $f\in\mathcal{D}(A_L)$ is also the derivative in the Schwartz's sense as a function on $D$

$$A(x, D_x)=\sum_{i,j=1}^{N} a_{ij}(x)D_{x_i}D_{x_j}f+\sum_{i=1}^{N} b_i(x)D_{x_i}f+c(x)f.$$

In fact, for each $h$ in $C_0^\infty(D)$ and a sequence $\{f_n\}$ in Definition 3.1,

$$\begin{aligned}(A_L f, h)=(A_L f, h)_\delta&=\lim_{n\to\infty}\{(Af_n, h)_\delta+\langle Lf_n, h\rangle\}=\lim_{n\to\infty}(Af_n, h)\\ &=\lim_{n\to\infty}(f_n, A^*h)=(f, A^*h).\end{aligned}$$

By Hille-Yosida Theorem, $\begin{pmatrix} u_t \\ v_t \end{pmatrix}=U_t\begin{pmatrix} f \\ g \end{pmatrix}$ for $\begin{pmatrix} f \\ g \end{pmatrix}\in M$ is given by

$$\text{(4.7)} \qquad \begin{pmatrix} u_t \\ v_t \end{pmatrix}=\text{strong}\lim_{n\to\infty}\begin{pmatrix} u_t^{(n)} \\ v_t^{(n)} \end{pmatrix}, \qquad \begin{pmatrix} u_t^{(n)} \\ v_t^{(n)} \end{pmatrix}=\sum_{m=0}^{\infty}\frac{t^m}{m!}(\mathfrak{G}_n)^m\begin{pmatrix} f \\ g \end{pmatrix}$$

where $\mathfrak{G}_n=n\mathfrak{G}(n-\mathfrak{G})^{-1}$. Since the both convergences in (4.7) are locally uniform in $t$ and $\frac{t^m}{m!}(\mathfrak{G}_n)^m\begin{pmatrix} f \\ g \end{pmatrix}$ are measurable in $(t, x)$, there are versions of $u_t$ and $v_t$ which are measurable in $(t, x)$. Clearly, these are locally square integrable as functions on $(-\infty, \infty)\times D$.

Hence, there is a version of $u_t$ such that $(D_t^2+A(x, D_x))^m u_t$ is locally square integrable as a function on $(-\infty, \infty)\times D$, and hence $u_t(x)$ belongs to $H_{\text{loc}}^{2m}((-\infty, \infty)\times D)$ since $(D_t^2+A(x, D_m))^m$ is elliptic. Thus, in order that $u_t(x)$ is of class $C^2$, it is enough to assume $2m\geqq\left[\frac{N}{2}\right]+3$ by the Sobolev's lemma.

If $f$, $g$ and the coefficients of $A$ are infinitely differentiable, there is an infinitely differentiable version of $u_t$.

The Remark 4.1 is clear, since we need the conditions in Theorem 4.1 only to derive $f, g\in\mathcal{D}(A_L^m)$, or $f, g\in\mathcal{D}(A_L^n)$ for all $n\geqq 1$.

THEOREM 4.3. *Let the coefficients of $A$ be in $C^{N+8}(\bar{D})$, $\nu_D(x, \cdot)\equiv 0$, and $f$ in $C_0^{N+8}(D)$. Then, a version $w(t, x)$ of $w_t=T_t f$ satisfies*

$$\frac{\partial u}{\partial t}(t, x)=Au(t, x), \qquad (t, x)\in(0, \infty)\times D. \tag{4.8}$$

*Moreover, if the coefficients of $A$ and $f$ are infinitely differentiable, so is a version of $w_t$.*

REMARK 4.2. If $f$ is in $\mathcal{D}(A_L^{N+8})$, then a version of $w_t=T_t f$ satisfies (4.8). If $f$ is in $\mathcal{D}(A_L^n)$ for all $n\geqq 1$, there is an infinitely differentiable version of $w_t$.

PROOF. Let the coefficients of $A$ be in $C^{4m}(\bar{D})$ and $f$ in $C_0^{4m}(D)$, and let $\nu_D(x, \cdot)\equiv 0$. Then, $f$ is in $\mathcal{D}(A_L^{2m})$ and $\left(\frac{d^2}{dt^2}+A_L\right)^k w_t=T_t(A_L+A_L^2)^k f$, $0\leqq k\leqq 2m$. Hence, there is a measurable version of $w_t$ such that $(D_t^2+A(x, D_x))^m w_t$ is locally square integrable on $(0, \infty)\times D$, as in the proof of Theorem 4.2. Since the version is in $H_{\mathrm{loc}}^{2m}((0, \infty)\times D)$ as before, it is enough to assume $2m\geqq\left[\frac{N}{2}\right]+3$ again. Remark 4.2 is clear, since we need the conditions in Theorem 4.3 only to derive $f\in\mathcal{D}(A_L^{2m})$, or $f\in\mathcal{D}(A_L^n)$ for all $n\geqq 1$.

## § 5. A semigroup and a group of operators on the boundary

It is known that there is a unique solution of the Dirichlet problem

$$Au(x)=0, \quad x\in D, \quad u(x)=\varphi(x), \quad x\in\partial D, \qquad \text{for } \varphi\in C(\partial D).$$

The solution $u$ is written as

$$u(x)=H\varphi(x)=\int_{\partial D} H(x, dy)\varphi(y)$$

by a bounded measure $H(x, dy)$[6] on $\partial D$, and

$$\|H\varphi\|\leqq C_\partial\|\varphi\|_\partial, \tag{5.1}$$

by a constant $C_\partial$. For $\varphi$ in $C^3(\partial D)$, $H\varphi$ belongs to $C^2(\bar{D})$, and hence $LH\varphi$ is in $C(\partial D)$. We define

$$B_\lambda\langle\varphi, \psi\rangle=\langle(\lambda-LH)\varphi, \psi\rangle \quad \text{for} \quad \varphi, \psi\in C^3(\partial D), \quad \lambda\geqq 0, \tag{5.2}$$

---

6) This measure is called the *hitting measure* to $\partial D$ and is the distribution of the first hit to $\partial D$ of the diffusion particle, which has started at $x$ and diffused in $D$ according to (0.3). In case $A=\Delta=\sum_1^N\frac{\partial^2}{\partial x_i^2}$, this is also known as the *harmonic measure* of $D$ with respect to $x$.

$$\langle\langle\varphi, \psi\rangle\rangle=\langle\varphi, \psi\rangle+\tilde{D}\langle\varphi, \psi\rangle+D(H\varphi, H\psi)+\nu(\varphi, \psi)^{7)}, \qquad \text{for} \quad \varphi, \psi\in C^3(\partial D)$$

$$|||\varphi|||_\partial=\langle\langle\varphi, \psi\rangle\rangle^{1/2}, \qquad \text{for} \quad \varphi\in C^3(\partial D).$$

Let $\tilde{H}$ and $\tilde{K}$ be the completions of $C^3(\partial D)$ by $\| \ \|_\partial$ and $||| \ |||_\partial$, respectively.

PROPOSITION 5.1. i) *There are constants $\underset{\sim}{C}$ and $\tilde{C}$ such that for $\varphi\in C^3(\partial D)$,*

$$\underset{\sim}{C}|||\varphi|||_\partial\leqq|||\varphi|||<\tilde{C}|||\varphi|||_\partial. \tag{5.3}$$

ii) *If a sequence $\{\varphi_n\}$ in $C^3(\partial D)$ satisfies $\lim_{n\to\infty}\|\varphi_n\|_\partial=0$ and $\lim_{m,n\to\infty}|||\varphi_m-\varphi_n|||_\partial=0$, then $\lim_{n\to\infty}|||\varphi_n|||_\partial=0$. Thus, $\tilde{K}$ can be imbedded in $\tilde{H}$ as a dense subset.*

PROOF. i) $|||H\varphi|||^2\equiv\|H\varphi\|^2+\langle\varphi, |\delta|\varphi\rangle+D(H\varphi, H\varphi)+D\langle\varphi, \varphi\rangle+\nu(\varphi, \varphi)$

$$\leqq(C_\partial^2+\bar{\delta})\|\varphi\|_\partial^2+D(H\varphi, H\varphi)+\tilde{D}\langle\varphi, \varphi\rangle+\nu(\varphi, \varphi)\leqq(1+C_\partial^2+\bar{\delta})|||\varphi|||_\partial^2,$$

where $\bar{\delta}=\sup_{x\in\partial D}|\delta(x)|$. Let $C_1=(2\bar{\delta})^{-1}$ when $\bar{\delta}>0$, and $C_1=0$ when $\bar{\delta}=0$. Then, by (1.16),

$$\begin{aligned}|||\varphi|||_\partial^2&=\|\varphi\|_\partial^2+D(H\varphi, H\varphi)+\tilde{D}\langle\varphi, \varphi\rangle+\nu(\varphi, \varphi)\\&=C_1\langle\varphi, |\delta|\varphi\rangle+\langle\varphi, (1+C_1\delta)\varphi\rangle+D(H\varphi, H\varphi)+\tilde{D}\langle\varphi, \varphi\rangle+\nu(\varphi, \varphi)\\&\leqq C_1\langle\varphi, |\delta|\varphi\rangle+C(\varepsilon)\|H\varphi\|^2+(1+\varepsilon)D(H\varphi, H\varphi)\\&\quad+\tilde{D}\langle\varphi, \varphi\rangle+\nu(\varphi, \varphi)\leqq(1+C_1+\varepsilon+C(\varepsilon))|||H\varphi|||^2.\end{aligned}$$

ii) By the assumption, we have $|||H\varphi_m-H\varphi_n|||\leqq\tilde{C}|||\varphi_m-\varphi_n|||_\partial\to0$ and $\|H\varphi_n\|_\partial^2=\|H\varphi_n\|^2+\langle\varphi_n, |\delta|\varphi_n\rangle\leqq(C_\partial^2+\bar{\delta})\|\varphi_n\|^2\to0$, as $m, n\to\infty$. Hence, by Proposition 1.3, $|||H\varphi_n|||\to0$, and hence $|||\varphi_n|||_\partial\to0$ by (5.3).

PROPOSITION 5.2. *$B_\lambda\langle\varphi, \psi\rangle$ can be extended to a bilinear form on $\tilde{K}$ uniquely. The extension, written by the same notation, satisfies the following, where $\lambda_1\geqq0$, $\underset{\sim}{k}$, $\tilde{k}$, $\tilde{k}_\lambda>0$ are constants.*

$$|B_\lambda\langle\varphi, \psi\rangle|\leqq\tilde{k}_\lambda|||\varphi|||_\partial\cdot|||\psi|||_\partial, \qquad \lambda\geqq0, \quad \varphi, \psi\in\tilde{K}. \tag{5.4}$$

$$B_\lambda\langle\varphi, \varphi\rangle\geqq\underset{\sim}{k}|||\varphi|||_\partial^2, \qquad \lambda\geqq\lambda_1, \quad \varphi\in\tilde{K}. \tag{5.5}$$

$$B_\lambda\langle\varphi, \varphi\rangle\geqq\|\varphi\|_\partial^2, \qquad \lambda\geqq\lambda_1, \quad \varphi\in\tilde{K}. \tag{5.6}$$

*When $\delta$ is strictly negative, or $A$ and $L$ are formally self-adjoint,*

$$|B_\lambda\langle\varphi, \psi\rangle-B_\lambda\langle\psi, \varphi\rangle|\leqq\tilde{k}|||\varphi|||_\partial\cdot\|\psi\|_\partial, \quad \lambda\geqq0, \quad \varphi, \psi\in\tilde{K}. \tag{5.7}$$

$$B_\lambda\langle\varphi, \psi\rangle=\langle(\lambda-LH)\varphi, \psi\rangle, \quad \lambda\geqq0, \quad \varphi\in C^3(\partial D), \quad \psi\in\tilde{K}. \tag{5.8}$$

7) $\langle \ , \ \rangle$, $\tilde{D}\langle \ \rangle$ and $\nu( \ , \ )$ are defined as in §1. These quantities are determined only by the values on $\partial D$.

PROOF. Let $\varphi$ and $\psi$ be in $C^3(\partial D)$. Since $H\varphi$ is in $C^2(\bar{D})$, $AH\varphi\equiv 0$ on $\bar{D}$, and hence,

$$B_{\lambda_0}(H\varphi, H\psi)=((\lambda_0-A)H\varphi, H\psi)_\delta-\langle LH\varphi, \psi\rangle$$
$$=\lambda_0(H\varphi, H\psi)_\delta-\lambda\langle\varphi, \psi\rangle+\langle(\lambda-LH)\varphi, \psi\rangle$$
$$=\lambda_0(H\varphi, H\psi)+\lambda_0\langle\varphi, |\delta|\psi\rangle-\lambda\langle\varphi, \psi\rangle+B_\lambda\langle\varphi, \psi\rangle.$$

By Proposition 2.2, (5.1) and (5.3),

$$|B_\lambda\langle\varphi, \psi\rangle|\leqq|B_{\lambda_0}(H\varphi, H\psi)|+\lambda_0\|H\varphi\|\cdot\|H\psi\|+(\lambda+\lambda_0\bar{\delta})\|\varphi\|_\partial\cdot\|\psi\|_\partial$$
$$\leqq k_{\lambda_0}\tilde{C}_2|||\varphi|||_\partial\cdot|||\psi|||_\partial+(\lambda_0C_\partial^2+\lambda+\lambda_0\bar{\delta})\|\varphi\|_\partial\cdot\|\psi\|_\partial$$
$$\leqq(k_{\lambda_0}\tilde{C}^2+\lambda_0C_\partial^2+\lambda+\lambda_0\bar{\delta})|||\varphi|||_\partial\cdot|||\psi|||_\partial.$$

$$B_\lambda\langle\varphi, \varphi\rangle\geqq B_{\lambda_0}(H\varphi, H\varphi)+\lambda\langle\varphi, \varphi\rangle-\lambda_0\{(H\varphi, H\varphi)+\langle\varphi, |\delta|\varphi\rangle\}$$
$$\geqq\underset{\sim}{k}\underset{\sim}{C}^2|||\varphi|||_\partial^2+(\lambda-\lambda_0C_\partial^2-\lambda_0\bar{\delta})\|\varphi\|_\partial^2.$$

Hence, (5.4)-(5.6) hold for $\lambda_1=\lambda_0(C_\partial^2+\bar{\delta})+1$ and $\underset{\sim}{k}=\underset{\sim}{k}\underset{\sim}{C}^2$. By ii) of Proposition 2.2,

$$|B_\lambda\langle\varphi, \psi\rangle-B_\lambda\langle\psi, \varphi\rangle|=|B_{\lambda_0}(H\varphi, H\psi)-B_{\lambda_0}(H\psi, H\varphi)|$$
$$\leqq\bar{k}|||H\varphi|||\cdot\|H\psi\|_\delta\leqq\bar{k}\cdot\tilde{C}\cdot C_\partial|||\varphi|||_\partial\cdot\|\psi\|_\partial.$$

It is clear that $B_\lambda\langle\varphi, \psi\rangle$ and the estimates (5.4)-(5.7) can be extended for $\varphi$, $\psi$ in $\tilde{K}$. (5.8) is also obtained by continuity.

PROPOSITION 5.3. *If* $\{\varphi_n\}$ *in* $C^3(\partial D)$ *and* $\psi$ *in* $\tilde{H}$ *satisfy* $\lim_{n\to\infty}|||\varphi|||_\partial=0$ *and* $\lim_{n\to\infty}\langle LH\varphi_n-\psi, \tau\rangle=0$ *for each* $\tau\in C^3(\partial D)$, *then* $\psi=0$.

PROOF. Since $|||\varphi_n|||_\partial\to 0$, we have $B_\lambda\langle\varphi_n, \tau\rangle\to 0$ and $\langle\varphi_n, \tau\rangle\to 0$, and hence,

$$0=\lim_{n\to\infty}\langle LH\varphi_n-\psi, \tau\rangle$$
$$=\lim_{n\to\infty}\{\langle\lambda\varphi_n-\psi, \tau\rangle-B_\lambda\langle\varphi_n, \tau\rangle=-\langle\psi, \tau\rangle, \qquad \text{for} \quad \tau\in C^3(\partial D).$$

Since $C^3(\partial D)$ is dense in $\tilde{H}$, $\psi=0$.

DEFINITION 5.1. *If, for* $\varphi$ *in* $\tilde{K}$, *there are a sequence* $\{\varphi_n\}$ *in* $C^3(\partial D)$ *and* $\psi$ *in* $\tilde{H}$ *such that* $\lim_{n\to\infty}|||\varphi_n-\varphi|||_\partial=0$ *and* $\lim_{n\to\infty}\langle LH\varphi_n-\psi, \tau\rangle=0$ *for* $\tau\in C^3(\partial D)$, *then we define* $\overline{LH}\varphi=\psi$. *We write* $\mathcal{D}(\overline{LH})$ *for the domain of* $\overline{LH}$.

PROPOSITION 5.4. *Let* $\lambda\geqq\lambda_1$. i) *For* $\psi$ *in* $\tilde{K}$ *and* $\varphi$ *in* $\tilde{H}$,

$$B_\lambda\langle\psi, \tau\rangle=\langle\varphi, \tau\rangle \quad \textit{for each } \tau\in\tilde{K}, \tag{5.9}$$

*if and only if* $\psi$ *is in* $\mathcal{D}(\overline{LH})$ *and*

(5.10) $$(\lambda-\overline{LH})\psi=\varphi\,.$$

ii) *For $\varphi$ in $\tilde{H}$, there is a unique solution of* (5.9) *and* (5.10) *with values in $\tilde{K}$ such that*

(5.11) $$|||(\lambda-\overline{LH})^{-1}\varphi|||_\partial \leqq \underset{\sim}{k}^{-1}\|\varphi\|_\partial\,.$$

PROOF. i) If $|||\psi_n-\psi|||_\partial\to 0$ for $\psi_n\in C^3(\partial D)$ and $\psi\in\tilde{K}$, we have, by (5.8),

(5.12) $$\begin{aligned}\langle\varphi,\tau\rangle-B_\lambda\langle\psi,\tau\rangle&=\lim_{n\to\infty}\{\langle\varphi,\tau\rangle-B_\lambda\langle\psi_n,\tau\rangle\}\\&=\lim_{n\to\infty}\langle LH\psi_n-(\lambda\psi-\varphi),\tau\rangle\,,\qquad \tau\in\tilde{K}\,.\end{aligned}$$

If $\varphi$ and $\psi$ satisfy (5.9), there is $\{\psi_n\}$ such that $|||\psi_n-\psi|||_\partial\to 0$ and (5.12) vanishes. Hence, $\overline{LH}\psi=\lambda\psi-\varphi$ by definition of $\overline{LH}$. Conversely, if $\varphi$ and $\psi$ satisfy (5.10), then there is $\{\psi_n\}$ such that $|||\psi_n-\psi|||_\partial\to 0$ and the right hand side of (5.12) vanishes, and hence $\varphi$ and $\psi$ satisfy (5.9).

ii) If $\varphi$ and $\psi$ satisfy (5.9), we have $|||\psi|||_\partial\leqq k^{-1}\|\varphi\|_\partial$ by taking $\tau=\psi$ in (5.9):

$$|||\psi|||_\partial^2\leqq\underset{\sim}{k}^{-1}B_\lambda\langle\psi,\psi\rangle=\underset{\sim}{k}^{-1}\langle\varphi,\psi\rangle\leqq\underset{\sim}{k}^{-1}\|\varphi\|_\partial\cdot\|\psi\|_\partial\leqq\underset{\sim}{k}^{-1}\|\varphi\|_\partial\cdot|||\psi|||_\partial\,.$$

Hence, the uniqueness of the solution of (5.9) or (5.10) is clear.

To prove the existence of the solution, we take $\varphi$ in $\tilde{K}$, and define $F(\tau)=\langle\varphi,\tau\rangle$ on $\tilde{K}$. Since $F(\tau)$ is continuous by $|F(\tau)|\leqq\|\varphi\|_\partial\cdot\|\tau\|_\partial\leqq\|\varphi\|_\partial\cdot|||\tau|||_\partial$, there is $\sigma$ in $\tilde{K}$ such that $F(\tau)=\langle\langle\sigma,\tau\rangle\rangle$. Then, (5.4)-(5.5) and Milgram-Lax Theorem, there is $\psi$ such that $B_\lambda\langle\psi,\tau\rangle=\langle\langle\sigma,\tau\rangle\rangle$. Hence, $\psi$ is the unique solution of (5.9)-(5.10). Since the mapping $\varphi\to\psi=(\lambda-\overline{LH})^{-1}\varphi$ satisfies $|||\psi|||_\partial\leqq\underset{\sim}{k}^{-1}\|\varphi\|_\partial$ for $\varphi\in\tilde{K}$, it can be extended over $\tilde{H}$ so that (5.9) and (5.11) hold.

DEFINITION 5.2. *For $\varphi\in\tilde{K}$, we define*

$$\|\varphi\|_{L,\partial}=B_{\lambda_1}\langle\varphi,\varphi\rangle^{1/2}\,.$$

PROPOSITION 5.5.

(5.13) $$\underset{\sim}{k}^{1/2}|||\varphi|||_\partial\leqq\|\varphi\|_{L,\partial}\leqq\tilde{k}_{\lambda_1}^{1/2}|||\varphi|||_\partial\,,\qquad\varphi\in\tilde{K}\,.$$

(5.14) $$\|\varphi\|_\partial\leqq\|\varphi\|_{L,\partial}\,,\qquad\varphi\in\tilde{K}\,.$$

(5.15) $$\|\varphi\|_{L,\partial}^2=\langle(\lambda_1-\overline{LH})\varphi,\varphi\rangle\,,\qquad\varphi\in\mathfrak{D}(\overline{LH})\,.$$

(5.16) $$|\langle\overline{LH}\varphi,\psi\rangle-\langle\overline{LH}\psi,\varphi\rangle|\leqq\tilde{k}|||\varphi|||_\partial\cdot\|\psi\|_\partial\,,\qquad\varphi,\psi\in\mathfrak{D}(\overline{LH})\,.$$

(5.17) $$(\lambda-\lambda_1)\|\varphi\|_\partial^2+\|\varphi\|_{L,\partial}^2=B_\lambda\langle\varphi,\varphi\rangle\,,\qquad\varphi\in\tilde{K}\,.$$

PROOF. (5.13) and (5.14) are clear by (5.4)-(5.6). By the definition of $\overline{LH}$ and (5.8), (5.18) $$\langle(\lambda-\overline{LH})\varphi, \psi\rangle=B_\lambda\langle\varphi, \psi\rangle, \quad \varphi\in\mathcal{D}(\overline{LH}), \quad \psi\in\tilde{K}.$$

Thus, we have (5.15) by $\lambda=\lambda_1$ and $\psi=\varphi$ in (5.18), and (5.16) by (5.7) and (5.8).

PROPOSITION 5.6. *For $\lambda\geqq\lambda_1$ and $\psi=(\lambda-\overline{LH})^{-1}\varphi$,*

(5.19) $$(\lambda-\lambda_1)\|\psi\|_\partial^2+\|\psi\|_{L,\partial}^2=\langle\psi, \varphi\rangle, \quad \varphi\in\tilde{H},$$

(5.20) $$\|\lambda\psi-\varphi\|_\partial^2+\lambda(\|\varphi\|_{L,\partial}^2-\lambda_1\|\varphi\|_\partial^2)=B_{\lambda_1}\langle\psi, \varphi\rangle-\lambda_1\langle\psi, \varphi\rangle, \quad \varphi\in\tilde{K},$$

(5.21) $$\|\lambda\psi-\psi\|_{L,\partial}^2+(\lambda-\lambda_1)\|\lambda\psi-\varphi\|_\partial^2=\langle LH\varphi, \lambda\psi-\varphi\rangle, \quad \varphi\in C^3(\partial D).$$

PROOF. These are proved by using (5.9) repeatedly, taking $\psi$ for $\tau$ to obtain (5.19) and $\lambda\psi-\varphi$ for $\tau$ to obtain (5.20) and (5.21). To prove (5.21), we also use (5.8).

$$\langle\varphi, \psi\rangle=B_\lambda\langle\psi, \psi\rangle=(\lambda-\lambda_1)\|\psi\|_\partial^2+B_{\lambda_1}\langle\psi, \psi\rangle=(\lambda-\lambda_1)\|\psi\|_\partial^2+\|\psi\|_{L,\partial}^2$$

For (5.20): $\|\lambda\psi-\varphi\|_\partial^2=\langle\lambda\psi-\varphi, \lambda\psi-\varphi\rangle$

$$=\lambda\langle\psi, \lambda\psi-\varphi\rangle-\langle\varphi, \lambda\psi-\varphi\rangle=\lambda\langle\psi, \lambda\psi-\varphi\rangle-B_\lambda\langle\psi, \lambda\psi-\varphi\rangle$$
$$=\lambda\langle\psi, \lambda\psi-\varphi\rangle-(\lambda-\lambda_1)\langle\psi, \lambda\psi-\varphi\rangle-B_{\lambda_1}\langle\psi, \lambda\psi-\varphi\rangle$$
$$=\lambda_1\cdot\lambda\|\psi\|_\partial^2-\lambda_1\langle\psi, \varphi\rangle-\lambda\|\psi\|_{L,\partial}^2+B_{\lambda_1}\langle\psi, \varphi\rangle.$$

For (5.21): $\|\lambda\psi-\varphi\|_{L,\partial}^2=B_{\lambda_1}\langle\lambda\psi-\varphi, \lambda\psi-\varphi\rangle$

$$=(\lambda_1-\lambda)\|\lambda\psi-\varphi\|_\partial^2+B_\lambda\langle\lambda\psi-\varphi, \lambda\psi-\varphi\rangle$$
$$=(\lambda_1-\lambda)\|\lambda\psi-\varphi\|_\partial^2+\lambda\langle\varphi, \lambda\psi-\varphi\rangle-B_\lambda\langle\varphi, \lambda\psi-\varphi\rangle$$
$$=(\lambda_1-\lambda)\|\lambda\psi-\varphi\|_\partial^2+\lambda\langle\varphi, \lambda\psi-\varphi\rangle-\lambda\langle\varphi, \lambda\psi-\varphi\rangle+\langle LH\varphi, \lambda\psi-\varphi\rangle.$$

PROPOSITION 5.7. *For $\lambda\geqq\lambda_1$,*

(5.22) $$(\lambda-\lambda_1)\|(\lambda-\overline{LH})^{-1}\varphi\|_\partial\leqq\|\varphi\|_\partial, \quad \varphi\in H.$$

(5.23) $$(\lambda-\lambda_1^2)\|(\lambda-\overline{LH})^{-1}\varphi\|_{L,\partial}\leqq\tilde{k}'\|\varphi\|_{L,\partial}, \quad \varphi\in\tilde{K}, \text{ for a constant } k'.$$

(5.24) $$\lim_{n\to\infty}\|\lambda(\lambda-\overline{LH})^{-1}\varphi-\varphi\|_\partial=0, \quad \varphi\in\tilde{H}.$$

(5.25) $$\lim_{n\to\infty}\|\lambda(\lambda-\overline{LH})^{-1}\varphi-\varphi\|_{L,\partial}=0, \quad \varphi\in\tilde{K}.$$

PROOF. We write $\psi$ for $(\lambda-\overline{LH})^{-1}\varphi$. (5.22) is clear by (5.19). We can prove (5.23) similarly as (3.16), by adding (5.20) to (5.19) multiplied by $\lambda_1$:

(5.26) $$\|\lambda\psi-\varphi\|_\partial^2+(\lambda+\lambda_1)\|\varphi\|_{L,\partial}^2-\lambda_1^2\|\psi\|_\partial^2=B_{\lambda_1}\langle\psi, \varphi\rangle\leqq\tilde{k}_{\lambda_1}\cdot\underset{\sim}{k}^{-1}\|\varphi\|_{L,\partial}\cdot\|\psi\|_{L,\partial},$$

$$(\lambda-\lambda_1^2)\|\psi\|_{L,\partial}^2\leqq\lambda\|\psi\|_{L,\partial}^2-\lambda_1^2\|\psi\|_\partial^2\leqq\tilde{k}_{\lambda_1}\cdot\underset{\sim}{k}^{-1}\|\varphi\|_{L,\partial}\cdot\|\psi\|_{L,\partial}.$$

(5.24) also follows from (5.26) and (5.23) for $\varphi\in\tilde{K}$:

$$\|\lambda\psi-\varphi\|_\partial^2 \leqq \tilde{k}_{\lambda_1}\cdot \underset{\sim}{k}^{-1}\|\varphi\|_{L,\partial}\cdot\|\psi\|_{L,\partial}$$
$$\leqq(\tilde{k}_{\lambda_1}\cdot\underset{\sim}{k}^{-1})^2(\lambda-\lambda_1^2)\|\varphi\|_{L,\partial}^2\to 0, \quad \text{as } \lambda\to\infty.$$

Since $\tilde{K}$ is dense in $\tilde{H}$ and by (5.22), (5.24) holds for $\varphi\in\tilde{H}$. By (5.21), we have

$$\|\lambda\psi-\varphi\|_{L,\partial}^2 \leqq \|\overline{LH}\varphi\|_\partial\cdot\|\lambda\psi-\varphi\|_\partial .$$

By (1.16) and (5.13),

$$\|\lambda\psi-\varphi\|_\partial \leqq (\varepsilon D(H(\lambda\psi-\varphi), H(\lambda\psi-\varphi))+C(\varepsilon)\|H(\lambda\psi-\varphi)\|^2)^{1/2}$$
$$\leqq(\varepsilon|||H(\lambda\psi-\varphi)|||^2+C(\varepsilon)\|H(\lambda\psi-\varphi)\|^2)^{1/2}$$
$$\leqq(\varepsilon/\underset{\sim}{k})^{1/2}\|\lambda\psi-\varphi\|_{L,\partial}+(C(\varepsilon)\cdot C_\partial)^{1/2}\|\lambda\psi-\varphi\|_\partial .$$

Hence, by (5.24), $\overline{\lim}_{\lambda\to\infty}\|\lambda\psi-\varphi\|_{L,\partial}$ is finite, and

$$(\overline{\lim}_{\lambda\to\infty}\|\lambda\psi-\varphi\|_{L,\partial})^2 \leqq (\varepsilon/\underset{\sim}{k})^{1/2}\,\overline{\lim}_{\lambda\to\infty}\|\lambda\psi-\varphi\|_{L,\partial}+0 .$$

But, since $\varepsilon>0$ is arbitrary, $\lim_{\lambda\to\infty}\|\lambda\psi-\varphi\|_{L,\partial}=0$, completing the proof.

THEOREM 5.1. *$\overline{LH}$ is the generator of a semigroup of linear operators $\{\tilde{T}_t, t\geqq 0\}$ on $\tilde{H}$, which is strongly continuous in $t\geqq 0$ and satisfies $\|\tilde{T}_t\varphi\|_\partial \leqq e^{\lambda_1 t}\|\varphi\|_\partial$ for each $\varphi\in\tilde{H}$. The domain $\mathfrak{D}(\overline{LH})$ is a dense subset of the space $\tilde{K}$.*

This is clear by the Proposition 5.4 and 5.7 and Hille-Yosida Theorem.

THEOREM 5.2. *Let $\tilde{M}$ be the product space $\begin{pmatrix}\tilde{K}\\ \tilde{H}\end{pmatrix}$ of $\tilde{K}$ and $\tilde{H}$ with norm $\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|=(\|\varphi\|_{L,\partial}^2+\|\psi\|_\partial^2)^{1/2}$, and let*

$$\tilde{\mathfrak{G}}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\begin{pmatrix}\psi\\ \overline{LH}\varphi\end{pmatrix}, \quad \textit{for} \quad \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\in\mathfrak{D}(\tilde{\mathfrak{G}})=\begin{pmatrix}\mathfrak{D}(\overline{LH})\\ \tilde{K}\end{pmatrix}.$$

*If $\delta(x)$ is strictly negative, or if $A$ and $L$ are formally self-adjoint, then $\tilde{\mathfrak{G}}$ is the generator of a group of linear operators $\{\tilde{U}_t, -\infty<t<\infty\}$ on $\tilde{M}$, which is strongly continuous in $t$, and satisfies $\left\|U_t\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|\leqq e^{\tilde{\mathfrak{z}}|t|}\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|$ for each $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\in\tilde{M}$ and a constant $\tilde{\mathfrak{z}}$. Thus, for $\varphi\in\mathfrak{D}(\overline{LH})$ and $\psi\in\tilde{K}$, $\varphi_t$ of $\begin{pmatrix}\varphi_t\\ \psi_t\end{pmatrix}=\tilde{U}_t\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ satisfies*

$$\frac{d^2}{dt^2}\varphi_t=\overline{LH}\varphi_t,\ -\infty<t<\infty, \quad \varphi_t\to\varphi \ \text{ in } \tilde{K}, \quad \frac{d}{dt}\varphi_t\to\psi \ \text{ in } \tilde{H}, \ \text{ as } t\to 0.$$

This is obtained by taking $\tilde{H}$, $\tilde{K}$, $\|\varphi\|_\partial$, $\|\varphi\|_{L,\partial}$, $\{\tilde{T}_t\}$ and $\overline{LH}$ for $H$, $K$, $\|\varphi\|_s$, $\|\varphi\|_l$, $\{T_t\}$ and $\mathfrak{A}$ in Lemma 4.1.

The operator $\overline{LH}$ on the boundary is essentially an integro-differential operator as in

PROPOSITION 5.8.[8] *For $\varphi$ in $C^3(\partial D)$,*

$$\overline{LH}\varphi(x)=LH\varphi(x)=\sum_{i,j=1}^{N-1}\tilde{\alpha}_{ij}(x)\frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j}(x)$$

$$+\sum_{i=1}^{N-1}\tilde{\beta}_i(x)\frac{\partial\varphi}{\partial\xi_i}(x)+\tilde{\gamma}(x)\varphi(x)$$

$$+\int_{\partial D}\left\{\varphi(y)-\varphi(x)-\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_i}(x)\xi_i^x(y)\right\}\tilde{\nu}(x, dy),$$

*where $\{\tilde{\alpha}_{ij}(x)\}$ is symmetric and non-negative definite, $\tilde{\gamma}(x)\leqq 0$ and $\nu(x, \cdot)$ is a measure on $\partial D$ such that, for each neighbourhood $U_x$ of $x$*

$$\int_{U_x\cap\partial D}\sum_{i=1}^{N-1}(\xi_i^x(y))^2\tilde{\nu}(x, dy)+\tilde{\nu}(x, U_x^c\cap\partial D)<\infty .$$

## §6. Comments on $\{\tilde{U}_t\}$ and the boundary condition

The diffusion equation with the boundary condition (0.3)-(0.2)

$$\frac{\partial}{\partial t}u(t, x)=Au(t, x),\quad x\in D;\quad Lu(x)=0,\quad x\in\partial D$$

determines the diffusion process on $\bar{D}$. The semigroup on the boundary with generator $\overline{LH}$, in the setup of [12], determines the *Markov process on the boundary.* This process is the trace on $\partial D$ of the diffusion described by a random time scale on $\partial D$—*the local time on the boundary*[9]. This interpretation of the Markov process on $\partial D$ was suggested by the following analytical fact.

The resolvents $\{G_\lambda\}$ of the diffusion semigroup $\{T_t\}$ are, roughly speaking, obtained by solving

(6.1) $$(\lambda-A)u(x)=v(x),\qquad x\in D,$$

(6.2) $$Lu(x)=0,\qquad x\in\partial D,$$

for sufficiently many $v$ on $\bar{D}$, and defining

$$G_\lambda: v\rightarrow u=G_\lambda v.$$

On the other hand, the resolvents $\{\tilde{G}_\lambda\}$ of the semigroup $\{\tilde{T}_t\}$ are obtained by

8) This was derived to construct the diffusion semigroup on $C(\bar{D})$ when the boundary condition (0.2) is given ([11], [12]). For the proof, the reader can consult [11], pp. 572-573.

9) This was conjectured and justified for a special case in [12] on the setup of $C(\partial D)$. The result was extended by K. Sato [10], P. Priouret [9] and others. There are also more probabilistic approaches as in M. Motoo [7], K. Kunita [6] and others.

solving

$$(6.3) \qquad Au(x)=0, \qquad x\in D,$$

$$(6.4) \qquad (\lambda-L)u(x)=\varphi(x), \qquad x\in\partial D,$$

for $\varphi$ on $\partial D$, and defining

$$\tilde{G}_\lambda : \varphi\to[u]_{\partial D}=\tilde{G}_\lambda\varphi,$$

where $[u]_{\partial D}$ is the restriction of $u$ to the boundary $\partial D$. In fact, we can write $u=H[u]_{\partial D}=H\phi$ by (6.3) with $\phi=[u]_{\partial D}$. Then, by (6.4),

$$(\lambda-LH)\phi=\varphi.$$

Hence, there is a kind of "*duality*" in appearance between this operation (6.3)-(6.4) of obtaining $\{\tilde{G}_\lambda\}$ and the operation (6.1)-(6.2) of obtaining $\{G_\lambda\}$ of the diffusion, suggesting the above interpretation of $\{\tilde{T}_t\}$.

Since, the relation between $\{\tilde{U}_t\}$ and $\{\tilde{T}_t\}$ is exactly the same with that of $\{U_t\}$ and $\{T_t\}$ as in §4-5, analogy with the case of diffusion leads to a conjecture that $\{\tilde{U}_t\}$ corresponds to the *wave propagation on the boundary described by some time scale on* $\partial D$[10].

For the diffusion equation, each term of $Lu(x)$ has a probabilistic meaning. $\sum_{i,j}\alpha_{ij}\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}$ corresponds to the diffusing effect along $\partial D$, $\sum_i\beta_i\frac{\partial u}{\partial\xi_i}$ to the drift along $\partial D$, $\gamma\cdot u$ to the absorption (or killing), $\mu\frac{\partial u}{\partial n}$ to the reflection at $\partial D$, and the last term to the jumps from boundary points when the particle arrives at $\partial D$. When $\delta(x)$ is strictly negative, the particle spends time on $\partial D$ comparably long as it stays in $D$, where the time spent on $\partial D$ is proportional to $|\delta(x)|$.

For the wave equation with boundary condition (0.2), it seems natural to consider a model of the wave propagation (sound wave, for instance) through a system of some medium in $D$ and a vessel $\partial D$ which has a mass distribution $|\delta(x)|dx$.[11] Here, the wave propagates through $\partial D$ according to $\sum_{i,j}\alpha_{ij}\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}+\sum_i\beta_i\frac{\partial u}{\partial\xi_i}+\gamma\cdot u$, just as it propagates through the medium in $D$ according to $Au$ of (0.1). The wave reflects at $\partial D$ according to $\mu\frac{\partial u}{\partial n}$, and it gives effect to the points in $\bar{D}$ according to $\nu(x,\cdot)$ as soon as it arrives at $\partial D$.

10) For a strictly positive function $\sigma(x)$ on $\partial D$, the boundary conditions $Lu(x)=0$ and $\sigma(x)Lu(x)=0$ are the same. But $\overline{LH}$ and $\overline{\sigma\cdot LH}$ yield different groups of operators on the boundary. This indirectly suggests that a suitable time scale should be chosen for $\{\tilde{U}_t\}$ derived by $A$ and the given $L$ in (0.2).

11) The interpretation of the term $\delta(x)Au(x)$ is suggested by the definition of $H_\delta$, as Feller [4] discussed for one-dimensional case, where $\delta_i'$s in (0.4) are the masses of the end points of a vibrating string with mass distribution $dm$.

But, a rigorous justification of these interpretations, as in the case of diffusion processes, is still an open problem.

In this paper, we started with Wentzell's boundary condition as given. But, it seems interesting to know the most general boundary condition for the wave equation in the sense of Wentzell (that is, as a necessary condition), or in some other sense.

## References


[1] Feller, W., The parabolic differential equations and the associated semigroup of transformations, Ann. of Math. **55** (1952), 468-519.

[2] Feller, W., On second order differential operators, Ann. of Math. **61** (1955), 90-105.

[3] Feller, W., Generalized second order differential operators and their lateral conditions, Illinois J. Math. **1** (1957), 459-504.

[4] Feller, W., On the equation of the vibrating string, J. Math. Mech. 8 (1959), 339-348.

[5] Freidlin, M.I., On the factorization of non-negative definite matrices, Teor. Verojatnost. i Primenen. **13** (1968), 375-378 (in Russian).

[6] Kunita, H., General boundary conditions for multi-dimensional diffusion processes, J. Math. Kyoto Univ. **10** (1970), 273-335.

[7] Motoo, M., Application of additive functionals to the boundary problem of Markov Processes (Levy's system of U-processes), Proc. 5-th Berkeley Symp., 1967, **2**, Part II, 75-110.

[8] Phillips, P.S. and L. Sarason, Elliptic-parabolic equations of the second order, J. Math. Mech. **17** (1968), 891-917.

[9] Priouret, P., Processus de Markov sur une variété a bord compacte, Ann. Inst. H. Poincaré **4** (1968), 193-253.

[10] Sato, K., Time change and killing for multi-dimentional reflecting diffusion, Proc. Japan Acad. **39** (1963), 68-73.

[11] Sato, K. and T. Ueno, Multi-dimensional diffusion and the Markov process on the boundary, J. Math. Kyoto Univ. **4** (1965), 529-603.

[12] Ueno, T., The diffusion satisfying Wentzell's boundary condition and the Markov process on the boundary I, II, Proc. Japan Acad. **36** (1960), 533-538, 625-629.

[13] Ueno, T., Wave equation with Wentzell's boundary condition and a related semigroup on the boundary I, II, Proc. Japan Acad. **49** (1973), 672-677, **50** (1974), 281-286.

[14] Wentzell, A.D., On lateral conditions for multi-dimensional diffusion processes, Teor. Verojatnost. i Primenen. **4** (1959), 172-185 (in Russian).

[15] Yosida, K., An operator-theoretical integration of the wave equation. J. Math. Soc. Japan 8 (1956), 79-92.

(Received November 1, 1978)

Department of Mathematics
College of General Education
University of Tokyo
Komaba, Tokyo
153 Japan