On the structure of real analytic foliations of codimension one

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Introduction

Let M be a compact C^∞ manifold. As is well known, M admits a C^∞ foliation of codimension one if and only if its Euler characteristic $\chi(M)$ vanishes (Thurston [14]). However in order to admit a real analytic foliation of codimension one, M must satisfy some further topological conditions. Up to now several papers have been appeared to seek such conditions ([2], [13], [9], [1]). In this paper we study this problem and moreover we investigate the influence of the topology of M upon the real analytic foliations of M.

In §1, we study topological properties of real analytic foliations of M satisfying $H^1(M; \boldsymbol{R})=0$ and show that if M has a fundamental group with nonexponential growth then M does not admit any real analytic foliations. This is considered to be a generalization of Plante-Thurston's theorem ([9]). In §2, we deal with the case where M is 3-dimensional and give more detailed results, which are closely related to S. Goodman's theorem ([1]). In §3, we assume $\pi_1(M)$ is a finite extension of an abelian group. Using the concept of homology secants, which was defined by C. Lamoureux in [5], we obtain results about the relation between $H_1(M)$ and $i_*H_1(L)$, where L is a leaf of a foliation of M and $i: L \to M$ is an inclusion map.

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§1. Real analytic foliations and underlying manifolds

In this section we consider some properties of real analytic codimension one foliations of manifolds whose first homology groups are zero. We begin by proving a preliminary lemma.

Let M be a C^∞ manifold and $\mathcal{F}$ a transversely oriented, C^∞, codimension one foliation of M. For a leaf L of $\mathcal{F}$, we denote by $\mathcal{H}(L)$ the holonomy group of L.

LEMMA 1.1. *Let μ be a holonomy invariant measure for $\mathcal{F}$ and L a leaf con-*

tained in the interior of the support of μ. *Then* $\mathcal{H}(L)$ *is trivial.*

For the definition of a holonomy invariant measure, see [8].

PROOF. Let x be a point of L and let T be an embedded arc through x which is transverse to $\mathcal{F}$ and contained in the interior of the support of μ. We identify T with $[-1, 1]$ by some diffeomorphism $\varphi: [-1, 1] \to T$ such that $\varphi(0)=x$.

Now suppose that $\mathcal{H}(L)$ is not trivial. Then there exists an element γ of the holonomy pseudogroup of T such that $\gamma(0)=0$ and $\gamma(t)<t$ for some $t>0$. Since μ is invariant under γ, we have $\mu([0, t])=\mu([0, \gamma(t)])$. Then $\mu((\gamma(t), t])=0$. This contradicts the assumption that $(\gamma(t), t]\subset T$ is contained in the support of μ. Lemma is proved.

As is well known, any C^∞ manifold admits a compatible real analytic structure. In the rest of this section we assume M is a closed C^ω manifold such that $H^1(M; \boldsymbol{R})=0$ and $\mathcal{F}$ is a transversely oriented, real analytic, codimension one foliation of M.

It is easily seen that each compact leaf of $\mathcal{F}$ separates M into two connected components and that the number of compact leaves of $\mathcal{F}$ is finite. Now let $L_1, L_2, \cdots, L_p$ be the compact leaves of $\mathcal{F}$ and let $V_1, V_2, \cdots, V_q$ be connected compact submanifolds of M such that

$$M-\bigcup_{i=1}^{p} L_i=\bigcup_{i=1}^{q} \operatorname{int} V_i\,,$$

where $\operatorname{int} V_i$ denotes the interior of V_i.

We have the following.

LEMMA 1.2. *Let* M, $\mathcal{F}$ *and* V_i*'s be as above. Then there exists some* i, $1\leq i\leq q$, *such that every leaf contained in* $\operatorname{int} V_i$ *has exponential growth.*

PROOF. Suppose the contrary. Then for each i, $\operatorname{int} V_i$ contains a leaf with nonexponential growth. It follows from [8] that $\operatorname{int} V_i$ admits a non-trivial holonomy invariant measure μ_i. Let Φ_i be the corresponding cohomology class in $H^1(V_i; \boldsymbol{R})$.

We require the following.

Assertion. For each j and k such that $V_j\cap V_k$ is a compact leaf, say L, the classes $i_j^*\Phi_j$ and $i_k^*\Phi_k$ are linearly dependent in $H^1(L; \boldsymbol{R})$, where $i_j: L\to V_j$ and $i_k: L\to V_k$ are inclusion maps.

If this assertion is proved, it follows from Plante-Thurston's lemma 7.3 [9] that $H^1(M; \boldsymbol{R})\neq 0$. This will contradict the assumption and the lemma will follow.

Proof of Assertion. In the case when $\mathscr{H}(L)$ is abelian, Plante-Thurston's proof goes through without change. So we assume $\mathscr{H}(L)$ is not abelian. Then by Hector's theorem 9 [4], there exists a neighborhood U of L in V_j such that every leaf intersecting $U-L$ is dense in U. We claim that $U\cap \operatorname{supp}\mu_j=\emptyset$, where $\operatorname{supp}\mu_j$ is the support of μ_j. Let x be a point of L and T a small transverse arc contained in U with one end point at x. We identify T with $[0, 1]$ by some diffeomorphism φ such that $\varphi(0)=x$. First it is seen that there is a non-trivial element f of $\mathscr{H}(L)$ such that $f(t)=t$ for some $t>0$. In fact, suppose no non-trivial elements have fixed points in $(0, 1]$. Then we can define an Archimedean ordering in $\mathscr{H}(L)$ by $f\leqq g$ if $f(t)\leqq g(t)$ for some $t>0\in T$. It follows from classical Hölder's theorem that $\mathscr{H}(L)$ is abelian, contradicting the assumption. Now if we take a leaf L' corresponding to the fixed point t of f, we see that $\mathscr{H}(L')$ is non-trivial. Suppose that $U\cap\operatorname{supp}\mu_j\neq\emptyset$. Since $\operatorname{supp}\mu_j$ is a closed saturated subset of int V_j and all leaves in $U-L$ is dense in U, it follows that $\operatorname{supp}\mu_j\supset U-L$, hence $\operatorname{int}(\operatorname{supp}\mu_j)\supset L'$. This is a contradiction by (1.1). Thus the claim is proved. By the definition of Φ_j, this implies that $i_j^*\Phi_j=0$. Hence $i_j^*\Phi_j$ and $i_k^*\Phi_k$ are linearly dependent. Assertion is proved. This completes the proof of (1.2).

The following is considered as a generalization of Plante-Thurston's theorem 7.1 in [9].

THEOREM 1.3. *Let M be a closed manifold such that $\pi_1(M)$ has nonexponential growth. If M admits a transversely orientable, real analytic, codimension one foliation, then $H^1(M;\boldsymbol{R})\neq 0$.*

PROOF. Since a real analytic foliation has no null homotopic closed transversal, it follows from Plante's lemma 7.2 [8] that every leaf has nonexponential growth. By (1.2), we obtain the conclusion.

THEOREM 1.4. *Let M be a closed manifold. If M admits a transversely orientable, real analytic, codimension one foliation with all leaves proper, then $H^1(M;\boldsymbol{R})\neq 0$.*

PROOF. We need the following result which is obtained by Tsuchiya [16].

THEOREM 1.5. *Let M be a compact manifold and $\mathscr{F}$ a real analytic, codimension one foliation of M with all leaves proper. Then every leaf of $\mathscr{F}$ has polynomial growth.*

(1.4) immediately follows from (1.5) and (1.2).

COROLLARY 1.6. *Let M be a closed manifold such that $H^1(M;\boldsymbol{R})=0$ and $\mathscr{F}$ a real analytic, transversely orientable, codimension one foliation of M. Then there exists a non-proper leaf with exponential growth.*

PROOF. It suffices to show that there exists a non-proper leaf in V_i of (1.2). This immediately follows from (1.5).

§ 2. Real analytic foliations of 3-manifolds

Let I denote the closed unit interval $[0, 1]$. $SL(2, \boldsymbol{Z})$ acts on $T^2(=\boldsymbol{R}^2/\boldsymbol{Z}^2)$ naturally. For $A\in SL(2, \boldsymbol{Z})$, we denote by M_A the mapping torus obtained from $T^2\times I$ by identifying $(x, 0)$ with $(Ax, 1)$, where $x\in T^2$. Note that if A is of the form $\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}$, M_A is diffeomorphic to a total space of an S^1-bundle over T^2.

First we sketch the proof of the following preliminary lemma.

LEMMA 2.1. *The fundamental group of* M_A *has exponential growth if and only if* $|\mathrm{trace}\, A|>2$.

OUTLINE OF PROOF. $\pi_1(M_A)$ is isomorphic to the semi-direct product $\boldsymbol{Z}^2\cdot\boldsymbol{Z}$ whose multiplication is given by

$$(a, m)\cdot(b, n)=(a+A^m b, m+n),$$

where $a, b\in\boldsymbol{Z}^2$ and $m, n\in\boldsymbol{Z}$.

We first prove the "if" part. Let $e=\begin{pmatrix}1\\ 0\end{pmatrix}$. If $|\mathrm{trace}\, A|>2$, words obtained by using only $(e, 1)$ and $(-e, 1)$ repeatedly are distinct as elements in $\pi_1(M_A)$. Thus $\pi_1(M_A)$ has exponential growth.

Next we prove the "only if" part. If $|\mathrm{trace}\, A|\leqq 2$, the eigenvalues of A are at most twelfth roots of unity. Now define a subgroup Γ by

$$\Gamma=\{(a, 12n)\in\boldsymbol{Z}^2\cdot\boldsymbol{Z}\,|\, a\in\boldsymbol{Z}^2, n\in\boldsymbol{Z}\}.$$

Then Γ is a nilpotent subgroup of finite index. By Wolf [17], $\pi_1(M_A)$ has polynomial growth. Lemma is proved.

The main result of this section is the following.

THEOREM 2.2. *Let M be a closed orientable 3-manifold such that $\pi_1(M)$ has nonexponential growth. Let $\mathcal{F}$ be an orientable, real analytic, codimension one foliation of M. Then,*

i) *M is either $S^1\times S^2$ or a T^2-bundle over S^1.*

ii) *If $\mathcal{F}$ has a non-proper leaf, then M is either an S^1-bundle over T^2 or $M_{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}}$.*

iii) *For each leaf L of $\mathcal{F}$, the holonomy group of L is abelian of rank $\leqq 2$.*

iv) *If some leaf has holonomy of rank 2, then M is either T^3 or $M_{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}}$.*

In this section we assume that M is a closed oriented 3-manifold whose fundamental group has nonexponential growth and $\mathcal{F}$ an oriented, real analytic, codimension one foliation of M. We first recall the notion "erasing the Reeb component" [12].

Let T be the boundary torus of a Reeb component R of $\mathcal{F}$. Since $\mathcal{F}$ is real analytic, it follows that T is not flat (i.e. any non-trivial element of the holonomy group $\mathcal{H}(T)$ is not infinitely tangent to the identity). Hence we can choose a neighborhood U of R which is isotopic to R and whose boundary is transverse to $\mathcal{F}$, inducing a foliation by circles, each null homotopic in U. We can change $\mathcal{F}$ by replacing the foliation $\mathcal{F}|U$ by a foliation by disks whose boundary is the induced foliation on ∂U by $\mathcal{F}$. We call this process "erasing the Reeb component".

LEMMA 2.3. *There is an oriented C^∞ codimension one foliation $\mathcal{F}'$ of M without Reeb components which is obtained from $\mathcal{F}$ by a finite number of erasing-Reeb-component processes.*

PROOF. If $\mathcal{F}$ has no compact leaves, then we can take $\mathcal{F}'=\mathcal{F}$. If $\mathcal{F}$ has infinitely many compact leaves, then every leaf of $\mathcal{F}$ must be compact because $\mathcal{F}$ is real analytic. Hence we can also take $\mathcal{F}'=\mathcal{F}$. So we may suppose that $\mathcal{F}$ has finitely many compact leaves. We erase the Reeb components of $\mathcal{F}$ one by one. If we erase one of these Reeb components, say R, then the resulting foliation does not have any new toral leaf that is the boundary of some Reeb component, because every modified leaf has a closed transversal to the new foliation which was the core transversal in R. Thus after finitely many steps, all the Reeb components are erased.

REMARK. It is easy to see that every leaf of the new foliation $\mathcal{F}'$ has still real analytic holonomy.

LEMMA 2.4. *Each leaf of $\mathcal{F}'$ is diffeomorphic to S^2, T^2, R^2 or $S^1\times R$.*

PROOF. Since $\mathcal{F}'$ has no Reeb components, by [6], $i^L_\#: \pi_1(L)\to\pi_1(M)$ is injective for every leaf L of $\mathcal{F}'$, where i^L is a natural inclusion map. Thus $\pi_1(L)$ has nonexponential growth. It follows from [10] that a 2-manifold whose fundamental group has nonexponential growth is either S^2, T^2, R^2 or $S^1\times R$ and the lemma is proved.

Let us recall that a foliation is called *almost without holonomy* if every non-compact leaf has trivial holonomy.

LEMMA 2.5. *$\mathcal{F}'$ is almost without holonomy.*

PROOF. Since $\pi_1(M)$ has nonexponential growth and $\mathcal{F}'$ has no null homotopic closed transversal, it follows from [8] and [7] that there exists a non-trivial holonomy invariant measure μ for $\mathcal{F}'$ and that there exist no exceptional minimal sets in M. First consider the case when $\mathcal{F}'$ has no compact leaves. By the above remark, the support of μ must be whole M. Then by (1.1), $\mathcal{F}'$ is without holonomy. Next we consider the case when $\mathcal{F}'$ has infinitely many compact leaves. Then by the remark after (2.3), all leaves of $\mathcal{F}'$ must be compact. Thus $\mathcal{F}'$ is without holonomy. Now we assume that $\mathcal{F}'$ has finitely many compact leaves. Note that in this case every compact leaf of $\mathcal{F}'$ is diffeomorphic to T^2. In fact, if there is a compact leaf which is not diffeomorphic to T^2, we see by (2.4) that it must be diffeomorphic to S^2. Then by the Reeb stability theorem, all leaves of $\mathcal{F}'$ are diffeomorphic to S^2 and hence compact. This contradicts the assumption. Now let L be a compact leaf of $\mathcal{F}'$. Then $\mathcal{H}(L)$ is isomorphic to $\boldsymbol{Z}$ or $\boldsymbol{Z}\oplus\boldsymbol{Z}$. We claim the following.

Assertion. Every non-compact leaf that contains L in its closure has trivial holonomy.

Proof of Assertion. Case i) $\mathcal{H}(L)\cong\boldsymbol{Z}$. We can choose a torus T near L, isotopic to L and transverse to $\mathcal{F}'$, inducing a foliation by circles. We see that each of these circles is not null homotopic in the leaf which contains it. In fact, if it is null homotopic, then by the standard argument [6], we can find a vanishing cycle between T and L and hence a Reeb component. This contradicts the fact that $\mathcal{F}'$ has no Reeb components. Thus by (2.4), every leaf near L is diffeomorphic to $S^1\times R$ and without holonomy.

Case ii) $\mathcal{H}(L)\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$. We can choose a torus T near L, isotopic to L and transverse to $\mathcal{F}'$, inducing a foliation with all leaves dense. We take a circle C in T transverse to the induced foliation. The holonomy pseudogroup Γ of $\mathcal{F}'$ acts on C. Since every leaf of $\mathcal{F}'$ has nonexponential growth, Γ also has nonexponential growth. Thus by [8], there is a Γ-invariant measure on C. Since every Γ-orbit is dense in C, the support of the measure is whole C. By (1.1), we see that every leaf that meets C, and hence every leaf near L, is without holonomy. The proof of Assertion is now complete.

Since $\mathcal{F}'$ has no exceptional minimal sets, every non-compact leaf of $\mathcal{F}'$ is asymptotic to some compact leaf. Then the above assertion implies that $\mathcal{F}'$ is almost without holonomy. Thus (2.5) is proved.

Now we are ready to prove the main result of this section.

PROOF OF (2.2). First we prove i). If all leaves of $\mathcal{F}'$ are compact, then by the Reeb stability theorem, M is either $S^1\times S^2$ or a T^2-bundle over S^1. So we assume $\mathcal{F}'$ has finitely many compact leaves, which we denote by L_1, L_2, $\cdots$, L_p. Let V_1, V_2, $\cdots$, V_q be the connected compact submanifolds of M defined by the same way as in § 1. By (2.4), $\mathcal{F}'|\mathrm{int}\, V_j$ is either a foliation by planes or a foliation by cylinders. Such foliations have been classified up to topological conjugacy [11], [3]. The following is the complete list of them.

A) foliation by planes

A1) T^3 with a foliation by irrational planes,

A2) $S^1\times D^2$—a Reeb component,

A3) $T^2\times I$ with a standard irrational foliation.

B) foliation by cylinders

B1) an S^1-bundle over T^2 with the pull-back foliation of a foliation by irrational lines of T^2,

B2) a non-trivial I-bundle over a Klein bottle, which we denote by $\mathcal{K}$, with a standard foliation by cylinders,

B3) $T^2\times I$ with a standard foliation by cylinders.

For detail, see [11] and [3]. (Remark that our foliations are always assumed to be transversely orientable.)

$\mathcal{F}'$ does not contain components of type A2 because $\mathcal{F}'$ has no Reeb components. If $\mathcal{F}'$ contains components of type A1 or B1, then M is an S^1-bundle over T^2.

The boundary leaves of components of type A3 have holonomy of rank 2 while the boundary leaves of components of type B2 and B3 have holonomy of rank 1. Therefore the components adjacent to a component of type A3 (resp. type B2 or B3) are also type A3 (resp. type B2 or B3). Hence if $\mathcal{F}'$ has a component of type A3, then M is an T^2-bundle over S^1. If $\mathcal{F}'$ consists of components of type B2 or B3, then M is either $M\left(\begin{smallmatrix}\pm 1 & q\\ 0 & \pm 1\end{smallmatrix}\right)$ or the union of two $\mathcal{K}$'s identified along their boundaries by the diffeomorphism $\begin{pmatrix}\pm 1 & q\\ 0 & \pm 1\end{pmatrix}$. We can easily see that the latter is diffeomorphic to $M\left(\begin{smallmatrix}-1 & \pm q\\ 0 & -1\end{smallmatrix}\right)$. Thus in every case M is an T^2-bundle over S^1. The proof of i) is complete.

Next we prove ii). Note that $\mathcal{F}$ has non-proper leaves if and only if $\mathcal{F}'$ does. So we may suppose that $\mathcal{F}'$ has non-proper leaves. By the list in the proof of i), we see that $\mathcal{F}'$ has a component of type A1, A3 or B1. If $\mathcal{F}'$ has a component of type A1 or B1, then M is an S^1-bundle over T^2. Now suppose $\mathcal{F}'$ has components of type A3. Then M is diffeomorphic to M_A for some $A\in SL(2, \boldsymbol{Z})$. Let L be a boundary leaf of a component of type A3. Then $\mathcal{H}(L)\cong \boldsymbol{Z}\oplus\boldsymbol{Z}$. We can define in $\mathcal{H}(L)$ an Archimedean ordering by $f\leqq g$ if $f(x)\leqq g(x)$ for some small x. By Hölder's theorem, there is a unique (up to constant multiple) injective order-preserving homomorphism $\phi:\mathcal{H}(L)\rightarrow R$. Let f, g be standard generators of $\mathcal{H}(L)$. Then $\phi(g)=\alpha\phi(f)$ for some irrational number α. Since the glueing map A preserves these holonomy structures, A must have

$\binom{1}{\alpha}$ as its eigenvector. Since $|\text{trace } A| \leqq 2$ by (2.1), it follows that $A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. This proves ii).

Finally we prove iii) and iv). By (2.3), we see that $\mathcal{F}$ has no leaves with holonomy of rank>2 if and only if $\mathcal{F}'$ does. On the other hand, by (2.4), $\mathcal{F}'$ has no leaves with holonomy of rank>2. Thus iii) is proved. To prove iv), suppose that $\mathcal{F}$, hence $\mathcal{F}'$, has a leaf with holonomy of rank 2. Then $\mathcal{F}'$ must have a component of type A3. By the proof of ii), M is T^3 or $M\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. This proves iv). The proof of Theorem 2.2 is completed.

§ 3. Real analytic foliations and homology secants

Let M be a C^∞ compact manifold and $\mathcal{F}$ a transversely oriented, C^∞, codimension one foliation of M. Let x_0 be a point in a leaf L of $\mathcal{F}$. The following definition was given by C. Lamoureux in [5].

DEFINITION 3.1. The *homotopy secant* of $\mathcal{F}$ at x_0, denoted by $\Pi S(x_0, \mathcal{F})$, is the set of elements of $\pi_1(M, x_0)$ which can be represented by oriented closed transversals to $\mathcal{F}$ containing the point x_0 and whose orientation coincides with the transverse orientation of $\mathcal{F}$. The *homology secant* of $\mathcal{F}$ at x_0, denoted by $HS(x_0, \mathcal{F})$, is the Hurewicz image of $\Pi S(x_0, \mathcal{F})$ in $H_1(M; \boldsymbol{Z})$.

Homotopy secants and homology secants are semi-groups. The isomorphism classes of these semi-groups do not depend on the choice of the point x_0 in the leaf L. So we may briefly write $\Pi S(L)$ (resp. $HS(L)$) instead of $\Pi S(x_0, \mathcal{F})$ (resp. $HS(x_0, \mathcal{F})$).

The homotopy secant of a non-compact leaf is non-empty. If $\mathcal{F}$ is real analytic or without holonomy, then the homotopy secant of any leaf of $\mathcal{F}$ does not contain the unit element.

Let x, y be two points in a leaf L. If α is a curve in L connecting x with y and β is a curve transverse to $\mathcal{F}$ connecting y with x, then the composition $\alpha*\beta$ is homotopic to a closed transversal through L. From this fact we easily have the following.

LEMMA 3.2. *If* $\alpha \in i_* \pi_1(L, x_0)$ *and* $\beta \in \Pi S(x_0, \mathcal{F})$, *then* $\alpha * \beta \in \Pi S(x_0, \mathcal{F})$.

DEFINITION 3.3. Let S be a semi-group. An element $s \in S$ is said to be *infinitely divisible* in S if for every positive integer N, there exist N elements $s_1, s_2, \cdots, s_N$ of S such that $s = s_1 s_2 \cdots s_N$. Elements which are not infinitely divisible are said to be *almost primitive.*

The following lemma is due to C. Lamoureux and is essentially proved in [5].

LEMMA 3.4. *Let* τ *be an oriented closed transversal through a point* x_0 *in* L.

If τ intersects L at infinitely many points, then the homotopy class $[\tau]\in\pi_1(M, x_0)$ of τ is infinitely divisible in $\Pi S(x_0, \mathcal{F})$.

PROOF. We parametrize τ by an orientation preserving map $\varphi: (I, \partial I)\to(\tau, x_0)$ with the canonical orientation of I. Let $x_0(=\varphi(0))$, $\varphi(t_1)$, $\varphi(t_2)$, $\cdots$ be infinitely many points in $\tau\cap L$ such that $0=t_0<t_1<t_2<\cdots<1$, reversing the orientation if necessary.

For $0\leqq i<j$, we fix a curve c_{ij} in L from $\varphi(t_i)$ to $\varphi(t_j)$. For $0\leqq j<i$, we take c_{ij} so that $c_{ij}=c_{ji}^{-1}$. For $0\leqq i<j$, τ_{ij} denotes $\varphi([t_i, t_j])$ and τ_{j0} denotes $\varphi([t_j, 1])$.

Then τ is homotopic to

$$(\tau_{01}*C_{10})*(C_{01}*\tau_{12}*C_{20})*(C_{02}*\tau_{23}*C_{30})*\cdots$$
$$*(C_{0i}*\tau_{i,i+1}*C_{i+1,0})*\cdots*(C_{0,N-2}*\tau_{N-2,N-1}*C_{N-1,0})*(C_{0,N-1}*\tau_{N-1,0}).$$

By the standard argument, $\tau_{01}*C_{10}$, $C_{0i}*\tau_{i,i+1}*C_{i+1,0}$ and $C_{0,N-1}*\tau_{N-1,0}$ are homotopic to closed transversals through x_0. Thus we have divided $[\tau]$ into N elements of $\Pi S(x_0, \mathcal{F})$. Lemma is proved.

The purpose of this section is to prove the following.

THEOREM 3.5. *Let M be a closed orientable manifold whose fundamental group is a finite extension of an abelian group. Let $\mathcal{F}$ be an orientable, C^∞, codimension one foliation of M which has no null homotopic transversals.*

i) *If L meets some closed transversal to $\mathcal{F}$, then corank $i_*^L\geqq 1$.*

ii) *If $\bar{L}$, the closure of L in M, contains a non-compact leaf besides L, then corank $i_*^L\geqq 2$.*

In the statement of this theorem, $i^L: L\to M$ is the inclusion map and $i_*^L: H_1(L)\to H_1(M)$ is the induced homomorphism. Corank $i_*^L=\mathrm{rank}_Z(H_1(M)/i_*^L H_1(L))$.

REMARK. Real analytic foliations satisfy (3.5) since they have no null-homotopic transversals.

REMARK. Standard arguments show that every non-compact leaf meets some closed transversal and that every non-proper leaf contains other non-proper leaves in its closure.

The following corollaries are immediate from (3.5).

COROLLARY 3.6. *Let M, $\mathcal{F}$ be as in* (3.5). *If $\mathcal{F}$ has a non-proper leaf, then* $\mathrm{rank}_Z H_1(M)\geqq 2$.

COROLLARY 3.7. *Let M, $\mathcal{F}$ be as in* (3.5) *If* $\mathrm{rank}_Z H_1(M)=1$, *then all leaves are proper and $\mathcal{F}$ is almost without holonomy.*

To prove (3.7), we have only to note that the holonomy homomorphism $\Phi: \pi_1(L)\to\mathcal{H}(L)$ factors through $i_*^L\pi_1(L)$.

PROOF OF (3.5). We need the following two elementary lemmas, whose proofs are left to the reader.

LEMMA 3.8. *Let S be a subsemi-group of $\boldsymbol{Z}^n$ without the unit element. Then there exists a non-zero element a of $\boldsymbol{R}^n$ such that*

$$S\subset\{x\in\boldsymbol{Z}^n \mid (a, x)\geqq 0\},$$

where $(\ ,\ )$ denotes the standard inner product in $\boldsymbol{R}^n$.

LEMMA 3.9. *Let S be as in* (3.8). *If there exists a non-zero element a of $\boldsymbol{Z}^n$ such that*

$$S\subset\{x\in\boldsymbol{Z}^n \mid (a, x)>0\},$$

then every element of S is almost primitive.

Let M and $\mathcal{F}$ be as in (3.5). Let L be a leaf of $\mathcal{F}$. It follows from the assumption that $HS(L)$ does not have the unit element. Therefore if $F_1(M)$ denotes the quotient group of $H_1(M)$ by its torsion subgroup and $q: H_1(M)\to F_1(M)$ the canonical projection, then $qHS(L)$ does not have the unit element. We identify $F_1(M)$ with $\boldsymbol{Z}^n$ by choosing an arbitrary generating set of $F_1(M)$, where n is the first Betti number of M. Let $p: \boldsymbol{R}^n-\{0\}\to S^{n-1}$ be the radial projection defined by $p(x)=x/|x|$.

LEMMA 3.10. *If $HS(L)\neq\varnothing$, then $pqi_*^L H_1(L)\subset\overline{pqHS(L)}$.*

PROOF. Let $\alpha\in i_*^L H_1(L)$ and $\beta\in HS(L)$. By (3.2), $n\alpha+\beta\in HS(L)$ for each $n\in\boldsymbol{Z}$. Therefore

$$pq(\alpha)=\lim_{n\to\infty} pq(n\alpha+\beta)\in\overline{pqHS(L)}.$$

We now prove i) of (3.5). Suppose corank $i_*^L=0$. Then $pqi_*^L H_1(L)$ is dense in S^{n-1}. Therefore by (3.10), we have

$$S^{n-1}=\overline{pqi_*^L H_1(L)}=\overline{pqHS(L)}.$$

But this is a contradiction by (3.8). This proves i).

LEMMA 3.11. *$i_*^L H_1(L)\cap HS(L)=\varnothing$.*

PROOF. Suppose there is an element $\gamma\in i_*^L H_1(L)\cap HS(L)$. Since $-\gamma\in i_*^L H_1(L)$, we have by (3.2),

$$0=(-\gamma)+\gamma\in HS(L).$$

This contradicts the assumption of (3.5).

LEMMA 3.12. *If the closure of L contains a non-compact leaf L' other than L, then $HS(L)$ has an infinitely divisible element.*

PROOF. Take any closed transversal τ that intersects L'. Then τ necessarily intersects L at infinitely many points. Thus by (3.4), $[\tau]$ is an infinitely divisible element of $HS(L)$.

Finally we prove ii) of (3.5). Suppose corank $i_*^L=1$. Then $\overline{pqi_*^L H_1(L)}$ is $S^{n-1}\cap\{x\in \boldsymbol{R}^n \mid (a, x)=0\}$ for some $a\in \boldsymbol{Z}^n$. Then from (3.9) and (3.11) it follows easily that every element of $HS(L)$ is almost primitive. But this contradicts (3.12). Thus ii) is proved. This completes the proof of the desired theorem.

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