(Received January 30, 1979)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan
Present address
Department of Mathematics
College of General Education
Chiba University
Yayoicho, Chiba
280 Japan

Lower semi-continuity of growth of leaves

By Nobuo TSUCHIYA

(Communicated by I. Tamura)

§ 1. Introduction

In this paper we study basic properties of the degree of growth of leaves in foliations of codimension one. Earlier, the growth of leaves was studied by J. F. Plante with respect to the existence of transverse invariant measures (see e. g. [1]). Our main result in this paper is the following. If a proper leaf F is contained in the limit set of another leaf F', then the degree of growth of F is not greater than that of F'.

In § 2, we define the degree of growth of leaves and state the theorems. In § 3, we observe some properties of a neighbourhood of a proper leaf in order to prove the theorems. The proof is finally carried out in § 4.

A paper in which the author deals with the growth types of codimension one foliations is in preparation. The author wishes to thank T. Inaba, K. Masuda, T. Tsuboi and I. Tamura for helpful comments.

§ 2. Statement of results

First of all we recall the definition of the growth function of a leaf ([1]). Let $\mathcal{F}$ be a codimension one foliation of class C^1 of a closed manifold M. Let $M=\bigcup U_i$ be a finite covering of M by regular distinguished charts (see [1], [2]). Thus for each i there exists a coordinate chart map $\phi_i: U_i \to \mathring{D}^{n-1}\times\mathring{D}^1$ which sends the leaves of $\mathcal{F}$ restricted to U_i to the leaves of the trivial codimension one foliation of $\mathring{D}^{n-1}\times\mathring{D}^1$. Each set of the form $\phi_i^{-1}(\mathring{D}^{n-1}\times\{x\})$ is called a *plaque*, and each set $\phi_i^{-1}(\{0\}\times\mathring{D}^1)$ is called an *axis*. If $x\in U_i$, $P_i(x)$ denotes the plaque containing x. A *chain of plaques* is a sequence $(\rho_1, \cdots, \rho_r)$ of plaques such that $\rho_i\cap\rho_{i+1}\neq\emptyset$ for $i=0, \cdots, r-1$.

Let F be a leaf of $\mathcal{F}$. We fix an initial plaque $\rho_0=P_{i_0}(x_0)$ contained in F.

DEFINITION. The growth function of the leaf F, $f: \boldsymbol{Z}^+\to\boldsymbol{Z}^+$, with respect to the regular distinguished covering $\{U_i\}$ and the initial plaque ρ_0 is defined as follows:

$f(n)$=the number of distinct plaques which can be reached from the initial plaque ρ_0 by a chain of length$\leqq n$.

Definition. The degree of growth of F, denoted $gr(F)$, is a non-negative integer or a symbol ∞ or exp defined as follows:

1) If the growth function f is dominated by a polynomial, then $gr(F)$ is the smallest integer degree of the polynomials which dominate f.

2) $gr(F)=\exp$, if the growth function f dominates an exponential function.

3) $gr(F)=\infty$, if the growth function f is not dominated by any polynomials and does not dominate exponential functions.

It is known ([1]) that $gr(F)$ does not depend on the choice of the initial plaque or the regular distinguished charts.

Let $M/\mathcal{F}$ be the space of leaves of $\mathcal{F}$ with the quotient topology. gr is a function from $M/\mathcal{F}$ to the linearly ordered set $\{0, 1, \cdots, \infty, \exp\}$. The first theorem asserts that this function is lower semi-continuous at a proper leaf.

Theorem 1. *Let M be a closed manifold and $\mathcal{F}$ a codimension one C^1 foliation of M. Let F be a proper leaf of $\mathcal{F}$. If for every neighbourhood U of F there exists a leaf F' such that $F'\neq F$, $F'\cap U\neq\emptyset$ and $gr(F')\leqq d$, then $gr(F)\leqq d$.*

Theorem 2. *Let M be a closed manifold and $\mathcal{F}$ a codimension one C^1 foliation of M. Let F be a proper leaf of $\mathcal{F}$ which is contained in the closure of another leaf F'. Then $gr(F')\geqq gr(F)+1$. Where, for convenience, we set $\infty+1=\infty$ and* $\exp+1=\exp$.

§3. Structures of a neighbourhood of a proper leaf

In this section we study the structure of a neighbourhood of a proper leaf. The first proposition asserts that a neighbourhood of a proper leaf has a structure of a foliated D^1-bundle outside a compact set. The second proposition says that a neighbourhood of a proper leaf which is contained in the closure of another leaf is unstable.

Let $\mathcal{F}$ be a transversely oriented codimension one foliation of class C^1 of a closed Riemannian manifold M. We choose a regular distinguished finite covering $M=\bigcup U_i$, $\phi_i: U_i\to \mathring{D}^{n-1}\times \mathring{D}^1$ and a flow $\varphi: M\times \boldsymbol{R}\to M$ transverse to $\mathcal{F}$.

Lemma. *Let F be a proper leaf of $\mathcal{F}$. Then for any $\varepsilon>0$, there exists a compact set $K\subset F$ such that if $x\in\overline{F-K}$, then $\bigcup_{0<t<\varepsilon}\varphi_t(x)\cap F\neq\emptyset$.*

Proof. Assume the contrary, then there exists a positive number δ and a sequence $\{x_i\}$ of points of F such that each x_i is contained in a distinguished chart U_a and the distance of $P_a(x_i)$ and $P_a(x_j)$ is greater than δ, if $i\neq j$. Then the axis of U_a must have infinite length. This contradicts the assumption that

$\{U_i\}$ is regular.

It is easy to choose the compact set K so that K and $\overline{F-K}$ are manifolds. In what follows we assume K is so chosen.

For $x\in\overline{F-K}$, we define

$$t_x=\inf\{t \mid t>0,\ \varphi_t(x)\in F\}\ .$$

Since F is proper, $t_x>0$ for any x.

The following proposition is clear from the construction. In the statement of the proposition, we use the same letter F for a leaf of $\mathcal{F}$ and for an abstract manifold.

PROPOSITION A. *We define the immersion* $\Phi:\overline{F-K}\times[0,1]\rightarrow M$ *by* $\Phi(x,t)=\varphi_{tt_x}(x)$. *Then Image*$(\Phi)$ *is a one side neighbourhood of* $\overline{F-K}$, *and satisfies the following properties*:

1) *The induced foliation* $\Phi^*\mathcal{F}$ *is transverse to the fibres of the projection* $\overline{F-K}\times[0,1]\rightarrow\overline{F-K}$.

2) $\Phi^{-1}(F)\subset\overline{F-K}\times\{0,1\}$.

3) *The length of the transverse arc* $\Phi(\{x\}\times[0,1])$ *is smaller than* ε.

REMARK. Obviously a locally dense leaf never has such neighbourhood. But a side of an exceptional leaf may have a neighbourhood of this type. So our theorems are true for such a proper side of an exceptional leaf. But we do not formulate it here explicitly.

Next we prove the second proposition. To state it, we recall the definition of holonomy homeomorphisms. Let $(M,\mathcal{F})$, $\{U_i\}$ be as before. For each i, let X_i denote the space of plaques in U_i. As a topological space, X_i is homeomorphic to the axis $\phi_i^{-1}(\{0\}\times\mathring{D}^1)$.

Let $c=(\rho_0,\cdots,\rho_r)$ be a chain of plaques with $\rho_k\subset U_{i_k}$. The chain c defines a local diffeomorphism γ_c: Dom$(\gamma_c)\rightarrow$Range(γ_c), where Dom$(\gamma_c)\subset X_{i_0}$, Range$(\gamma_c)\subset X_{i_r}$ and $\gamma_c(x)=y$ if and only if there is a plaque chain $(P_{i_0}(x_0),\cdots,P_{i_r}(x_r))$ such that $P_{i_k}(x_k)\subset U_{i_k}$ for $k=0,\cdots,r$, $x_0=x$ and $x_r=y$. γ_c is well-defined since the covering $\{U_i\}$ is regular (see [2]). We call γ_c the *holonomy homeomorphism associated with the chain c*.

Let F be a proper leaf of $\mathcal{F}$ which is contained in the closure of another leaf F'. We assume that $\bar{F}$ does not pass the chart ends of $\{U_i\}$ and $F\cap U_0\neq\emptyset$. Let $x_0\in U_0\cap F$ be a base point of F and $P_0(x_0)$ the plaque containing x_0. Fix a parametrization of the plaque space X_0 so that x_0 corresponds to 0. We assume F' accumulates to $x_0\in F$ from the positive side. Choose a sequence $\{y_i\}$ of points of $F'\cap X_0$ which converges to 0 from the positive side. We further assume that the interval $(0,y_1]$ is disjoint from F.

PROPOSITION B. *For any $N>0$, there exists an integer $l\geqq N$ and a closed plaque chain $c=(\sigma_0, \cdots, \sigma_r)$, $\sigma_0=\sigma_r=P_0(x_0)$ such that $\gamma_c(y_l)\neq y_l$.*

This proposition is closely related to a theorem of Sacksteder-Schwartz ([3]) which says that a proper leaf contained in the limit set of a leaf has locally infinite holonomy pseudogroup.

PROOF. For each i, choose a chain d_i of minimal length connecting y_i and y_{i+1}:

$$d_i=(\rho_0^i, \cdots, \rho_{m_i}^i), \quad \rho_0^i=P_0(y_i) \quad \text{and} \quad \rho_{m_i}^i=P_0(y_{i+1}).$$

Let $n_i=\max\{k\,;\, k\leqq m_i,\ \mathrm{Dom}(\gamma_{(\rho_0^i,\dots,\rho_k^i)})\supset[0, y_i]\}$, $\sigma_k^i=\gamma_{(\rho_0^i,\dots,\rho_k^i)}(0)$ and $c_i=(\sigma_0^i, \cdots, \sigma_{n_i}^i)$. There are two cases.

Case 1) For some $l\geqq N$, $n_l=m_l$. In this case, $\sigma_{n_l}^l=P_0(x_0)$ since $F\cap(0, y_1]=\emptyset$, and c_l is a required chain.

Case 2) For each $i\geqq N$, n_i is smaller than m_i. In this case, taking a subsequence if necessary, we may assume there is a chart U_a such that every $\sigma_{n_i}^i$ is contained in U_a and the plaques $\sigma_{n_i}^i$ converge to a plaque σ in U_a. From the maximality of the chain c_i, there is a positive number δ such that the length of $\gamma_{c_i}((0, y_i])$ is greater than δ, for any i. Since $\mathcal{F}$ is transversely oriented and $(0, y_1]\cap F=\emptyset$, if $\gamma_{c_i}([0, y_i])\cap\gamma_{c_j}([0, y_j])\neq\emptyset$ then $\gamma_{c_i}(0)=\gamma_{c_j}(0)$. It follows that there is $i\geqq N$ such that for any $j>i$, $\sigma_{n_j}^j=\sigma$, and one can consider the holonomy homeomorphism associated with the chain $c_i^{-1}\# c_j$. If there is $l>i$ such that $\gamma_{c_i^{-1}\# c_l}(y_l)\neq y_l$, we are done. Suppose $\gamma_{c_i^{-1}\# c_l}(y_l)=y_l$ for any $l>i$. Then there is $j>i$ such that $n_j>n_i$ and $\gamma_{c_i^{-1}\# c_j}(y_j)=y_j$. This contradicts the minimality of length of d_j. Thus the proof is complete.

§4. Proof of theorems

Since $gr(F)=0$ for a compact leaf F, Theorems 1, 2 are clearly true if F is compact. So we assume that F is a non-compact proper leaf. Taking a finite cover of M if necessary, we further assume $\mathcal{F}$ is transversely oriented. Let $x_0\in F\cap U_0$ be a base point of F. We denote by A_n the set of plaques which can be reached from $P_0(x_0)$ by a plaque chain of length$\leqq n$. For each plaque $\rho\in A_n$, we choose a plaque chain c_ρ of minimal length connecting $P_0(x_0)$ and ρ. We choose a parametrization of the plaque space X_0 so that $P_0(x_0)$ corresponds to 0.

LEMMA 1. *There exists a positive number δ such that*

$$\bigcap_{\substack{\rho\in A_n\\ n\in\mathbf{N}}} \mathrm{Dom}(\gamma_{c_\rho})\supset[0, \delta].$$

PROOF. We use the structure of the neighbourhood of F described in Proposition A. We assume there is a positive number ε such that $d(F, \phi_i(\mathring{D}^{n-1}\times\partial D^1))>\varepsilon$ for each i. Using this ε, take a compact subset K of F and the immersion

$$\Phi: \overline{F-K}\times[0, 1]\to M$$

as in Proposition A. Note that $\mathrm{Image}(\Phi)\cap\phi_i(\mathring{D}^{n-1}\times\partial D^1)=\emptyset$, for each i. This implies that if c is a chain which is a join $c=c_1\# c_2$ and c_2 is contained in $\overline{F-K}$, then $\mathrm{Dom}(\gamma_c)\cap[0, 1]=\mathrm{Dom}(\gamma_{c_1})\cap[0, 1]$.

Since the chains c_ρ have minimal length, there exists a positive integer N such that if $n>N$ and $\rho\in A_n-A_N$, then $\gamma_c(P_0(x_0))$ is contained in $F-K$. It follows that there exists a positive number δ_1 such that

$$\bigcap_{\substack{\rho\in A_n-A_N\\ n>N}}\mathrm{Dom}(\gamma_{c_\rho})\supset[0, \delta_1].$$

Choose $\delta<\delta_1$ so small that $[0, \delta]$ is contained in the common domain of γ_{c_ρ}, $\rho\in A_N$. Then δ satisfies the required property.

PROOF OF THEOREM 1. Let δ be the number in the proof of Lemma 1. By the assumption of Theorem 1 there exists a leaf F' such that $F'\neq F$, $F'\cap[0, \delta]\neq\emptyset$ and $gr(F')\leqq d$. Let $y_0\in F'\cap[0, \delta]$ and let g_{x_0} and g_{y_0} be the growth functions of F and F' at x_0 and y_0 respectively. Then for each ρ, $\mathrm{Dom}(\gamma_{c_\rho})\ni y_0$ by Lemma 1, and so we get $g_{x_0}(n)\leqq g_{y_0}(n)$. Thus $gr(F)\leqq gr(F')\leqq d$.

Next we prove Theorem 2. We assume F' accumulates to F from the positive side. Let δ and N be the numbers in Lemma 1. By Proposition B, there exist a point y_0 of $F'\cap[0, \delta]$ and a closed plaque chain $c=(\rho_0, \cdots, \rho_h)$, $\rho_0=\rho_h=P_0(x_0)$ such that $\gamma_c(y_0)<y_0$. We set $\gamma=\gamma_c$.

Let B_n be the set of plaques of F' which can be reached from the initial plaque $P_0(y_0)$ by a chain of length$\leqq n$. For large n, we define $i_l: A_{n-lh}-A_N\to B_n$ by $i_l(\rho)=\gamma_{c_\rho}\circ\gamma^l(P_0(y_0))$ $(l=0, 1, \cdots, k)$, where k is the largest integer which satisfies $n-kh\geqq N$.

LEMMA 2. *i_l is one to one and* $\mathrm{Image}(i_l)\cap\mathrm{Image}(i_{l'})=\emptyset$, *if* $l\neq l'$.

PROOF. Assume i_l is not one to one. Then there are two different plaques $\rho, \rho'\in A_{n-lh}-A_N$ such that $i_l(\rho)=i_l(\rho')$. By Lemma 1 the holonomy homeomorphism $\gamma_{c_\rho}\circ\gamma_{c_{\rho'}}^{-1}$ contains $\gamma_{c_{\rho'}}(P_0(x_0))$ in its domain and $\gamma_{c_\rho}\circ\gamma_{c_{\rho'}}^{-1}(\gamma_{c_{\rho'}}(P_0(x_0))\neq\gamma_{c_{\rho'}}(P_0(x_0))$. This contradicts the property 2) of the neighbourhood $\Phi(\overline{F-K}\times[0, 1])$ of $\overline{F-K}$. The second assertion is proved similarly using the holonomy homeomorphism $\gamma_{c_\rho}\circ\gamma^{l'-l}\circ\gamma_{c_{\rho'}}^{-1}$.

Let g_{x_0} and g_{y_0} be the growth functions of F and F' at $P_0(x_0)$ and $P_0(y_0)$

respectively. Then by Lemma 2,

$$g_{y_0}(n)\geqq\{g_{x_0}(n)-g_{x_0}(N)\}+\{g_{x_0}(n-h)-g_{x_0}(N)\}+\cdots+\{g_{x_0}(n-kh)-g_{x_0}(N)\}.$$

Adding this inequality for n, $n-1$, $\cdots$, $n-h+1$, we get

$$g_{y_0}(n)\geqq 1/h\sum_{l=N}^{n}\{g_{x_0}(l)-g_{x_0}(N)\}. \tag{*}$$

LEMMA 3. *Let $f: Z^+\to Z^+$ be an increasing function and*

$$g(n)=\sum_{l=1}^{n}f(l).$$

Then f is dominated by a polynomial of degree $d>0$, if and only if $g(n)$ is dominated by a polynomial of degree $d+1$.

PROOF. The necessity is obvious, so we prove the sufficiency. Assume $g(n)\leqq Cn^{d+1}$, $C>0$, we shall prove $f(n)\geqq Dn^d$ where $D=\max\{f(2d^2), 6C(d+1)\}$.

Assume the contrary, then there exists an integer n such that $f(n)\leqq Dn^d$. Note that since $D\geqq f(2d^2)$, $n\geqq 2d^2$. Let $n=n_1d+h$, $0\leqq h<d$. From the assumption

$$g(n_1(d+1))\leqq C(n_1(d+1))^{d+1}.$$

From the definition of g

$$\begin{aligned}g(n_1(d+1))&\geqq f(n_1d+h)+f(n_1d+h+1)+\cdots+f(n_1d+n_1)\\&\geqq(n_1-h)f(n_1d+h)\\&\geqq(n_1-h)D(n_1d)^d.\end{aligned}$$

From these two inequalities we get

$$C\{n_1(d+1)\}^{d+1}\geqq(n_1-h)6C(d+1)(n_1d)^d.$$

So

$$(1+1/d)^d\geqq 6(1-h/n_1).$$

Since $1-h/n_1\geqq 1/2$, we have $(1+1/d)^d\geqq 3$, which is a contradiction.

By Lemma 3 and the inequality (*), $gr(F')\geqq gr(F)+1$. And the proof of Theorem 2 is complete.

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(Received March 14, 1979)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan

Present address
Department of Mathematics
Faculty of Science
Tokyo Institute of Technology
Ōokayama, Meguro-ku, Tokyo
152 Japan