# *Growth and depth of leaves*

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## Introduction

The notion of growth of leaves in foliations was first introduced by J. F. Plante with respect to his study of transverse invariant measures. Since then, it proved to be a useful tool for the qualitative study of non-compact leaves in foliations of codimension one. Generally speaking, the degree of growth of a leaf measures the complexity of the injective immersion of the leaf to the ambient foliated manifold. Besides the usual definitions such as polynomial growth, we say a leaf $F$ has exact polynomial growth if $F$ grows as fast as a polynomial. The main purpose of this paper is to give conditions under which a leaf has exact polynomial growth.

There is another concept which measures the complexity of a leaf. In [Ni 3], Nishimori defined the notion of depth of a leaf and used it to the study of asymptotic behaviour of ends of non-compact leaves. These two notions are mutually related. In Theorem 1, we show that a proper leaf has exact polynomial growth of degree equal to the depth if each leaf contained in the limit set of the leaf has abelian holonomy.

To treat non-proper leaves, we introduce the concept of nice saturated sets. This is a generalization of a component of an almost without holonomy foliation. Leaves contained in a nice saturated set are shown to have exact polynomial growth under the assumption of abelian holonomy (Theorem 2). Again we give a topological interpretation to the degree of growth of leaves.

These results are applied to the case where the foliation is almost without holonomy (Theorem 4) and to the case where the foliation is transversely analytic (Theorem 5, 6). Polynomial growth leaves in these foliations are shown to have exact polynomial growth.

The growth of a foliation is the set of degrees of growth of leaves in this foliation. A foliation of codimension one has finite growth if it is transversely analytic (Theorem 6) or if it is almost without holonomy (Theorem 4). In Theorem 8 and 9, we give characterization of codimension one foliations whose growth consist of one or two elements.

In §1 we state the main results of this paper. In §2 we introduce a notion which will be used in the proofs of Theorem 1 and 2 — a uniform transverse manifold for a leaf. It measures the growth of a leaf in terms of the growth of the holonomy pseudogroup which acts on this manifold. In § 3 we study basic properties of depth of leaves. In particular we prove that a leaf $F$ has finite depth if and only if the limit set of $F$ consists of finitely many leaves. § 4 is devoted to the proof of Theorem 1. For the proof we use a method of Nishimori to decompose a foliated manifold into simpler ones. In § 5 we study the structure of a minimal nice saturated set. In § 6 we prove Theorem 2. We again use the Nishimori decomposition in a generalized form. In § 7 we study leaves with linear growth. In § 8 we prove Theorem 4. In § 9 we prove Theorem 5 and 6 which treat transversely analytic foliations. Finally in § 10 we prove Theorem 7 and 8.

Unless otherwise specified, $\mathcal{F}$ will denote a transversely oriented $C^2$ foliation of codimension one on a compact manifold $M$ which is tangent to the boundary.

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## § 1. Main results

We begin with some notations and definitions. Choose a Riemannian metric of $M$ and relativise it to a leaf $F$ of $\mathcal{F}$. The resulting distance function on $F$ is denoted by $d_F$. For a point $x$ of $F$, the metric ball of radius $R$ centered at $x$ is defined by

$$D_x(R)=\{y\in F \mid d_F(x, y)<R\} .$$

The growth function $f_x(R)$ of $F$ at $x$ is defined to be the Riemannian volume of $D_x(R)$.

DEFINITION 1. (A) The degree of growth of $F$, denoted $gr(F)$, is a nonnegative integer or a symbol $\infty$ or exp defined as follows.

1) If the growth function $f_x(R)$ is dominated by a polynomial, we say $F$ has polynomial growth. In this case $gr(F)$ is the smallest integer degree of polynomials which dominate $f_x(R)$.

2) If $f_x(R)$ dominates an exponential function, we say $F$ has exponential growth. In this case we define $gr(F)=\exp$.

3) Finally $gr(F)=\infty$, if $F$ has neither polynomial nor exponential growth.

(B) If $F$ has polynomial growth of degree $d$ and the growth function $f_x(R)$ dominates a polynomial of degree $d$, we say $F$ has *exact polynomial growth* of degree $d$.

It is known that $gr(F)$ does not depend on the choice of the base point $x$

or the Riemannian metric of the compact manifold $M$ ([P 3]). Let $M/\mathcal{F}$ denote the set of leaves of $\mathcal{F}$. We consider $gr$ as a function from $M/\mathcal{F}$ to the linearly ordered set $\{0, 1, 2, \cdots, \infty, \exp\}$.

DEFINITION 2. The growth of $\mathcal{F}$, denoted by $gr(\mathcal{F})$, is the image of the function

$$gr: M/\mathcal{F} \to \{0, 1, 2, \cdots, \infty, \exp\}.$$

Let $F$ be a leaf. $L(F)$ denotes the *limit set* of $F$. That is, $L(F)$ is the set of points $y$ of $M$ such that there exists a sequence $\{x_n\}$ of points of $F$ which is discrete with respect to the topology of $F$ as a manifold, and which converges to $y$ with respect to the topology of $M$.

DEFINITION 3. Let $F$ be a leaf of $\mathcal{F}$. Let $d(F)=\sup\{k \mid$ there exists a sequence $F_0, F_1, \cdots, F_k$ of leaves such that $F_i \subset L(F_{i+1})$, $F_i \neq F_{i+1}$ for $i<k$ and $F_k=F\}$. Secondly let $pd(F)=\sup\{k \mid$ there exists a sequence $F_0, F_1, \cdots, F_k$ of leaves such that $F_i \subset L(F_{i+1})$, $F_i \neq F_{i+1}$, $F_i$ is proper for $i<k$ and $F_k=F\}$. We call $d(F)$ and $pd(F)$ the *depth* and *proper depth* of $F$ respectively.

In [T], we have proved that $gr(F) \geqq pd(F)$ for any leaf $F$.

THEOREM 1. *Let $F$ be a leaf with the following properties.*

(1) *$d(F)$ is finite.*

(2) *Let $F'$ be a leaf contained in $L(F)$. Then the holonomy group of $F'$ on the side approached by $F$ is abelian.*

*Then $F$ has exact polynomial growth of degree $d(F)$.*

Next we give a condition for a non-proper leaf to have exact polynomial growth.

DEFINITION 4. A *saturated* set is a set which is a union of leaves. Let $U$ be a saturated open connected subset of $M$. We say $U$ is *nice* if the saturated closed set $\bar{U}-U$ consists of a finite number of proper leaves and all leaves in $U$ have trivial holonomy groups. A saturated open set $U$ is *minimal* if $U$ contains no non-empty saturated relatively closed proper subset.

In § 5 we study the structure of a minimal nice saturated set $U$. In particular, we define the Novikov transformation. It is a homomorphism from the fundamental group of $U$ to the group of diffeomorphisms of a transverse circle. Its image is abelian.

THEOREM 2. *Let $U$ be a minimal nice saturated set and $F$ a leaf contained in $U$. Let $G$ be the image of the Novikov transformation of $\mathcal{F}_U$. Assume the* *following condition. If $F'$ is a leaf contained in $\bar{U}$, then the holonomy group of $F'$ on the side approached by $U$ is abelian. Then $G$ is finitely generated and $F$ has exact polynomial growth of degree $pd(F)+\mathrm{rank}(G)$.*

COROLLARY 1. *Let $F$ be a leaf. Assume the following condition. If $F'$ is a leaf contained in the limit set of $F$, then the holonomy group of $F'$ on the side approached by $F$ is abelian. Then $F$ has polynomial growth if and only if $d(F)$ is finite or the set $U=\bar{F}-\bigcup\{F'|F'$ is a proper leaf contained in $L(F)\}$ is a minimal nice saturated set.*

We can characterize leaves with linear growth. In this case we do not need the assumption of abelian holonomy.

THEOREM 3. *Let $F$ be a leaf. Then $gr(F)=1$ if and only if one of the following two cases occurs.*

(1) *The limit set of $F$ consists of a finite number of compact leaves.*

(2) *All leaves in $M$ are everywhere dense and have trivial holonomy groups. And the rank of the image of the Novikov transformation is one.*

Recall that a foliation is almost without holonomy if the holonomy groups of non-compact leaves are trivial ([He 1], [I 2]).

THEOREM 4.[1] *Assume $\mathcal{F}$ is almost without holonomy. Then each leaf of $\mathcal{F}$ has exact polynomial growth and the set $gr(\mathcal{F})$ is a finite set.*

If the foliation is transversely analytic, we can apply Theorem 1 and 2 satisfactorily.

THEOREM 5. *Let $F$ be a leaf of a transversely analytic foliation of codimension one on a compact manifold. Then $F$ has polynomial growth if and only if either $d(F)$ is finite or the set $\bar{F}-\bigcup\{F'|F'$ is a proper leaf contained in $L(F)\}$ is a minimal nice saturated set.*

COROLLARY 2. *If $F$ is a leaf with polynomial growth in a transversely analytic codimension one foliation on a compact manifold, then $F$ has exact polynomial growth of degree$\geqq pd(F)$. The equality holds if and only if $F$ is proper.*

THEOREM 6. *Let $\mathcal{F}$ be a transversely analytic codimension one foliation on a compact manifold $M$. Then the set $gr(\mathcal{F})$ is a finite set.*

In the proof of Theorem 4 we see that the growth of a foliation without

1) This theorem was obtained independently by G. Hector ([He 3]).

holonomy consists of a single element. Using the results of Plante, we can characterize such foliations.

THEOREM 7. *Let $\mathcal{F}$ be a transversely orientable codimension one foliation of class $C^2$ on a closed manifold. Then*

(1) *$gr(\mathcal{F})=\{d\}$, $0\leqq d<\infty$, if and only if $\mathcal{F}$ is without holonomy.*

(2) *There are no foliations $\mathcal{F}$ with $gr(\mathcal{F})=\{\infty\}$.*

(3) *$gr(\mathcal{F})=\{\exp\}$, if and only if all leaves of $\mathcal{F}$ are non-compact and there exists a leaf of $\mathcal{F}$ with non-trivial holonomy.*

THEOREM 8. *Assume $gr(\mathcal{F})=\{d_1, d_2\}$, $0\leqq d_1\leqq d_2<\infty$. Then $\mathcal{F}$ is almost without holonomy.*

## §2. Uniform transversals

First we recall the definition of the *holonomy pseudogroup* of a foliation $\mathcal{F}$. Let $X$ be the disjoint union of all one-dimensional submanifolds of $M$ which are transverse to $\mathcal{F}$. Let $c$ be a curve in $M$ from a point $x$ of $X$ to a point $y$ of $X$ which is contained in a single leaf. The basic property of a foliation is that some neighbourhood $U$ of $x$ in $X$ is translated along $c$ to a neighbourhood $V$ of $y$ in $X$. The resulting local diffeomorphism of $X$ is denoted by $\gamma_c$. The collection $\Gamma_X$ of all these local diffeomorphisms forms a pseudogroup and is called *the holonomy pseudogroup* of $\mathcal{F}$. If $Y$ is a subset of $X$, $\Gamma_Y$ denotes the restriction of $\Gamma_X$ to $Y$. We call $\Gamma_Y$ *the holonomy pseudogroup of $\mathcal{F}$ restricted to $Y$.*

Let $F$ be a leaf and $R$ a one-dimensional submanifold through $F$ which is transverse to $\mathcal{F}$. Let $\Gamma$ be a finitely generated subpseudogroup of $\Gamma_{R\cap F}$ and $S=\{\gamma_1, \cdots, \gamma_k\}$ a finite generating set of $\Gamma$. For an element $\gamma$ of $\Gamma$, $l(\gamma)$ denotes the length of $\gamma$ with respect to $S$. As before, $d_F$ denotes the distance function on $F$ induced from a Riemannian metric of $M$.

DEFINITION 2.1. Let $F$, $R$ and $\Gamma$ be as above.

(1) We say the pair $(R, \Gamma)$ is *dispersed* for $F$ if

a) $K=\sup\{d_F(x, \gamma_i^{\pm 1}(x))\mid \gamma_i\in S,\ x\in R\cap F\}$ is finite and

b) $\delta=\inf\{d_F(x, x')\mid x, x'\in R\cap F,\ x\neq x'\}$ is positive.

(2) We say $(R, \Gamma)$ or $R$ is *finitely dense* for $F$ if $\Gamma=\Gamma_{R\cap F}$,

a) $\rho=\sup\{d_F(x, R-\{x\})\mid x\in F\}$ is finite and

b) let $a_n=\inf\{d_F(x, \gamma(x))\mid x\in R\cap F,\ \gamma\in\Gamma,\ l(\gamma)>n$ and there is no $\gamma'\in\Gamma$ such that $l(\gamma')\leqq n$ and $\gamma'(x)=\gamma(x)\}$, then the sequence $\{a_n\}$ is unbounded.

(3) If $(R, \Gamma)$ is both dispersed and finitely dense for $F$, we say $R$ is *uniform* for $F$, or $F$ admits a *uniform transversal* $R$.

*Example.* Let $\mathcal{F}$ be a codimension one foliation of a total space of a circle bundle over a compact manifold which is transverse to fibres. Then each fibre is uniform for every leaf.

Choose a base point $x_0$ of $F\cap R$. We define the growth function of $\Gamma$ at $x_0$.

DEFINITION 2.2. The growth function $g_{x_0}$: $Z^+\to Z^+$ of $\Gamma$ at $x_0$ is defined as follows. $g_{x_0}(n)$ is the number of distinct points of $F\cap R$ which are mapped from $x_0$ by elements of $\Gamma$ with length$\leqq n$.

The main theorem of this section is the following. $f_{x_0}(R)$ denotes the growth function at $x_0$ of $F$ defined by the volume of the metric ball.

THEOREM 2.3. *Let $F$, $R$ and $\Gamma$ be as in* (2.1).

(1) *If $(R, \Gamma)$ is dispersed for $F$, then the growth function $f_{x_0}$ of $F$ dominates the growth function $g_{x_0}$ of $\Gamma$.*

(2) *If $R$ is finitely dense for $F$, then $g_{x_0}$ dominates $f_{x_0}$.*

(3) *If $R$ is uniform for $F$, then $f_{x_0}$ and $g_{x_0}$ have the same growth type.*

PROOF. Let $A_n=\{\gamma(x_0)\,;\,\gamma\in\Gamma,\ l(\gamma)\leqq n\}$. Assume $(R, \Gamma)$ is dispersed for $F$. Then by the definition of $K$ and $\delta$, it is easy to see that

$$D_{x_0}(Kn+\delta)\supset\bigcup_{x\in A_n}D_x(\delta/2)$$

and if $x$, $x'$ are distinct points of $R\cap F$, then $D_x(\delta/2)\cap D_{x'}(\delta/2)=\varnothing$. If $\sigma=\inf\{\text{volume } D_x(\delta/2)\,|\,x\in F\cap R\}$, then we have

$$f_{x_0}(Kn+\delta)\geqq\sigma g_{x_0}(n)\,.$$

Thus we have proved the first assertion.

Assume $R$ is finitely dense for $F$. Then, by the unboudedness of $\{a_n\}$, there is an integer $N$ such that, if $x$, $x'\in F\cap R$, $d_F(x, x')\leqq 3\rho$, then there exists $\gamma\in\Gamma$ for which $x'=\gamma(x)$ and $l(\gamma)\leqq N$ hold. We show

$$D_{x_0}(n\rho)\subset\bigcup_{x\in A_{Nn}}D_x(3\rho)\,.$$

In fact, if $x$ is a point of $F$ with $d_F(x, x_0)<n\rho$, then from the definition of $\rho$ there exists a sequence $x_0, x_1, \cdots, x_n$ of points of $F\cap R$ such that $d_F(x_i, x_{i+1})\leqq 3\rho$ for $i=0, 1, \cdots, n-1$ and $d_F(x_n, x)\leqq 2\rho$. For each $i\leqq n-1$, there exists $\gamma_i\in\Gamma$ such that $\gamma_i(x_i)=x_{i+1}$ and $l(\gamma_i)\leqq N$. It follows that $x_n=\gamma_{n-1}\circ\cdots\circ\gamma_1\circ\gamma_0(x_0)\in A_{Nn}$ and $x\in\bigcup_{y\in A_{Nn}}D_y(3\rho)$.

Let $\tau=\sup\{\text{volume } D_x(3\rho)\,|\,x\in F\cap R\}$, then

$$f_{x_0}(n\rho)\leqq\tau g_{x_0}(Nn)\,.$$

Thus we have proved the second assertion. This completes the proof.

## § 3. Depth of leaves

In this section we study some basic properties of depth of leaves. Let $F_1$ and $F_2$ be leaves of $\mathcal{F}$. By $F_1 < F_2$ we mean $F_1 \subset L(F_2)$ and $F_1 \neq F_2$. The proof of the following proposition is due to K. Yano.

PROPOSITION 3.1. *If $F$ is a non-proper leaf, then $d(F)=\infty$.*

PROOF. Let $T$ be a closed line segment transverse to $\mathcal{F}$ which intersects with $F$. Since $F$ is non-proper, $T \cap \bar{F}$ is a perfect set. Let $\{U_j\}$ be a countable basis of open sets of $T \cap \bar{F}$. Let $\Gamma$ denote the holonomy pseudogroup of $\mathcal{F}$ restricted to $T \cap \bar{F}$. We consider the set $X = \bigcap_{j=1}^{\infty} \bigcup_{\gamma \in \Gamma} \gamma(U_j)$. It is easy to see that for each $x \in X$, the leaf through $x$ contains $F$ in its closure. By the Baire category theorem $X$ is residual in $T \cap \bar{F}$. So $X$ contains uncountably many points. On the other hand it is easily seen that for any leaf $G$, $G \cap T$ is a countable set. So $X-F$ is non-empty. Choose a point $x_1 \in X-F$. Let $F_1$ be the leaf through $x_1$. Then we have $F > F_1$. Choose a point $x_2 \in X-F-F_1$ and let $F_2$ be the leaf through $x_2$. Then we have $F > F_1 > F_2$. In this way we can choose an infinite sequence $F > F_1 > F_2 > \cdots$. Thus we have $d(F)=\infty$.

Secondly we analyze the structure of the limit set of a leaf with finite depth. By (3.1), the closure of such a leaf contains no non-proper leaves.

LEMMA 3.2. *If $F$ is a leaf with finite depth and $G$ is a leaf contained in $L(F)$, then the holonomy group of $G$ on the side approached by $F$ contains a contracting element.*

PROOF. Fix a transverse arc through $G$ and a parametrization $T \approx [-1, +1]$ such that $T \cap G = \{0\}$ and $F \cap (0, 1)$ accumulates on 0. Let $\Gamma$ be the holonomy pseudogroup on $T$ defined by the holonomy along $G$. By a theorem of Sacksteder-Schwartz ([S-S], Theorem 1, [C-C], Lemma 1) there is $\varepsilon > 0$ such that for each $t_0 \in (0, \varepsilon) \subset T$, there exists an element $\gamma \in \Gamma$ such that $\gamma(t) < t_0$ for $0 < t < \varepsilon$.

Suppose $\Gamma$ does not contain a contracting element. Then there is $t_1 \in (0, \varepsilon) \cap F$ and $\gamma_1 \in \Gamma$ such that $\gamma_1(t_1) < t_1$. By assumption, $\lim_{n \to \infty} \gamma_1^n(t_1) = t_2 > 0$. Choose $\gamma_2 \in \Gamma$ such that $\gamma_2(t_2) < t_2$, and set $\lim_{n \to \infty} \gamma_2^n(t_2) = t_3$. In this way we can choose infinite sequences $\{\gamma_i\} \subset \Gamma$ and $\{t_i\} \subset (0, \varepsilon)$ such that $\gamma_i(t_i) < t_i$ and $\lim_{n \to \infty} \gamma_i^n(t_i) = t_{i+1}$. This is a contradiction since $F$ was assumed to have finite depth.

Let $F$ be a leaf of $\mathcal{F}$. Assume that for each $G\subset L(F)$, the holonomy group of $G$ on the side approached by $F$ contains a contracting element.

LEMMA 3.3. *For each $k$, let $C_k=\{G\subset L(F)\mid d(G)\leqq k\}$. Then, under the above assumption, each $C_k$ consists of a finite number of proper leaves.*

PROOF. $C_0$ coincides with the set of compact leaves contained in $L(F)$. And it is well-known that the number of compact leaves contained in the limit set of a leaf is finite. Suppose $C_j$ consists of finitely many leaves for $j=0, 1, \cdots, k-1$ and $C_k$ contains infinitely many leaves $\{G_i\}$. Without loss of generality we can assume that each $G_i$ accumulates to a fixed leaf $G_0\in C_{k-1}$ from one side. Choose a transverse arc $T$ through $G_0$, $T=[-1, +1]$, $G_0\cap T=\{0\}$, $G_i\cap(0, 1)$ accumulates to 0 for each $i$, $T\cap C_j=\varnothing$ for $j<k-1$ and $T\cap C_{k-1}=T\cap G_0$. Using the contracting holonomy of $G_0$, we can find a compact subarc $J\subset(0, 1)$ such that $G_i\cap J\ni x_i$ for each $i>0$. Let $x$ be a cluster point of $\{x_i\}$ and let $G$ be the leaf through $x$. Since $G$ is contained in $L(F)$, the holonomy group of $G$ contains a contracting element. So $G$ is contained in the limit set of some $G_i$. From the choice of $T$, $d(G)\geqq k$ and it follows that $d(G_i)\geqq k+1$. This cotradiction completes the proof.

THEOREM 3.4. *Let $F$ be a leaf. Then $d(F)$ is finite if and only if the limit set of $F$ consists of a finite number of leaves.*

PROOF. This follows easily from (3.2) and (3.3).

COROLLARY 3.5. *If $d(F)$ is finite, then $d(F)=pd(F)$.*

In a later section, we shall have an occasion to make use of the following lemma.

LEMMA 3.6. *Suppose there exists an infinite sequence $F_0, F_1, \cdots$ of proper leaves such that $F_0<F_1<F_2<\cdots$. Then there exists a non-proper leaf.*

PROOF. Choose a transverse arc $T$ through $F_0$, $T\approx[-1, 1]$, $T\cap F_0=\{0\}$. For simplicity we assume that $F_{j+1}\cap(0, 1)$ accumulates on $F_j\cap(0, 1)$ from the positive side for $j=0, 1, 2, \cdots$. Let $a_0=0$. By the theorem of Sacksteder-Schwartz (see (3.2)), we can choose $\gamma_0\in\Gamma_T$, $b_0\in F_1\cap(0, 1)$ such that $\gamma_0(a_0)=a_0$ and $\gamma_0(b_0)<b_0$. Let $a_1=\gamma_0(b_0)$ and choose $\gamma_1\in\Gamma_T$, $b_1\in F_2\cap(0, 1)$ such that $b_0>b_1>a_1$, $\gamma_1(a_1)=a_1$ and $\gamma_1(b_1)<b_1$. In this way we can choose sequences $\{\gamma_i\}\subset\Gamma_T$, $\{a_i\}$ and $\{b_i\}$ such that $a_0<a_1<a_2<\cdots<b_2<b_1<b_0$, $\gamma_i(a_i)=a_i$, $\gamma_i(b_i)=a_{i+1}$ and $a_i, b_i\in F_i$. Let $a_\infty=\lim a_i$. Then for each $i$, we have $a_i<\gamma_i(a_\infty)<a_{i+1}$. This shows that the leaf through $a_\infty$ is non-proper. The lemma is proved.

## §4. Proper leaves with exact polynomial growth

In this section we prove Theorem 1. For the proof we use Nishimori's method to decompose a subset of a foliated manifold into simply foliated ones. We need some preliminaries. Let $A$ be a compact manifold with or without boundary, and let $B$ be a codimension one submanifold of $A$ such that $A-B$ is connected. $C(A, B)$ denotes the compact manifold obtained from $A-B$ by attaching two copies $B_1$ and $B_2$ of $B$. Let $f: [0, \varepsilon)\to[0, \delta)$, $0<\delta<\varepsilon$ be a contracting diffeomorphism. $X(A, B, f)$ denotes the manifold obtained from $C(A, B)\times[0, \varepsilon)$ by identifying $(x, t)\in B_1\times[0, \varepsilon)$ with $(x, f(t))\in B_2\times[0, \delta)$. And $\mathcal{F}(A, B, f)$ denotes the codimension one foliation of $X(A, B, f)$ induced from the product foliation of $C(A, B)\times[0, \varepsilon)$.

We consider a certain special type of foliated manifolds with corner. Let $X$ be a compact manifold such that the boundary $\partial X$ is divided into two parts $Y_1$ and $Y_2$ by the corner, that is $\partial X=Y_1\cup Y_2$ and $\partial Y_1=\partial Y_2=Y_1\cap Y_2$. Let $\mathcal{F}$ be a transversely oriented codimension one $C^2$ foliation of $X$ such that $\mathcal{F}$ is tangent to $Y_1$, transverse to $Y_2$ and the induced foliation $\mathcal{F}|_{Y_2}$ is without holonomy. Such a foliation will be called *admissible*.

The following theorem is proved in [Ni 2] when $Y_2$ is empty. But the proof easily generalizes to our case. So we omit the proof (see also [Ni 3] and [Ni 4]).

THEOREM 4.1 (Nishimori). *Let $(X, \mathcal{F})$ be as above and let $K$ be a connected component of $Y_1$. Suppose $K$ is contained in the limit set of a proper leaf of $\mathcal{F}$ and the holonomy group of $K$ is abelian. Then there exist codimension one submanifold $N$ of* Int$(K)$, *a contracting diffeomorphism $f: [0, \varepsilon)\to[0, \delta)$, $0<\delta<\varepsilon$ and an embedding $h: X(K, N, f)\to X$ such that $h(x, 0)=x$ for $x\in K$ and $h^*\mathcal{F}=\mathcal{F}(K, N, f)$.*

Now we are in a position to explain Nishimori's method. Let $F$ be a leaf such that $d=d(F)$ is finite and the holonomy groups of leaves in $L(F)$ on the side approached by $F$ are abelian. As we have seen in §3, $L(F)$ consists of a finite number of proper leaves. Let $\Omega^1$ be the connected component of

$$M-\bigcup\{K|K \text{ is a compact leaf of } \mathcal{F}\}$$

containing $F$. Since the closure of $\Omega^1$ contains a finite number of compact leaves, we can consider the compact manifold $\bar{\Omega}^1$ obtained from $\Omega^1$ by attaching the boundary. $\bar{\Omega}^1$ is naturally immersed in $M$. Then $F$ can be considered as a leaf of the foliation $\mathcal{F}|_{\bar{\Omega}^1}$ of $\bar{\Omega}^1$ induced from $\mathcal{F}$ by the immersion. Let $K_j^1$, $j=1, \cdots, \nu_1$, be the compact leaves of $\mathcal{F}|_{\bar{\Omega}^1}$ contained in $F$. By (4.1), we have codimension one submanifolds $N_j^1$ of $K_j^1$, contracting diffeomorphisms

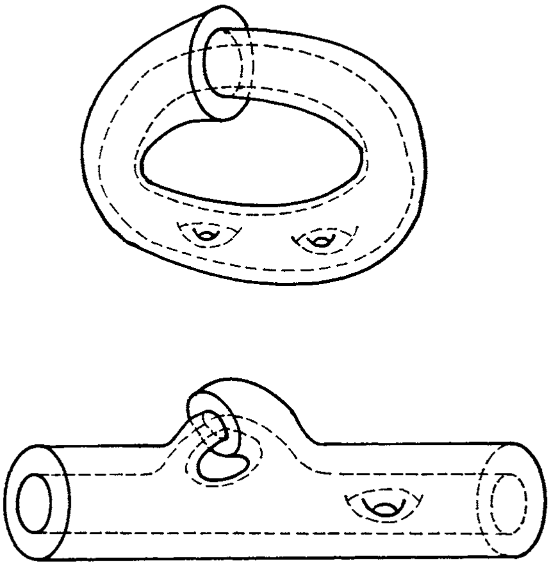

Figure 1

$f_j^1: [0, \varepsilon_j^1) \to [0, \delta_j^1)$ and imbeddings $h_j^1: X(K_j^1, N_j^1, f_j^1) \to \bar{\Omega}^1$ such that $h_j^1(C(K_j^1, N_j^1) \times \{0\}) = K_j^1$ and $h_j^{1*}\mathcal{F} = \mathcal{F}(K_j^1, N_j^1, f_j^1)$, for $j=1, \cdots, \nu_1$.

Let

$$\Omega_0^1 = \bar{\Omega}^1 - \bigcup_{j=1}^{\nu_1} h_j^1(\mathrm{Int}\, X(K_j^1, N_j^1, f_j^1)) - \bigcup_{j=1}^{\nu_1} K_j^1 .$$

Then $\Omega_0^1$ is a compact manifold with corner equipped with an admissible foliation $\mathcal{F}|_{\Omega_0^1}$. If $K$ is a compact leaf of $\mathcal{F}|_{\Omega_0^1}$ contained in $\overline{F \cap \Omega_0^1}$, then it has the form $G \cap \Omega_0^1$ where $G$ is a leaf of $\mathcal{F}$ contained in $\bar{F}$ and $d(G)=1$. Let $\Omega^2$ be the connected component of

$$\Omega_0^1 - \bigcup \{K \mid K \text{ is a compact leaf of } \mathcal{F}|_{\Omega_0^1}\}$$

containing $F \cap \Omega_0^1$. As before the closure of $\Omega^2$ contains a finite number of compact leaves of $\mathcal{F}|_{\Omega_0^1}$ and we can consider the compact manifold $\bar{\Omega}^2$ obtained from $\Omega^2$ by attaching the boundary and the admissible foliation $\mathcal{F}|_{\bar{\Omega}^2}$. Let $K_j^2$, $j=1, \cdots, \nu_2$, be the compact leaves of $\mathcal{F}|_{\bar{\Omega}^2}$ contained in $\overline{F \cap \Omega_0^1}$. By (4.1), we obtain $N_j^2 \subset K_j^2$, $f_j^2: [0, \varepsilon_j^2) \to [0, \delta_j^2)$ and $h_j^2: X(K_j^2, N_j^2, f_j^2) \to \bar{\Omega}^2$ such that

$$h_j^2(C(K_j^2, N_j^2) \times \{0\}) = K_j^2, \quad h_j^{2*}\mathcal{F} = \mathcal{F}(K_j^2, N_j^2, f_j^2)$$

and

$$h_j^2(\partial K_j^2 \times [0, \varepsilon_j^2)) \subset \bigcup_{k=1}^{\nu_1} h_k^1(N_k^1 \times [\delta_k^1, \varepsilon_k^1)) .$$

Let

$$\Omega_0^2=\bar{\Omega}^2-\bigcup_{j=1}^{\nu_2} h_j^2(\mathrm{Int}\, X(K_j^2, N_j^2, f_j^2))-\bigcup_{j=1}^{\nu_2} K_j^2 .$$

Then $\Omega_0^2$ is a compact manifold with corner.

We can repeat this process and obtain a sequence $\Omega_0^1, \Omega_0^2, \cdots, \Omega_0^d$ of submanifolds of $M$. Note that $K$ is a compact leaf of $\mathcal{F}|_{\Omega_0^i}$ if and only if there is a leaf $F'$ of depth $i$ of $\mathcal{F}$ which is contained in $\bar{F}$ and $F'\cap\Omega_0^i=K$. Since $d(F)=d$, $F\cap\Omega_0^d$ is a compact leaf of $\mathcal{F}|_{\Omega_0^d}$ and our process finishes at the $d$-th step. For notational conveniency, we formulate all of this in the following proposition.

PROPOSITION 4.2 (Nishinori decomposition theorem). *Let $F$ be a leaf of $\mathcal{F}$ such that $d=d(F)$ is finite. Suppose that the holonomy groups of leaves of $F'\subset L(F)$ on the side approached by $F$ is abelian. Then there exist sequences $(\Omega_0^0, \Omega_0^1, \Omega_0^2, \cdots, \Omega_0^d)$, $(\Omega^1, \Omega^2, \cdots, \Omega^d)$ and $(\bar{\Omega}^1, \bar{\Omega}^2, \cdots, \bar{\Omega}^d)$ of manifolds with the following properties.*

(1) *$\Omega_0^0=M$. $\Omega_0^{i-1}$ is a compact submanifold of $M$ with corner and $\mathcal{F}|_{\Omega_0^{i-1}}$ is admissible for $i\geqq 2$.*

(2) *$\Omega^i$ is the connected component containing $F\cap\Omega_0^{i-1}$ of*
*$\Omega_0^{i-1}-\bigcup\{K|K$ is a compact leaf of $\mathcal{F}|_{\Omega_0^{i-1}}\}$.*

(3) *$\bar{\Omega}^i$ is the compact manifold obtained from $\Omega^i$ by attaching the boundary. Although $\bar{\Omega}^i$ is not a submanifold of $M$, it naturally immerses to $M$ and we can consider the induced foliation which is simply denoted by $\mathcal{F}|_{\bar{\Omega}^i}$.*

(4) *The compact leaves of $\mathcal{F}|_{\bar{\Omega}^i}$ contained in the limit set of $F\cap\bar{\Omega}^i$ are finite in number. Let $K_j^i$, $j=1, 2, \cdots, \nu_i$, be such leaves. For each $j$, there are a submanifold $N_j^i\subset K_j^i$, a contracting diffeomorphism $f_j^i: [0, \varepsilon_j^i)\to[0, \delta_j^i)$ and an embedding $h_j^i: X(K_j^i, N_j^i, f_j^i)\to\bar{\Omega}^i$ such that $h_j^i(C(K_j^i, N_j^i)\times\{0\})=K_j^i$, $h_j^{i*}\mathcal{F}=\mathcal{F}(K_j^i, N_j^i, f_j^i)$ and $h_j^i(\partial K_j^i\times[0, \varepsilon_j^i))\subset\bigcup_{k=1}^{\nu_{i-1}} h_k^{i-1}(N_k^{i-1}\times(\delta_k^{i-1}, \varepsilon_k^{i-1}))$.*

(5) *$\Omega_0^i=\Omega^i-\bigcup_{j=1}^{\nu_i} h_j^i(\mathrm{Int}\, X(K_j^i, N_j^i, f_j^i))-\bigcup_{j=1}^{\nu_i} K_j^i$.*

(6) *$F\cap\Omega_0^d$ is compact.*

Now we are ready to prove Theorem 1. Let $F$ be a leaf which satisfies the assumptions of Theorem 1. We decompose a neighbourhood of the limit set of $F$ as in (4.2). For simplicity, we assume each $N_j^i$ is connected. For each compact leaf $K_j^i$ of $\mathcal{F}|_{\bar{\Omega}^i}$, choose a base point $x_j^i\in N_j^i$, and choose a base point $x_0\in\mathrm{Int}(F\cap\Omega_0^d)$. Let $R_j^i=h_j^i(x_j^i\times(0, \varepsilon_j^i))$ and let $R_0$ be a small transverse arc through $x_0$, $R_0\cap F=\{x_0\}$. Let $R=\cup R_j^i\cup R_0$. We shall prove $R$ is uniform for $F$. For this purpose we study the holonomy pseudogroup of $\mathcal{F}$ restricted to $R\cap F$, which is denoted by $\Gamma$.

$\Gamma$ contains the following three types of elements. First there are $\gamma_j^i$: $R_j^i\cap F\to R_j^i\cap F$, $i\leqq d$, $j=1, 2, \cdots, \nu_i$, which is the conjugate of $f_j^i$ by $h_j^i$.

Secondly, let $K_j^i$ be a compact leaf of $\mathcal{F}|_{\bar{\Omega}^i}$ and $K_k^{i-1}$ a compact leaf of $\mathcal{F}|_{\bar{\Omega}^{i-1}}$. By (4.2), $h_j^i(\partial K_j^i \times (0, \varepsilon_j^i)) \cap R_k^{i-1}$ consists of a finite number of open intervals $R_{j,k,l}^{i,i-1}$, $l=1, 2, \cdots, a_{j,k}^{i,j-1}$. One of the end points of $R_{j,k,l}^{i,j-1}$ lies on $\partial K_j^i$. Choose a curve $c$ in $K_j^i$ from this endpoint to $x_j^i$ such that $c \cap N_j^i = \{x_j^i\}$. It is easy to see that $\mathcal{F}|_{h_j^i(c\times(0,\varepsilon_j^i))}$ is the trivial foliation. So we can define an element of the holonomy pseudogroup

$$\gamma_{j,k,l}^{i,i-1} : R_j^i \to R_{j,k,l}^{i,i-1}.$$

Note that $\gamma_{j,k,l}^{i,i-1}$ is independent of the choice of the curve $c$ such that $c \cap N_j^i = \{x_j^i\}$.

There remains one more type of elements of $\Gamma$. For each $(i, j)$, $F \cap (R_j^i - \bigcup_{j'=1}^{\nu_{i+1}} R_{j',j,l}^{i+1,i})$ consists of a finite number of points $x_{j,k}^i$, $k=1, 2, \cdots, b_j^i$. We define $\gamma_{x_{j,k}^i}$ to be the unique map from $x_0$ to $x_{j,k}^i$.

LEMMA 4.3. *$\Gamma$ is generated by $S=\{\gamma_j^i, \gamma_{j,k,l}^{i,i-1}, \gamma_{x_{j,k}^i}\}$.*

The proof is straight-forward from the detailed description of the structure of the limit set of $F$ in (4.2), so we omit it.

LEMMA 4.4. *$R$ is uniform for $F$.*

PROOF. It is easy to see that $(R, \Gamma)$ is dispersed for $F$. We show that $R$ is finitely dense for $F$. The number $\rho$ is bounded, approximately, by

$$\max\{\text{diameter of } F \cap \Omega_0^d, \text{ diameter of } (K_j^i - N_j^i) \mid i \leqq d,\ j=1, \cdots, \nu_i\}.$$

Let $m$ be the cardinality of the set $\{\gamma_{j,k,l}^{i,i-1}, \gamma_{x_{j,k}^i}\}$. If $\gamma$ is an element of $\Gamma$ with $l(\gamma)=n>m$, then it is easy to see that $\gamma$ contains at least $(n-m)$-letters in the $\gamma_j^i$'s. Let

$$\varepsilon=\inf\{d_F(h_j^i(x, t), h_j^i(x, f_j^i(t)) \mid h_j^i(x, t) \in F,\ i \leqq d,\ j=1, \cdots, \nu_i\}.$$

Then $d_F(x, \gamma(x))>(n-m)\varepsilon$, so $a_n>(n-m)\varepsilon$ and the sequence $\{a_n\}$ is unbounded. This completes the proof.

By (2.3), $F$ has the same growth type as the growth function $g_{x_0}(n)$ of $\Gamma$ at $x_0$. First we prove that the growth function $g_{x_0}(n)$ dominates a polynomial of degree $d$. To see this, choose a sequence $F_0, F_1, \cdots, F_d$ of leaves such that $F_i \subset L(F_{i+1})$, $F_i \neq F_{i+1}$ and $F_d=F$. For each $i$, a connected component of $F_i \cap \bar{\Omega}^i$ is a compact leaf $K_{j_i}^i$ of $\mathcal{F}|_{\bar{\Omega}^i}$, and there exists $\gamma_{j_i,j_{i-1},a}^{i,i-1} : R_{j_i}^i \to R_{j_i,j_{i-1}}^{i,i-1}$, for some $a$. Fix one of them. Let $g'_{x_0}(n)$ be the growth function of the sub-pseudogroup of $\Gamma$ generated by $\gamma_{j_i,j_{i-1},a}^{i,i-1}$, $\gamma_{j_i}^i$ and $\gamma_{x_{j_d,1}^d}$ at $x_0$. It is easy to see that for $n>d$, we have

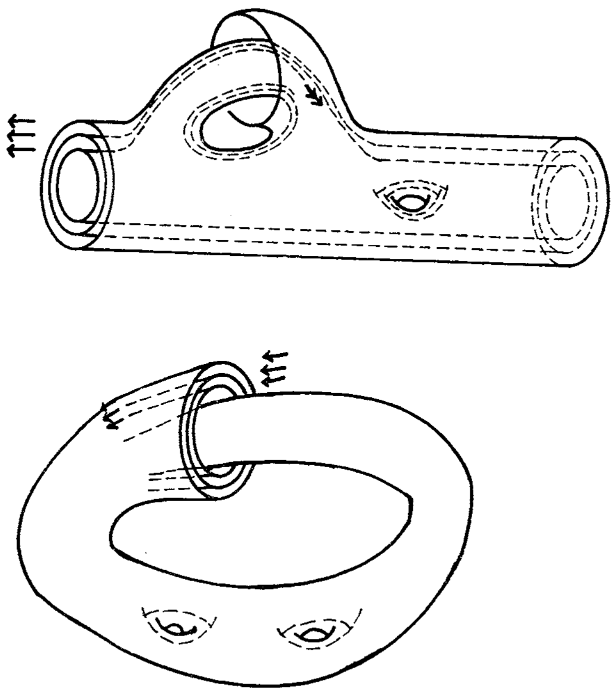

Figure 2

$$g'_{x_0}(n)-g'_{x_0}(n)\geqq(n-d)^{d-1}.$$

So $g'_{x_0}(n)$, hence $g_{x_0}(n)$, dominates a polynomial of degree $d$.

Secondly we show that $g_{x_0}(n)$ is dominated by a polynomial of degree $d$. To see this, let $\Gamma_{d-a}$ be the subpseudogroup of $\Gamma$ generated by $\gamma^i_j$, $i\geqq a$, $\gamma^{i+1,j}_{j,k,l}$, $i\geqq a$ and $\gamma_{x^i_{j,k}}$. Thus we get a sequence

$$\Gamma_0\subset\Gamma_1\subset\cdots\subset\Gamma_d=\Gamma.$$

Let $g^a_{x_0}(n)$ be the growth function of $\Gamma_a$ at $x_0$, that is,

$$g^a_{x_0}(n)=\sharp A^a_n \quad \text{where} \quad A^a_n=\{\gamma(x_0)\,;\,\gamma\in\Gamma_a,\ l(\gamma)\leqq n\}\,.$$

We prove inductively that $g^a_{x_0}(n)$ is bounded by a polynomial of degree $a$. It is clear that $g^0_{x_0}$ is bounded. Assume that $g^a_{x_0}(n)$ is bounded by a polynomial of degree $a$. If $\gamma(x_0)\in A^{a+1}_n-A^a_n$, then it is easy to see that $\gamma$ has the form $\gamma=\gamma_1\circ\gamma_2$ where $\gamma_2\in\Gamma_a$, $l(\gamma_2)=m$ and $\gamma_1$ is a word in $\{\gamma^{d-a-1}_j,\ \gamma^{d-a-1,d-a}_{j,k,l}\}$ of length $\leqq n-m$. Let

$$C=\max\{\text{the number of elements of } \Gamma \text{ which is a word in}$$

$$\gamma^{d-a-1}_j,\ \gamma^{d-a-1,d-a}_{j,k,l} \text{ of length } n\,|\,n=1,2,\cdots\}\,.$$

Then it follows that

$$g_{x_0}^{a+1}(n) \leqq C\{g_{x_0}^a(1)+g_{x_0}^a(2)+\cdots+g_{x_0}^a(n)\}.$$

So $g_{x_0}^{a+1}(n)$ is bounded by a polynomial of degree $a+1$. By induction, $g_{x_0}(n)$ is bounded by a polynomial of degree $d$.

Thus $g_{x_0}$, hence $f_{x_0}$, has the same growth type as a polynomial of degree $d$. This completes the proof of Theorem 1.

## §5. Nice saturated sets

If all leaves of $\mathcal{F}$ contained in the interior of $M$ have trivial holonomy groups, then the interior of $M$ is a nice saturated set. And there are many results related to the structure of such foliations ([No], [S], [He 1], [I 1, 2]). In this section we show that some of these results can be generalized to nice saturated sets.

Let $U$ be a nice saturated set, $X$ a non-singular $C^2$ vector field transverse to $\mathcal{F}$ and $\varphi_0$ its flow. Let $f: M \to \boldsymbol{R}$ be a bump function such that $f(x)>0$ if $x \in U$ and $f(x)=0$ if $x \in M-U$. Let $\varphi: U \times \boldsymbol{R} \to U$ be the flow generated by the vector field $f \cdot X|_U$. A curve $c: [0, 1] \to U$ is a *leaf curve* when its image is contained in a single leaf. If $c$ is a leaf curve, $\mathcal{F}_c$ denotes the foliation of $[0, 1] \times \boldsymbol{R}$ induced from $\mathcal{F}|_U$ by the composed map

$$[0, 1] \times \boldsymbol{R} \xrightarrow{c \times \mathrm{id.}} U \times \boldsymbol{R} \xrightarrow{\varphi} U.$$

The leaves of $\mathcal{F}_c$ are transverse to the lines $\{t\} \times \boldsymbol{R}$, for $0 \leqq t \leqq 1$.

LEMMA 5.1. *Let $c: [0, 1] \to U$ be a leaf curve. Then for any $t \in \boldsymbol{R}$, the leaf of $\mathcal{F}_c$ through $(0, t)$ intersects with $\{1\} \times \boldsymbol{R}$. In other words, all leaves of $\mathcal{F}_c$ are compact.*

PROOF. Let $t_0=\sup\{t \mid t>0$, the leaf of $\mathcal{F}_c$ through $(0, s)$ intersects with $\{1\} \times \boldsymbol{R}$ for $0 \leqq s < t\}$ and $-t_0'=\inf\{t \mid t<0$, the leaf of $\mathcal{F}_c$ through $(s, 0)$ intersects with $\{1\} \times \boldsymbol{R}$ for $t < s \leqq 0\}$. If $t_0$ is finite, then by a theorem of Imanishi ([I 1], Theorem 3.1), the leaf of $\mathcal{F}$ through $\varphi(c(0), t_0)$ is a holonomy limit leaf, that is, it is the limit of leaves which have non-trivial holonomy groups. Since $U$ is nice, this is absurd. So $t_0$ is infinite. Similarly $t_0'$ is infinite. This completes the proof.

LEMMA 5.2. *Let $c: [0, 1] \to U$ be a curve in $U$ from $c(0)=x$ to $c(1)=y$. Then there is a real number $t_c$ such that $c$ is homotopic relative $\{x, y\}$ to a curve which is a join of $\varphi|\{x\} \times [0, t_c]$ (or $(\varphi|\{x\} \times [t_c, 0])^{-1}$) with a leaf curve from $\varphi(x, t_c)$ to $y$. Moreover the number $t_c$ is uniquely determined by the homotopy* *class of $c$ relative $\{x, y\}$.*

PROOF. Using the distinguished charts of $U$, we can homotope $c$ to a curve in the form $a_1 \# b_1 \# \cdots \# a_k \# b_k$ where each $a_i$ is contained in $\varphi(a_i(0), \boldsymbol{R})$ and $b_i$ is a leaf curve from $a_i(1)$ to $a_{i+1}(0)$. Consider the foliation $\mathcal{F}_{b_1}$. From (5.1) it is easy to see that $a_1 \# b_1 \# a_2 \# b_2$ is homotopic to a curve in the form $a_1 \# a_2' \# b_1' \# b_2$ where $a_1 \# a_2'$ is contained in $\varphi(a_1(0), \boldsymbol{R})$ and $b_1' \# b_2$ is a leaf curve. By induction, $c$ is homotopic to a join of a transverse curve with a leaf curve.

Suppose there are $t_c' > t_c \geqq 0$ such that $(\varphi | \{x\} \times [0, t_c]) \# c_1$ and $(\varphi | \{x\} \times [0, t_c']) \# c_2$ are homotopic where $c_1$ and $c_2$ are leaf curves. Then the null-homotopic curve $(\varphi | \{x\} \times [t_c, t_c']) \# c_2 \# c_1^{-1}$ can be deformed to a loop which is everywhere transverse to $\mathcal{F}$. This implies there exists a leaf which has non-trivial holonomy ([No]). This contradiction completes the proof.

Now assume $U$ is a minimal nice saturated set. Then there is a simple closed curve $C$ in $U$ transverse to $\mathcal{F}$. And we can assume that the flows $\varphi_0$ and $\varphi$ have $C$ as a closed orbit of period one. By a theorem of Sacksteder-Schwartz ([S-S], Theorem 4), $C$ intersects with every leaf of $\mathcal{F}|_U$. Choose a base point $x_0 \in C$.

Let PerDiff($\boldsymbol{R}$) denote the group of periodic $C^2$ diffeomorphisms of $\boldsymbol{R}$ of period one and let Diff($C$) denote the group of $C^2$ diffeomorphisms of $C$. There is a natural map $p_*$: PerDiff($\boldsymbol{R}$)$\to$Diff($C$) which is induced from the projection $p: \boldsymbol{R} \to C$, $p(t) = \varphi(x_0, t)$. We define two maps

$$\tilde{q}: \pi_1(U, x_0) \to \mathrm{PerDiff}(\boldsymbol{R})$$

and

$$q: \pi_1(U, x_0) \to \mathrm{Diff}(C)$$

as follows. Let $\alpha$ be an element of $\pi_1(U, x_0)$, $c: (S^1, 0) \to (U, x_0)$ a representative of $\alpha$ and $t$ a positive real number. By (5.2) there exists uniquely a real number $\tilde{q}(\alpha)(t)$ such that the closed curve $(\varphi | \{x_0\} \times [0, t])^{-1} \# c \# (\varphi | \{x_0\} \times [0, t])$ is homotopic relative $\{\varphi(x_0, t)\}$ to a curve in the form $(\varphi | \{x_0\} \times [t, \tilde{q}(\alpha)(t)]) \# c_1$ where $c_1$ is a leaf curve. Similarly we define $\tilde{q}(\alpha)(t)$ when $t \leqq 0$. It is easy to see that $\tilde{q}(\alpha)$ belongs to PerDiff($\boldsymbol{R}$). We define $q(\alpha): C \to C$ by $q(\alpha) = p_* \tilde{q}(\alpha)$. The map $q$ is called the *Novikov transformation* of $\mathcal{F}|_U$ with respect to a transverse circle $C$ ([No]), [I 1]). The proof of the following lemma is easy and we omit it.

LEMMA 5.3. (1) *$\tilde{q}$ and $q$ are group homomorphisms.*

(2) *Let $\tilde{G}$ and $G$ be the image of $\tilde{q}$ and $q$ respectively. Then the kernel of the homomorphism $p_*: \tilde{G} \to G$ is the infinite cyclic group generated by $\tilde{q}([C])$.*

(3) *$\tilde{G}$ and $G$ are free, i.e., no element of $\tilde{G}$ or $G$ other than identity has fixed points.*

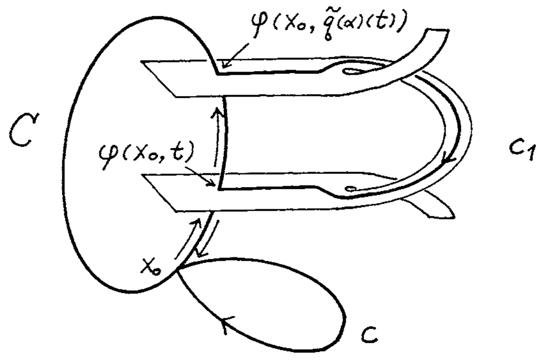


Figure 3

(4) *$G$ coincides with the holonomy pseudogroup of $\mathcal{F}$ restricted to $C$.*

According to a theorem of Imanishi ([I 1], Theorem 2.1), a finitely generated free subgroup of PerDiff($\boldsymbol{R}$) is topologically conjugate to translations. So we get the following corollary.

COROLLARY 5.4. *$\tilde{G}$ and $G$ are abelian. If $G$ is finitely generated, then there are a homeomorphism $h: S^1 \to C$ and a lift $\tilde{h}: \boldsymbol{R} \to \boldsymbol{R}$ of $h$ such that $hGh^{-1}$ is contained in the group of rotations and $\tilde{h}\tilde{G}\tilde{h}^{-1}$ is contained in the group of translations.*

PROPOSITION 5.5. *Assume $G$ is finitely generated. Then there is a reparametrization map $a: U \times \boldsymbol{R} \to \boldsymbol{R}$ of the flow $\varphi$ such that the reparametrized flow $\phi$: $U \times \boldsymbol{R} \to U$, $\phi(x, t) = \varphi(x, a(x, t))$ maps each leaf of $\mathcal{F}|_U$ diffeomorphically to a leaf of $\mathcal{F}|_U$.*

PROOF. We use the homeomorphism $\tilde{h}: \boldsymbol{R} \to \boldsymbol{R}$ of (5.4). Let $(x, t)$ be a point of $U \times \boldsymbol{R}$, $t > 0$. We define $a(x, t)$ as follows. Choose a leaf curve $c$ from $x$ to a point $y$ of $C$. By (5.2), the curve $(\varphi|\{x\} \times [0, t])^{-1} \# c$ can be homotoped relative the endpoints to a curve of the form $c_1 \# (\varphi|\{y\} \times [0, t'])$ where $t'$ is a real number and $c_1$ is a leaf curve from $\varphi(x, t)$ to $\varphi(y, t')$. We define $a(x, t)$ by $a(x, t) = \text{length}(\tilde{h}([0, t']))$. Similarly we can define $a(x, t)$ when $t$ is negative. Since $hGh^{-1}$ is a subgroup of the group of rotations, $a(x, t)$ does not depend on the choice of the leaf curve $c$. It is easy to see that $a$ is a continuous reparametrization map and the reparametrized flow $\phi$ is foliation preserving. This completes the proof.

Let $F$ be a leaf contained in $\bar{U} - U$. We define the *proper height* of $F$, denoted $ph(F)$, as follows:

$$ph(F)=\max\{k \mid \text{there is a sequence } F_0, F_1, \cdots, F_k \text{ of leaves in } \bar{U}-U \text{ such that } F_0<F_1<\cdots<F_k \text{ and } F_0=F\}.$$

Let $K$ be a leaf in $\bar{U}-U$ with $ph(K)=0$. We study the relations between the holonomy group of $K$ on the side bordered by $U$ and the group $\tilde{G}$. Choose a base point $y_0$ of $K$ and a real number $\tau$ such that $\varphi_0(y_0, (0, \tau))$ is contained in $U$ and $\varphi_0(y_0, \tau)$ lies on the leaf through $x_0$. Let $T=\varphi_0(y_0, (0, \tau))$ and let $\Gamma_T$ denote the holonomy pseudogroup of $\mathcal{F}$ restricted to $T$. We define a homomorphism $\Phi_T\colon \tilde{G}\to\Gamma_T$ as follows. Choose a leaf curve $c$ from $x_0$ to $\varphi_0(y_0, \tau)$. Let $\alpha$ be an element of $\pi_1(U, x_0)$ and $c_1$ a loop representing $\alpha$. Then $\Phi_T(\tilde{q}(\alpha))$ is the element of $\Gamma_T$ associated with the curve $c\sharp c_1\sharp c^{-1}$. If $G$ is finitely generated, we can express $\Phi_T(\tilde{q}(\alpha))$ in terms of the linearlization map $\tilde{h}\colon \boldsymbol{R}\to\boldsymbol{R}$ and the reparametrized flow $\psi$. If $\tilde{h}\tilde{q}(\alpha)\tilde{h}^{-1}$ is the translation of $\boldsymbol{R}$ by a number $t_\alpha$, then it is easy to see that $\Phi_T(\tilde{q}(\alpha))$ is given by

$$\Phi_T(\tilde{q}(\alpha))(\psi(\varphi_0(y_0, \tau)), t))=\psi(\varphi_0(y_0, \tau), t+t_\alpha).$$

PROPOSITION 5.6. *$\Gamma_T$ is generated by $\Phi_T(\tilde{G})$. In particular, $\Gamma_T$ is abelian.*

The proof is left to the reader.

COROLLARY 5.7. *The holonomy group of $K$ on the side bordered by $U$ is abelian.*

Till now we fixed a transverse circle $C$ in considering the Novikov transformation. We study what happens when we use another transversal. Let $C'$ be a transverse circle, $G(C')$ denotes the image of the Novikov transformation with respect to $C'$.

LEMMA 5.8. *If $G=G(C)$ is finitely generated, then for any transverse circle $C'$, $G(C')$ is finitely generated and* $\operatorname{rank}(G(C'))=\operatorname{rank}(G)$. *And there exists a transverse circle $C_0$ such that $G(C_0)$ is torsion free.*

The proof is not difficult.

## §6. Non proper leaves with exact polynomial growth

The purpose of the present section is to prove Theorem 2. Assume $U$ is a nice saturated set and the holonomy groups of leaves in $\bar{U}-U$ on the side approached by $U$ are abelian. Let $d$ be the proper depth of a leaf contained in $U$. We remark that all leaves in $U$ have the same proper depth. As in (4.2), we can decompose a neighbourhood of non-isolated ends of $U$.

PROPOSITION 6.1. *Under the above hypotheses, there exist sequences* $(\Omega_0^0, \Omega_0^1, \Omega_0^2, \cdots, \Omega_0^d)$, $(\Omega^1, \Omega^2, \cdots, \Omega^d)$ *and* $(\bar{\Omega}^1, \bar{\Omega}^2, \cdots, \bar{\Omega}^d)$ *of immersed submanifolds of* $M$ *with the following properties.*

(1) $\Omega_0^0=M$. $\Omega_0^{i-1}$ *is a compact immersed submanifold of* $M$ *with corner and the induced foliation* $\mathcal{F}|_{\Omega_0^{i-1}}$ *is admissible for* $i\geqq 2$.

(2) $\Omega^i$ *is the connected component containing* $U\cap\Omega_0^{i-1}$ *of*

$$\Omega_0^{i-1}-\bigcup\{K|K \text{ is a compact leaf of } \mathcal{F}|_{\Omega_0^{i-1}}\}.$$

(3) $\bar{\Omega}^i$ *is the compact manifold obtained from* $\Omega^i$ *by attaching the boundary.* $\bar{\Omega}^i$ *naturally immerses to* $M$ *and we can consider the induced foliation* $\mathcal{F}|_{\bar{\Omega}^i}$.

(4) *Let* $K_j^i$, $j=1, 2, \cdots, \nu_i$ *be the compact leaves of* $\mathcal{F}|_{\Omega^i\cap(\bar{U}-U)}$ *with positive proper height. For each* $j$, *there are a submanifold* $N_j^i$ *of* $K_j^i$, *a contracting diffeomorphism* $f_j^i: [0, \varepsilon_j^i)\to[0, \delta_j^i)$ *and an embedding* $h_j^i: X(K_j^i, N_j^i, f_j^i)\to\bar{\Omega}^i$ *such that* $h_j^i(C(K_j^i, N_j^i)\times\{0\})=K_j^i$, $h_j^{i*}\mathcal{F}=\mathcal{F}(K_j^i, N_j^i, f_j^i)$, $h_j^i(\partial K_j^i\times[0, \varepsilon_j^i))\subset\bigcup_{k=1}^{\nu_{i-1}} h_k^{i-1}(N_k^{i-1}\times(\delta_k^{i-1}, \varepsilon_k^{i-1}))$ *and there exists* $\eta_j^i$ *with* $\delta_j^i<\eta_j^i<\varepsilon_j^i$, *for which* $h_j^i(C(K_j^i, N_j^i)\times(\eta_j^i, \varepsilon_j^i))\cap U=\emptyset$ *holds.*

(5) $\Omega_0^i=\Omega^i-\bigcup_{j=1}^{\nu_i} h_j^i(\operatorname{Int} X(K_j^i, N_j^i, f_j^i))-\bigcup_{j=1}^{\nu_i} K_j^i$.

The proof is much the same as that of (4.2).

COROLLARY 6.2. *Let* $U$ *be a minimal nice saturated set. Assume that the holonomy groups of leaves in* $\bar{U}-U$ *on the sides approached by* $U$ *are abelian. Then the image of the Novikov transformation of* $\mathcal{F}|_U$ *is finitely generated.*

PROOF. We use the decomposition given in (6.1). Choose a transverse circle $C$ and a base point $x_0\in C$. We may assume $C$ is contained in $\operatorname{Int}(\Omega_0^d)$. Let $q: \pi_1(U, x_0)\to\operatorname{Diff}(C)$ denote the Novikov transformation and let $i: \operatorname{Int}(\Omega_0^d)\to U$ denote the inclusion map. We show $q(\pi_1(U, x_0))=q(i_*\pi_1(\operatorname{Int}\Omega_0^d, x_0))$. To see this, it is sufficient to prove the following assertion. Let $a: [0, 1]\to U$ be a leaf curve from a point $a(0)$ of $C$ to a point $a(1)$ of $C$, then there exists a leaf curve $a': [0, 1]\to\operatorname{Int}(\Omega_0^d)$ such that $a(0)=a'(0)$ and $a(1)=a'(1)$. But this assertion is easily proved using our decomposition, since each $\mathcal{F}(K_j^i, N_j^i, f_j^i)$ is induced from the trivial foliation. Since $\Omega_0^d$ is compact, the image of $q$ is finitely generated. This completes the proof.

PROOF OF THEOREM. 2. Let $U$ be as in Theorem 2 and let $F$ be a leaf of $\mathcal{F}|_U$. Choose a transverse circle $C$, a base point $x_0\in F\cap C$ and a transverse flow $\varphi_0$ on $M$ which has $C$ as a closed orbit of period one. Let $G$ be the image of the Novikov transformation with respect to $C$. By (6.2) and (5.8), we may assume $G$ is free abelian of rank $r$. We decompose a neighbourhood of non-isolated ends of $U$ as in (6.1). We assume $C$ is contained in $\mathrm{Int}(\Omega_0^d)$. For simplicity, we assume each $N_j^i$ is connected.

We define a uniform transversal for $F$. For each compact leaf $K_j^i$, $i=1, \cdots, d-1$, $j=1, \cdots, \nu_i$, choose a base point $x_j^i \in \mathrm{Int}(N_j^i)$ and set $R_j^i = h_j^i(\{x_j^i\} \times (0, \varepsilon_j^i))$. Let $K_j^d$, $j=1, \cdots, \nu_d$ (resp. $K_j^d$, $j=\nu_d+1, \cdots, \mu_d$) be the compact leaves of $\mathcal{F}|_{\Omega_0^d}$ with empty (resp. non-empty) boundary. We remark that each $K_j^d$, $j=1, \cdots, \nu_d$ (resp. $j=\nu_d+1, \cdots, \mu_d$) corresponds to a compact (resp. non-compact) leaf of $\mathcal{F}|_{\bar{U}-U}$ whose proper height is zero. For each $K_j^d$, $j=1, \cdots, \nu_d$, choose a base point $x_j^d \in K_j^d$ and a real number $t_j^d$ such that $\varphi_0(x_j^d, t_j^d) \in F$ and $R_j^d = \varphi_0(x_j^d, (0, t_j^d)) \subset U$. For each $K_j^d$, $j=\nu_d+1, \cdots, \mu_d$, let $N_{j,k}^d$, $k=1, \cdots, c_j^d$ be the connected components of $\partial K_j^d$. For each $N_{j,k}^d$ choose a base point $x_{j,k}^d \in N_{j,k}^d$ and a real number $t_{j,k}^d$ such that $\varphi_0(x_{j,k}^d, t_{j,k}^d) \in F$ and $R_{j,k}^d = \varphi_0(x_{j,k}^d, (0, t_{j,k}^d)) \subset U$. Let

$$R = \bigcup_{\substack{i<d \\ j=1,\cdots,\nu_i}} R_j^i \cup \bigcup_{j=1,\cdots,\nu_d} R_j^d \cup \bigcup_{\substack{j=\nu_d+1,\cdots,\mu_d \\ k=1,\cdots,c_j^d}} R_{j,k}^d \cup C .$$

We shall prove $R$ is uniform for $F$. Let $\Gamma$ be the holonomy pseudogroup of $\mathcal{F}$ restricted to $R$. We define a finite generating set of $\Gamma$. First we define $\gamma_j^i$: $R_j^i \to R_j^i$, $i<d$, $j=1, \cdots, \nu_i$ and $\gamma_{j,k,l}^{i,i-1}$: $R_j^i \to R_{j,k,l}^{i-1}$, $i \leqq d$ in the same way as in §4. Secondly the image of the Novikov transformation $G$ acts on $C$. We choose a basis $\gamma_1, \cdots, \gamma_r$ of $G$. Thirdly, for each $R_j^d$, $j=1, \cdots, \nu_d$, we defined a map $\Phi_{R_j^d}$: $\tilde{G} \to \Gamma_{R_j^d}$ in §5. Let $\gamma_{j,k}^d = \Phi_{R_j^d}(\gamma_k)$ and $\gamma_j^d = \Phi_{R_j^d}(\tilde{q}([C]))$. We define a map $\lambda_{R_j^d}: C \to R_j^d$ as follows. Choose a leaf curve $c$ from $x_0$ to $\varphi_0(x_j^d, t_j^d)$. $\lambda_{R_j^d}$ is the element of the holonomy pseudogroup of $\mathcal{F}$ associated with the leaf curve $c$. To define $\lambda_{R_j^d}$ uniquely, we cut $C$ at $x_0$. Using the leaf preserving flow $\psi$ of (6.5), $\lambda_{R_j^d}$ is expressed as follows. If $t_j^d$ is positive (resp. negative), $\lambda_{R_j^d}$: $\psi(x_0, [-1, 0)) \to R_j^d$ (resp. $\lambda_{R_j^d}$: $\psi(x_0, (0, 1]) \to R_j^d$) is defined by $\lambda_{R_j^d}(\psi(x_0, t)) = \psi(\varphi_0(x_j^d, t_j^d), t)$. For each $R_{j,k}^d$, $j=\nu_d+1, \cdots, \mu_d$, $k=1, \cdots, c_j^d$, we define $\gamma_{j,k,l}^d = \Phi_{R_{j,k}^d}(\gamma_l)$, $\gamma_{j,k}^d = \Phi_{R_{j,k}^d}(\tilde{q}([C]))$ and $\lambda_{R_{j,k}^d}: C \to R_{j,k}^d$ in a similar way. From (5.3), (4) and (5.6), it follows easily that $\Gamma$ is generated by the elements listed above.

LEMMA 6.3. *$R$ is uniform for $F$.*

PROOF. It is easy to see that $(R, \Gamma)$ is dispersed for $F$. We prove $R$ is finitely dense for $F$. Let $x$ be a point of $U$. $F_x$ denotes the leaf through $x$ and $d_{F_x}$ denotes the distance function on $F_x$ induced from a Riemannian metric of $M$. We show $\sup\{d_{F_x}(x, R-\{x\}) \mid x \in U\}$ is finite. At first we prove that $\sup\{d_{F_x}(x, R-\{x\}) \mid x \in R\}$ is finite. Assume the contrary. Then there exists a sequence $x_1, x_2, \cdots$ of points of $R$ such that $d_{F_{x_n}}(x_n, R-\{x_n\}) \geqq n$. Taking a subsequence if necessary, we can assume that $\{x_n\}$ converges to a point $x_\infty \in \bar{R}$. If $x_\infty$ is contained in $U$, an easy compactness argument leads to a contradic-

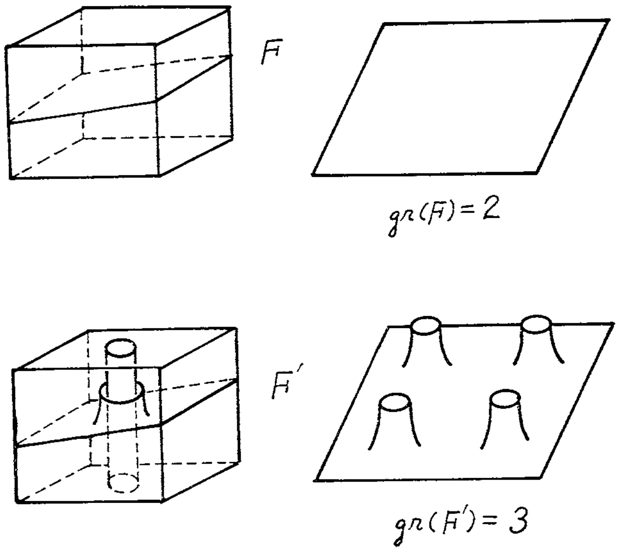


Figure 4

tion. So we assume $x_\infty = x_j^i \in K_j^i$. Take a closed curve $c$ in $K_j^i$ through $x_j^i$ and consider a fence over $c$. That is, consider a map $f\colon S^1\times[0, 1]\to\bar{U}\subset M$ such that $f(S^1\times\{0\})=c$, $f(\{x\}\times[0, 1])$ is transverse to $\mathcal{F}$ for each $x\in S^1$ and $f(\{0\}\times(0, 1))=R_j^i$. It is easy to see that, for large $n$, $d_{F_{x_n}}(x_n, R-\{x_n\})$ is bounded approximately by the length of $c$. This is a contradiction.

Next we prove that $\sup\{d_{F_x}(x, R)\mid x\in U-R\}$ is finite. Assume the contrary. Then there exists a sequence $x_1, x_2, \cdots$ of points of $U-R$ such that $d_{F_{x_n}}(x_n, R)\geqq n$ and $\{x_n\}$ converges to $x_\infty\in\bar{U}$. Choose a leaf curve $c$ from $x_\infty$ to a point $y$ of $F_{x_\infty}\cap\bar{R}$ and consider a fence over $c$ on the side of $F_{x_\infty}$ from which $\{x_n\}$ accumulates on $x_\infty$. That is, consider a map $f\colon[0, 1]\times[0, 1]\to\bar{U}$ such that $f([0, 1]\times\{0\})=c$ and $f(\{t\}\times[0, 1])$ is transverse to $\mathcal{F}$, $0\leqq t\leqq 1$. Again it is easy to see that, for large $n$, $d_{F_{x_n}}(x_n, R)$ is bounded approximately by the length of $c$. Thus we have proved $\sup\{d_{F_x}(x, R-\{x\})\mid x\in U\}$ is finite.

Finally we prove the unboundedness of the sequence $\{a_n\}$ of (2.1). If the sequence $\{a_n\}$ is bounded, there exist a real number $L$ and sequences $\{x_n\}$ of points of $F\cap R$ and $\{\gamma_n\}$ of elements of $\Gamma$ such that $l(\gamma_n)\geqq n$, $d_F(x_n, \gamma_n(x_n))\leqq L$, $\gamma(x_n)\neq\gamma_n(x_n)$ if $l(\gamma)\leqq n$ and the sequences $\{x_n\}$ and $\{\gamma_n(x_n)\}$ converge to $x_\infty$ and $y_\infty$ respectively. It is easy to see that the points $x_\infty$ and $y_\infty$ lie on the same leaf $F_{x_\infty}$. Assume $F_{x_\infty}$ is contained in $U$. Then there exist a positive number $\varepsilon$ and an integer $n_0$ such that the restricted flow $\varphi_0\colon D_{x_\infty}(L+1)\times(-\varepsilon, \varepsilon)\to M$ is an embedding and the points $x_n$ and $\gamma_n(x_n)$ are contained in $\varphi_0(D_{x_\infty}(L+1)\times(-\varepsilon, \varepsilon))$ for $n\geqq n_0$. From the compactness, the number of connected components of $\varphi_0(D_{x_\infty}(L+1)\times(-\varepsilon, \varepsilon))\cap R$ is finite. Let $R_i$, $i=1, \cdots, s$ be these components where $R_1$ is the component containing $x_\infty$. For each $i\geqq 2$, there are finitely many elements $\gamma_{i_1}, \cdots, \gamma_{i_{j_i}}$ of $\Gamma$ such that $\bigcup_{j=1}^{j_i}\gamma_{i_j}(R_1)\supset R_i$. Let $N$ be the maximum of the length of the $\gamma_{i_j}$'s. Then $l(\gamma_n)$ is bounded by $N$ for $n\geqq n_0$. This is a contradiction. Similarly we get a contradiction when $x_\infty$ is contained in $\bar{U}-U$. Thus we have proved $R$ is finitely dense for $F$. This completes the proof of (6.3).

We continue the proof of Theorem 2. Let $g_{x_0}(n)$ be the growth function of the pseudogroup $\Gamma$ at $x_0$. By (2.3), $F$ has the same growth type as $g_{x_0}(n)$. We prove $g_{x_0}(n)$ dominates, and is dominated by, a polynomial of degree $d+r$. Let $\Gamma_a$, $a=0, 1, \cdots, d$, be the subpseudogroup of $\Gamma$ generated by $G$ and the remaining elements with $i>d-a$. Thus we have

$$G=\Gamma_0\subset\Gamma_1\subset\cdots\subset\Gamma_d=\Gamma.$$

Let $g^a_{x_0}(n)$ be the growth function of $\Gamma_a$ at $x_0$. $g^0_{x_0}(n)$ coincides with the growth function of the free abelian group $G$. So it grows as fast as a polynomial of degree $r$. As in §4, we can inductively prove that $g^a_{x_0}(n)$ has the same growth type as a polynomial of degree $r+a$. So $F$ has exact polynomial growth of degree $d+r$. Since the leaf $F$ was chosen arbitrarily, this completes the proof of Theorem 2.

PROOF OF COROLLARY 1. The "if" part is the content of Theorem 1 and 2. On the other hand, the "only if" part is proved in ([C-C], Proposition 2 and 3). This completes the proof.

## §7. Leaves with linear growth

In this section we study leaves with linear growth. At first we prove the following easy lemma.

LEMMA 7.1. *Let $F$ be a non-compact leaf and $x_0$ a point of $F$. Then the growth function of $F$ at $x_0$ dominates a linear function.*

PROOF. As before $D_x(R)$ denotes the metric ball in $F$ of radius $R$ centered at $x_0$. Since the metric of $F$ is induced from that of a compact manifold $M$, there exists a positive number $\varepsilon$ such that for each $x\in F$ the exponential map is a diffeomorphism from the ball in the tangent space of $F$ at $x$ of radius $\varepsilon$ to $D_x(\varepsilon)$. Let $\delta=\inf\{\text{volume } D_x(\varepsilon)\,|\, x\in F\}$. Then it is easy to see that $f_{x_0}(2n\varepsilon)\geqq\delta n$. Thus the lemma is proved.

COROLLARY 7.2. *If $F$ has linear growth, then $F$ has exact linear growth.*

PROOF OF THEOREM 3. In [T], we proved that if a leaf $F$ with polynomial growth contains a proper leaf $F'$ in its limit set, then $gr(F) \geqq gr(F')+1$. So if $F$ is a leaf with linear growth, then each proper leaf contained in $L(F)$ is compact. If $F$ is proper, then $L(F)$ consists of finitely many compact leaves, since the number of compact leaves contained in the limit set of a leaf is finite. If $F$ is non-proper, then by ([C-C], Proposition 3), $\bar{F}-\{F' \mid F' \text{ is a proper leaf contained in } \bar{F}\}$ is a minimal nice saturated set. By (5.7), we can apply Theorem 2. It follows that $M$ itself is a minimal nice saturated set and the rank of the image of the Novikov transformation is equal to one.

Conversely assume $F$ is proper and $L(F)$ consists of finitely many compact leaves $K_1, \cdots, K_r$. We shall prove that $F$ has linear growth. For an end $\varepsilon$ of $F$, $L_\varepsilon(F)$ denotes the $\varepsilon$-limit set of $F$, that is, $L_\varepsilon(F)=\bigcap_{U \in \varepsilon} L(U)$ (see [Ni 1]). By a theorem of Nishimori ([Ni 4], Theorem 1), the number of ends of $F$ is finite and each end of $F$ is a tame end of depth 1. Let $\varepsilon_i$, $i=1, \cdots, s$, be the ends of $F$. There is a surjective map $j: \{1, \cdots, s\} \to \{1, \cdots, r\}$ such that $L_{\varepsilon_i}(F)=K_{j(i)}$. There are submanifolds $U_i \in \varepsilon_i$, $i=1, \cdots, s$, such that $U_i \cap U_{i'}=\emptyset$ if $i \neq i'$ and $F-\bigcup_i U_i$ is compact. Let $f_i(R)=\text{volume}(D_{x_0}(R) \cap U_i)$ where $x_0$ is a base point of $F$. Then we have $f_{x_0}(R)=\text{volume}(F-\bigcup U_i)+\sum_i f_i(R)$. Since the end $\varepsilon_i$ approaches to $K_{j(i)}$ tamely ([Ni 1]), it is easy to see that $f_i$ has linear growth. So $f_{x_0}$ also has linear growth.

Finally assume that the closed manifold $M$ itself is a nice saturated set and the image of the Novikov transformation has rank one. Then each leaf has linear growth by Theorem 2. This completes the proof of Theorem 3.

## § 8. Growth of almost without holonomy foliations

In order to study almost without holonomy foliations, it is convenient to consider models of them.

DEFINITION 8.1 ([He 1], [I 2]). We say that a pair $(M, \mathcal{F})$ is a model of type 1) or 2) if $\mathcal{F}$ is a codimension one foliation on a compact manifold $M$ and if the following condition is satisfied.

type 1) $M=V \times [0, 1]$ where $V$ is a closed manifold and $\mathcal{F}$ is the product foliation.

type 2) $\mathcal{F}$ is tangent to the boundary $\partial M$ and the leaves in the interior of $M$ are non-compact with trivial holonomy groups.

The following theorem ([He 1]) is an easy consequence of the Reeb stability theorem.

THEOREM 8.2 (Hector). *Let $M$ be a compact manifold and $\mathcal{F}$ a codimension one foliation on $M$ which is almost without holonomy. Then there exists a foliated manifold $(M', \mathcal{F}')$ which is a disjoint countable union of models $(M', \mathcal{F}') = \bigcup_i (M_i, \mathcal{F}_i)$ and a foliation preserving immersion $p$ of $M'$ onto $M$ such that $p|_{\mathrm{Int}(M')}$ is an embedding of $\mathrm{Int}(M')$ and $p|_{\partial M'}$ is a two fold cover onto $M - p(\mathrm{Int}(M'))$.*

*Moreover there exists a finite number of closed manifolds $N_1, \cdots, N_q$ and a natural number $i_0$ such that if $i \geqq i_0$, then $M_i$ is diffeomorphic to some $N_{j_i} \times [0, 1]$.*

By this theorem, to consider the growth of an almost without holonomy foliation, it is sufficient to consider those of models of type 2). Assume $(M, \mathcal{F})$ is a model of type 2) and $\mathcal{F}$ is of class $C^2$. Then it is known ([Sa], [He 1], [I 2]) that the interior of $M$ is a nice saturated set. In general $\mathrm{Int}(M)$ is not minimal, but it is easy to see that there exists a transverse circle $C$ and we can consider the Novikov transformation $q: \pi_1(\mathrm{Int}(M)) \to \mathrm{Diff}(C)$.

THEOREM 8.3. *Let $(M, \mathcal{F})$ be a model of type 2) and let $r$ be the rank of the image of the Novikov transformation of $\mathcal{F}|_{\mathrm{Int}(M)}$. Then all leaves of $\mathcal{F}$ have exact polynomial growth. If $\partial M = \emptyset$ (resp. $\partial M \neq \emptyset$), then $gr(\mathcal{F}) = \{r\}$ (resp. $gr(\mathcal{F}) = \{0, r+1\}$). In both cases, the degree of growth of a leaf is bounded by the first Betti number of $M$.*

PROOF. The first and the second assertion follow directly from Theorem 1 and 2. Let $\tilde{q}: \pi_1(\mathrm{Int}(M)) \to \mathrm{PerDiff}(\boldsymbol{R})$ be the lift of the Novikov transformation (§5). Then we have $r+1 = \mathrm{rank}(\mathrm{Image}(\tilde{q})) \leqq b_1(\mathrm{Int}(M)) = b_1(M)$. This completes the proof.

PROOF OF THEOREM 4. By (8.2) and (8.3), each leaf of $\mathcal{F}$ has exact polynomial growth and the degree of growth of a leaf is bounded by $\max\{b_1(M_i), b_1(N_j) \mid i < i_0, j = 1, \cdots, q\}$. Thus we have proved Theorem 4.

## §9. Growth of leaves in transversely analytic foliations

In this section we apply our theorems to the case where the foliation is transversely analytic. We use the following theorem of Hector ([He 2], Theorem 9).

THEOREM 9.1 (Hector). *Let $F$ be a proper leaf in a transversely oriented transversely analytic foliation $\mathcal{F}$ of codimension one. If the holonomy group of $F$ is non-abelian, then all leaves sufficiently near to $F$ are locally dense. More precisely, there exists a transverse arc $T \approx [0, 1]$ such that $T \cap F = \{0\}$ and if a leaf $F'$ intersects with $T - \{0\}$ then $\bar{F}' \cap T = T$.*

PROOF OF THEOREM 5 AND COROLLARY 2. The only if part of Theorem 5 follows from a theorem of Cantwell-Conlon ([C-C], Proposition 2 and 3). Assume $d(F)$ is finite. Then $F$ is proper by (3.1), and each leaf contained in the limit set of $F$ has abelian holonomy by (9.1). So $F$ has exact polynomial growth by Theorem 1. Assume $\bar{F}$ is the closure of a minimal nice saturated set. Let $F'$ be a leaf contained in $F$. If the proper height $ph(F')$ of $F'$ is positive (resp. $ph(F')=0$), then the holonomy group of $F'$ is abelian by (9.1) (resp. by (5.7)). So we can apply Theorem 2 and we conclude $F$ has exact polynomial growth. This completes the proof of Theorem 5 and Corollary 2.

In order to prove Theorem 6, we need a filtration theorem of Dippolito ([D]). Let $U$ be a saturated open subset of a foliated compact Riemannian manifold. $\hat{U}$ denotes the completion of $U$ with respect to the induced metric. A triple $(E, B, \mathcal{F})$ is called a foliated $I$-bundle if $E$ is the total space of an $I$-bundle over $B$ and $\mathcal{F}$ is a codimension one foliation of $E$ such that each fibre is transverse to $\mathcal{F}$.

THEOREM 9.2 (Dippolito). *Let $\mathcal{F}$ be a codimension one foliation of a compact manifold $M$. Then there exists a finite filtration*

$$\emptyset=U_{2m+1}\subset U_{2m}\subset\cdots\subset U_1\subset U_0=M$$

*by saturated open sets such that for each $i=0, 1, \cdots, m$,*

a) *$U_{2i}-U_{2i+1}$ is a relative minimal set in $U_{2i}$, and*

b) *the foliation on $\widehat{U_{2i+1}-U_{2i+2}}$ induced from $\mathcal{F}$ admits a structure of a foliated I-bundle.*

Let $(E, B, \mathcal{F})$ be a foliated $I$-bundle. It is well-known ([Ha]) that $\mathcal{F}$ is determined by the *total holonomy map* $\Pi: \pi_1(B, x_0)\to \mathrm{Diff}(I)$, where $x_0$ is a base point of $B$ and $\mathrm{Diff}(I)$ denotes the group of diffeomorphisms of the interval. Let $G$ denote the image of $\Pi$. We call $G$ the *total holonomy group* of $\mathcal{F}$. If $\mathcal{F}$ is transversely orientable, each element of $G$ is orientation preserving and $E$ is a trivial $I$-bundle over $B$. We say $(E, B, \mathcal{F})$ is *irreducible* if $\mathcal{F}$ is transversely orientable and $G$ has no common fixed points other than $\{0, 1\}$. If $\mathcal{F}$ is transversely orientable, transversely analytic and $G\neq\{1\}$, then $\mathcal{F}$ is decomposed into a finite number of irreducible foliated $I$-bundles.

LEMMA 9.3. *Let $(E, B, \mathcal{F})$ be an irreducible, transversely analytic foliated I-bundle. If there exists a leaf with finite depth in Int(E), then the total holonomy group $G$ of $\mathcal{F}$ is an infinite cyclic group generated by a contracting diffeomorphism.*

PROOF. We choose a trivialization of $E$, $E=B\times[0, 1]$. Let $B_0$ be the leaf $B\times\{0\}$ of $\mathcal{F}$ and $\mathcal{H}(B_0)$ the holonomy group of $B_0$. There is a natural map $p: G\to\mathcal{H}(B_0)$. Since $\mathcal{F}$ is transversely analytic, $p$ is an isomorphism. Let $F$ be a leaf with finite depth in $\mathrm{Int}(E)$. Then $F$ is proper and the closure of $F$ must contain $B\times\{0\}$ or $B\times\{1\}$. We assume $F$ contains $B_0$ in its limit set. By (9.1), $\mathcal{H}(B_0)=G$ is abelian. We prove $G$ is cyclic. If $\mathcal{H}(B_0)$ contains an abelian subgroup of rank$\geqq 2$, it is easy to see that all leaves sufficiently near to $B_0$ are locally dense. So each finitely generated subgroup of $G$ is cyclic. Assume $G$ is not finitely generated. Then from the irreducibility of $\mathcal{F}$, there exists a contraction $f\in G$, a sequence of integers $\{m_i\}$ and a sequence of diffeomorphisms $\{g_i\}$ such that $\lim_{i\to\infty} m_i=\infty$, $g_i\in G$ and $g_i^{m_i}=f$. It follows that each leaf of $\mathcal{F}|_{\mathrm{Int}(E)}$ is non-proper. This proves $G$ is cyclic. Since $\mathcal{F}$ is irreducible, $G$ is generated by a contraction. Thus we have proved the lemma.

PROOF OF THEOREM 6. Taking a finite cover if necessary, we may assume $\mathcal{F}$ is transversely orientable. We use the filtration theorem of Dippolito (9.2). Since $U_{2i}-U_{2i+1}$ is a relative minimal set, $i=0, \cdots, m$, it follows from Theorem 5 that either all leaves in $U_{2i}-U_{2i+1}$ have non-polynomial growth or all leaves in $U_{2i}-U_{2i+1}$ have polynomial growth of the same degree. So the set $\{gr(F)|F$ is a leaf contained in some $U_{2i}-U_{2i+1}$, $i=0, 1, \cdots, m\}$ is a finite set. We consider growth of leaves in $U_{2i+1}-U_{2i+2}$. If $U_{2i}-U_{2i+1}$ is a relative exceptional minimal set, then each leaf in $U_{2i+1}-U_{2i+2}$ has non-polynomial growth by Theorem 5. Assume $U_{2i}-U_{2i+1}$ is not exceptional. Then $\widehat{U_{2i+1}-U_{2i+2}}$ has finitely many connected components. Let $E_j$, $j=1, 2, \cdots, k$, be such components. We can assume each $E_j$ is irreducible. We claim that if there exists a leaf $F$ with polynomial growth in $\mathrm{Int}(E_j)$, then all leaves in $\mathrm{Int}(E_j)$ have polynomial growth of the same degree. If $F$ is proper, then each leaf in $\mathrm{Int}(E_j)$ is proper and has polynomial growth of degree $d(F)$ from (9.3) and Theorem 5. If $F$ is non-proper, then $\mathrm{Int}(E_j)$ is a minimal nice saturated set again by (9.3) and Theorem 5. So each leaf in $\mathrm{Int}(E_j)$ has polynomial growth of the same degree. Thus we have proved our claim. As a result, the set $\{gr(F)|F$ is a leaf contained in $U_{2i+1}-U_{2i+2}$, $i=0, 1, \cdots, m\}$ is a finite set. This completes the proof of Theorem 6.

COROLLARY 9.4. *Let $\mathcal{F}$ be a transversely analytic codimension one foliation on a compact manifold. Then there exists an integer $d$ such that for each leaf $F$ of $\mathcal{F}$, either $d(F)$ is infinite or $d(F)$ is smaller than $d$.*

The proof is left to the reader.

COROLLARY 9.5. *Let $\mathcal{F}$ be a transversely analytic codimension one foliation on a compact manifold $M$. Let $C$ be the union of proper leaves with non-trivial* *holonomy. Assume each leaf of $\mathcal{F}$ has polynomial growth. Then $C$ is closed, $C$ consists of a finite number of leaves and each connected component of $M-C$ is a nice saturated set.*

PROOF. For each $k$, let $C_k=\{F|F\in C,\ d(F)\leqq k\}$. Assume there exists an integer $k$ such that $C_k$ contains infinitely many leaves. Then an irreducible component of some $\widehat{U_{2i+1}-U_{2i+2}}$ must contain infinitely many leaves in $C_k$. This contradicts (9.3). So each $C_k$, hence $C$ by (9.4), consists of finitely many leaves. The remaining assertions are easy to prove.

From the existence of a leaf preserving flow (5.5), it is easy to see that all leaves in a nice saturated set are diffeomorphic. So we get the following corollary.

COROLLARY 9.6. *Let $\mathcal{F}$ be a transversely analytic codimension one foliation on a compact manifold $M$. Assume each leaf of $\mathcal{F}$ has polynomial growth. Then all leaves of $\mathcal{F}$ are classified into a finite number of diffeomorphic classes of manifolds.*

We remark that the assumption of (9.5) or (9.6) is satisfied when the fundamental group of $M$ has polynomial growth ([P 1]). We prove that the assumption is satisfied if we assume all leaves of $\mathcal{F}$ are proper.

THEOREM 9.7. *Let $\mathcal{F}$ be a transversely analytic codimension one foliation on a compact manifold. Assume all leaves of $\mathcal{F}$ are proper. Then each leaf of $\mathcal{F}$ has exact polynomial growth and there exists a non-negative integer $d$ such that $gr(\mathcal{F})=\{0, 1, \cdots, d\}$.*

PROOF. For each $k$, let $C_k=\{F|\,d(F)\leqq k$ and the holonomy group of $F$ is non-trivial$\}$. By the proof of (9.5), each $C_k$ consists of finitely many leaves. Assume $C_k-C_{k-1}\neq\emptyset$ for any $k\geqq 1$. Then we can find a sequence $\{F_i\}$ of leaves such that $F_1<F_2<F_3<\cdots$. It follows from (3.6) that there exists a non-proper leaf. So there exists an integer $d$ such that $C_0\subsetneqq C_1\subsetneqq\cdots\subsetneqq C_{d-1}=C_d=C_{d+1}=\cdots$. It is easy to see that $gr(\mathcal{F})=\{0, 1, \cdots, d\}$. This completes the proof.

## §10. Foliations with simple growth

In this section we prove Theorem 7 and Theorem 8 and obtain some corollaries from them.

PROOF OF THEOREM 7. Assume $gr(\mathcal{F})=\{d\}$, $0\leqq d<\infty$. If $d=0$, then all leaves of $\mathcal{F}$ are compact and $\mathcal{F}$ is without holonomy. Assume $d$ is positive. Then all leaves of $\mathcal{F}$ are non-compact. Hence a minimal set of $M$ is either an exceptional minimal set or all of $M$. But a leaf contained in an exceptional minimal set has exponential growth ([P 2]). So the only minimal set is all of $M$. This is equivalent to saying all leaves of $\mathcal{F}$ are everywhere dense. Assume there exists a non-trivial holonomy. Then there exists a (one-side) contracting holonomy $\gamma$ of a leaf $F$ say at $x_0$. Let $T$ be a compact transverse arc through $x_0$ which is contained in the domain of $\gamma$. Since all leaves of $\mathcal{F}$ have polynomial growth, there exists a holonomy invariant measure $\mu$ on $T$ which is finite on compact sets ([P 3]). On the other hand, from the existence of the contracting holonomy $\gamma$, $\mu(T)=\mu(\bigcap_n \gamma^n(T))=\mu(\{x_0\})$. This contradiction shows that $\mathcal{F}$ is without holonomy.

Conversely if $\mathcal{F}$ is without holonomy, then $gr(\mathcal{F})$ consists of one element by (8.3).

The second assertion follows from ([P 3], 6.4).

If $gr(\mathcal{F})=\{\exp\}$, then $\mathcal{F}$ has no compact leaves and by 1), $\mathcal{F}$ must have a leaf with non-trivial holonomy.

Finally assume all leaves of $\mathcal{F}$ are non-compact and there exists a leaf with non-trivial holonomy. If there exists a leaf with non-exponential growth, then there exists a transverse invariant measure $\mu$ ([P 3]). A minimal set contained in the support of $\mu$ is either an exceptional minimal set or all of $M$. It cannot be exceptional again by ([P 3]). If it is all of $M$, we can show $\mathcal{F}$ is without holonomy by the same argument used to prove the first assertion. Thus all leaves of $\mathcal{F}$ have exponential growth. This completes the proof.

Theorem 7 implies, in particular, that if all leaves of $\mathcal{F}$ are non-compact, then $gr(\mathcal{F})$ consists of only one element. So we have the following corollaries.

COROLLARY 10.1. *If there exist two leaves $F$ and $F'$ such that $gr(F)\neq gr(F')$, then there exists a compact leaf.*

COROLLARY 10.2. *If there exists a leaf such that $gr(F)=\infty$, then there exists a compact leaf.*

If $\mathcal{F}$ is transversely orientable and without holonomy, it is known ([Sa]) that all leaves of $\mathcal{F}$ are diffeomorphic. So we get the following corollary which generalizes a theorem of Plante ([P 2], 1.6).

COROLLARY 10.3. *Assume $\mathcal{F}$ is transversely orientable and $\mathcal{F}$ has no compact leaves. If there exist two leaves of $\mathcal{F}$ which are not diffeomorphic, then all leaves of $\mathcal{F}$ have exponential growth.*

PROOF OF THEOREM 8. From ([T], Theorem 2), the holonomy group of each non-compact proper leaf is trivial. From ([C-C], Proposition 3), the holonomy group of each non-proper leaf is trivial. This completes the proof.

COROLLARY 10.4. *$gr(\mathcal{F})=\{0, 1\}$ if and only if there is a non-compact leaf, all leaves of $\mathcal{F}$ are proper and $\mathcal{F}$ is almost without holonomy.*

The proof is left to the reader.

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