Markov fields and local operators

By Shigeo KUSUOKA

1. Introduction

The purpose of this paper is to investigate the transformations preserving Markov property of random distributions and to give a remarkable difference between a Gaussian white noise and a Poisson white noise from a viewpoint of Markov property. Before stating our results, we shall give a short account of results of J. L. Doob [1], N. Levinson and H. P. McKean Jr. [5] and Y. Okabe [8], [9]. By Markov property we mean the definition in H. P. McKean Jr. [7].

J. L. Doob [1] considered a stationary Gaussian process $X(t)$ satisfying a stochastic differential equation

$$P\left(\frac{d}{dt}\right)X(t)=Q\left(\frac{d}{dt}\right)B'(t),$$

where $P(\xi)$ and $Q(\xi)$ are polynomials without common factors such that $P(i\xi)\neq 0$ for any $\xi\in\boldsymbol{R}$ and $B'(t)$ is a Gaussian white noise. Then he showed that if Q is constant, $X(t)$ has Markov property, and that if Q is not constant, $X(t)$ has not Markov property, but the so-called finite multiple Markov property. We note that $P(d/dt)$ and $Q(d/dt)$ operate on the space $\mathscr{S}'(\boldsymbol{R})$ of tempered distributions as linear local operators.

Giving a refinement of results of N. Levinson and H. P. McKean Jr. [5] concerning Markov property, Y. Okabe [8], [9] showed that a stationary linear process $X(t)$ satisfying a stochastic differential equation

$$P\left(\frac{d}{dt}\right)X(t)=Z'(t)$$

has Markov property, where $P(\xi)$ is an entire function of infra-exponential type and $Z'(t)$ is an additive white noise. We also note that $P(d/dt)$ operates on the space $\mathscr{B}$ of hyperfunctions as a linear local operator.

Since the polynomial P with $P(i\xi)\neq 0$ for any $\xi\in\boldsymbol{R}$ induces a linear local and invertible operator $P(d/dt)$ from $\mathscr{S}'(\boldsymbol{R})$ to $\mathscr{S}'(\boldsymbol{R})$, and the additive white noise $Z'(t)$ is a typical Markov random distribution, we are led to the following question: Let Y be a Markov random distribution included in $\mathscr{S}'(\boldsymbol{R}^d)$ and $P(\xi_1, \cdots, \xi_d)$ be a polinomial with $P(i\xi_1, \cdots, i\xi_d)\neq 0$ for any $(\xi_1, \cdots, \xi_d)\in\boldsymbol{R}^d$. Then, does the random distribution X defined by

$$X=P\left(\frac{\partial}{\partial X_1}, \cdots, \frac{\partial}{\partial X_d}\right)^{-1} Y$$

possess Markov property?

Our main aim in this paper is to answer this question. In Theorem 1 in Section 4, we shall show in a more general situation that X has Markov property, if Y has 0-Markov property due to V. Mandrekar [6], which is defined in Section 2 and actually it will be seen to be stronger than Markov property for random distributions.

The next purpose is to apply Theorem 1 to investigate a Gaussian white noise and a Poisson white noise from a viewpoint of 0-Markov property. We shall show in Theorem 2 in Section 6 that in a class of mean 0, stationary Gaussian random distributions with independent values at every point, a Gaussian white noise is characterized as a random distribution having 0-Markov property, under a certain condition. In Section 7, we shall consider a stationary linear random distribution X satisfying a differential equation

$$P\left(\frac{\partial}{\partial X_1}, \cdots, \frac{\partial}{\partial X_d}\right)X=Q\left(\frac{\partial}{\partial X_1}, \cdots, \frac{\partial}{\partial X_d}\right)W,$$

where $P(\xi_1, \cdots, \xi_d)$ and $Q(\xi_1, \cdots, \xi_d)$ are polynomials without common factors such that $P(i\xi_1, \cdots, i\xi_d)\neq 0$ for any $(\xi_1, \cdots, \xi_d)\in \boldsymbol{R}^d$, and W is a white noise. By virtue of Theorem 2, we see that $Q(\partial/\partial X_1, \cdots, \partial/\partial X_d)W$ has not 0-Markov property, if W is a Gaussian white noise and Q is not constant. From J. L. Doob's result, we also see that X has not 0-Markov property in that case. On the other hand, we will see in Example 1 that $Q(\partial/\partial X_1, \cdots, \partial/\partial X_d)W$ has 0-Markov property, if W is a Poisson white noise. Therefore we can apply Theorem 1 to prove that X has 0-Markov property in the case of W being a Poisson white noise. These facts show a remarkable difference between a Gaussian white noise and a Poisson white noise.

The author expresses his gratitude to Professor Y. Okabe for his valuable advices and hearty encouragement.

2. The definition of 0-Markov property

To begin with, let us introduce some preliminary notations. We denote by $(\Omega, \mathscr{B}, P)$ a given complete probability space. The trivial sub-σ-fields of $\mathscr{B}$ is denoted by $\mathscr{N}$, i.e. $\mathscr{N}=\{B\in\mathscr{B}; P(B)=0 \text{ or } 1\}$. A d-dimensional Euclidean space is denoted by $\boldsymbol{R}^d$. For any domain D in $\boldsymbol{R}^d$, we denote by $C_0^\infty(D)$ a space of real-valued infinitely differentiable functions on D with compact supports and we denote by $\mathscr{D}'(D)$ a space of real-valued distributions on D. We denote by τ_M the topological Borel field of M for any topological space M. For a family of random variables $\{f_\lambda; \lambda \in \Lambda\}$, we denote by $\sigma\{f_\lambda; \lambda \in \Lambda\}$ the least sub-σ-field of $\mathscr{B}$ with respect to which the f_λ's are measurable. We denote by $\mathscr{B}_1 \vee \mathscr{B}_2$ the least σ-field that contains $\mathscr{B}_1$ and $\mathscr{B}_2$. We will write $\mathscr{B}_1 \underset{\mathscr{B}_3}{\perp\!\!\!\perp} \mathscr{B}_2$, if $\mathscr{B}_1$ and $\mathscr{B}_2$ are independent under the conditional probability with respect to $\mathscr{B}_3$ for any sub-σ-fields $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ of $\mathscr{B}$.

Now let X be a random distribution, i.e. X is a measurable mapping from $(\Omega, \mathscr{B})$ to $(\mathscr{D}'(\boldsymbol{R}^d), \tau_{\mathscr{D}'(\boldsymbol{R}^d)})$. For any domain D in $\boldsymbol{R}^d$, we define a sub-σ-field $\mathscr{B}_D$ of $\mathscr{B}$ by

$$\mathscr{B}_D = \sigma\{{}_{\mathscr{D}}\langle \varphi, X(\omega)\rangle_{\mathscr{D}'}; \varphi \in C_0^\infty(\boldsymbol{R}^d), \text{support}(\varphi) \subset D\} \vee \mathscr{N}.$$

DEFINITION 1. Let ε be a non-negative number. We say that a random distribution X is ε-Markov, if $\mathscr{B}_{D_1} \underset{\mathscr{B}_{D_0}}{\perp\!\!\!\perp} \mathscr{B}_{D_2}$ for any pair of domains $\{D_1, D_2\}$ such that $D_1 \cup D_2 = \boldsymbol{R}^d$ and $\mathrm{dis}(D_1^c, D_2^c) > \varepsilon$, where $D_0 = D_1 \cap D_2$, $\mathrm{dis}(\cdot, \cdot)$ is an ordinary distance function on $\boldsymbol{R}^d$ and D^c is the complement of D.

For any closed set C in $\boldsymbol{R}^d$, we define a sub-σ-field $\mathscr{B}_C$ of $\mathscr{B}$ by $\mathscr{B}_C = \bigcap_{\delta>0} \mathscr{B}_{U_\delta}$, where U_δ is a δ-neighborhood of C.

Following H. P. McKean Jr. [7], we define the Markov property.

DEFINITION 2. We say that a random distribution X is Markov, if $\mathscr{B}_D \underset{\mathscr{B}_{\partial D}}{\perp\!\!\!\perp} \mathscr{B}_{D^e}$ for any domain D in $\boldsymbol{R}^d$, where ∂D is the boundary of D and D^e is the exterior of D.

REMARK 1. If X is 0-Markov, then X is Markov. Conversely if X has the Markov property, and if $X = X(x, \omega)$, $x \in \boldsymbol{R}^d$ and $\omega \in \Omega$, is a continuous function of X with probability one, then X is 0-Markov. These facts are essentially proved by V. Mandrekar [6].

REMARK 2. In general the 0-Markov property is strictly stronger than the Markov property (see Theorem 2 and Remark 6).

3. A random variable inducing a Radon measure

Let M and N be Hausdorff topological spaces, and let S be a continuous mapping from M to N. Let X be an M-valued random variable, i.e. X is a measurable mapping from $(\Omega, \mathscr{B})$ to (M, τ_M), and put $Y = SX$, an N-valued random variable. We define σ-fields $\mathscr{B}_X$ and $\mathscr{B}_Y$ by

$$\mathscr{B}_X = \{X^{-1}(A); A \in \tau_M\} \vee \mathscr{N} \quad \text{and} \quad \mathscr{B}_Y = \{Y^{-1}(A); A \in \tau_N\} \vee \mathscr{N}.$$

PROPOSITION 1. *Suppose that S is a one-to-one mapping, and that the proba-*

bility measure $\mu_X = XP$ *induced on M by X and P is Radon, i.e. for any* $A \in \tau_M$, $\mu_X(A) = \sup\{\mu_X(K);$ *K is a compact subset of M,* $K \subset A\}$ *and* $\mu_X(A) = \inf\{\mu_X(O);$ *O is an open subset of M,* $A \subset O\}$. *Then* $\mathscr{B}_X = \mathscr{B}_Y$.

The proof of Proposition 1 is obvious by the following proposition due to L. Schwartz [11].

PROPOSITION 2. *Suppose that* μ *is a Radon measure on M, then* $\nu = S\mu$ *induced by S and* μ *is also Radon. Furthermore, for any subset A of N, A belongs to the completion of* τ_N *with respect to* ν, *if and only if* $S^{-1}(A)$ *belongs to the completion of* τ_M *with respect to* μ.

Next let M_1 and M_2 be Hausdorff topological spaces, and let $(X_1(\omega), X_2(\omega))$ be an $M_1 \times M_2$-valued random variable. So X_1 (resp. X_2) is an M_1 (resp. M_2)-valued random variable. We define σ-fields $\mathscr{B}_{X_1}$, $\mathscr{B}_{X_2}$ and $\mathscr{B}_{(X_1,X_2)}$ by $\mathscr{B}_{X_1} = \{X_1^{-1}(A); A \in \tau_{M_1}\} \vee \mathscr{N}$, $\mathscr{B}_{X_2} = \{X_2^{-1}(A); A \in \tau_{M_2}\} \vee \mathscr{N}$ and $\mathscr{B}_{(X_1,X_2)} = \{(X_1(\cdot), X_2(\cdot))^{-1}A; A \in \tau_{M_1 \times M_2}\} \vee \mathscr{N}$.

PROPOSITION 3. *Suppose that the measure* μ *induced on* $M_1 \times M_2$ *by* $(X_1(\cdot), X_2(\cdot))$ *and P is Radon. Then* $\mathscr{B}_{(X_1,X_2)} = \mathscr{B}_{X_1} \vee \mathscr{B}_{X_2}$.

PROOF. Set $\tilde{\tau} = \{A \in \tau_{M_1 \times M_2}; (X_1(\cdot), X_2(\cdot))^{-1}A \in \mathscr{B}_{X_1} \vee \mathscr{B}_{X_2}\}$, then $\tilde{\tau}$ is a σ-field. If $\mu(A) = 0$ for $A \in \tilde{\tau}_{M_1 \times M_2}$, then $(X_1(\cdot), X_2(\cdot))^{-1}A \in \mathscr{N}$, and so $A \in \tilde{\tau}$. We denote by $\mathscr{O}_1$ (resp. $\mathscr{O}_2$) the family of open subsets of M_1 (resp. M_2). Then it is obvious that $\{U_1 \times U_2; U_1 \in \mathscr{O}_1, U_2 \in \mathscr{O}_2\} \subset \tilde{\tau}$. But $\{U_1 \times U_2; U_1 \in \mathscr{O}_1, U_2 \in \mathscr{O}_2\}$ is an open base of $M_1 \times M_2$. Since μ is Radon, it is easy to see that $\mu(K) = \inf\{\mu(A); K \subset A, A \in \tilde{\tau}\}$ for any compact subset K of $M_1 \times M_2$. So every compact subset of $M_1 \times M_2$ belongs to $\tilde{\tau}$. Therefore it is easy to see that $\tilde{\tau} = \tau_{M_1 \times M_2}$, which completes the proof.

Next let E be a locally convex Hausdorff vector space, and let E' denote the dual space of E. Now let X be an E-valued random variable, and we define σ-fields $\mathscr{B}_X$ and $\mathscr{B}'_X$ by $\mathscr{B}_X = \{X^{-1}(A); A \in \tau_E\} \vee \mathscr{N}$ and $\mathscr{B}'_X = \sigma\{{}_E\langle X(\omega), u\rangle_{E'}; u \in E'\} \vee \mathscr{N}$.

PROPOSITION 4. *Suppose that the probability measure* $\mu_X = XP$ *is a Radon measure on E, then* $\mathscr{B}_X = \mathscr{B}'_X$.

It is not difficult to see that Proposition 4 follows from Proposition 1, Proposition 2 and the fact that the cylindrical open sets are basis of the weak topology of E.

4. The inheritance of the 0-Markov property

Let E_1 and E_2 be locally convex Hausdorff vector spaces which are continuously included in $\mathscr{D}'(\boldsymbol{R}^d)$, and we suppose that E_k $(k=1, 2)$ satisfies the following conditions (C-1) and (C-2): Let $\{D_1, D_2\}$ be an arbitrary pair of domains in $\boldsymbol{R}^d$ such that $D_1 \cup D_2 = \boldsymbol{R}^d$ and $\mathrm{dis}\,(D_1^c, D_2^c)>0$, and set $D_0=D_1\cap D_2$.

(C-1) For any $f, g \in E_k$ which satisfy $f|_{D_0}=g|_{D_0}$, there uniquely exists $h\in E_k$ such that $h|_{D_1}=f|_{D_1}$ and $h|_{D_2}=g|_{D_2}$, where $f|_{D_0}$ is the restriction of f to D_0 as a distribution, etc. We shall write ${}_{D_1}(f, g)_{D_2}$ for h.

(C-2) Put $H_k=\{f\in E_k; f|_{D_0}=0\}$. The mapping from $H_k\times H_k$ to H_k corresponding $(f, g)\in H_k\times H_k$ to ${}_{D_1}(f, g)_{D_2}\in H_k$ is continuous.

Let L be a one-to-one, onto and linear mapping from E_1 to E_2, and suppose that L is bicontinuous and local, i.e. for any $f, g\in E_1$ and for any domain D in $\boldsymbol{R}^d$, if $f|_D=g|_D$, then $Lf|_D=Lg|_D$. Put $G=L^{-1}$, a continuous linear mapping from E_2 to E_1.

THEOREM 1. *Let ε be a non-negative number, and let Y be an E_2-valued random variable. Suppose that the probability measure on E_2 induced by Y and P is Radon, and that Y is ε-Markov, when we regard Y as a random distribution. (Notice that $E_1, E_2\subset\mathscr{D}'(\boldsymbol{R}^d)$.) Then $X=GY$, an E_1-valued random variable, is also ε-Markov as a random distribution.*

REMARK 3. We can adopt a space of ultra-distributions and a random ultra-distribution in place of $\mathscr{D}'(\boldsymbol{R}^d)$ and a random distribution almost without changing the proof. Then we can take an ultra-differential operator as an example of L.

REMARK 4. If E_1 and E_2 are Fréchet spaces, then the condition (C-2) holds automatically by the condition (C-1) and the closed-graph theorem.

For any domain D in $\boldsymbol{R}^d$, we define an equivalence relation $\underset{D}{\sim}$ on $\mathscr{D}'(\boldsymbol{R}^d)$ by $f\underset{D}{\sim}g$, iff $f|_D=g|_D$ for $f, g\in\mathscr{D}'(\boldsymbol{R}^d)$. We write $E_{k,D}$ for $E_k/\underset{D}{\sim}$, and we denote by $[u]_{k,D}$ for an element of $E_{k,D}$ corresponding to $u\in E_k$ $(k=1, 2)$. $E_{1,D}$ and $E_{2,D}$ are locally convex Hausdorff vector spaces continuously included in $\mathscr{D}'(D)$. Since L is local, we can regard L as a continuous mapping from $E_{1,D}$ to $E_{2,D}$.

For any domain D in $\boldsymbol{R}^d$, we define σ-fields $\mathscr{F}_D$ and $\mathscr{I}_D$ by

$$\mathscr{F}_D=\sigma\{{}_{\mathscr{D}}\langle\varphi, X(\omega)\rangle_{\mathscr{D}'};\, \varphi\in C_0^\infty(\boldsymbol{R}^d),\ \text{support}\,(\varphi)\subset D\}\vee\mathscr{N}$$

and

$$\mathscr{I}_D=\sigma\{{}_{\mathscr{D}}\langle\varphi, Y(\omega)\rangle_{\mathscr{D}'};\, \varphi\in C_0^\infty(\boldsymbol{R}^d),\ \text{support}\,(\varphi)\subset D\}\vee\mathscr{N}.$$

The operator G and the canonical mapping from E_1 to $E_{1,D}$ are continuous and the measure on E_2 induced by Y and P is Radon. So it is obvious by Proposition 2 that the measure on $E_{1,D}$ induced by $[X(\cdot)]_{1,D}=[GY(\cdot)]_{1,D}$ and P is Radon. Since the canonical mapping from $E_{1,D}$ to $\mathscr{D}'(D)$ is continuous, the measure on $\mathscr{D}'(D)$ induced by $X(\cdot)|_D$ and P is also Radon. Therefore it is obvious by Proposition 4 that $\mathscr{F}_D=\sigma\{{}_{\mathscr{D}(D)}\langle\varphi, X(\omega)|_D\rangle_{\mathscr{D}'(D)};\ \varphi\in C_0^\infty(D)\}\vee\mathscr{N}=\{X(\cdot)|_D^{-1}A;\ A\in\tau_{\mathscr{D}'(D)}\}\vee\mathscr{N}$. Applying Proposition 1 to $[X(\cdot)]_{1,D}$ and $X(\cdot)|_D$, we get $\mathscr{F}_D=\{[X(\cdot)]_{1,D}^{-1}A;\ A\in\tau_{E_{1,D}}\}\vee\mathscr{N}$. Similarly we get $\mathscr{I}_D=\{[Y(\cdot)]_{2,D}^{-1}A;\ A\in\tau_{E_{2,D}}\}\vee\mathscr{N}$. Since $L[X(\cdot)]_{1,D}=[LX(\cdot)]_{2,D}=[Y(\cdot)]_{2,D}$, $\mathscr{I}_D\subset\mathscr{F}_D$.

So we obtain the following claim.

Claim 1. $\mathscr{F}_D=\{[X(\cdot)]_{1,D}^{-1}A;\ A\in\tau_{E_{1,D}}\}\vee\mathscr{N}$, $\mathscr{I}_D=\{[Y(\cdot)]_{2,D}^{-1}A;\ A\in\tau_{E_{2,D}}\}\vee\mathscr{N}$ and $\mathscr{I}_D\subset\mathscr{F}_D$ for each domain D in $\boldsymbol{R}^d$.

Now let $\{D_1, D_2\}$ be a pair of domains in $\boldsymbol{R}^d$ such that $D_1\cup D_2=\boldsymbol{R}^d$ and $\mathrm{dis}\,(D_1^c, D_2^c)>\varepsilon$, and set $D_0=D_1\cap D_2$.

LEMMA 1. *Let* $\mathscr{F}_1=\mathscr{F}_{D_0}\cap\mathscr{I}_{D_1}$ *and* $\mathscr{F}_2=\mathscr{F}_{D_0}\cap\mathscr{I}_{D_2}$. *Then* $\mathscr{F}_{D_0}=\mathscr{F}_1\vee\mathscr{F}_2$.

We shall prove Lemma 1 in the next section 5.

LEMMA 2. $\mathscr{F}_{D_1}=\mathscr{F}_{D_0}\vee\mathscr{I}_{D_1}$ and $\mathscr{F}_{D_2}=\mathscr{F}_{D_0}\vee\mathscr{I}_{D_2}$.

PROOF. Let S be a mapping from E_{1,D_1} to $E_{1,D_0}\oplus E_{2,D_1}$ which corresponds $[u]_{1,D_1}$ to $[u]_{1,D_0}\oplus[Lu]_{2,D_1}$ for any $u\in E_1$. S is well-defined and continuous. We claim that S is one-to-one. For this purpose, it is enough to show that $u|_{D_1}=v|_{D_1}$ for any $u, v\in E_1$ such that $u|_{D_0}=v|_{D_0}$ and $Lu|_{D_1}=Lv|_{D_1}$. But set $w={}_{D_1}(u, v)_{D_2}$, then $Lw|_{D_1}=Lu|_{D_1}=Lv|_{D_1}$ and $Lw|_{D_2}=Lv|_{D_2}$ by the locality of L. So $Lw=Lv$. Since L is one-to-one, $w=v$. Therefore $u|_{D_1}=w|_{D_1}=v|_{D_1}$.

Notice that $S[X(\omega)]_{1,D_1}=[X(\omega)]_{1,D_0}\oplus[LX(\omega)]_{2,D_1}=[X(\omega)]_{1,D_0}\oplus[Y(\omega)]_{2,D_1}$. It is obvious by Proposition 2 that the measure on $E_{1,D_0}\oplus E_{2,D_1}$ induced by $[X(\cdot)]_{1,D_0}\oplus[Y(\cdot)]_{2,D_1}$ and P is Radon. By Proposition 3 and Claim 1, we get $\mathscr{F}_{D_0}\vee\mathscr{I}_{D_1}=\{([X(\cdot)]_{1,D_0}\oplus[Y(\cdot)]_{2,D_1})^{-1}A;\ A\in\tau_{E_{1,D_0}\oplus E_{2,D_1}}\}\vee\mathscr{N}$. Therefore by Proposition 1, Claim 1 and the injectivity of S, we obtain $\mathscr{F}_{D_1}=\mathscr{F}_{D_0}\vee\mathscr{I}_{D_1}$. Similarly $\mathscr{F}_{D_2}=\mathscr{F}_{D_0}\vee\mathscr{I}_{D_2}$ is proved.

The following lemma is due to F. Knight [3].

LEMMA 3. *Let* $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$ *and* $\mathscr{B}_4$ *be sub-σ-fields of* $\mathscr{B}$.

(1) *Suppose* $\mathscr{B}_1\underset{\mathscr{B}_3}{\perp\!\!\!\perp}\mathscr{B}_2$ *and* $\mathscr{B}_3\subset\mathscr{B}_4\subset\mathscr{B}_3\vee\mathscr{B}_1$, *then* $\mathscr{B}_1\underset{\mathscr{B}_4}{\perp\!\!\!\perp}\mathscr{B}_2$.

(2) *Suppose* $\mathscr{B}_1\underset{\mathscr{B}_3}{\perp\!\!\!\perp}\mathscr{B}_2$ *and* $\mathscr{B}_4\subset\mathscr{B}_1\vee\mathscr{B}_3$, *then* $\mathscr{B}_4\underset{\mathscr{B}_3}{\perp\!\!\!\perp}\mathscr{B}_2$.

Now we shall prove Theorem 1. The ε-Markov property of Y implies $\mathscr{I}_{D_1}\underset{\mathscr{I}_{D_0}}{\perp\!\!\!\perp}\mathscr{I}_{D_2}$. Noticing $\mathscr{I}_{D_0}\subset\mathscr{F}_1\subset\mathscr{I}_{D_1}$ and using Lemma 3-(1), we obtain $\mathscr{I}_{D_1}\underset{\mathscr{F}_1}{\perp\!\!\!\perp}\mathscr{I}_{D_2}$. Since $\mathscr{F}_1\subset\mathscr{F}_{D_0}=\mathscr{F}_1\vee\mathscr{F}_2\subset\mathscr{F}_1\vee\mathscr{I}_{D_2}$ by Lemma 1, we get $\mathscr{I}_{D_1}\underset{\mathscr{F}_{D_0}}{\perp\!\!\!\perp}\mathscr{I}_{D_2}$ by Lemma

3-(1). It follows from Lemma 2 and Lemma 3-(2) that $\mathscr{F}_{D_1} \underset{\mathscr{F}_{D_0}}{\perp\!\!\!\perp} \mathscr{F}_{D_2}$, which implies the ε-Markov property of X. This completes the proof of Theorem 1.

5. The proof of Lemma 1

For any $f_1, f_2, g_1, g_2 \in E_2$ such that $f_1|_{D_0}=f_2|_{D_0}=g_1|_{D_0}=g_2|_{D_0}$, it is easy to see that ${}_{D_1}(f_1, g_1)_{D_2}+{}_{D_1}(f_2, g_2)_{D_2}={}_{D_1}(f_1, g_2)_{D_2}+{}_{D_1}(f_2, g_1)_{D_2}$. By the linearity of G, we obtain $G({}_{D_1}(f_1, g_1)_{D_2})|_{D_0}+G({}_{D_1}(f_2, g_2)_{D_2})|_{D_0}=G({}_{D_1}(f_1, g_2)_{D_2})|_{D_0}+G({}_{D_1}(f_2, g_1)_{D_2})|_{D_0}$.

Therefore the following condition holds.

(A-1) For any $f_1, f_2, g_1, g_2 \in E_2$ such that $f_1|_{D_0}=f_2|_{D_0}=g_1|_{D_0}=g_2|_{D_0}$,

(1) if $G({}_{D_1}(f_1, g_1)_{D_2})|_{D_0}=G({}_{D_1}(f_2, g_1)_{D_2})|_{D_0}$, then $G({}_{D_1}(f_1, g_2)_{D_2})|_{D_0}=G({}_{D_1}(f_2, g_2)_{D_2})|_{D_0}$.

(2) if $G({}_{D_1}(f_1, g_1)_{D_2})|_{D_0}=G({}_{D_1}(f_1, g_2)_{D_2})|_{D_0}$, then $G({}_{D_1}(f_2, g_1)_{D_2})|_{D_0}=G({}_{D_1}(f_2, g_2)_{D_2})|_{D_0}$.

We define two relations R_1 and R_2 on E_2 as follows.

For $f_1, f_2 \in E_2$, we define $f_1 \underset{R_1}{\sim} f_2$, if $f_1|_{D_0}=f_2|_{D_0}$ and there exists some $g \in E_2$ such that $g|_{D_0}=f_1|_{D_0}=f_2|_{D_0}$ and $G({}_{D_1}(f_1, g)_{D_2})|_{D_0}=G({}_{D_1}(f_2, g)_{D_2})|_{D_0}$. For $g_1, g_2 \in E_2$, we define $g_1 \underset{R_2}{\sim} g_2$, if $g_1|_{D_0}=g_2|_{D_0}$ and there exists some $f \in E_2$ such that $f|_{D_0}=g_1|_{D_0}=g_2|_{D_0}$ and $G({}_{D_1}(f, g_1)_{D_2})|_{D_0}=G({}_{D_1}(f, g_2)_{D_2})|_{D_0}$.

By (A-1) it is easy to see that R_1 and R_2 are equivalence relations. We write $[f]_{R_1}$ (resp. $[f]_{R_2}$) for an equivalence class of f with respect to R_1 (resp. R_2) for each $f \in E_2$.

Since $[0]_{R_1}=\{f \in E_2; f|_{D_0}=0, G({}_{D_1}(f, 0)_{D_2})|_{D_0}=0\}$ and $[0]_{R_2}=\{g \in E_2; g|_{D_0}=0, G({}_{D_1}(0, g)_{D_2})|_{D_0}=0\}$, we can see that $[0]_{R_1}$ and $[0]_{R_2}$ are closed linear subspaces of E_2 by the condition (C-1). Therefore

(A-2) E_2/R_1 and E_2/R_2 are Hausdorff topological spaces.

Next we define two relations Q_1 and Q_2 on E_1 as follows.

For $u_1, u_2 \in E_1$, we define $u_1 \underset{Q_1}{\sim} u_2$, if there exist some $f, g_1, g_2 \in E_2$ such that $f|_{D_0}=g_1|_{D_0}=g_2|_{D_0}$, $u_1|_{D_0}=G({}_{D_1}(f, g_1)_{D_2})|_{D_0}$ and $u_2|_{D_0}=G({}_{D_1}(f, g_2)_{D_2})|_{D_0}$. For $v_1, v_2 \in E_1$, we define $v_1 \underset{Q_2}{\sim} v_2$, if there exist some $f_1, f_2, g \in E_2$ such that $f_1|_{D_0}=f_2|_{D_0}=g|_{D_0}$, $v_1|_{D_0}=G({}_{D_1}(f_1, g)_{D_2})|_{D_0}$ and $v_2|_{D_0}=G({}_{D_1}(f_2, g)_{D_2})|_{D_0}$.

It is not difficult to show that Q_1 and Q_2 are equivalence relations. We demonstrate only the transitive law of Q_1. Suppose that $u_1 \underset{Q_1}{\sim} u_2$ and $u_2 \underset{Q_1}{\sim} u_3$ for some u_1, u_2 and $u_3 \in E_1$. Then there exist f, g_1, g_2, h, k_1 and $k_2 \in E_2$ such that $f|_{D_0}=g_1|_{D_0}=g_2|_{D_0}$, $h|_{D_0}=k_1|_{D_0}=k_2|_{D_0}$, $u_1|_{D_0}=G({}_{D_1}(f, g_1)_{D_2})|_{D_0}$, $u_2|_{D_0}=G({}_{D_1}(f, g_2)_{D_2})|_{D_0}$, $u_2|_{D_0}=G({}_{D_1}(h, k_1)_{D_2})|_{D_0}$ and $u_3|_{D_0}=G({}_{D_1}(h, k_2)_{D_2})|_{D_0}$. Put $v_1=G({}_{D_1}(f, g_2)_{D_2})$ and $v_2=G({}_{D_1}(h, k_1)_{D_2})$, then $v_1|_{D_0}=v_2|_{D_0}=u_2|_{D_0}$. So set $v_3={}_{D_1}(v_1, v_2)_{D_2}$, then $v_3|_{D_1}=v_1|_{D_1}$ and $v_3|_{D_2}=v_2|_{D_2}$. By the locality of L, we get $Lv_3|_{D_1}=Lv_1|_{D_1}=f|_{D_1}$ and $Lv_3|_{D_2}=Lv_2|_{D_2}=k_1|_{D_2}$. Therefore $f|_{D_0}=k_1|_{D_0}$ and $Lv_3={}_{D_1}(f, k_1)_{D_2}$, which shows that $v_3=G({}_{D_1}(f, k_1)_{D_2})$. Since $v_3|_{D_0}=v_2|_{D_0}$, we obtain $G({}_{D_1}(f, k_1)_{D_2})|_{D_0}=G({}_{D_1}(h, k_1)_{D_2})|_{D_0}$. By (A-1)-(2), we get $G({}_{D_1}(f, k_2)_{D_2})|_{D_0}=G({}_{D_1}(h, k_2)_{D_2})|_{D_0}$. So $u_1|_{D_0}=G({}_{D_1}(f, g_1)_{D_2})|_{D_0}$ and $u_3|_{D_0}=G({}_{D_1}(f, k_2)_{D_2})|_{D_0}$, which implies $u_1 \underset{Q_1}{\sim} u_3$.

We write $[u]_{Q_1}$ (resp. $[u]_{Q_2}$) for an equivalence class of u with respect to Q_1 (resp. Q_2) for each $u \in E_1$. By the definitions of R_1, R_2, Q_1 and Q_2, it is easy to prove the following claim.

Claim 2.
(1) (i) If $f \underset{D_1}{\sim} g$, then $f \underset{R_1}{\sim} g$ for any $f, g \in E_2$.
(ii) If $f \underset{D_2}{\sim} g$, then $f \underset{R_2}{\sim} g$ for any $f, g \in E_2$.
(2) If $u \underset{D_0}{\sim} v$, then $u \underset{Q_1}{\sim} v$ and $u \underset{Q_2}{\sim} v$ for any $u, v \in E_1$.

Therefore there exist the following continuous mappings:

$$h_1: E_{2,D_1} \to E_2/R_1, \quad h_1[f]_{2,D_1}=[f]_{R_1} \quad \text{for} \quad f \in E_2\,,$$
$$h_2: E_{2,D_2} \to E_2/R_2, \quad h_2[g]_{2,D_2}=[g]_{R_2} \quad \text{for} \quad g \in E_2\,,$$
$$j_1: E_{1,D_0} \to E_1/Q_1, \quad j_1[u]_{1,D_0}=[u]_{Q_1} \quad \text{for} \quad u \in E_1\,,$$
and
$$j_2: E_{1,D_0} \to E_1/Q_2, \quad j_2[v]_{1,D_0}=[v]_{Q_2} \quad \text{for} \quad v \in E_1\,.$$

The relations between R_1 and Q_1, and between R_2 and Q_2 are clarified by the following claim.

Claim 3.
(1) (i) If $u \underset{Q_1}{\sim} v$, then $Lu \underset{R_1}{\sim} Lv$ for any $u, v \in E_1$.
(ii) If $u \underset{Q_2}{\sim} v$, then $Lu \underset{R_2}{\sim} Lv$ for any $u, v \in E_1$.
(2) (i) If $f \underset{R_1}{\sim} g$, then $Gf \underset{Q_1}{\sim} Gg$ for any $f, g \in E_2$.
(ii) If $f \underset{R_2}{\sim} g$, then $Gf \underset{Q_2}{\sim} Gg$ for any $f, g \in E_2$.

Therefore the following canonical mappings are homeomorphisms.

$$E_1/Q_1 \underset{G}{\overset{L}{\rightleftarrows}} E_2/R_1\,, \qquad L[u]_{Q_1}=[Lu]_{R_1}\,, \qquad G[f]_{R_1}=[Gf]_{Q_1}\,,$$
$$E_1/Q_2 \underset{G}{\overset{L}{\rightleftarrows}} E_2/R_2\,, \qquad L[u]_{Q_2}=[Lu]_{R_2}\,, \qquad G[f]_{R_2}=[Gf]_{Q_2}\,,$$

where $u \in E_1$ and $f \in E_2$.

PROOF. (1) Suppose that $u \underset{Q_1}{\sim} v$ for some $u, v \in E_1$. Then there exist f, g_1 and $g_2 \in E_2$ such that $f|_{D_0}=g_1|_{D_0}=g_2|_{D_0}$, $u|_{D_0}=G({}_{D_1}(f, g_1)_{D_2})|_{D_0}$ and $v|_{D_0}=G({}_{D_1}(f, g_2)_{D_2})|_{D_0}$. Put $w_1=G({}_{D_1}(f, g_1)_{D_2})$, then $w_1|_{D_0}=u|_{D_0}$. Let $w_2={}_{D_1}(u, w_1)_{D_2}$, then we get $Lw_2|_{D_1}=Lu|_{D_1}$ by the locality of L, which implies $Lu \underset{R_1}{\sim} Lw_2$ by Claim 2. It is easy to see that $Lw_2|_{D_2}=Lw_1|_{D_2}=g_1|_{D_2}$ and $w_1|_{D_0}=w_2|_{D_0}$. Put $h=Lw_2$, then $h|_{D_2}=g_1|_{D_2}$. Since

$w_1|_{D_0}=G({}_{D_1}(f, g_1)_{D_2})|_{D_0}=G({}_{D_1}(f, h)_{D_2})|_{D_0}$ and $w_2|_{D_0}=G({}_{D_1}(h, h)_{D_2})|_{D_0}$, we obtain $G({}_{D_1}(f, h)_{D_2})|_{D_0}=G({}_{D_1}(h, h)_{D_2})|_{D_0}$, which implies $Lw_2=h \underset{R_1}{\sim} f$. Therefore we get $Lu \underset{R_1}{\sim} f$. Similarly we get $Lv \underset{R_1}{\sim} f$. So we have $Lu \underset{R_1}{\sim} Lv$, which completes the proof of (i). The proof of (ii) is similar. (2) Suppose $f \underset{R_1}{\sim} g$ for some $f, g \in E_2$, then $f|_{D_0}=g|_{D_0}$ and there exists some $h \in E_2$ such that $h|_{D_0}=f|_{D_0}=g|_{D_0}$ and $G({}_{D_1}(f, h)_{D_2})|_{D_0}=G({}_{D_1}(g, h)_{D_2})|_{D_0}$. Using Claim 2, we obtain $G({}_{D_1}(f, h)_{D_2}) \underset{Q_1}{\sim} G({}_{D_1}(g, h)_{D_2})$. Since $Gf|_{D_0}=G({}_{D_1}(f, f)_{D_2})|_{D_0}$ and $Gg|_{D_0}=G({}_{D_1}(g, g)_{D_2})|_{D_0}$, we get $Gf \underset{Q_1}{\sim} G({}_{D_1}(f, h)_{D_2})$ and $Gg \underset{Q_1}{\sim} G({}_{D_1}(g, h)_{D_2})$. So we have $Gf \underset{Q_1}{\sim} Gg$, which completes the proof of (i). The proof of (ii) is similar.

Claim 4. If $u \underset{Q_1}{\sim} v$ and $u \underset{Q_2}{\sim} v$, then $u \underset{D_0}{\sim} v$ for any $u, v \in E_1$.

PROOF. From the assumptions of Q_1 and Q_2, there exist some f, g_1 and $g_2 \in E_2$ such that $f|_{D_0}=g_1|_{D_0}=g_2|_{D_0}$, $u|_{D_0}=G({}_{D_1}(f, g_1)_{D_2})|_{D_0}$ and $v|_{D_0}=G({}_{D_1}(f, g_2)_{D_2})|_{D_0}$, and there exist some h_1, h_2 and $k \in E_2$ such that $h_1|_{D_0}=h_2|_{D_0}=k|_{D_0}$, $u|_{D_0}=G({}_{D_1}(h_1, k)_{D_2})|_{D_0}$ and $v|_{D_0}=G({}_{D_1}(h_2, k)_{D_2})|_{D_0}$. Put $w_1=G({}_{D_1}(f, g_1)_{D_2})$ and $w_2=G({}_{D_1}(h_1, k)_{D_2})$, then $w_1|_{D_0}=w_2|_{D_0}=u|_{D_0}$. Set $w_3={}_{D_1}(w_1, w_2)_{D_2}$. Using the locality of L, we obtain $Lw_3|_{D_1}=Lw_1|_{D_1}=f|_{D_1}$ and $Lw_3|_{D_2}=Lw_2|_{D_2}=k|_{D_2}$. So $f|_{D_0}=k|_{D_0}$, and we have $Lw_3={}_{D_1}(f, k)_{D_2}$, which implies $w_3=G({}_{D_1}(f, k)_{D_2})$. Since $w_3|_{D_0}=u|_{D_0}$, we get $u|_{D_0}=G({}_{D_1}(f, k)_{D_2})|_{D_0}$. Similarly we get $v|_{D_0}=G({}_{D_1}(f, k)_{D_2})|_{D_0}$. Therefore $u|_{D_0}=v|_{D_0}$, which completes the proof.

Thus we have the following diagram which shows the relations between the spaces and the mappings introduced till now.

$$
\begin{tikzcd}
 & E_{2,D_1} \arrow[r, "h_1"] & E_2/R_1 \arrow[r, shift left, "G"] & E_1/Q_1 \arrow[l, shift left, "L"] & & \\
E_2 \arrow[ur] \arrow[dr] & & & & E_{1,D_0} \arrow[ul, "j_1"'] \arrow[dl, "j_2"] & E_1 \arrow[l] \\
 & E_{2,D_2} \arrow[r, "h_2"] & E_2/R_2 \arrow[r, shift left, "G"] & E_1/Q_2 \arrow[l, shift left, "L"] & &
\end{tikzcd}
$$

Now we establish Lemma 1. Let $X_1(\omega)=[X(\omega)]_{Q_1}$ and $X_2(\omega)=[X(\omega)]_{Q_2}$. Since $X_1(\omega)=j_1[X(\omega)]_{1,D_0}$, $X_1(\omega)$ is $\mathscr{F}_{D_0}$-measurable by Claim 1. We, however, get $X_1(\omega)=G[Y(\omega)]_{R_1}=Gh_1[Y(\omega)]_{2,D_1}$. So $X_1(\omega)$ is also $\mathscr{F}_{D_1}$-measurable by Claim 1. Therefore $X_1(\omega)$ is $\mathscr{F}_1$-measurable. Similarly $X_2(\omega)$ is $\mathscr{F}_2$-measurable. Since the mapping $j_1 \otimes j_2$ from E_{1,D_0} to $E_1/Q_1 \times E_1/Q_2$ corresponding $u \in E_{1,D_0}$ to $(j_1 u, j_2 u)$ is continuous and the measure on E_{1,D_0} induced by $[X(\cdot)]_{1,D_0}$ and P is Radon, the measure on $E_1/Q_1 \times E_1/Q_2$ induced by $(X_1(\cdot), X_2(\cdot))$ and P is also Radon by Proposition 2. Using Proposition 3, $\{(X_1(\cdot), X_2(\cdot))^{-1}A;\ A \in \tau_{E_1/Q_1 \times E_1/Q_2}\} \vee \mathscr{N} \subset \mathscr{F}_1 \vee \mathscr{F}_2$. Furthermore $j_1 \otimes j_2$ is one-to-one by Claim 4. So by Proposition 1 and Claim 1, $\{(X_1(\cdot), X_2(\cdot))^{-1}A;\ A \in \tau_{E_1/Q_1 \times E_1/Q_2}\} \vee \mathscr{N}=\{[X(\cdot)]_{1,D_0}^{-1}A;\ A \in \tau_{E_{1,D_0}}\} \vee \mathscr{N}=\mathscr{F}_{D_0}$. Therefore $\mathscr{F}_{D_0} \subset \mathscr{F}_1 \vee \mathscr{F}_2$. $\mathscr{F}_{D_0} \supset \mathscr{F}_1 \vee \mathscr{F}_2$ is obvious. This completes the proof.

REMARK 5. We need not assume in Theorem 1 that L and G is linear, if the conditions (A-1) and (A-2) are satisfied. Indeed we have used the linearity of L and G only to prove the conditions (A-1) and (A-2).

6. A Gaussian white noise and the 0-Markov property

We say that a random distribution W is a Gaussian white noise, if W satisfies $E[\exp(i_{\mathscr{D}}\langle\varphi, W(\omega)\rangle_{\mathscr{D}'})]=\exp\left(-\frac{1}{2}\int_{R^d}|\varphi(x)|^2\,dx\right)$ for every $\varphi\in C_0^\infty(\boldsymbol{R}^d)$, where $E[\cdot]$ denotes the expectation and $i=\sqrt{-1}$. It is obvious that W is 0-Markov. By Minlos-Sazanov-Kolmogorov's Theorem, we can regard W as an $\mathscr{S}'(\boldsymbol{R}^d)$-valued random variable, where $\mathscr{S}'(\boldsymbol{R}^d)$ is a space of real-valued tempered distributions on $\boldsymbol{R}^d$. We denote by $\mathscr{S}(\boldsymbol{R}^d)$ a space of real-valued rapidly decreasing C^∞-functions on $\boldsymbol{R}^d$.

Let $P(\xi)=\sum_{|\alpha|\leqq m}a_\alpha\xi^\alpha$ be a polynomial of $\xi=(\xi_1,\cdots,\xi_d)$ with degree m, where $\alpha=(\alpha_1,\cdots,\alpha_d)$ is a multi-index of non-negative integers with length $|\alpha|=\alpha_1+\cdots+\alpha_d$, ξ^α denotes $\xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}$ and a_α's are real numbers. We denote by $P(D_x)$ a differential operator $\sum_{|\alpha|\leqq m}a_\alpha\frac{\partial^{|\alpha|}}{\partial X_1^{\alpha_1}\cdots\partial X_d^{\alpha_d}}$.

PROPOSITION 5. *Let W be a Gaussian white noise, and let $X=P(D_x)W$. If $P(\xi)$ is not constant, then X is not 0-Markov.*

PROOF. For any domain D in $\boldsymbol{R}^d$, we denote by $\mathscr{H}_D$ (resp. $\mathscr{K}_D$) the closed linear hull in $L^2(\Omega,\mathscr{B},P)$ of $\{_{\mathscr{S}}\langle\varphi, W(\omega)\rangle_{\mathscr{S}'};\varphi\in\mathscr{S}(\boldsymbol{R}^d),\text{support}(\varphi)\subset D\}$ (resp. $\{_{\mathscr{S}}\langle\varphi, X(\omega)\rangle_{\mathscr{S}'};\varphi\in\mathscr{S}(\boldsymbol{R}^d),\text{support}(\varphi)\subset D\}$), where $L^2(\Omega,\mathscr{B},P)$ is a space of real-valued random variables whose squares are integrable. Since $_{\mathscr{S}}\langle\varphi, X\rangle_{\mathscr{S}'}={}_{\mathscr{S}}\langle P(-D_x)\varphi, W\rangle_{\mathscr{S}'}$, it follows that $\mathscr{K}_D\subset\mathscr{H}_D$.

Let $L^2{}_*(\boldsymbol{R}^d_\xi)=\left\{f;\ f(\xi)\text{ is a complex-valued function on }\boldsymbol{R}^d,\ \int_{R^d}|f(\xi)|^2\,d\xi<+\infty\right.$ and $\left.f(\xi)=\overline{f(-\xi)}\text{ for every }\xi\in\boldsymbol{R}^d\right\}$, and we define the inner product on $L^2{}_*(\boldsymbol{R}^d_\xi)$ by $(f,g)=\left(\frac{1}{2\pi}\right)^d\int_{R^d}f(\xi)\overline{g(\xi)}\,d\xi$ for each $f,g\in L^2{}_*(\boldsymbol{R}^d_\xi)$, where $\bar{z}$ denotes a conjugate number of z for each complex number z. Then $L^2{}_*(\boldsymbol{R}^d_\xi)$ is a Hilbert space. For any domain D in $\boldsymbol{R}^d$, we denote by H_D (resp. K_D) the closed linear hull in $L^2{}_*(\boldsymbol{R}^d_\xi)$ of $\{\mathscr{F}[\varphi](\xi);\varphi\in\mathscr{S}(\boldsymbol{R}^d),\text{ support }(\varphi)\subset D\}$ (resp. $\{P(-i\xi)\mathscr{F}[\varphi](\xi);\varphi\in\mathscr{S}(\boldsymbol{R}^d),\text{ support }(\varphi)\subset D\}$), where $\mathscr{F}$ is a Fourier transform.

We define a linear mapping Φ from $\{_{\mathscr{S}}\langle\varphi, W(\omega)\rangle_{\mathscr{S}'};\varphi\in\mathscr{S}(\boldsymbol{R}^d)\}$ to $L^2{}_*(\boldsymbol{R}^d_\xi)$ by $\Phi(_{\mathscr{S}}\langle\varphi, W(\omega)\rangle_{\mathscr{S}'})=\mathscr{F}[\varphi]$ for each $\varphi\in\mathscr{S}(\boldsymbol{R}^d)$, then Φ can be extended to an isometric operator from $\mathscr{H}_{R^d}$ onto $L^2{}_*(\boldsymbol{R}^d_\xi)=H_{R^d}$. It is easy to see that $\Phi(\mathscr{H}_D)=H_D$ and $\Phi(\mathscr{K}_D)=K_D$ for each domain D. So $K_D\subset H_D$ for any domain D.

Now suppose that $P(\xi)\not\equiv$constant and X is 0-Markov. Since the set of roots of $P(-i\xi)=0$ in $\boldsymbol{R}^d$ is a closed set in $\boldsymbol{R}^d$ and its Lebesgue measure is zero, we obtain $K_{\boldsymbol{R}^d}=H_{\boldsymbol{R}^d}=L^2{}_*(\boldsymbol{R}^d)$.

Let $\{D_1, D_2\}$ be a pair of domains such that $D_1\cup D_2=\boldsymbol{R}^d$, $\mathrm{dis}\,(D_1^c, D_2^c)>0$, D_1 is a bounded set and D_2 is not dense in $\boldsymbol{R}^d$, and set $D_0=D_1\cap D_2$. Since X is 0-Markov and Gaussian, $K_{D_1}\cap K_{D_0}^{\perp}$ is orthogonal to $K_{D_2}\cap K_{D_0}^{\perp}$, where $K^{\perp}$ denotes the orthogonal complement of some subspace K in $L^2{}_*(\boldsymbol{R}^d)$. It is obvious that $K_{\boldsymbol{R}^d}=K_{D_1}+K_{D_2}$, so $K_{\boldsymbol{R}^d}=(K_{D_1}\cap K_{D_0}^{\perp})\oplus K_{D_0}\oplus(K_{D_2}\cap K_{D_0}^{\perp})=(K_{D_1}\cap K_{D_0}^{\perp})\oplus K_{D_2}$. Since $K_{\boldsymbol{R}^d}=H_{\boldsymbol{R}^d}=H_{D_2}^{\perp}\oplus H_{D_2}$ and $K_{D_2}\subset H_{D_2}$, we get $H_{D_2}^{\perp}\subset K_{D_1}\cap K_{D_0}^{\perp}\subset K_{D_1}$. So $H_{D_2}^{\perp}\subset K_{D_1}\subset H_{D_1}$. But we shall show that it is impossible.

Since D_1 is a bounded set, $L^2(D_1, dx)$ is continuously included in $\mathscr{E}'(\boldsymbol{R}^d)$, where $\mathscr{E}'(\boldsymbol{R}^d)$ is a space of distributions with compact supports. So by Paley-Wiener's Theorem, a Fourier transform is a continuous mapping from $L^2(D_1, dx)$ to $\mathscr{A}(\boldsymbol{C}^d)$, where $\mathscr{A}(\boldsymbol{C}^d)$ is a space of entire functions on $\boldsymbol{C}^d$ and its topology is introduced by uniform convergence in wider sense. Therefore H_{D_1} and so K_{D_1} are continuously included in $\mathscr{A}(\boldsymbol{C}^d)$. Since $P(\xi)\not\equiv$constant, there exists $\xi_0\in\boldsymbol{C}^d$ such that $P(-i\xi_0)=0$. By our assumption $\{P(-i\xi)\mathscr{F}[\varphi](\xi);\ \varphi\in\mathscr{S}(\boldsymbol{R}^d), \mathrm{support}\,(\varphi)\subset D_1\}$ is dense in K_{D_1}, so we get $K_{D_1}\subset\{f\in\mathscr{A}(\boldsymbol{C}^d);\ f(\xi_0)=0\}$.

We assumed that D_2 is not dense in $\boldsymbol{R}^d$. So there exists $x_0=(x_0^1, \cdots, x_0^d)\in\boldsymbol{R}^d$ such that the δ-neighborhood U of x_0 is disjoint with D_2. It is obvious that H_U is orthogonal to H_{D_2}, which implies $H_U\subset K_{D_1}$. There exists $\varphi\in C_0^{\infty}(\boldsymbol{R}^d)$ such that $\mathrm{support}\,(\varphi)\subset\{x\in\boldsymbol{R}^d;\ \mathrm{dis}\,(0, x)<1\}$ and $\int_{\boldsymbol{R}^d}\varphi(x)\,dx=1$. Set $\varphi_\varepsilon(x)=\varepsilon^{-d}(\varepsilon^{-1}(x-x_0))$ for $0<\varepsilon<\delta$, then $\mathscr{F}[\varphi_\varepsilon]\in H_U$. Since $\varphi_\varepsilon(x)\to\delta(x-x_0)$ in $\mathscr{E}'(\boldsymbol{R}^d)$ as $\varepsilon\to0$, $\mathscr{F}[\varphi_\varepsilon](\xi)\to\exp(-ix_0\cdot\xi)$ in $\mathscr{A}(\boldsymbol{C}^d)$ as $\varepsilon\to0$, where $\delta(\cdot)$ is a Dirac distribution and $x_0\cdot\xi=x_0^1\cdot\xi_1+\cdots+x_0^d\cdot\xi_d$. Since $\lim_{\varepsilon\downarrow0}\mathscr{F}[\varphi_\varepsilon](\xi_0)=\exp(-ix_0\cdot\xi_0)\neq0$, there exists ε_0 such that $\mathscr{F}[\varphi_{\varepsilon_0}](\xi_0)\neq0$, which contradicts to the fact that $H_U\subset K_{D_1}$. This completes the proof.

Next let Y be a mean 0, stationary Gaussian random distribution with independent values at every point. By I. M. Gelfand-N. Ja Vilenkin [2], there uniquely exists a polynomial $P(\xi)$ of $\xi=(\xi_1, \cdots, \xi_d)$ of degree $2m$ with real coefficients such that $P(-\xi)=P(\xi)$, $P(i\xi)\geqq0$ for each $\xi\in\boldsymbol{R}^d$ and $E[\exp(i_{\mathscr{D}}\langle\varphi, Y(\omega)\rangle_{\mathscr{D}'})]=\exp\left(-\dfrac{1}{2}\int_{\boldsymbol{R}^d}(P(D_x)\varphi)(x)\cdot\varphi(x)\,dx\right)$ for every $\varphi\in C_0^{\infty}(\boldsymbol{R}^d)$.

THEOREM 2. (1) *If there exists a polynomial $r(\xi)$ with real coefficients such that $P(\xi)=r(\xi)\cdot r(-\xi)$, and if $P(\xi)$ is not constant, then Y is not 0-Markov.*

(2) *If $P(i\xi)\neq0$ for any $\xi\in\boldsymbol{R}^d$ and $P(\xi)$ is not constant, then Y is not 0-Marov.*

REMARK 6. Since each mean 0, stationary Gaussian random distribution with independent values at every point is Markov, Theorem 2 gives an example of a random distribution which is Markov but not 0-Markov.

REMARK 7. We conjecture that a mean 0, stationary Gaussian random distribution with independent values at every point is 0-Markov, iff it is a Gaussian white noise up to a constant factor.

Theorem 2-(1) is obvious by Propositon 5, since there exists a Gaussian white noise W such that $Y=r(D_x)W$.

Notice that we can regard Y as an $\mathscr{S}'(\boldsymbol{R}^d)$-valued random variable and the measure on $\mathscr{S}'(\boldsymbol{R}^d)$ induced by Y and P is Radon by Minlos-Sazanov-Kolmogorov's theorem.

To prove Theorem 2-(2), we prepare a lemma.

LEMMA 4. *$\mathscr{S}'(\boldsymbol{R}^d)$ satisfies conditions* (C-1) *and* (C-2).

PROOF. Let $\{D_1, D_2\}$ be an arbitrary pair of domains in $\boldsymbol{R}^d$ such that $D_1\cup D_2=\boldsymbol{R}^d$ and $\mathrm{dis}(D_1^c, D_2^c)=\delta>0$, and set $D_0=D_1\cap D_2$. There exists some $\varphi\in C_0^\infty(\boldsymbol{R}^d)$ such that $\int_{R^d}\varphi(x)\,dx=1$ and support $(\varphi)\subset\{x\in\boldsymbol{R}^d;\ \mathrm{dis}(0,x)<\delta/6\}$, and there exists a continuous function ψ on $\boldsymbol{R}^d$ such that $\psi(x)=1$ if $\mathrm{dis}(x, D_2^c)<\delta/3$, $\psi(x)=0$ if $\mathrm{dis}(x, D_1^c)<\delta/3$ and $0\leqq\psi\leqq1$. Now let $\eta_1=\varphi*\psi$ and $\eta_2=\varphi*(1-\psi)$, where $*$ means a convolution. Then η_1 and η_2 are infinitely differentiable and their derived functions of each order are bounded. Setting ${}_{D_1}(f, g)_{D_2}=\eta_1 f+\eta_2 g$ for any $f, g\in\mathscr{S}'(\boldsymbol{R}^d)$ such that $f|_{D_0}=g|_{D_0}$, we find that the conditions (C-1) and (C-2) are satisfied.

Now we shall prove Theorem 2-(2). Suppose that Y is 0-Markov. It suffices to prove that $P(\xi)\equiv$constant. Suppose $P(\xi)\not\equiv$constant. Let $D(\xi, c)=-(\xi_1^2+\cdots+\xi_d^2)+c$ for each $c>0$. Then there exists a positive number c_0 such that $D(\xi, c_0)$ and $P(\xi)$ are relatively prime as polynomials of ξ. Put $q(\xi)=D(\xi, c_0)^{m+d}$. Then $q(i\xi)^2/P(i\xi)$ is not a polynomial. Let $Lu=q(D_x)u$ and $Gu=\bar{\mathscr{F}}[(1/q(i\xi))\hat{u}(\xi)]$ for each $u\in\mathscr{S}'(\boldsymbol{R}^d)$, where $\hat{u}$ is a Fourier transform of u and $\bar{\mathscr{F}}$ is an inverse Fourier transform. Then L and G are continuous linear mapping from $\mathscr{S}'(\boldsymbol{R}^d)$ to $\mathscr{S}'(\boldsymbol{R}^d)$, and L is local.

Put $E_1=E_2=\mathscr{S}'(\boldsymbol{R}^d)$. Since E_1, E_2, L, G and Y satisfy the assumptions of Theorem 1, $X=GY$ is 0-Markov as a random distribution. It is easy to see that $E[\exp(i_{\mathscr{D}}\langle\varphi, X\rangle_{\mathscr{D}'})]=\exp\left[-\frac{1}{2}\left(\frac{1}{2\pi}\right)^d\int_{R^d}\frac{P(i\xi)}{q(i\xi)^2}|\hat{\varphi}(\xi)|^2\,d\xi\right]$ for each $\varphi\in C_0^\infty(\boldsymbol{R}^d)$. Since there is a constant A such that $|P(i\xi)/q(i\xi)^2|\leqq A/(1+\xi_1^2+\cdots+\xi_d^2)^{d+1}$ for every $\xi\in\boldsymbol{R}^d$, we can regard X as a continuous function on $\boldsymbol{R}^d$ with probability one (see M. Reed-L. Rosen [10]). So X is regarded as an ordinary stationary Gaussian random field whose spectral density is $(1/2\pi)^d(P(i\xi)/q(i\xi)^2)$ and its reciprocal is locally integrable by our assumption. By Remark 1 X is a Markov field. But this contradicts to the result of S. Kotani [4], because $q(i\xi)^2/P(i\xi)$ is not a polynomial. So $P(\xi)$ must be a constant. This completes the proof of Theorem 2-(2).

REMARK 8. The proof of Theorem 2 shows that we cannot adopt the Markov property in Theorem 1 instead of the 0-Markov property.

7. Examples

Example 1. Let Z be a Poisson random measure on $\boldsymbol{R}^d$, i.e. $E[\exp(i_{\mathscr{D}}\langle\varphi, Z(\omega)\rangle_{\mathscr{D}'})]=\exp\left(\int_{R^d}(e^{i\varphi(x)}-1)\,dx\right)$ for every $\varphi\in C_0^\infty(\boldsymbol{R}^d)$. It is obvious that Z is 0-Markov. It is well-known that support $(Z(\omega))$ is a countable set with no cluster point and $Z(\omega)=\sum_y \delta(\cdot\,-y):\ y\in\text{support}\,(Z(\omega))$ for almost every ω. Let $Q(\xi)$ be a polynomial of $\xi=(\xi_1,\cdots,\xi_d)$ with real coefficients. Suppose $Q(\xi)$ is not zero, and let $Y=Q(D_x)Z$. Then it is obvious that $Y(\omega)=\sum_y Q(D_x)\delta(\cdot\,-y):\ y\in\text{support}\,(Z(\omega))$ and so support $(Y(\omega))=$support $(Z(\omega))$ for almost every ω. Since $Z(\omega)|_D=\sum_y \delta(-y):\ y\in\text{support}\,(Z(\omega))\cap D$ and $Y(\omega)|_D=\sum_y Q(D_x)\delta(\cdot\,-y):\ y\in\text{support}\,(Y(\omega))\cap D$ for almost every ω, $\sigma\{_{\mathscr{D}}\langle\varphi, Y(\omega)\rangle_{\mathscr{D}'};\ \varphi\in C_0^\infty(\boldsymbol{R}^d),\ \text{support}\,(\varphi)\subset D\}\vee\mathscr{N}=\sigma\{\text{support}\,(Y(\omega))\cap D\}\vee\mathscr{N}=\sigma\{\text{support}\,(Z(\omega))\cap D\}\vee\mathscr{N}=\sigma\{_{\mathscr{D}}\langle\varphi, Z(\omega)\rangle_{\mathscr{D}'};\ \varphi\in C_0^\infty(\boldsymbol{R}^d),\ \text{support}\,(\varphi)\subset D\}\vee\mathscr{N}$ for any domain D in $\boldsymbol{R}^d$. So not only Z but also $Y=Q(D_x)Z$ is 0-Markov. This fact is in a striking contrast to Proposition 5.

Now let $E_1=E_2=\mathscr{S}'(\boldsymbol{R}^d)$ and let $P(\xi)$ be a polynomial of $\xi=(\xi_1,\cdots,\xi_d)$ with real coefficients such that $P(i\xi)\neq 0$ for every $\xi\in\boldsymbol{R}^d$. Let $Lu=P(D_x)u$ and $Gu=\bar{\mathscr{F}}[(1/P(i\xi))\hat{u}(\xi)]$ for each $u\in\mathscr{S}'(\boldsymbol{R}^d)$. Then by Lemma 4 we find that E_1, E_2, L and G satisfy the assumptions of Theorem 1. So, if Y is an $\mathscr{S}'(\boldsymbol{R}^d)$-valued random variable inducing a Radon measure on $\mathscr{S}'(\boldsymbol{R}^d)$, and if Y is 0-Markov as a random distribution, then $X=GY$ is also 0-Markov.

Example 2. The random distribution Y in Example 1 satisfies $E[\exp(i_{\mathscr{D}}\langle\varphi, Y(\omega)\rangle_{\mathscr{D}'})]=\exp\left[\int_{R^d}(\exp(i\cdot Q(-D_x)\varphi(x))-1)\,dx\right]$ for every $\varphi\in C_0^\infty(\boldsymbol{R}^d)$. So it follows from Minlos–Sazanov–Kolmogorov's theorem that we can regard Y as an $\mathscr{S}'(\boldsymbol{R}^d)$-valued random variable and the measure on $\mathscr{S}'(\boldsymbol{R}^d)$ induced by Y and P is Radon. Therefore $X=GY=\bar{\mathscr{F}}[(Q(i\xi)/P(i\xi))\hat{Z}]$ is 0-Markov.

Example 3. Let Y be a white noise, i.e. there exist a non-negative number c and a measure m on $\boldsymbol{R}$ such that $m(\{0\})=0$, $\int_R \frac{|\lambda|^2}{1+|\lambda|}m(d\lambda)<+\infty$ and Y satisfies $E[\exp(i_{\mathscr{D}}\langle\varphi, Y(\omega)\rangle_{\mathscr{D}'})]=\exp\left[-\frac{c}{2}\int_{R^d}|\varphi(x)|^2\,dx+\int_{R^d\times R}(e^{i\lambda\varphi(x)}-1-i\lambda\varphi(x))\,dx\,m(d\lambda)\right]$ for every $\varphi\in C_0^\infty(\boldsymbol{R}^d)$. It follows from Minlos–Sazanov–Kolmogorov's theorem that

Y can be regarded as an $\mathscr{S}'(\boldsymbol{R}^d)$-valued random variable and the measure on $\mathscr{S}'(\boldsymbol{R}^d)$ induced by Y and P is Radon. It is obvious that Y is 0-Markov. So $X=GY$ is 0-Markov.

References

[1] Doob, J. L., The elementary Gaussian processes, Ann. Math. Statist. **15** (1944), 229–282.

[2] Gelfand, I. M. and N. Ia. Vilenkin, Generalized function, Academic Press, 1934.

[3] Knight, F., A remark on Markovian germ fields, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. **15** (1970), 291–296.

[4] Kotani, S., On a Markov property for stationary Gaussian processes with a multidimensional parameter, Proc. 2nd Japan-USSR Symp. on Prob., Lecture notes in Math. vol. 330, Springer, 1937, 239–250.

[5] Levinson, N. and McKean Jr., H. P., Weighted trigonometrical approximation on R^1 with application to the germ field of a stationary Gaussian noise, Acta Math. **112** (1964), 99–143.

[6] Mandrekar, V., Germ field property for multiparameter processes, Sem. pro. Univ. Strasbourg, Lecture Notes in Math. vol. 511, Springer, 1967, 78–85.

[7] McKean Jr., H. P., Brownian motion with a several dimensional time, Teor. Verojatnost. i. Primenen, 8 (1963), 357–378.

[8] Okabe, Y., Stationary Gaussian processes with Markovian property and M. Sato's hyperfunctions, Japan. J. Math. **41** (1973), 66–122.

[9] Okabe, Y., On the germ fields of stationary Gaussian processes with Markovian property, J. Math. Soc. Japan, **28** (1976), 86–95.

[10] Reed, M. and Rosen, L., Support properties of the free measure for Boson fields, Comm. Math. Phys. **36** (1974), 123–132.

[11] Schwartz, L., Radon measure on arbitrary Topological spaces and Cylindrical Measures, Tata-Oxford, 1973.

A note added in proof

Professor S. Kotani of Kyoto University informed me of the following proposition due to L. D. Pitt [Some problems in the spectral theory of stationary process on $\boldsymbol{R}^d$, Indiana Univ. Math. J. **23** (1973), 343–365].

PROPOSITION. *Suppose* $\{X(x)\}_{x\in \boldsymbol{R}^d}$ *is a mean* 0, *stationary Gaussian random field whose spectral density function* $\Delta(\xi)$ *is a rational function* ($\xi\in\boldsymbol{R}^d$). *If* $\{X(x)\}_{x\in\boldsymbol{R}^d}$ *has the Markov property with respect to all half spaces, then* $\Delta(\xi)^{-1}$ *is a polynomial.*

Using this proposition, we can prove our conjecture in Remark 7 similarly as the proof of Theorem 2-(2).

(Received September 20, 1978)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan