# *Singular hyperbolic systems, I*
## *Existence, uniqueness and differentiability*

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Let us consider a differential equation of the form

$$\text{(S)} \qquad t^{\sigma}\frac{du}{dt}+A(t)u-t^{\rho}B(t)u=f(t)\,, \qquad 0<t\leqq T \quad (T>0)$$

in a Hilbert space $X$, where $A(t)$ and $B(t)$ are densely defined, closed linear operators in $X$ and $u=u(t)$, $f(t)$ are functions on $[0, T]$ with values in $X$. We assume the following conditions on (S):

(1) $\sigma\geqq 1$,

(2) $\rho-\sigma+1>0$.

Then we say that the equation (S) is *a singular system of type* $(\sigma, \rho)$ *with respect to* $t$. If $\sigma=1$, the equation (S) has a regular singularity at $t=0$ and if $\sigma>1$, the equation (S) has a irregular singularity at $t=0$. If $\sigma=\rho=1$, then the equation (S) is nothing but a modification of a Fuchsian system in Tahara [10]. Alinhac [1] also treated the case $\sigma=\rho=1$.

The purpose of this paper is to establish the existence, the uniqueness and the differentiability of solutions of a singular system of type $(\sigma, \rho)$ with some hyperbolicity conditions, that we call *a singular hyperbolic system*. Our theory is motivated by Friedrichs [2] and is based on a new energy inequality for a certain class of symmetric positive systems. Mizohata's argument in [6] is also adapted in our theory.

In Section 1, using the theory of semi-groups we solve the equation (S) when $A$ and $B$ are independent of $t$. This is a model study of singular systems. The equation (S) in the general case will be discussed in Sections 2, 3, 4, and 5. In Section 2, we establish a new energy inequality for a certain class of positive systems and in Section 3, combining the energy inequality with Cauchy's polygon method in Mizohata [6], we prove the existence and the differentiability of solutions of the equation (S) when $X=L^2(\boldsymbol{R}^n)$ and (S) is a symmetric positive system. In Section 4, the results are generalized to symmetrizable positive systems and in Section 5, we discuss the asymptotic behaviour as $t\to 0$ of solutions with some singularities at $t=0$. Thus, our program is as follows:

1. Abstract model,

2. Energy inequality,
3. Existence and differentiability,
4. Generalization,
5. Asymptotic behaviour as $t \to 0$.

Moreover, in Appendix we discuss the existence of solutions of the non-characteristic Cauchy problem, which we shall utilize in Sections 3 and 4.

This paper is the first part of a series of my papers and the results in this paper will be applied to the study of various problems, especially, of the initial value problem for concrete partial differential equations.

## 1. Abstract model

Let us consider a singular system of type $(\sigma, \rho)$

$$t^{\sigma}\frac{du}{dt}+Au-t^{\rho}Bu=f(t)\,, \qquad 0<t\leqq T \tag{A}$$

in a Hilbert space $X$. We assume the following conditions (A-1)~(A-5) on (A):

(A-1) $D(A)\cap D(B)$ is dense in $X$, where $D(A)$ and $D(B)$ are domains of the definitions of $A$ and $B$ respectively.

(A-2) There exists a positive constant $a$ satisfying $\mathrm{Re}\,(Au, u)\geqq a\|u\|^2$ for any $u\in D(A)$ and $\mathrm{Re}\,(A^*v, v)\geqq a\|v\|^2$ for any $v\in D(A^*)$.

(A-3) There exists a constant $b$ satisfying $\mathrm{Re}\,(Bu, u)\leqq b\|u\|^2$ for any $u\in D(B)$ and $\mathrm{Re}\,(B^*v, v)\leqq b\|v\|^2$ for any $v\in D(B^*)$.

(A-4) $ABx=BAx$ holds for any $x\in D(A)\cap D(B)$ satisfying $Ax\in D(B)$ and $Bx\in D(A)$.

(A-5) $(\lambda+A)^{-1}D(A)\cap D(B)\subset D(A)\cap D(B)$, $(\mu-B)^{-1}D(A)\cap D(B)\subset D(A)\cap D(B)$, $(\mu-B)(\lambda+A)^{-1}D(A)\cap D(B)\subset D(A)$ and $(\lambda+A)(\mu-B)^{-1}D(A)\cap D(B)\subset D(B)$ hold for any $\lambda$ and $\mu$ such that $\lambda>-a$ and $\mu>b$.

Under these assumptions, we have the next theorem.

THEOREM 1.1. *For an arbitrary* $f(t)\in C^0([0, T], X)$ *satisfying* $f(t)\in D(A)\cap D(B)$, $Af(t)\in C^0([0, T], X)$ *and* $Bf(t)\in C^0([0, T], X)$, *there exists a unique solution of the equation* (A) *such that it satisfies* $u(t)\in C^0([0, T], X)\cap C^1((0, T], X)$, $t^{\sigma}(du/dt)(t)\in C^0([0, T], X)$, $u(t)\in D(A)\cap D(B)$, $Au(t)\in C^0([0, T], X)$ *and* $Bu(t)\in C^0([0, T], X)$. *Moreover the solution* $u(t)$ *is expressed in the form*

$$u(t)=\int_0^{\infty} T(s)S(\theta_{\sigma}(t, s, \rho))f(\phi_{\sigma}(t, s))ds\,, \tag{1.1}$$

*where* $\{T(t)\}_{t\geqq 0}$ *and* $\{S(t)\}_{t\geqq 0}$ *are continuous semi-groups generated by* $-A$ *and* $B$ *respectively. Here the functions* $\theta_{\sigma}(t, s, \rho)$ *and* $\phi_{\sigma}(t, s)$ *are defined by*

$$\theta_\sigma(t,s,\rho)=\begin{cases}\dfrac{t^\rho}{\rho}(1-e^{-\rho s})\,, & when \quad \sigma=1\,,\\ \dfrac{t^{\rho-\sigma+1}}{\rho-\sigma+1}\left[1-\left(\dfrac{1}{(\sigma-1)st^{\sigma-1}+1}\right)^{(\rho-\sigma+1)/(\sigma-1)}\right], & when \quad \sigma>1\end{cases}$$

*and*

$$\phi_\sigma(t,s)=\begin{cases}te^{-s}\,, & when \quad \sigma=1\,,\\ t\left(\dfrac{1}{(\sigma-1)st^{\sigma-1}+1}\right)^{1/(\sigma-1)}\,, & when \quad \sigma>1\end{cases}$$

*respectively.*

REMARK 1.2. The integral formula (1.1) immediately leads us to the following energy inequality

$$\|u(t)\|\leqq\int_0^\infty e^{-as}e^{b\theta_\sigma(t,s,\rho)}\|f(\phi_\sigma(t,s))\|ds$$

for $0\leqq t\leqq T$. This is the very energy inequality on which our theory is based.

For the proof of this theorem, we must prepare several lemmas.

LEMMA 1.3. (i) (A-2) *is equivalent to the condition that there exists a resolvent* $(\lambda+A)^{-1}$ *of* $-A$ *for* $\lambda>-a$ *satisfying* $\|(\lambda+A)^{-1}\|\leqq(\lambda+a)^{-1}$ $(\lambda>-a)$. (ii) (A-3) *is equivalent to the condition that there exists a resolvent* $(\mu-B)^{-1}$ *of* $B$ *for* $\mu>b$ *satisfying* $\|(\mu-B)^{-1}\|\leqq(\mu-b)^{-1}$ $(\mu>b)$.

PROOF. Note that the following equality

$$\|(\lambda+A)u\|^2=\|(A-a)u\|^2+2(\lambda+a)\{\mathrm{Re}\,(Au,u)-a\|u\|^2\}+(\lambda+a)^2\|u\|^2 \tag{1.2}$$

holds for $u\in D(A)$. If $A$ satisfies the condition (A-2), we have $\|(\lambda+A)u\|\geqq(\lambda+a)\|u\|$ for $u\in D(A)$. Since $A^*$ satisfies the same condition as $A$, we also have $\|(\lambda+A^*)v\|\geqq(\lambda+a)\|v\|$ for $v\in D(A^*)$. Therefore $(\lambda+A)$ is a bijection from $D(A)$ onto $X$. Hence $(\lambda+A)^{-1}$ exists and $\|(\lambda+A)^{-1}\|\leqq(\lambda+a)^{-1}$. Conversely, if $A$ satisfies the condition in (i), we have the inequality

$$0\leqq\|(A-a)u\|^2+2(\lambda+a)\{\mathrm{Re}\,(Au,u)-a\|u\|^2\} \tag{1.3}$$

for $u\in D(A)$ and $\lambda>-a$. If $\mathrm{Re}\,(Au,u)<a\|u\|^2$ for some $u$, then (1.3) does not hold for a sufficiently large $\lambda$. This is a contradiction. Hence $\mathrm{Re}\,(Au,u)\geqq a\|u\|^2$ for any $u\in D(A)$. Since $(\lambda+A^*)^{-1}=(\lambda+A)^{-1*}$ and $\|(\lambda+A^*)^{-1}\|=\|(\lambda+A)^{-1*}\|=\|(\lambda+A)^{-1}\|\leqq(\lambda+a)^{-1}$, we have $\mathrm{Re}\,(A^*v,v)\geqq a\|v\|^2$ for any $v\in D(A^*)$. Thus (i) is proved. The proof of (ii) is the same as (i). Q.E.D.

From Lemma 1.3 and Hille-Yosida's theorem, $-A$ and $B$ are infinitesimal generators of continuous semi-groups $\{T(t)\}_{t\geqq0}$ and $\{S(t)\}_{t\geqq0}$ such that $\|T(t)\|\leqq e^{-at}$,

$\|S(t)\| \leqq e^{bt}$ for $t \geqq 0$ respectively. We now investigate the commutativity among $A$, $B$, $T(t)$ and $S(t)$.

LEMMA 1.4 *If* $x \in D(A) \cap D(B)$, *we have* (i) $B(\lambda+A)^{-1}x=(\lambda+A)^{-1}Bx$, (ii) $A(\mu-B)^{-1}x=(\mu-B)^{-1}Ax$ *and* (iii) $(\lambda+A)^{-1}(\mu-B)^{-1}x=(\mu-B)^{-1}(\lambda+A)^{-1}x$.

PROOF. Set $y=(\lambda+A)^{-1}x$. From (A-5), $y \in D(A) \cap D(B)$, $Ay \in D(B)$ and $By \in D(A)$. Therefore from the commutativity of (A-4) we have $ABy=BAy$, that is, $(\lambda+A)(\mu-B)y=(\mu-B)(\lambda+A)y$. Operating with $(\lambda+A)^{-1}$ on both sides, we have $(\mu-B)(\lambda+A)^{-1}x=(\lambda+A)^{-1}(\mu-B)x$. This means (i). The proof of (ii) is the same as (i). Next, we set $z=(\mu-B)^{-1}y=(\mu-B)^{-1}(\lambda+A)^{-1}x$. Since $z \in D(A) \cap D(B)$, $Az \in D(B)$ and $Bz \in D(A)$, we have $ABz=BAz$, that is, $(\lambda+A)(\mu-B)z=(\mu-B)(\lambda+A)z$. Operating with $(\lambda+A)^{-1}(\mu-B)^{-1}$ on both sides, we have $(\lambda+A)^{-1}(\mu-B)^{-1}x=(\mu-B)^{-1}(\lambda+A)^{-1}x$. This is (iii). Q.E.D.

LEMMA 1.5. (i) $T(t)S(t')=S(t')T(t)$ *holds for* $t, t' \geqq 0$. (ii) *If* $x \in D(A) \cap D(B)$, *then we have* $T(t)x \in D(A) \cap D(B)$, $AT(t)x=T(t)Ax$ *and* $BT(t)x=T(t)Bx$. (iii) *If* $x \in D(A) \cap D(B)$, *then we have* $S(t)x \in D(A) \cap D(B)$, $AS(t)x=S(t)Ax$ *and* $BS(t)x=S(t)Bx$.

PROOF. For the proof of this lemma, we may assume that $a=b=0$ without loss of generality. In this case, $T(t)$ and $S(t)$ are given by

$$T(t)x=\lim_{n\to\infty}\left(1+\frac{t}{n}A\right)^{-n}x\,, \qquad S(t)x=\lim_{m\to\infty}\left(1-\frac{t}{m}B\right)^{-m}x$$

respectively. From Lemma 1.4, $(1+(t/n)A)^{-n}(1-(t'/m)B)^{-m}x=(1-(t'/m)B)^{-m}(1+(t/n)A)^{-n}x$ holds for $x \in D(A) \cap D(B)$. Therefore we have $T(t)S(t')x=S(t')T(t)x$ for $x \in D(A) \cap D(B)$. Since $D(A) \cap D(B)$ is dense in $X$, we have $T(t)S(t')=S(t')T(t)$ on $X$. This is (i). Next we will show (ii). $T(t)x \in D(A)$ and $AT(t)x=T(t)Ax$ are clear. From Lemma 1.4, we have $(1+(t/n)A)^{-n}Bx=B(1+(t/n)A)^{-n}x$. The left hand side converges to $T(t)Bx$ as $n\to\infty$ and $(1+(t/n)A)^{-n}x$ converges to $T(t)x$ as $n\to\infty$. Since $(1+(t/n)A)^{-n}x \in D(A) \cap D(B)$ and $B$ is a closed operator, we have $T(t)x \in D(B)$ and $BT(t)x=T(t)Bx$. This is (ii). The proof of (iii) is the same as (ii). Q.E.D.

PROOF OF THEOREM 1.1. First we assume that $\sigma=1$. We shall show that (1.1) is in fact a solution of (A). Since $\|T(t)\| \leqq e^{-at}$ and $a>0$, $u(t) \in C^0([0, T], X)$ is clear. We note that $u(t) \in D(A)$ and

$$Au(t)=\int_0^\infty T(s)S(\theta_\sigma(t, s, \rho))Af(\phi_\sigma(t, s))ds\,.$$

To prove this, first recall the definition of integration and note that $A$ is a closed operator. Here we also used Lemma 1.5. Therefore $Au(t)\in C^0([0, T], X)$. Using the same argument we have $u(t)\in D(B)$ and $Bu(t)\in C^0([0, T], X)$. On the other hand, (1.1) is rewritten in the form

$$u(t)=\int_0^t T(\log t-\log\tau)S\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)f(\tau)\frac{1}{\tau}d\tau$$

by the change of variables: $s=\log t-\log\tau$. For $t>0$ and $\eta>0$,

$$\frac{u(t+\eta)-u(t)}{\eta}=\frac{1}{\eta}\int_t^{t+\eta}T\Big(\log\frac{t+\eta}{\tau}\Big)S\Big(\frac{(t+\eta)^\rho-\tau^\rho}{\rho}\Big)f(\tau)\frac{1}{\tau}d\tau$$
$$+\frac{T\Big(\log\frac{t+\eta}{t}\Big)-I}{\eta}S\Big(\frac{(t+\eta)^\rho-t^\rho}{\rho}\Big)u(t)+\frac{S\Big(\frac{(t+\eta)^\rho-t^\rho}{\rho}\Big)-I}{\eta}u(t)\,.$$

So that as $\eta\to+0$, we obtain

$$\Big(\frac{d}{dt}\Big)_+u(t)=\frac{1}{t}f(t)-\frac{1}{t}Au(t)+t^{\rho-1}Bu(t)\,.$$

Since the right hand side terms are continuous functions of $t$ on $t>0$, we have $u(t)\in C^1((0, T], X)$ and

$$t\frac{du}{dt}(t)+Au(t)-t^\rho Bu(t)=f(t)\,,\qquad 0<t\leqq T\,.$$

We now prove the uniqueness. It is sufficient to show that

$$t\frac{du}{dt}(t)+Au(t)-t^\rho Bu(t)=0 \tag{1.4}$$

implies $u(t)\equiv 0$. To do so, we introduce Yosida's approximations $A_\lambda$, $B_\lambda$ of $A$, $B$ respectively. $A_\lambda$ and $B_\lambda$ satisfy the following conditions: (i) $A_\lambda$ and $B_\lambda$ are bounded operators in $X$, (ii) $\|e^{-tA_\lambda}\|\leqq e^{-at}$, $\|e^{tB_\lambda}\|\leqq e^{bt}$ for $t\geqq 0$, (iii) $\|A_\lambda x-Ax\|\to 0$ as $\lambda\to\infty$ for $x\in D(A)$, (iv) $\|B_\lambda x-Bx\|\to 0$ as $\lambda\to\infty$ for $x\in D(B)$, and (v) there exists a positive constant $c$ such that $\|A_\lambda x-Ax\|\leqq c\|Ax\|$ for $x\in D(A)$ and $\|B_\lambda x-Bx\|\leqq c\|Bx\|$ for $x\in D(B)$. Moreover using the same argument as in the proof of Lemma 1.5, we have $A_\lambda B_\lambda=B_\lambda A_\lambda$ for a sufficiently large $\lambda$. We set $f_\lambda(t)=(A_\lambda u(t)-Au(t))-t^\rho(B_\lambda u(t)-Bu(t))$. Then (1.4) is rewritten in the form

$$t\frac{du}{dt}(t)+A_\lambda u(t)-t^\rho B_\lambda u(t)=f_\lambda(t)\,,\qquad 0<t\leqq T\,. \tag{1.5}$$

Since $A_\lambda$ and $B_\lambda$ are bounded operators and $A_\lambda B_\lambda=B_\lambda A_\lambda$ holds on $X$, the solution $u(t)$ of (1.5) is uniquely determined and is given by

$$u(t)=\int_0^\infty e^{-sA_\lambda}e^{\theta_\sigma(t,s,\rho)B_\lambda}f_\lambda(\phi_\sigma(t,s))ds\ .$$

From the properties (iii), (iv) and (v), $\|f_\lambda(t)\|$ is uniformly bounded for $0\leqq t\leqq T$ and a sufficiently large $\lambda$. Since $\|f_\lambda(t)\|\to 0$ as $\lambda\to\infty$ for a fixed $t$, by Lebesgue's convergence theorem we have $u(t)\equiv 0$. Therefore the solution is unique. Thus we have proved the theorem when $\sigma=1$. If $\sigma>1$, we put $s=\xi(\tau)-\xi(t)$, where $\xi(t)=1/((\sigma-1)t^{\sigma-1})$. Then (1.1) is rewritten in the form

$$u(t)=\int_0^t T(\xi(\tau)-\xi(t))S\left(\frac{t^\mu-\tau^\mu}{\mu}\right)f(\tau)\frac{1}{\tau^\sigma}d\tau\ ,$$

where $\mu=\rho-\sigma+1$. Hence the argument in the case $\sigma=1$ is also valid in this case. Therefore we may omit the details. Q.E.D.

COROLLARY 1.6. *If $\rho\geqq 1$ and $f(t)\in C^1([0,T],X)$, then we have $u(t)\in C^1([0,T],X)$.*

PROOF. By derivating (1.1) directly, we have

$$\frac{du}{dt}(t)=\int_0^\infty\left(\frac{d}{dt}\theta_\sigma\right)(t,s,\rho)T(s)S(\theta_\sigma(t,s,\rho))Bf(\phi_\sigma(t,s))ds$$
$$+\int_0^\infty\left(\frac{d}{dt}\phi_\sigma\right)(t,s)T(s)S(\theta_\sigma(t,s,\rho))\frac{df}{dt}(\phi_\sigma(t,s))ds\ .$$

Since $|(d/dt)\theta_\sigma(t,s,\rho)|=O(t^{\rho-1})$ and $|(d/dt)\phi_\sigma(t,s)|=O(t^{\sigma-1})$ as $t\to+0$, we have $(du/dt)(t)\in C^0([0,T],X)$ by Lebesgue's convergence theorem. This means $u(t)\in C^1([0,T],X)$. Q.E.D.

REMARK 1.7. If the assumptions in Corollary 1.6 are not satisfied, we can not necessarily obtain $u(t)\in C^1([0,T],X)$ in general. We only have $\|(du/dt)(t)\|=O(t^{-\sigma})$ as $t\to+0$. This situation will be illustrated in the following examples. Let us consider an ordinary differential equation of the form

$$t^\sigma\frac{du}{dt}+\alpha u-t^\rho\beta u=1\ ,$$

where $\alpha>0$ and $\beta\neq 0$. Then $\rho\geqq 1$ is the necessary and sufficient condition for $u(t)\in C^1([0,T])$. Next we consider the equation

$$t^\sigma\frac{du}{dt}+\alpha u=t^\gamma\ ,$$

where $\alpha>0$. Then $\gamma\geqq 1$ is the necessary and sufficient condition for $u(t)\in C^1([0,T])$.

REMARK 1.8. By changing $t$ into $-t$, we can discuss the same argument as above in the domain $t\leqq 0$. Therefore if $\sigma$ is an odd integer, we can solve the equation (A) on $-T \leqq t \leqq T$ under suitable assumptions. If fact, since $t^{\sigma}(d/dt)+A$ is invariant under $t \to -t$, the positivity of $A$ in (A-2) is also valid in $t \leqq 0$. If $\sigma$ is an even integer, $t^{\sigma}(d/dt)+A$ is transformed into $-(t^{\sigma}(d/dt)-A)$ by $t \mapsto -t$. Then the positivity in $t \geqq 0$ is not compatible with the positivity in $t \leqq 0$. In this case, we can not deal with the equation (A) on $-T \leqq t \leqq T$.

At the end of this section, we shall give an interesting example of a singular system of parabolic type. Examples of hyperbolic type will be discussed in the later sections.

EXAMPLE 1.9. Consider the following partial differential equation of the form

$$\left(t^{\sigma}\frac{\partial}{\partial t}+c-a\frac{\partial^2}{\partial x^2}+bt^{\rho}\frac{\partial^2}{\partial x^2}\right)u(t, x)=f(t, x)$$

on $[0, T]\times \boldsymbol{R}$ in the space $X=L^2(\boldsymbol{R})$, where $a \geqq 0$, $b \geqq 0$ and $c>0$. Then this equation satisfies our conditions in Theorem 1.1 and the unique solution $u(t, x)$ is given by

$$u(t, x)=\int_0^{\infty} ds \int_0^{\infty} e^{-cs}K_{a,b}(t, s, x-y, \sigma, \rho)f(\phi_{\sigma}(t, s), y)dy \ ,$$

$$K_{a,b}(t, s, x, \sigma, \rho)=\frac{1}{\sqrt{4\pi(as+b\theta_{\sigma}(t, s, \rho))}}e^{-x^2/4(as+b\theta_{\sigma}(t,s,\rho))}$$

for $f(t, x) \in C^0([0, T], L^2(\boldsymbol{R}))$. As is well known, we can give the meaning to $K_{0,0}(t, s, x, \sigma, \rho)$ as Dirac's delta function.

## 2. Energy inequality

In this section, we establish an energy inequality for a singular system of type $(\sigma, \rho)$

$$\text{(B)} \qquad t^{\sigma}\frac{du}{dt}+A(t)u-t^{\rho}B(t)u=f(t) \ , \qquad 0<t \leqq T$$

in a Hilbert space $X$. We assume the following conditions on (B):

(B-1) There exists a positive constant $a$ satisfying $\mathrm{Re}\,(A(t)u, u) \geqq a\|u\|^2$ for any $u \in D(A(t))$ and $0 \leqq t \leqq T$.

(B-2) There exists a constant $b$ satisfying $\mathrm{Re}\,(B(t)u, u) \leqq b\|u\|^2$ for any $u \in D(B(t))$ and $0 \leqq t \leqq T$.

Then we have the following:

PROPOSITION 2.1 (Energy inequality). *Let* $f(t) \in C^0([0, T], X)$ *in* (B) *and let* $u(t) \in C^0([0, T], X)$ *for a solution* $u(t)$. *Further, we assume that* $u(t) \in C^1((0, T], X)$ *and* $u(t) \in D(A(t)) \cap D(B(t))$. *Then, for* $0 \leqq t \leqq T$ *we have an energy inequality*

$$\|u(t)\| \leqq \int_0^\infty e^{-as} e^{b\theta_\sigma(t,s,\rho)} \|f(\phi_\sigma(t, s))\| ds . \tag{2.1}$$

PROOF. Clearly, for $t>0$

$$t^\sigma \frac{d}{dt} \|u(t)\|^2 = 2 \operatorname{Re} \left( t^\sigma \frac{du}{dt}(t), u(t) \right)$$
$$= -2 \operatorname{Re} (A(t)u, u) + 2t \operatorname{Re} (B(t)u, u) + 2 \operatorname{Re} (f, u) .$$

From (B-1) and (B-2), we have the inequality

$$t^\sigma \frac{d}{dt} \|u(t)\| + a\|u(t)\| - bt^\rho \|u(t)\| \leqq \|f(t)\| . \tag{2.2}$$

In the case $\sigma=1$, multiplying both sides of (2.2) by the function $t^{-1} e^{a \log t} e^{-(b/\rho) t^\rho}$ and integrating from $\varepsilon$ $(>0)$ to $t$, we obtain

$$[e^{a \log t} e^{-(b/\rho) t^\rho} \|u(t)\|]_\varepsilon^t \leqq \int_\varepsilon^t e^{a \log \tau} e^{-(b/\rho) \tau^\rho} \|f(\tau)\| \frac{1}{\tau} d\tau .$$

Since $a>0$, making $\varepsilon \to +0$ we obtain

$$\|u(t)\| \leqq \int_0^t e^{-a(\log t - \log \tau)} e^{(b/\rho)(t^\rho - \tau^\rho)} \|f(\tau)\| \frac{1}{\tau} d\tau . \tag{2.3}$$

Put $s = \log t - \log \tau$. Then (2.3) is transformed into (2.1). If $\sigma>1$, we multiply both sides of (2.2) by the function $t^{-\sigma} e^{-a\xi(t)} e^{-(b/\mu) t^\mu}$, where $\xi(t) = 1/((\sigma-1)t^{\sigma-1})$ and $\mu = \rho - \sigma + 1$. By integrating from 0 to $t$, we obtain

$$\|u(t)\| \leqq \int_0^t e^{-a(\xi(\tau) - \xi(t))} e^{(b/\mu)(t^\mu - \tau^\mu)} \|f(\tau)\| \frac{1}{\tau^\sigma} d\tau . \tag{2.4}$$

Put $s = \xi(\tau) - \xi(t)$. Then (2.4) is transformed into (2.1). Q.E.D.

THEOREM 2.2 (Uniqueness). *The solution $u(t)$ of the equation* (B) *is uniquely determined under the assumptions that $u(t) \in C^0([0, T], X) \cap C^1((0, T], X)$ and $u(t) \in D(A(t)) \cap D(B(t))$.*

PROOF. From Proposition 2.1, the proof is clear. Q.E.D.

We now investigate the influence on the energy inequality (2.1) of the perturbation by bounded operators.

PROPOSITION 2.3. *Assume that the following inequality*

$$\|u(t)\| \leqq C \int_0^\infty e^{-as} e^{b\theta_\sigma(t,s,\rho)} \|f(\phi_\sigma(t, s))\| ds \qquad (a>0) \tag{2.5}$$

*holds for a solution of the singular system of type $(\sigma, \rho)$*

$$t^{\sigma}\frac{du}{dt}+A(t)u-t^{\rho}B(t)u=f(t)\ ,\qquad 0<t\leqq T\ .$$

*Let $T(t)$ and $S(t)$ be bounded operators in $X$. We assume that $a-C\alpha>0$ and $\beta<+\infty$, where $\alpha=\sup\{\|T(t)\|; 0\leqq t\leqq T\}$ and $\beta=\sup\{\|S(t)\|; 0\leqq t\leqq T\}$. Then, for a solution $u(t)$ of the equation*

$$t^{\sigma}\frac{du}{dt}+(A(t)+T(t))u-t^{\rho}(B(t)+S(t))u=f(t)\ ,\qquad 0<t\leqq T\ , \tag{2.6}$$

*we have the following new energy inequality*

$$\|u(t)\|\leqq C\int_0^{\infty}e^{-a_1 s}e^{b_1\theta_{\sigma}(t,s,\rho)}\|f(\phi_{\sigma}(t, s))\|ds\ , \tag{2.7}$$

*where $a_1=a-C\alpha$ and $b_1=b+C\beta$.*

PROOF. From (2.5) and (2.6), we have

$$\|u(t)\|\leqq C\left[\int_0^{\infty}e^{-as}e^{b\theta_{\sigma}(t,s,\rho)}\{(\alpha+\beta\phi_{\sigma}(t, s)^{\rho})\|u(\phi_{\sigma}(t, s))\|+\|f(\phi_{\sigma}(t, s))\|\}ds\right].$$

We denote the integral term in the bracket [ ] by $\varphi(t)$. Then $\varphi(t)$ satisfies

$$t^{\sigma}\frac{d}{dt}\varphi(t)=\alpha\|u(t)\|+\beta t^{\rho}\|u(t)\|+\|f(t)\|-a\varphi(t)+bt^{\rho}\varphi(t)\ .$$

Since $\|u(t)\|\leqq C\varphi(t)$, we obtain the inequality

$$t^{\sigma}\frac{d}{dt}\varphi(t)+a_1\varphi(t)-b_1 t^{\rho}\varphi(t)\leqq\|f(t)\|\ .$$

Hence by the same argument as in the proof of Proposition 2.1 and $\|u(t)\|\leqq C\varphi(t)$, we have the energy inequality (2.7). Q.E.D.

REMARK 2.4. In the case $\sigma=\rho=1$, the analogous energy inequalities are treated by several authors, Alinhac [1], Oleinik [8], Menikoff [5] etc.

At the end of this section, we will give some examples.

EXAMPLE 2.5. Consider the equation

$$t^{\sigma}\frac{du}{dt}+A(t)u-t^{\rho}B(t)u=f(t)\ ,\qquad 0<t\leqq T\ . \tag{2.8}$$

We assume that $A(t)$ is a bounded operator satisfying $\mathrm{Re}\,(A(t)u, u)\geqq a\|u\|^2$ for some $a>0$ and $B(t)^*+B(t)$ is also a bounded operator in $X$. Then the equation (2.8) satisfies our conditions. This is a symmetric positive system in the sense of Friedrichs [2].

EXAMPLE 2.6. Consider the equation

$$t^{\sigma}\frac{\partial u}{\partial t}+A(t, x)u-t^{\rho}\sum_{j=1}^{n}B_j(t, x)\frac{\partial u}{\partial x_j}=f(t, x)\ , \tag{2.9}$$

where $A(t, x)$, $B_j(t, x)$ $(1\leqq j\leqq n)$ are $N\times N$ matrices. We assume the conditions:

(i) Let $\lambda_i(t, x)$ $(1\leqq i\leqq N)$ be eigen-values of the matrix $A(t, x)+{}^t\overline{A(t, x)}$. Then there exists a positive constant $a$ satisfying $\mathrm{Re}\,\lambda_i(t, x)\geqq a$ for any $(t, x)$ and $1\leqq i\leqq N$.

(ii) $B_j(t, x)$ is a Hermitian matrix.

Then the equation (2.9) satisfies our conditions. We call the system (2.9) *a symmetric singular hyperbolic system of type* $(\sigma, \rho)$.

EXAMPLE 2.7. Consider the equation

$$t^\sigma\frac{du}{dt}+A(t)u-t^\rho B(t)u=f(t)\ , \qquad 0<t\leqq T\ , \tag{2.10}$$

$$A(t)=-\sum_{i,j=1}^n a_{ij}(t, x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n a_i(t, x)\frac{\partial}{\partial x_i}+a_0(t, x)\ ,$$

$$B(t)=\sum_{i,j=1}^n b_{ij}(t, x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i(t, x)\frac{\partial}{\partial x_i}+b_0(t, x)$$

in the space $X=L^2(\boldsymbol{R}^n)$. We assume the following conditions:

(i) There exists a positive constant $c$ satisfying

$$\sum_{i,j=1}^n a_{ij}(t, x)\xi_i\xi_j\geqq c|\xi|^2 \qquad \text{and} \qquad \sum_{i,j=1}^n b_{ij}(t, x)\xi_i\xi_j\geqq c|\xi|^2$$

for any $(t, x)\in[0, T]\times\boldsymbol{R}^n$.

(ii) $a_{ij}(t, x)$, $a_i(t, x)$ and $b_{ij}(t, x)$ are real valued functions and $a_{ij}(t, x)=a_{ji}(t, x)$, $b_{ij}(t, x)=b_{ji}(t, x)$ hold.

(iii) There exists a positive constant $a$ satisfying

$$\left(a_0+\bar{a}_0-\sum_{i=1}^n\frac{\partial a_i}{\partial x_i}-\sum_{i,j=1}^n\frac{\partial^2 a_{ij}}{\partial x_i\partial x_j}\right)\geqq a$$

for any $(t, x)\in[0, T]\times\boldsymbol{R}^n$.

Then the equation (2.10) satisfies our conditions. This is an example of a singular evolution equation. Compare this equation with that in Example 1.9.

## 3. Existence and differentiability

In this section, we shall discuss $L^2$-theory of a singular system of type $(\sigma, \rho)$ and establish the existence and the differentiability of solutions of it. Let $X=L^2(\boldsymbol{R}^n)$ and let $\Lambda$ be a closed operator in $L^2(\boldsymbol{R}^n)$ defined by the pseudo-differential operator with a symbol $(1+|\xi^2|)^{1/2}$. We consider a singular system of the form

$$\text{(C)} \qquad \begin{cases} t^\sigma\dfrac{du}{dt}+A(t)u-t^\rho B(t)u=f(t)\ , & 0<t\leqq T\ , \\ A(t)=A_0(t) \quad \text{and} \quad B(t)=\Lambda B_0(t)-B_1(t) \end{cases}$$

in the space $X=L^2(\boldsymbol{R}^n)$. We assume the following conditions on (C):

(C-1) There exist bounded operators $A_j(t)$, $B_j(t)$ ($j=0, 1, \cdots$ and $0\leqq t\leqq T$) in $X$ such that (i) $A_j(t)u$, $B_j(t)u\in C^0([0, T], X)$ for any $u\in X$ and $j=0, 1, \cdots$, (ii) $(\Lambda A_j(t)-A_{j+1}(t))u=A_j(t)\Lambda u$ holds for any $u\in D(\Lambda)$, $0\leqq t\leqq T$ and $j=0, 1, \cdots, m-1$, and (iii) $(\Lambda B_j(t)-B_{j+1}(t))u=B_j(t)\Lambda u$ holds for any $u\in D(\Lambda)$, $0\leqq t\leqq T$ and $j=0, 1, \cdots, m$.

Then, we have the following differentiability theorem.

THEOREM 3.1 (Differentiability). *Assume that the energy inequality*

$$\|u(t)\|\leqq C\int_0^\infty e^{-as}e^{b\theta_\sigma(t,s,\rho)}\|f(\phi_\sigma(t, s))\|ds \qquad (a>0) \tag{3.1}$$

*holds for a solution of the equation* (C). *Let $u(t)$ be a solution of* (C) *satisfying $u(t)\in C^0([0, T], X)\cap C^1((0, T], X)$ and $u(t)\in D(A(t))\cap D(B(t))$. If the right hand side $f(t)$ of* (C) *satisfies the conditions that $f(t)\in D(\Lambda^m)$ and $\Lambda^m f(t)\in C^0([0, T], X)$ ($m\geqq 1$), then $u(t)$ satisfies the following conditions*: (i)$_m$ *$u(t)\in D(\Lambda^m)$ and $\Lambda^m u(t)\in C^0([0, T], X)$,* (ii)$_m$ *$\Lambda^{m-1}u(t)\in C^1((0, T], X)$,* (iii)$_m$ *$t^\sigma(d/dt)(\Lambda^{m-1}u(t))\in C^0([0, T], X)$,* (iv)$_m$ *the energy inequality*

$$\|\Lambda^k u(t)\|\leqq C\int_0^\infty e^{-as}e^{b_k\theta_\sigma(t,s,\rho)}\{\|\Lambda^k f(\phi_\sigma(t, s))\|+\sum_{j=1}^{k}c_j\|\Lambda^{k-j}u(\phi_\sigma(t, s))\|\}ds \tag{3.2}$$

*holds for $0\leqq t\leqq T$ and $k=0, 1, \cdots, m$, where $b_k$ and $c_j$ are positive constants, and* (v)$_m$ *the following differential equation*

$$t^\sigma\frac{d}{dt}\Lambda^k u(t)+\sum_{j=0}^{k}\binom{k}{j}A_j(t)\Lambda^{k-j}u(t)-t^\rho\sum_{j=0}^{k}\binom{k}{j}B_j(t)\Lambda^{k-j+1}u(t)=\Lambda^k f(t) \tag{3.3}$$

*is satisfied for $0\leqq t\leqq T$ and $k=0, 1, \cdots, m-1$.*

REMARK 3.2. From the energy inequality (3.2), we obtain

$$\|\Lambda^k u(t)\|\leqq C_k(T)\int_0^\infty e^{-a_k s}\|\Lambda^k f(\phi_\sigma(t, s))\|ds$$

for $0\leqq t\leqq T$ and $k=0, 1, \cdots, m$, where $C_k(T)$ and $a_k$ are positive constants. This is verified by an easy calculation.

Before the proof of Theorem 3.1, we prepare some lemmas.

LEMMA 3.3 (Muramatu [7]). *Let $S_0$ and $S_1$ be bounded operators in $X$ satisfying $(\Lambda S_0-S_1)u=S_0\Lambda u$ for any $u\in D(\Lambda)$. We put $T=\Lambda S_0-S_1$ and $R(\zeta)=(I-\zeta\Lambda)^{-1}$ for $\zeta\in\boldsymbol{C}$ such that* $\operatorname{Im}\zeta\neq 0$. *Then we have the following*: (i) *$\Lambda R(\zeta)=R(\zeta)\Lambda$ and this is bounded in $X$,* (ii) *$R(\zeta)S_0u=S_0R(\zeta)u+\zeta R(\zeta)S_1R(\zeta)u$ holds for any $u\in X$,* (iii) *$\Lambda R(\zeta)S_0u=S_0\Lambda R(\zeta)u+R(\zeta)S_1R(\zeta)u$ holds for any $u\in D(\Lambda)$, and* (iv) *$\Lambda R(\zeta)Tu=T\Lambda R(\zeta)u+R(\zeta)S_1\Lambda R(\zeta)u$ holds for any $u\in D(T)$.*

PROOF. Since $\mathrm{Im}\,\zeta\neq 0$, (i) is clear. For $u\in D(\Lambda)$, $RS_0u=RS_0(I-\zeta\Lambda)Ru=RS_0Ru-\zeta R(\Lambda S_0-S_1)Ru$ (by (i) and the assumption on $T$)$=(I-\zeta\Lambda)RS_0Ru+\zeta RS_1Ru=S_0Ru+\zeta RS_1Ru$. Since $D(\Lambda)$ is dense in $X$ and the both sides of (ii) are bounded, we obtain (ii). For $u\in D(\Lambda)$, $\Lambda RS_0u=\zeta^{-1}(RS_0-(I-\zeta\Lambda)RS_0)u=\zeta^{-1}(R-I)S_0u=\zeta^{-1}(S_0Ru+\zeta RS_1Ru)-\zeta^{-1}S_0u$ (by (ii)) $=\zeta^{-1}S_0(R-I)u+RS_1Ru=\zeta^{-1}S_0(R-R(I-\zeta\Lambda))u+RS_1Ru=S_0\Lambda Ru+RS_1Ru$. Hence we obtain (iii). For $u\in D(T)$, $\Lambda RTu=\Lambda R(\Lambda S_0-S_1)u=\Lambda S_0\Lambda Ru+\Lambda RS_1Ru-\Lambda RS_1u$ (by (iii)) $=T\Lambda Ru+S_1\Lambda Ru+\Lambda RS_1Ru-\Lambda RS_1u=T\Lambda Ru+\zeta^{-1}\{S_1(I-R)+(I-R)S_1R-(I-R)S_1\}u=T\Lambda Ru+\zeta^{-1}RS_1(I-R)u=T\Lambda Ru+RS_1\Lambda Ru$. This is (iv). Q.E.D.

LEMMA 3.4. *Put $R_\lambda=R(i\lambda)$ for $\lambda>0$. If $g(t)\in C^0([0,T],X)$, then $R_\lambda g(t)$ converges to $g(t)$ uniformly for $0\leqq t\leqq T$ as $\lambda\to+0$, that is,* $\max\{\|R_\lambda g(t)-g(t)\|;\ 0\leqq t\leqq T\}\to 0$ *as $\lambda\to+0$.*

PROOF. Since $X=L^2(\boldsymbol{R}^n)$, we can prove this by use of the Fourier transform. Therefore we may omit the details. Q.E.D.

PROOF OF THEOREM 3.1. The proof is by induction on $m$. Suppose that $m=1$. Then $(\mathrm{ii})_1$ and $(\mathrm{v})_1$ are trivial. Operating with $\Lambda R_\lambda$ on both sides of (C) and applying Lemma 3.3, we obtain

$$t^\sigma\frac{d}{dt}\Lambda R_\lambda u+A\Lambda R_\lambda u-t^\rho(B+R_\lambda B_1)\Lambda R_\lambda u=\Lambda R_\lambda f-R_\lambda A_1R_\lambda u\,. \tag{3.4}$$

From Proposition 2.3, we have the energy inequality

$$\|\Lambda R_\lambda u(t)\|\leqq C\int_0^\infty e^{-as}e^{b_1\theta_\sigma(t,s,\rho)}\{\|\Lambda R_\lambda f(\phi_\sigma(t,s))\|+c_1\|u(\phi_\sigma(t,s))\|\}ds\,, \tag{3.5}$$

where $b_1$ and $c_1$ are positive constants. This implies that $\|\Lambda R_\lambda u(t)\|$ is uniformly bounded for $\lambda>0$ and $0\leqq t\leqq T$. In fact, this is verified by Lemma 3.4 and the conditions: $u(t)$, $\Lambda f(t)\in C^0([0,T],X)$. On the other hand, from (3.4) and Lemma 3.3 we have

$$\begin{aligned}\left\{t^\sigma\frac{d}{dt}+A-t^\rho(B+B_1R_\lambda+B_1R_\mu)\right\}&(\Lambda R_\lambda u-\Lambda R_\mu u)\\ &=t^\rho(\lambda R_\lambda B_2R_\lambda\Lambda R_\lambda u-\mu R_\mu B_2R_\mu\Lambda R_\mu u)+(\Lambda R_\lambda f-\Lambda R_\mu f)\\ &\quad-(R_\lambda A_1R_\lambda u-R_\mu A_1R_\mu u)\,.\end{aligned} \tag{3.6}$$

We denote the right hand side of (3.6) by $f_{\lambda,\mu}(t)$. Then we have

$$\|\Lambda R_\lambda u(t)-\Lambda R_\mu u(t)\|\leqq C\int_0^\infty e^{-as}e^{b_2\theta_\sigma(t,s,\rho)}\|f_{\lambda,\mu}(\phi_\sigma(t,s))\|ds\,. \tag{3.7}$$

Since $\|\Lambda R_\lambda u(t)\|$ is uniformly bounded, from Lemma 3.4 we have $\|f_{\lambda,\mu}(t)\|\to 0$ uniformly for $0\leqq t\leqq T$ as $\lambda,\mu\to+0$. Hence (3.7) implies that $\|\Lambda R_\lambda u(t)-\Lambda R_\mu u(t)\|\to 0$

uniformly for $0\leqq t\leqq T$ as $\lambda, \mu\to+0$, that is, $\{\Lambda R_\lambda u(t)\}_{\lambda>0}$ forms a Cauchy sequence in $C^0([0, T], X)$. Therefore $\Lambda R_\lambda u(t)$ converges to a function $v(t)\in C^0([0, T], X)$ uniformly for $0\leqq t\leqq T$ as $\lambda\to+0$. Since $R_\lambda u(t)\in D(\Lambda)$, $R_\lambda u(t)\to u(t)$ as $\lambda\to+0$ and $\Lambda$ is a closed operator, we have $u(t)\in D(\Lambda)$ and $\Lambda u(t)=v(t)$. This is (i)$_1$. From (i)$_1$ and (v)$_1$, (iii)$_1$ is clear. Making $\lambda\to 0$ in (3.5), we have the energy inequality (iv)$_1$. Thus the theorem is proved when $m=1$. The assertion being supposed true for $m$, the passage to $m+1$ is established in the following way. We set $U(t)=\Lambda^{m-1}u(t)$ and

$$F(t)=-\sum_{j=1}^{m-1}\binom{m-1}{j}A_j(t)\Lambda^{m-1-j}u(t)+t^{\rho}\sum_{j=2}^{m-1}\binom{m-1}{j}B_j(t)\Lambda^{m-j}u(t)+\Lambda^{m-1}f(t)\,.$$

Then from (v)$_m$, we have the following equation

$$t^{\sigma}\frac{d}{dt}U(t)+A(t)U(t)-t^{\rho}(B(t)+(m-1)B_1(t))U(t)=F(t)\,. \tag{3.8}$$

Note that $U(t)\in D(\Lambda)$, $F(t)\in D(\Lambda^2)$ and $\Lambda U(t)$, $\Lambda^2F(t)\in C^0([0, T], X)$. Operating with $\Lambda^2R_\lambda R_\mu$ on both sides of (3.8) and making use of the energy inequality in the same way as in the proof of the case with $m=1$, we can conclude that $U(t)\in D(\Lambda^2)$ and $\Lambda^2U(t)\in C^0([0, T], X)$. Moreover we can obtain the energy inequality

$$\|\Lambda^2U(t)\|\leqq C\int_0^{\infty}e^{-as}e^{b'\theta_\sigma(t,s,\rho)}\{\|\Lambda^2F(\phi_\sigma(t, s))\|+c_1\|U(\phi_\sigma(t, s))\|+c_2\|\Lambda U(\phi_\sigma(t, s))\|\}ds\,.$$

Hence (i)$_{m+1}$ and (iv)$_{m+1}$ are proved. From (3.8), we have

$$\begin{aligned}t^{\sigma}\frac{d}{dt}\Lambda R_\lambda U=&-A_0\Lambda R_\lambda U+t^{\rho}B_0R_\lambda\Lambda^2U+t^{\rho}(R_\lambda B_1+(m-1)B_1)\Lambda R_\lambda U\\&+\Lambda R_\lambda F-R_\lambda A_1R_\lambda U+t^{\rho}(m-1)R_\lambda B_2R_\lambda U\,.\end{aligned} \tag{3.9}$$

Since the right hand side of (3.9) converges to a continuous function of $t$, $t^{\sigma}(d/dt)\Lambda U(t)$ exists and it is continuous on $[0, T]$. Hence (ii)$_{m+1}$ and (iii)$_{m+1}$ are proved. Making $\lambda\to+0$ in (3.9), we obtain the equation (v)$_{m+1}$. Q.E.D.

As for the differentiability in $t$, we have the next theorem.

THEOREM 3.5 (Differentiability in $t$). *Assume the same conditions as in Theorem* 3.1. *Further we assume the following conditions*: (i) $t^{\sigma k}\Lambda^{m-1-k}f(t)\in C^k([0, T], X)$ *for* $k=0, 1, \cdots, m-1$, (ii) $t^{\sigma k}A_j(t)u\in C^k([0, T], X)$ *for any* $u\in X$ *and* $(d/dt)^k(t^{\sigma k}A_j(t))$ *is a bounded operator in* $X$ *for* $j=0, 1, \cdots, m-1$ *and* $k=0, 1, \cdots, m-1-j$, *and* (iii) $t^{\sigma k}B_j(t)u\in C^k([0, T], X)$ *for any* $u\in X$ *and* $(d/dt)^k(t^{\sigma k}B_j(t))$ *is a bounded operator in* $X$ *for* $j=0, 1, \cdots, m-1$ *and* $k=0, 1, \cdots, m-1-j$. *Then we have* $t^{\sigma k}\Lambda^{m-k}u(t)\in C^k([0, T], X)$ *for* $k=0, 1, \cdots, m$.

REMARK 3.6. Let $A(t)$ be a bounded operator in $X$. If $A(t)u \in C^1([0, T], X)$ for any $u \in X$, then $(d/dt)A(t)$ is defined by

$$\left(\frac{d}{dt}A(t)\right)u=\lim_{h\to 0}\frac{A(t+h)u-A(t)u}{h}, \qquad u\in X.$$

$(d/dt)^k(t^{\sigma k}A_j(t))$ and $(d/dt)^k(t^{\sigma k}B_j(t))$ in (ii) and (iii) are defined in this sense.

For the proof of Theorem 3.5, we need the following lemma.

LEMMA 3.7. *Let $A(t)$ be a bounded operator in $X$ satisfying the following*: (i) $t^{\sigma k}A(t)u \in C^k([0, T], X)$ *for any* $u \in X$ *and* $k=0, 1, \cdots, m$ *and* (ii) $(d/dt)^k(t^{\sigma k}A(t))$ *is bounded for* $k=0, 1, \cdots, m$. *If $f(t)$ satisfies* $t^{\sigma k}f(t) \in C^k([0, T], X)$ *for* $k=0, 1, \cdots, m$, *then we have* $t^{\sigma k}A(t)f(t) \in C^k([0, T], X)$ *for* $k=0, 1, \cdots, m$.

PROOF. Take any $t_0 \in [0, T]$. Then, $\|A(t)f(t)-A(t_0)f(t_0)\| \leqq \|A(t)\|\|f(t)-f(t_0)\|+\|A(t)f(t_0)-A(t_0)f(t_0)\|$. Since $\|A(t)\|$ is uniformly bounded, we have $\|A(t)f(t)-A(t_0)f(t_0)\|\to 0$ as $t\to t_0$. This is the case $k=0$. Operating with $d/dt$ on $t^\sigma A(t)f(t)$ formally, we have $(d/dt)(t^\sigma A(t)f(t))=A(t)(d/dt)(t^\sigma f(t))+(d/dt)(t^\sigma A(t))\cdot f(t)-\sigma t^{\sigma-1}A(t)f(t)$. Since $\sigma\geqq 1$, we obtain $t^\sigma A(t)f(t) \in C^1([0, T], X)$. Thus the case $k=1$ is proved. Since we can prove the general case in the same way as in the case with $k=1$ or by induction on $k$, we may omit the details. Q.E.D.

PROOF OF THEOREM 3.5. The proof is by induction on $k$. The proof for $k=0$ is trivial from Theorem 3.1. Assume that the conclusion of Theorem 3.5 is true for $k=0, 1, \cdots, l$. This means $t^{\sigma k}u(t) \in C^k([0, T], X)$ for $k=0, 1, \cdots, l$. Multiplying by $t^{\sigma l}$ both sides of the equation in (v) in Theorem 3.1, we have

$$t^{\sigma(l+1)}\frac{d}{dt}(\Lambda^{m-l-1}u(t))=-\sum_{j=0}^{m-l-1}\binom{m-l-1}{j}t^{\sigma l}A_j(t)\Lambda^{m-l-j}u(t)$$
$$+t^{\rho}\sum_{j=0}^{m-l-1}\binom{m-l-1}{j}t^{\sigma l}B_j(t)\Lambda^{m-l-j}u(t)+t^{\sigma l}\Lambda^{m-l-1}f(t).$$

By the induction hypothesis and Lemma 3.7, we obtain $t^{\sigma(l+1)}(d/dt)(\Lambda^{m-l-1}u(t)) \in C^l([0, T], X)$, that is, $(d/dt)(t^{\sigma(l+1)}\Lambda^{m-l-1}u(t)) \in C^l([0, T], X)$. This means $t^{\sigma(l+1)}\Lambda^{m-l-1}u(t) \in C^{l+1}([0, T], X)$. Therefore the conclusion of Theorem 3.5 is also true for $k=l+1$. Q.E.D.

We now establish the existence theorem. For this purpose, we shall assume the following conditions on (C):

(C-2) There exists a positive constant $a$ satisfying $\mathrm{Re}\,(A(t)u, u)\geqq a\|u\|^2$ for any $u \in X$.

(C-3) There exist bounded operators $S(t)$ for $0\leqq t\leqq T$ such that $B(t)^*+B(t)=S(t)$ and $S(t)u \in C^0([0, T], X)$ for any $u \in X$.

Under these assumptions (C-1), (C-2) and (C-3), we have the following theorem.

THEOREM 3.8 (Existence). *For an arbitrary* $f(t) \in C^0([0, T], X)$ *satisfying* $f(t) \in D(A)$ *and* $Af(t) \in C^0([0, T], X)$, *there exists a unique solution* $u(t) \in C^0([0, T], X) \cap C^1((0, T], X)$ *of the equation* (C) *such that it satisfies* $u(t) \in D(A)$ *and* $Au(t) \in C^0([0, T], X)$.

PROOF. First, we assume that $f(t) \in D(A^2)$ and $A^2 f(t) \in C^0([0, T], X)$. Let $U(t)$ be a solution in $C^0([0, T], X)$ of the equation

$$t^\sigma \frac{d}{dt} U + A(0)U = f(t) , \qquad 0 < t \leqq T .$$

From Theorem 1.1, the solution $U(t)$ is given by the integral

$$U(t) = \int_0^\infty T(s) f(\phi_\sigma(t, s)) ds ,$$

where $\{T(t)\}_{t \geqq 0}$ is the semi-group generated by $-A(0)$. For a positive integer $m$, we define $u_m(t)$ as follows. For $0 \leqq t \leqq 1/m$ we define $u_m(t)$ by $U(t)$ and for $1/m \leqq t \leqq T$ we define $u_m(t)$ by

$$\begin{cases} t^\sigma \dfrac{d}{dt} u_m + A(t) u_m - t^\rho B(t) u_m = f(t) , & \dfrac{1}{m} \leqq t \leqq T , \\ u_m|_{t=1/m} = U\left(\dfrac{1}{m}\right) . & \end{cases} \tag{3.10}$$

The existence of the solution of (3.10) is guaranteed by Theorem II in Appendix. From Theorem 3.1, $u_m(t) \in D(A^2)$ and $A^2 u_m(t) \in C^0([0, T], X)$. Moreover we have the next lemma.

LEMMA 3.9. *The above sequence* $\{u_m(t)\}$ *satisfies the following conditions*: (i) $\|u_m(t)\|$, $\|Au_m(t)\|$ *and* $\|A^2 u_m(t)\|$ *are uniformly bounded for* $0 \leqq t \leqq T$ *and* $m \geqq 1$, (ii) $\|u_m(t) - u_n(t)\| \to 0$ *uniformly for* $0 \leqq t \leqq T$ *as* $m, n \to \infty$, *and* (iii) $\|Au_m(t) - Au_n(t)\| \to 0$ *uniformly for* $0 \leqq t \leqq T$ *as* $m, n \to \infty$.

PROOF OF LEMMA 3.9. Since $f(t) \in D(A^2)$ and $A^2 f(t) \in C^0([0, T], X)$, the proof of (i) is clear from Theorem 3.1. Next we shall show (ii). We put $A_j^{(m)}(t)$ and $B_j^{(m)}(t)$ as follows.

$$A_j^{(m)}(t) = \begin{cases} A_j(0) , & 0 \leqq t < \dfrac{1}{m} , \\ A_j(t) , & \dfrac{1}{m} \leqq t \leqq T \end{cases}$$

and

$$B_j^{(m)}(t)=\begin{cases}0\,, & 0\leqq t<\dfrac{1}{m}\,,\\ B_j(t)\,, & \dfrac{1}{m}\leqq t\leqq T\,.\end{cases}$$

Then $u_m(t)$ satisfies the following equation

$$t^\sigma\frac{d}{dt}u_m+A^{(m)}(t)u_m-t^\rho B^{(m)}(t)u_m=f(t)\,,\qquad 0\leqq t\neq\frac{1}{m}\leqq T\,,\tag{3.11}$$

where $A^{(m)}(t)=A_0^{(m)}(t)$ and $B^{(m)}(t)=\Lambda B_0^{(m)}(t)-B_1^{(m)}(t)$. From (3.11), we have

$$\begin{aligned}&\left(t^\sigma\frac{d}{dt}+A^{(m)}(t)-t^\rho B^{(m)}(t)\right)(u_m-u_n)\\&\qquad=-(A_0^{(m)}(t)-A_0^{(n)}(t))u_n+t^\rho(B_0^{(m)}(t)-B_0^{(n)}(t))\Lambda u_n\,.\end{aligned}\tag{3.12}$$

We denote the right hand side of (3.12) by $f_{m,n}(t)$. Then we have

$$\|u_m(t)-u_n(t)\|\leqq C\int_0^\infty e^{-as}\|f_{m,n}(\phi_\sigma(t,s))\|ds\tag{3.13}$$

for $0\leqq t\leqq T$. If $m>n$, $f_{m,n}(t)$ is expressed in the form

$$f_{m,n}(t)=\begin{cases}0\,, & 0\leqq t<\dfrac{1}{m}\,,\\ -(A_0(t)-A_0(0))U(t)+t^\rho B_0(t)\Lambda u_n(t)\,, & \dfrac{1}{m}\leqq t<\dfrac{1}{n}\,,\\ 0\,, & \dfrac{1}{n}\leqq t\leqq T\,.\end{cases}$$

Since $\rho>0$, we have $\|f_{m,n}(t)\|\to 0$ uniformly for $0\leqq t\leqq T$ as $m>n\to\infty$. Combining this fact with (3.13), we can easily obtain (ii). Also, from (3.11) we have

$$\begin{aligned}&\left(t^\sigma\frac{d}{dt}+A^{(m)}(t)-t^\rho(B^{(m)}(t)+B_1^{(m)}(t))\right)(\Lambda u_m-\Lambda u_n)\\&\qquad=-(A_0^{(m)}(t)-A_0^{(n)}(t))\Lambda u_n-(A_1^{(m)}(t)-A_1^{(n)}(t))u_n\\&\qquad\quad+t^\rho(B_0^{(m)}(t)-B_0^{(n)}(t))\Lambda^2u_n+t^\rho(B_1^{(m)}(t)-B_1^{(n)}(t))\Lambda u_n\,.\end{aligned}$$

Since $\|\Lambda^2u_n(t)\|$ is uniformly bounded, the above argument is also valid in this case. Thus we obtain (iii). Q.E.D.

We now continue the proof of Theorem 3.8. The above lemma implies that $\{u_m(t)\}$ and $\{\Lambda u_m(t)\}$ form Cauchy sequences in $C^0([0,T],X)$. Therefore, there exists a function $u(t)\in C^0([0,T],X)$ such that it satisfies the following: (i) $u(t)\in D(\Lambda)$ and $\Lambda u(t)\in C^0([0,T],X)$, and (ii) $\|\Lambda u_m(t)-\Lambda u(t)\|\to 0$ uniformly for $0\leqq t\leqq T$ as $m\to\infty$. Moreover, making $m\to\infty$ in (3.11) we obtain

$$t^\sigma\frac{du}{dt}+A(t)u-t^\rho B(t)u=f(t)\,,\qquad 0<t\leqq T\,.$$

Thus we have proved this theorem when $f(t) \in D(\Lambda^2)$ and $\Lambda^2 f(t) \in C^0([0, T], X)$. The proof of the general case is established in the following way. Let $\{f^{(n)}(t)\}$ be a sequence in $C^0([0, T], X)$ such that it satisfies $f^{(n)}(t) \in D(\Lambda^2)$, $\Lambda^2 f^{(n)}(t) \in C^0([0, T], X)$ and $\|\Lambda f^{(n)}(t) - \Lambda f(t)\| \to 0$ uniformly for $0 \leqq t \leqq T$ as $n \to \infty$. Then we can find a solution $u^{(n)}(t)$ of the equation

$$t^\sigma \frac{d}{dt} u^{(n)} + A(t) u^{(n)} - t^\rho B(t) u^{(n)} = f^{(n)}(t) , \qquad 0 < t \leqq T . \tag{3.14}$$

Since $f^{(n)}(t) \in D(\Lambda^2)$ and $\Lambda^2 f^{(n)}(t) \in C^0([0, T], X)$ are valid, we have $u^{(n)}(t) \in D(\Lambda^2)$ and $\Lambda^2 u^{(n)}(t) \in C^0([0, T], X)$. From (3.14), we have

$$\left(t^\sigma \frac{d}{dt} + A(t) - t^\rho B(t)\right)(u^{(m)} - u^{(n)}) = f^{(m)}(t) - f^{(n)}(t) .$$

Using the energy inequality for $u^{(m)} - u^{(n)}$ and making $m, n \to \infty$, we obtain $\|u^{(m)}(t) - u^{(n)}(t)\| \to 0$ uniformly for $0 \leqq t \leqq T$ as $m, n \to \infty$. Furthermore we can obtain $\|\Lambda u^{(m)}(t) - \Lambda u^{(n)}(t)\| \to 0$ uniformly for $0 \leqq t \leqq T$ as $m, n \to \infty$ by the same way. Hence we can find a function $u(t) \in C^0([0, T], X)$ such that it satisfies the following: (i) $u(t) \in D(\Lambda)$ and $\Lambda u(t) \in C^0([0, T], X)$, and (ii) $\|\Lambda u^{(n)}(t) - \Lambda u(t)\| \to 0$ uniformly for $0 \leqq t \leqq T$ as $n \to \infty$. Making $n \to \infty$ in (3.14), we obtain

$$t^\sigma \frac{du}{dt} + A(t) u - t^\rho B(t) u = f(t) , \qquad 0 < t \leqq T .$$

Thus the existence of the solution is proved. The uniqueness is clear from the energy inequality. Q.E.D.

REMARK 3.10. If $A_j(t), B_j(t) \in C^0([0, T], \mathscr{L}(X))$ hold, we can prove Theorem 3.8 more easily. Here $\mathscr{L}(X)$ means the Banach space of bounded operators in $X$. However, for the application to the study of partial differential equations in my subsequent papers, such assumptions are too strong. In fact, we shall apply Theorem 3.8 to operators such as

$$(A(t)u)(t, x) = \int e^{i(x,\xi)} \left(\frac{1}{1 + t|\xi|^2}\right) \hat{u}(\xi) d\xi .$$

This operator $A(t)$ is bounded in $X$ and $A(t)u \in C^0([0, T], X)$ for any $u \in X$, but $A(t) \notin C^0([0, T], \mathscr{L}(X))$.

REMARK 3.11. It is clear that $\Lambda^k u(t) \in C^l([0, T], X)$ is equivalent to $u(t) \in C^l([0, T], H^k(\boldsymbol{R}^n))$, where $H^k(\boldsymbol{R}^n)$ is Sobolev's space on $\boldsymbol{R}^n$. Therefore, we can obtain a classical solution by use of Sobolev's lemma.

EXAMPLE 3.12. Consider the following symmetric singular hyperbolic system of the form

$$t^{\sigma}\frac{\partial u}{\partial t}+A(t, x)u-t^{\rho}\sum_{j=1}^{n}B_j(t, x)\frac{\partial u}{\partial x_j}=f(t, x)$$

on $[0, T]\times \boldsymbol{R}^n$, where $A(t, x)$ and $B_j(t, x)$ $(1\leqq j\leqq n)$ are $N\times N$ matrices whose coefficients are $C^{\infty}$ functions with bounded derivatives on $[0, T]\times \boldsymbol{R}^n$. We assume the same conditions as in Example 2.6. Then the theorems in this section are valid for this equation.

## 4. Generalization

In this section, we shall generalize the results in Section 3 to a certain class of symmetrizable positive systems. Let us consider a singular system of the form

$$(\mathrm{D})_{\alpha} \qquad \begin{cases} t^{\sigma}\dfrac{du}{dt}+(\alpha+A(t))u-t^{\rho}B(t)u=f(t)\ , & 0<t\leqq T\ , \\ A(t)=A_0(t) \quad \text{and} \quad B(t)=\Lambda B_0(t)-B_1(t) \end{cases}$$

in the space $X=L^2(\boldsymbol{R}^n)$, where $\alpha$ is a real parameter. We assume the following conditions on $(\mathrm{D})_{\alpha}$:

(D-1) $A(t)$ and $B(t)$ satisfy the same condition as (C-1).

(D-2) $A_0(t)^*u$, $B_0(t)^*u\in C^0([0, T], X)$ for any $u\in X$. (See Appendix.)

(D-3) (Symmetrizability of $B(t)$). $B(t)$ satisfies the following conditions. There exist linear operators $N(t)$, $D(t)$, $S(t)$, $T(t)$ and $\Lambda(t)$ in $X$ for $0\leqq t\leqq T$ such that they satisfy the conditions: (i) $N(t)$, $S(t)$, $T(t)$, $\Lambda(t)$, $t^{\sigma}N_t'(t)$, $t^{\sigma}\Lambda_t'(t)$ and $\Lambda(t)B(t)$ are bounded operators in $X$ and their operator norms are uniformly bounded for $0\leqq t\leqq T$, (ii) $D(t)$ is a densely defined, closed operator in $X$ such that $D(t)^*+D(t)=S(t)$, (iii) $N(t)B(t)=D(t)N(t)+T(t)$, and (iv) there exist positive constants $\beta$, $c_1$ and $c_2$ such that $c_1\|u\|\leqq\|N(t)u\|+\beta\|\Lambda(t)u\|\leqq c_2\|u\|$ is valid for any $u\in X$ and $0\leqq t\leqq T$.

(D-4) (Symmetrizability of $B(t)^*$). $B(t)^*$ satisfies the same conditions as in (D-3).

Under these assumptions, we define $\alpha_0$ by

$$\alpha_0=\sup_{0\leqq t\leqq T}\left\{\frac{1}{c_1}(\|t^{\sigma}N_t'\|+\|NA\|)+\frac{\beta}{c_1}(\|t^{\sigma}\Lambda_t'\|+\|\Lambda A\|)\right\}.$$

PROPOSITION 4.1 (Energy inequality). *If $\alpha>\alpha_0$, the equation $(\mathrm{D})_{\alpha}$ has an energy inequality. Namely, if $u(t)$ and $f(t)$ satisfy the conditions that $f(t)\in C^0([0, T], X)$, $u(t)\in C^0([0, T], X)\cap C^1((0, T], X)$ and $u(t)\in D(B(t))$, then the solution $u(t)$ satisfies the following inequality*

$$\|u(t)\|\leqq C\int_0^{\infty}e^{-\alpha s}e^{b\theta_{\sigma}(t,s,\rho)}\|f(\phi_{\sigma}(t, s))\|ds \tag{4.1}$$

*for* $0 \leqq t \leqq T$, *where* $b$ *and* $C$ *are positive constants and* $a=\alpha-\alpha_0$.

PROOF. Operating with $N(t)$ on both sides of $(D)_\alpha$ and using the conditions in (D-3), we have

$$t^\sigma \frac{d}{dt}(Nu)-t^\rho N_t' u+\alpha(Nu)+NAu-t^\rho D(Nu)-t^\rho Tu=Nf.$$

Hence, we obtain

$$t^\sigma \frac{d}{dt}\|Nu\|+\alpha\|Nu\| \leqq (\|t^\sigma N_t'\|+\|NA\|)\|u\|+t^\rho(b_1\|Nu\|+b_2\|u\|)+\|Nf\| \tag{4.2}$$

for some constants $b_1$ and $b_2$. Moreover, operating with $\Delta(t)$ on both sides of $(D)_\alpha$, we obtain

$$t^\sigma \frac{d}{dt}\|\Delta u\|+\alpha\|\Delta u\| \leqq (\|t^\sigma \Delta_t'\|+\|\Delta A\|)\|u\|+t^\rho b_3\|u\|+\|\Delta f\| \tag{4.3}$$

for some constant $b_3$. We define the norm $|||u|||$ by $\|Nu\|+\beta\|\Delta u\|$. Then from (4.2), (4.3) and (iv) in (D-3), we have

$$t^\sigma \frac{d}{dt}|||u|||+(\alpha-\alpha_0)|||u||| \leqq t^\rho b|||u|||+|||f|||$$

for some constant $b$. Since $\alpha-\alpha_0>0$, the argument in the proof of Proposition 2.1 is also valid in this case. Thus, we obtain

$$|||u(t)||| \leqq \int_0^\infty e^{-as} e^{b\theta_\sigma(t,s,\rho)} |||f(\phi_\sigma(t,s))|||ds \tag{4.4}$$

for $0 \leqq t \leqq T$. Since the norm $|||\cdot|||$ is equivalent to the norm $\|\cdot\|$, (4.4) immediately leads us to (4.1). Q.E.D.

THEOREM 4.2 (Differentiability). *Let* $u(t)$ *be a solution of the equation* $(D)_\alpha$. *If* $\alpha>\alpha_0$, $f(t) \in D(\Lambda^m)$ *and* $\Lambda^m f(t) \in C^0([0,T],X)$ *are valid, then* $u(t)$ *satisfies the conditions* $(i)_m \sim (v)_m$ *in Theorem* 3.1.

PROOF. Trivial. Q.E.D.

THEOREM 4.3 (Differentiability in $t$). *Assume the same conditions as in Theorem* 3.5. *If* $\alpha>\alpha_0$, *then we have* $t^{\sigma k}\Lambda^{m-k}u(t) \in C^k([0,T],X)$ *for* $k=0,1,\cdots,m$.

PROOF. Trivial. Q.E.D.

THEOREM 4.4 (Existence). *We assume* $\alpha>\max\{\|A(0)\|,\alpha_0\}$. *Then for an arbitrary* $f(t) \in C^0([0,T],X)$ *satisfying* $f(t) \in D(\Lambda)$ *and* $\Lambda f(t) \in C^0([0,T],X)$, *there exists a unique solution* $u(t) \in C^0([0,T],X) \cap C^1((0,T],X)$ *of the equation* $(D)_\alpha$ *such that it satisfies* $u(t) \in D(\Lambda)$ *and* $\Lambda u(t) \in C^0([0,T],X)$.

PROOF. From Theorem III in Appendix, the argument in the proof of Theorem 3.8 are also valid in this case. Therefore the proof is clear. Note that the condition $\alpha>\|A(0)\|$ is used to solve the equation

$$t^{\sigma}\frac{d}{dt}U(t)+(\alpha+A(0))U(t)=f(t)\ , \qquad 0<t\leqq T\ .$$

For the details, follow the proof of Theorem 3.8. Q.E.D.

REMARK 4.5. The above argument is a modification of that in Mizohata [6]. He has treated the non-characteristic Cauchy problems.

REMARK 4.6. The real parameter $\alpha$ will play an important role in the application to the study of partial differential equations, which will be done in my subsequent papers. A part of its study is announced in Tahara [11].

EXAMPLE 4.7. Consider the following equation

$$t^{\sigma}\frac{\partial u}{\partial t}+(\alpha+A(t,x))u-t^{\rho}\sum_{j=1}^{n}B_j(t,x)\frac{\partial u}{\partial x_j}=f(t,x)$$

on $[0,T]\times\boldsymbol{R}^n$, where $A(t,x)$ and $B_j(t,x)$ $(1\leqq j\leqq n)$ are $N\times N$ matrices whose coefficients are $C^{\infty}$ functions with bounded derivatives on $[0,T]\times\boldsymbol{R}^n$. We assume the following conditions: (i) the eigen-values $\lambda_i(t,x,\xi)$ $(1\leqq i\leqq N)$ of the matrix $\sum_{j=1}^{n}B_j(t,x)\xi_j$ are real valued functions on $[0,T]\times\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0)$, and (ii) there exists a positive constant $c$ such that $|\lambda_i(t,x,\xi)-\lambda_j(t,x,\xi)|\geqq c|\xi|$ holds for any $(t,x,\xi)\in[0,T]\times\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0)$ and $i\neq j$. Then the above equation satisfies our conditions.

## 5. Asymptotic behaviour as $t\to+0$

In the previous sections, we have treated only the solutions with no singularities at $t=0$. However, if we admit a solution with some singularities at $t=0$, the situation comes to be somewhat different. For example, the general solution of the equation

$$t^{\sigma}\frac{du}{dt}+\alpha u=f(t) \qquad (\alpha>0)\ ,\quad 0<t\leqq T \tag{5.1}$$

is expressed in the form

$$u(t)=c\Phi_{\sigma}(t,\alpha)+\int_0^{\infty}e^{-\alpha s}f(\phi_{\sigma}(t,s))ds\ , \tag{5.2}$$

where $c$ is an arbitrary constant and $\Phi_{\sigma}(t,\alpha)$ is defined by

$$\Phi_{\sigma}(t,\alpha)=\begin{cases}\dfrac{1}{t^{\alpha}}\ , & \text{when } \sigma=1\ ,\\ e^{\alpha/((\sigma-1)t^{\sigma-1})}\ , & \text{when } \sigma>1\ .\end{cases}$$

The first term of the right hand side of (5.2) is a homogeneous solution of (5.1) and the second term is nothing but the solution discussed in Sections 3 and 4. In general, $u(t)$ has a singularity at $t=0$ and its asymptotic behaviour as $t\to+0$ is expressed in the form

$$u(t)=O(\Phi_\sigma(t,\alpha)) \quad \text{as} \quad t\to+0 .$$

This asymptotic property can be easily generalized to our singular hyperbolic systems in the following way. Let us consider a singular hyperbolic system of the form

$$\text{(E)} \qquad \begin{cases} t^\sigma \dfrac{du}{dt}+A(t)u-t^\rho B(t)u=f(t) , & 0<t\leqq T , \\ u|_{t=T}=u_T \end{cases}$$

in the space $X=L^2(\boldsymbol{R}^n)$. We assume that $A(t)$ and $B(t)$ satisfy the same conditions as in Section 3 or 4. Then from the theorems in Appendix, for an arbitrary $f(t)\in C^0((0,T],X)$ satisfying $f(t)\in D(\Lambda)$ and $\Lambda f(t)\in C^0((0,T],X)$ and an arbitrary $u_T\in D(\Lambda)$, we can find a unique solution $u(t)\in C^1((0,T],X)$ of the equation (E) such that it satisfies $u(t)\in D(\Lambda)$ and $\Lambda u(t)\in C^0((0,T],X)$. Then, we have:

PROPOSITION 5.1. *Let $\psi(t)$ be a decreasing function on $(0,T]$ such that $\psi(t)>0$ is valid for $t>0$. We assume the same conditions as in Section 4. Then if $\|f(t)\|=O(\psi(t))$ as $t\to+0$, we have $\|u(t)\|=O(\Phi_\sigma(t,\alpha)\psi(t))$ as $t\to+0$ for any positive $\alpha$ such that $\alpha\geqq\alpha_0$.*

PROOF. We define $|||u|||$ by $\|Nu\|+\beta\|\Lambda u\|$. By the same argument as in the proof of Proposition 4.1, we have

$$-t^\sigma\frac{d}{dt}|||u(t)|||-\alpha|||u(t)|||-t^\rho b|||u(t)|||\leqq|||f(t)||| \tag{5.3}$$

for some positive constant $b$. In the case $\sigma=1$, multiplying both sides of (5.3) by the function $t^{-1}e^{\alpha\log t}e^{(b/\rho)t^\rho}$ and integrating from $t$ to $T$, we obtain

$$[-(e^{\alpha\log t}e^{(b/\rho)t^\rho}|||u(t)|||)]_t^T\leqq\int_t^T e^{\alpha\log\tau}e^{(b/\rho)\tau^\rho}|||f(\tau)|||\frac{1}{\tau}d\tau . \tag{5.4}$$

Multiplying the function $1/\psi(t)$ on both sides of (5.4) and using the fact that $(1/\psi(t))|||f(\tau)|||=(\psi(\tau)/\psi(t))|||(f(\tau)/\psi(\tau)|||\leqq M$ is valid for some constant $M$, we obtain

$$\begin{aligned}\frac{1}{\psi(t)}e^{\alpha\log t}|||u(t)|||\leqq&\frac{1}{\psi(t)}e^{\alpha\log T}e^{(b/\rho)T^\rho}|||u_T||| \\ &+M\int_{-\log T}^{-\log t}e^{-\alpha s}e^{(b/\rho)e^{-\rho s}}ds ,\end{aligned}$$

where $s=-\log\tau$. Since $\alpha>0$, we have $(1/\psi(t))e^{\alpha\log t}|||u(t)|||=O(1)$ as $t\to+0$. In the case $\sigma>1$, we multiply both sides of (5.3) by the function $t^{-\sigma}e^{-\alpha\xi(t)}e^{(b/\mu)t^{\mu}}$, where $\xi(t)=1/((\sigma-1)t^{\sigma-1})$ and $\mu=\rho-\sigma+1$. Then the above argument is also valid in this case. Thus, we obtain

$$\frac{|||u(t)|||}{\psi(t)\Phi_\sigma(t,\alpha)}\leqq\frac{1}{\psi(t)\Phi_\sigma(T,\alpha)}e^{(b/\mu)T^{\mu}}|||u_T|||$$
$$+M\int_{\xi(T)}^{\xi(t)}e^{-\alpha s}e^{(b/\mu)((\sigma-1)s)^{-\mu/(\sigma-1)}}ds\,.$$

Since $\alpha>0$, we have $|||u(t)|||=O(\Phi_\sigma(t,\alpha)\psi(t))$ as $t\to+0$. Q.E.D.

COROLLARY 5.2. *If* $f(t)\equiv0$, *then we have* $\|u(t)\|=O(\Phi_\sigma(t,\alpha))$ *as* $t\to+0$.

REMARK 5.3. In the case $\sigma=1$, the order $\alpha$ of the singularity of $\Phi_\sigma(t,\alpha)=t^{-\alpha}$ at $t=0$ is closely related to the values of so-called characteristic exponents, which are defined for the equations with regular singularities. See Tahara [9] [10].

REMARK 5.4. Corollary 5.2 says that any homogeneous solution of the equation (E) on $(0,T]$ can be extended to a function on $[-T,T]$ as a distribution (when $\sigma=1$) or a ultradistribution of class $(\sigma/(\sigma-1))$ (when $\sigma>1$) on $[-T,T]$ with values in $X$. See Komatsu [3].

## Appendix

The Cauchy problem for the hyperbolic system of the first order of the form

$$\text{(F)}\qquad\begin{cases}\dfrac{du}{dt}-A(t)u=f(t)\,, & 0\leqq t\leqq T\,,\\ u|_{t=0}=u_0\,, & \\ A(t)=\Lambda A_0(t)-A_1(t) \quad\text{and} & X=L^2(\boldsymbol{R}^n)\end{cases}$$

has been investigated in details by many authors under the assumption that $A(t)$ is symmetrizable, modulo bounded operators in $X$, and we can refer to Mizohata [6], Muramatu [7] for the fundamental results of this equation. However, we shall remark the following: the condition $A_0(t)\in C^0([0,T],\mathscr{L}(X))$ is assumed and this assumption is too strong to utilize in our theory. For the application to our case, we must establish the existence theorem under the assumption that $A_j(t)u\in C^0([0,T],X)$ is valid for any $u\in X$. (See Remark 3.10). Since we have no good references to suit our purpose, we shall sketch the discussion of this case in brief in this section. Our assumptions are as follows.

(F-1) There exist bounded operators $A_j(t)$ ($j=0,1,\cdots$ and $0\leqq t\leqq T$) in $X$ such that (i) $A_j(t)u\in C^0([0,T],X)$ for any $u\in X$ and $j=0,1,\cdots,m+1$, and (ii) $(\Lambda A_j(t)-A_{j+1}(t))u=A_j(t)\Lambda u$ for any $u\in D(\Lambda)$ and $j=0,1,\cdots,m$.

(F-2) There exist bounded operators $S(t)$ for $0\leqq t\leqq T$ such that $A(t)^*+A(t)=S(t)$ and $S(t)u\in C^0([0, T], X)$ for any $u\in X$.

REMARK. From the above assumptions, we have $A_0(t)^*u\in C^0([0, T], X)$ for any $u\in X$. In fact, $-(\Lambda A_0(t)-A_1(t))u+S(t)u=A(t)^*u=\Lambda A_0(t)^*u$ for $u\in D(\Lambda)$. Hence we have $A_0(t)^*u=-A_0(t)u+\Lambda^{-1}A_1(t)u+\Lambda^{-1}S(t)u$. This implies $A_0(t)^*u\in C^0([0, T], X)$ for any $u\in X$. This fact will play an important role in the proof of Theorem II.

THEOREM I (Differentiability, Muramatu [7]). *Let $u(t)$ be a solution of the equation* (F) *satisfying $u(t)\in C^1([0, T], X)$ and $u(t)\in D(A(t))$. If the right hand side $f(t)$ of* (F) *satisfies the conditions that $f(t)\in D(\Lambda^m)$ and $\Lambda^m f(t)\in C^0([0, T], X)$ $(m\geqq 1)$, then $u(t)$ satisfies the following conditions:* $(\mathrm{i})_m$ *$u(t)\in D(\Lambda^m)$ and $\Lambda^m u(t)\in C^0([0, T], X)$,* $(\mathrm{ii})_m$ *$\Lambda^{m-1}u(t)\in C^1([0, T], X)$,* $(\mathrm{iii})_m$ *the energy inequality*

$$\|\Lambda^k u(t)\|\leqq C[e^{\gamma_k t}\|\Lambda^k u_0\|+\int_0^t e^{\gamma_k(t-s)}\{\|\Lambda^k f(s)\|+\sum_{j=1}^{k-1}c_j\|\Lambda^{k-j+1}u(s)\|\}ds] \tag{6.1}$$

*holds for $0\leqq t\leqq T$ and $k=0, 1, \cdots, m$, where $C$, $\gamma_k$ and $c_j$ are positive constants, and* $(\mathrm{iv})_m$ *the following differential equation*

$$\frac{d}{dt}\Lambda^k u(t)-\sum_{j=0}^{k}\binom{k}{j}A_j(t)\Lambda^{k-j+1}u(t)=\Lambda^k f(t)\ , \qquad 0\leqq t\leqq T \tag{6.2}$$

*is satisfied for $k=0, 1, \cdots, m-1$.*

PROOF. We can prove this theorem by the same argument as in the proof of Theorem 3.1. For the details, see Muramatu [7]. Q.E.D.

THEOREM II (Existence). *For an arbitrary $f(t)\in C^0([0, T], X)$ satisfying $f(t)\in D(\Lambda)$ and $\Lambda f(t)\in C^0([0, T], X)$ and an arbitrary $u_0\in D(\Lambda)$, there exists a unique solution $u(t)\in C^1([0, T], X)$ of the equation* (F) *such that it satisfies $u(t)\in D(\Lambda)$ and $\Lambda u(t)\in C^0([0, T], X)$.*

PROOF. By the argument in the proof of Theorem 3.8, we may assume that $f(t)\in D(\Lambda^2)$ and $\Lambda^2 f(t)\in C^0([0, T], X)$. For a positive integer $m$. we define $u_m(t)$ as follows. Let $u_m^{(k)}(t)$ be a solution of the equation

$$\begin{cases}\dfrac{d}{dt}u_m^{(k)}-A\left(\dfrac{k}{m}\right)u_m^{(k)}=f(t)\ , & \dfrac{k}{m}\leqq t<\dfrac{k+1}{m}\ ,\\ u_m^{(k)}|_{t=k/m}=u_m^{(k-1)}\left(\dfrac{k}{m}-0\right)\ , & \end{cases} \tag{6.3}$$

where $k=0, 1, \cdots$ and $u_m^{(-1)}(-0)=u_0$. By the theory of semi-groups, we can find a solution $u_m^{(k)}(t)$ inductively on $k$. For $k/m\leqq t<(k+1)/m$, we define $u_m(t)$ by $u_m^{(k)}(t)$. Then, from Theorem I we obtain $u_m(t)\in D(\Lambda^2)$ and $\Lambda^2 u_m(t)\in C^0([0, T], X)$. Moreover, we have

LEMMA II-1. *The above sequence* $\{u_m(t)\}$ *satisfies the following*: (i) $\|u_m(t)\|$, $\|\Lambda u_m(t)\|$ *and* $\|\Lambda^2 u_m(t)\|$ *are uniformly bounded for* $0\leqq t\leqq T$ *and* $m\geqq 1$, *and* (ii) *there exists a positive constant c such that* $\|u_m(t)-u_m(t')\|\leqq c|t-t'|$ *and* $\|\Lambda u_m(t)-\Lambda u_m(t')\|\leqq c|t-t'|$ *hold for* $0\leqq t, t'\leqq T$ *and* $m\geqq 1$.

PROOF OF LEMMA II-1. (i) is clear from the energy inequality (6.1). We shall show (ii). We define $A_j^{(m)}(t)$ by $A_j^{(m)}(t)=A_j(k/m)$ for $k/m\leqq t<(k+1)/m$. Then $u_m(t)$ satisfies the equation

$$\frac{d}{dt}u_m - A_0^{(m)}(t)\Lambda u_m = f(t)\ , \qquad 0\leqq t\neq \frac{1}{m}\ , \frac{2}{m}\ , \cdots \leqq T\ . \tag{6.4}$$

Since $\|\Lambda u_m(t)\|$ is uniformly bounded, $\|(d/dt)u_m(t)\|$ is also uniformly bounded. Hence, we have

$$\|u_m(t)-u_m(t')\|\leqq \left|\int_{t'}^{t}\left\|\frac{d}{dt}u_m(\tau)\right\|d\tau\right|\leqq c|t-t'|\ .$$

The proof of the rest part of (ii) is the same as above. Q.E.D.

Let $J=\{t_k\}_{k\geqq 1}$ be a dense subset of $[0, T]$. Since $\{\|u_m(t_k)\|\}$ is uniformly bounded, we can take a subsequence $\{u_{m'}(t)\}$ of $\{u_m(t)\}$ and a sequence $\{u(t_k)\}$ such that $u_{m'}(t_k)$ converges weakly to $u(t_k)$ as $m'\to\infty$ for any $t_k\in J$. From Lemma II-1, we have

$$\|u(t_k)-u(t_{k'})\|\leqq c|t_k-t_{k'}| \tag{6.5}$$

for any $t_k, t_{k'}\in J$. Now, we shall construct a solution $u(t)$. For any $t\in[0, T]$, we can take a subsequence $\{t_{k_n}\}_{n\geqq 1}$ of $J$ such that $t_{k_n}\to t$ as $n\to\infty$. Then (6.5) implies that $\{u(t_{k_n})\}_{n\geqq 1}$ is a Cauchy sequence in $X$. Therefore we can find $u(t)\in X$ such that $u(t_{k_n})\to u(t)$ as $n\to\infty$. Note that $u(t)$ is independant of the choice of $\{t_{k_n}\}_{n\geqq 1}$. Thus we have defined $u(t)$ on $[0, T]$. We shall show that $u(t)$ is in fact a solution.

LEMMA II-2. $u(t)$ *satisfies the following conditions*: (i) $\|u(t)-u(t')\|\leqq c|t-t'|$ *for* $0\leqq t, t'\leqq T$, (ii) $u_{m'}(t)$ *converges weakly to* $u(t)$ *as* $m'\to\infty$ *for a fixed t*, (iii) $u(t)\in D(\Lambda)$ *and* $\Lambda u_{m'}(t)$ *converges weakly to* $\Lambda u(t)$ *as* $m'\to\infty$ *for a fixed t, and* (iv) $\|\Lambda u(t)-\Lambda u(t')\|\leqq c|t-t'|$ *for* $0\leqq t, t'\leqq T$.

PROOF OF LEMMA II-2. (i) is clear from (6.5). If $t\in J$, then (ii) is also clear from the definition of $u(t)$. For a general $t$, we take a sequence $\{t_{k_n}\}_{n\geqq 1}\subset J$ such that $t_{k_n}\to t$ as $n\to\infty$. Then for any $g\in X$, we have $|(u_{m'}(t)-u(t), g)|\leqq|(u_{m'}(t)-u_{m'}(t_{k_n}), g)|+|(u_{m'}(t_{k_n})-u(t_{k_n}), g)|+|(u(t_{k_n})-u(t), g)|\leqq 2c\|g\||t_{k_n}-t|+|(u_{m'}(t_{k_n})-u(t_{k_n}), g)|$. Hence, we obtain (ii). Since $\{\Lambda u_m(t)\}$ satisfies the same conditions as $\{u_m(t)\}$ in Lemma II-1, we can discuss the same argument as above on $\{\Lambda u_m(t)\}$. Therefore we can take a subsequence $\{\Lambda u_{m'}(t)\}$ of $\{\Lambda u_m(t)\}$ and a function $v(t)$ such that $\|v(t)-v(t')\|\leqq c|t-t'|$ for $0\leqq t, t'\leqq T$ and that $\Lambda u_{m'}(t)$ converges weakly to $v(t)$ as $m'\to\infty$ for a fixed $t$. Thus, it is sufficient to show that $u(t) \in D(\Lambda)$ and $\Lambda u(t)=v(t)$. For $g \in D(\Lambda^*)$, we have $(\Lambda u_{m'}(t), g)=(u_{m'}(t), \Lambda^* g)$. Making $m' \to \infty$, we have $(v(t), g)=(u(t), \Lambda^* g)$ for $g \in D(\Lambda^*)$. Since $\Lambda$ is a self-adjoint operator in $X$, we obtain $u(t) \in D(\Lambda^{**})= D(\Lambda)$ and $\Lambda u(t)=v(t)$. Q.E.D.

By Lemma II-2, $u(t) \in D(\Lambda)$ and $\Lambda u(t) \in C^0([0, T], X)$ are proved. Next, we will show that $u(t)$ satisfies the equation (F). Take any $g \in X$ and fix it from now on. We put $h_m(t)=(u_m(t), g)$, $h(t)=(u(t), g)$, $w_m(t)=(A_0^{(m)}(t)\Lambda u_m(t)+f(t), g)$ and $w(t)=(A_0(t)\Lambda u(t)+f(t), g)$, where we write $m$ instead of $m'$ for simplicity. They are complex valued functions on $[0, T]$. From Lemma II-2, $h_m(t)$ converges to $h(t)$ as $m \to \infty$ for a fixed $t$. Since $(A_0^{(m)}(t)\Lambda u_m(t), g)=(A_0(t)\Lambda u_m(t), g)+((A_0^{(m)}(t)- A_0(t))\Lambda u_m(t), g)=(\Lambda u_m(t), A_0(t)^* g)+(\Lambda u_m(t), (A_0^{(m)}(t)^*-A_0(t)^*)g)$ and $|(\Lambda u_m(t), (A_0^{(m)}(t)^*- A_0(t)^*)g)| \leqq \|\Lambda u_m(t)\| \|A_0^{(m)}(t)^* g - A_0(t)^* g\|$, $(A_0^{(m)}(t)\Lambda u_m(t), g)$ converges to $(A_0(t)\Lambda u(t), g)$ as $m \to \infty$ for a fixed $t$. Here we used the fact that $A_0(t)^* g \in C^0([0, T], X)$. Hence, $w_m(t)$ also converges to $w(t)$ as $m \to \infty$ for a fixed $t$. From the equation (6.4), we have

$$h_m(t)=h_m(0)+\int_0^t w_m(s)ds .$$

Making $m \to \infty$ in both sides and applying Lebesgue's convergence theorem, we obtain

$$h(t)=h(0)+\int_0^t w(s)ds .$$

Note that $w_m(t)$ is a complex valued function and it is continuous on $[0, T] \setminus \{k/m; k=1, 2, \cdots\}$. Since $g$ is an arbitrary element in $X$, we obtain

$$u(t)=u_0+\int_0^t (A_0(s)\Lambda u(s)+f(s))ds$$

for $0 \leqq t \leqq T$. This implies that (i) $u(t) \in C^1([0, T], X)$, (ii) $u(0)=u_0$, and (iii) $u(t)$ satisfies the equation (F). Thus, the existence of a solution is proved. The uniqueness is clear from the energy inequality (6.1). Q.E.D.

REMARK. The above proof is a modification of the arguments in Mizohata [6], Muramatu [7] and Kumano-go [4]. They have assumed that $A_j(t) \in C^0([0, T], \mathscr{L}(X))$, but we have not used this fact.

As for symmetrizable hyperbolic systems, we can formulate the existence theorem as follows.

(F-3) $A_0(t)^* u \in C^0([0, T], X)$ for any $u \in X$.

(F-4) (Symmetrizability of $A(t)$). $A(t)$ satisfies the following conditions. There exist linear operators $N(t)$, $D(t)$, $S(t)$, $T(t)$ and $\Delta(t)$ in $X$ for $0 \leqq t \leqq T$ such that they satisfy the conditions: (i) $N(t)$, $S(t)$, $T(t)$, $\Delta(t)$, $N_t'(t)$, $\Delta_t'(t)$ and $\Delta(t)A(t)$

are bounded operators in $X$ and their operator norms are uniformly bounded for $0 \leqq t \leqq T$, (ii) $D(t)$ is a densely defined, closed operator in $X$ such that $D(t)^* + D(t) = S(t)$, (iii) $N(t)A(t) = D(t)N(t) + T(t)$, and (iv) there exist positive constants $\beta$, $c_1$ and $c_2$ such that $c_1\|u\| \leqq \|N(t)u\| + \beta\|\Lambda(t)u\| \leqq c_2\|u\|$ holds for $0 \leqq t \leqq T$ and $u \in X$.

(F-5) (Symmetrizability of $A(t)^*$). $A(t)^*$ satisfies the same conditions as in (F-4).

Then, we have the following.

THEOREM III (Existence). *We assume the conditions* (F-1), (F-3), (F-4) *and* (F-5). *Then, for an arbitrary* $f(t) \in C^0([0, T], X)$ *satisfying* $f(t) \in D(\Lambda)$ *and* $\Lambda f(t) \in C^0([0, T], X)$ *and an arbitrary* $u_0 \in D(\Lambda)$, *there exists a unique solution* $u(t) \in C^1([0, T], X)$ *of the equation* (F) *such that it satisfies* $u(t) \in D(\Lambda)$ *and* $\Lambda u(t) \in C^0([0, T], X)$.

PROOF. Note that Theorem I is also valid in this case. Therefore, combining the argument in Mizohata [6] with the proof of Theorem II, we can easily obtain this theorem. Q.E.D.

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