Finite groups with a standard subgroup isomorphic to ${}^3D_4(2^{3n})$

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1. Introduction.

A subgroup K of a finite group G is said to be *tightly embedded* if $|K|$ is even but $|K\cap K^g|$ is odd for each $g\in G-N_G(K)$. A quasisimple subgroup L of G is said to be a *standard* subgroup if $C_G(L)$ is tightly embedded in G, $N_G(L)=N_G(C_G(L))$, and $[L, L^g]\neq 1$ for each $g\in G$. After the fundamental work of Aschbacher [1] it becomes important to classify all finite groups which possess a standard subgroup of known type. In this paper we study a finite group G with a standard subgroup L isomorphic to the Steinberg group ${}^3D_4(q^3)$, $q=2^n$, and such that $C_G(L)$ has cyclic Sylow 2-subgroups. More precisely, the following theorem is proved.

THEOREM. *Let G be a finite group. Assume that L is a standard subgroup of G with $L\cong{}^3D_4(q^3)$, where $q=2^n\geq 2$, and that $C_G(L)$ has cyclic Sylow 2-subgroups. Assume furthermore that every section X of G satisfies the following condition*:

(*) *In $X/O(X)$, the 2-layer of every 2-local subgroup is semisimple.*

Then one of the following holds.

(1) $LO(G)\lhd G$.

(2) $E(G)\cong{}^3D_4(q^6)$.

(3) $E(G)\cong{}^3D_4(q^3)\times{}^3D_4(q^3)$.

The condition (*) is a consequence of the $B(G)$-conjecture of J. Thompson, the proof of which seems to be steadily nearing completion. We require the condition (*) in Lemma (5.4) to apply the main theorem of [13], in Lemma (7.29), and in Section 8 to apply [14, Lemma (2.7)]. All other parts of the proof are free from the condition.

The proof of the theorem is essentially a careful analysis of 2-local subgroups and proceeds along nearly the same line of argument as in our previous paper [16]. However, when $q=2$ some difference occurs. In this paper proofs are omitted in cases where they closely resemble the proofs of the corresponding results in [16].

The organization of the paper is as follows. Section 2 consists of two preliminary lemmas. In Section 3 we describe some properties of the group ${}^3D_4(q^3)$

and its automorphisms which will be used in later sections. In Section 4 the proof begins. Under the hypothesis that $LO(G) \not\trianglelefteq G$, it is shown that a Sylow 2-subgroup of $C_G(L)$ has order 2. In Section 5 we construct two 2-local subgroups which correspond to certain parabolic subgroups of ${}^3D_4(q^6)$ or ${}^3D_4(q^3) \times {}^3D_4(q^3)$. In Section 6 we deal with the case for $E(G) \cong {}^3D_4(q^6)$ and in Sections 7 and 8 the case for $E(G) \cong {}^3D_4(q^3) \times {}^3D_4(q^3)$.

Our notation will be fairly standard. Some possible exceptions are as follows: For a 2-group Q, $\mathscr{E}^*(Q)$ is the set of maximal elementary abelian subgroups of Q, $\mathscr{A}(Q)$ is the set of abelian subgroups of Q of maximal order, $J(Q)$ is the subgroup generated by all members of $\mathscr{A}(Q)$, and $J_r(Q)$ is the subgroup generated by abelian subgroups of Q of maximal rank. Also, $m(X)$ denotes the 2-rank of a group X and $X^{(\infty)}$ is the final term of the derived series of X. Moreover, for a subset U of a group, $I(U)$ denotes the set of involutions in U.

Two figures are attached at the end of this paper to illustrate the relations of subgroups which will appear in Section 5.

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2. Preliminaries.

(2.1) *Let P be a 2-subgroup of a group G and let M_i, $i=1,2$, be normal subgroups of P. Assume that for each i there is a subgroup L_i of $N_G(M_i)$ such that*

(1) $L_1/M_1 \cong SL(2, 2^{3n})$ *and* $L_2/M_2 \cong SL(2, 2^n)$ *for some* $n \geq 1$,

(2) $P \in \mathrm{Syl}_2(L_i)$, *and*

(3) $N_{L_i}(P) \leq N_G(L_j)$ *for* $\{i, j\} = \{1, 2\}$.

Assume furthermore that there is an involution $r_i \in L_i - N_{L_i}(P)$ such that

(4) $|r_1 r_2| = 6$,

(5) $P = M_i(P \cap P^{r_j} \cap P^{r_i r_j} \cap P^{r_j r_i r_j} \cap P^{r_i r_j r_i r_j} \cap P^{r_j r_i r_j r_i r_j})$ *for* $\{i, j\} = \{1, 2\}$, *and*

(6) $P \cap P^{(r_1 r_2)^3} = 1$.

Then $X = \langle L_1, L_2 \rangle$ is isomorphic to ${}^3D_4(2^{3n})$ and $P \in \mathrm{Syl}_2(X)$.

PROOF. Set $B = N_{L_1}(P) N_{L_2}(P)$ and $W = \langle r_1, r_2 \rangle$. Then, it follows from (5) and (6) that $P \neq P^w$ for each $w \in W^{\sharp}$, whence $B \cap W = 1$. Now proceed as in [6, Lemma (1L)].

(2.2) *Let t be an involution of a group H. Assume that $C_H(t) = \langle t \rangle \times K \times O(C_H(t))$ with $K \cong SL(3, 2)$. Then $E(H)/Z(E(H))$ is isomorphic to*

$$PSL(3, 4)\,, \qquad PSL(2, 7^2)\,, \qquad \textit{or} \qquad SL(3, 2) \times SL(3, 2)$$

and $Z(E(H))$ has odd order.

PROOF. The sectional 2-rank of H is at most 4 by Theorem 2 of [9]. Put $\bar{H}=H/O(H)$. Then $\bar{K}$ is a standard subgroup of $\bar{H}$ and hence $F^*(\bar{H})$ is simple or isomorphic to $SL(3,2)\times SL(3,2)$. Thus $F^*(\bar{H})$ is determined by [4] and [7]. Now (2.2) follows from [16, (2.10)].

3. Properties of ${}^3D_4(q^3)$.

We will give a summary of some of the properties of the Steinberg group $L={}^3D_4(q^3)$, where $q=2^n\geq 2$, and its automorphisms. An excellent description of ${}^3D_4(q^3)$ can be found in Thomas [15]. Proofs will be omitted because they can be found in [15] or else the assertions are consequences of direct computation.

Set $\Gamma=GF(q^3)$ and $\Gamma_0=GF(q)$. Let θ be the trace from Γ to Γ_0 and Γ_1 the kernel of θ. We write $\bar{\alpha}=\alpha^q$ for $\alpha\in\Gamma$.

Let Σ^+ be the set of positive roots of the simple Lie algebra of type D_4. The elements of Σ^+ can be written as

$$a\,,\quad b\,,\quad c\,,\quad d\,,\quad a+b\,,\quad a+c\,,\quad a+d\,,\quad a+c+d\,,$$
$$a+b+d\,,\quad a+b+c\,,\quad a+b+c+d\,,\quad \text{and}\quad 2a+b+c+d\,,$$

where a, b, c, d are the fundamental roots. The set Σ of roots consists of the elements of Σ^+ and their negatives. Denote by $\tilde{w}_r$ the reflection with respect to the root r. Then

$$\tilde{w}_a(b)=\tilde{w}_b(a)=a+b\,,\quad \tilde{w}_a(c)=\tilde{w}_c(a)=a+c\,,\quad \tilde{w}_a(d)=\tilde{w}_d(a)=a+d\,,$$
$$\text{and}\quad \tilde{w}_r(s)=s$$

for distinct elements r and s of $\{b, c, d\}$.

For each $r\in\Sigma$ there exists a nontrivial homomorphism φ_r: $SL(2, q^3)\to D_4(q^3)$. Set

$$x_r(\alpha)=\varphi_r\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix},\qquad w_r=\varphi_r\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad h_r(\beta)=\varphi_r\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix},$$

where $\alpha\in\Gamma$ and $\beta\in\Gamma^{\times}$. Define

$$U_1=\{x_1(\alpha)|x_1(\alpha)=x_b(\alpha)x_c(\bar{\alpha})x_d(\bar{\bar{\alpha}}),\ \alpha\in\Gamma\}\,,$$
$$U_2=\{x_2(\alpha)|x_2(\alpha)=x_a(\alpha),\ \alpha\in\Gamma_0\}\,,$$
$$U_3=\{x_3(\alpha)|x_3(\alpha)=x_{a+b}(\alpha)x_{a+c}(\bar{\alpha})x_{a+d}(\bar{\bar{\alpha}}),\ \alpha\in\Gamma\}\,,$$
$$U_4=\{x_4(\alpha)|x_4(\alpha)=x_{a+c+d}(\alpha)x_{a+b+d}(\bar{\alpha})x_{a+b+c}(\bar{\bar{\alpha}}),\ \alpha\in\Gamma\}\,,$$
$$U_5=\{x_5(\alpha)|x_5(\alpha)=x_{a+b+c+d}(\alpha),\ \alpha\in\Gamma_0\}\,,$$
$$U_6=\{x_6(\alpha)|x_6(\alpha)=x_{2a+b+c+d}(\alpha),\ \alpha\in\Gamma_0\}\,.$$

(3.1) The group $S=U_1U_2U_3U_4U_5U_6$ is a Sylow 2-subgroup of L and every element x in S has a unique expression as

$$x=x_1(\alpha_1)x_2(\alpha_2)x_3(\alpha_3)x_4(\alpha_4)x_5(\alpha_5)x_6(\alpha_6)\ .$$

The structure of S is determined by the following formulas:

$$\begin{aligned}
&x_i(\alpha)x_i(\beta)=x_i(\alpha+\beta)\ , \qquad 1\leq i\leq 6\ ,\\
&[x_1(\alpha),\, x_2(\beta)]=x_3(\alpha\beta)x_4(\bar{\alpha}\bar{\bar{\alpha}}\beta)x_5(\alpha\bar{\alpha}\bar{\bar{\alpha}}\beta)\ ,\\
&[x_1(\alpha),\, x_3(\beta)]=x_4(\bar{\alpha}\bar{\bar{\beta}}+\bar{\bar{\alpha}}\bar{\beta})x_5(\theta(\alpha\bar{\alpha}\bar{\bar{\beta}}))x_6(\theta(\alpha\bar{\beta}\bar{\bar{\beta}}))\ ,\\
&[x_1(\alpha),\, x_4(\beta)]=x_5(\theta(\alpha\beta))\ ,\\
&[x_2(\alpha),\, x_5(\beta)]=x_6(\alpha\beta)\ ,\\
&[x_3(\alpha),\, x_4(\beta)]=x_6(\theta(\alpha\beta))\ ,\\
&[x_i(\alpha),\, x_j(\beta)]=1 \qquad \text{otherwise.}
\end{aligned}$$

(3.2) Let $J=\langle h_a(\alpha),\, h_b(\beta)h_c(\bar{\beta})h_d(\bar{\bar{\beta}})|\alpha\in\Gamma_0^\times,\, \beta\in\Gamma^\times\rangle$. Then $N_L(S)=SJ$. There is an isomorphism ψ between J and the group of all characters χ from a free abelian group on the four generators $a,\ b,\ c,\ d$ into $\Gamma^\times$ which satisfy $\chi(a)\in\Gamma_0$ and $\overline{\overline{\chi(b)}}=\overline{\chi(c)}=\chi(d)$. If ψ_h denotes the image of $h\in J$ under ψ, then h acts on S according to the formulas:

$$\begin{aligned}
&hx_1(\alpha)h^{-1}=x_1(\psi_h(b)\alpha)\ ,\\
&hx_2(\alpha)h^{-1}=x_2(\psi_h(a)\alpha)\ ,\\
&hx_3(\alpha)h^{-1}=x_3(\psi_h(a+b)\alpha)\ ,\\
&hx_4(\alpha)h^{-1}=x_4(\psi_h(a+c+d)\alpha)\ ,\\
&hx_5(\alpha)h^{-1}=x_5(\psi_h(a+b+c+d)\alpha)\ ,\\
&hx_6(\alpha)h^{-1}=x_6(\psi_h(2a+b+c+d)\alpha)\ .
\end{aligned}$$

(3.3) Set $u=w_bw_cw_d$ and $v=w_a$. Then $\langle u,v\rangle$ is dihedral of order 12 and it normalizes J. The action of $\langle u,v\rangle$ on J is given by the formulas: $\psi_{uhu}(r)=\psi_h(\tilde{u}(r))$ and $\psi_{vhv}(r)=\psi_h(\tilde{v}(r))$ for $h\in J$ and $r=a,b,c,d$ where $\tilde{u}=\tilde{w}_b\tilde{w}_c\tilde{w}_d$ and $\tilde{v}=\tilde{w}_a$. Moreover

$$\begin{aligned}
&ux_2(\alpha)u=x_5(\alpha)\ , \qquad ux_3(\alpha)u=x_4(\alpha)\ , \qquad ux_6(\alpha)u=x_6(\alpha)\ .\\
&vx_1(\alpha)v=x_3(\alpha)\ , \qquad vx_4(\alpha)v=x_4(\alpha)\ , \qquad vx_5(\alpha)v=x_6(\alpha)\ .
\end{aligned}$$

(3.4) Each element g of L has a unique expression as $g=xhwy$, where $x\in S$, $h\in J$, $w\in\langle u,v\rangle$, and $y\in\langle U_i|U_i\not\leq S^w\rangle$.

(3.5) Let $J_i=C_J(U_i)$, $1\leq i\leq 6$, and $J_0=J^{q-1}$. Then

(1) $J_1=\{h\in J|\psi_h(b)=1\}$ and $J_2=\{h\in J|\psi_h(a)=1\}$.

The groups J_1 and J_2 are cyclic of order respectively $q-1$ and q^3-1.

(2) $J_4=\{h_a(\beta)|\beta\in\Gamma_0^\times\}$ and $J_6=\{h_b(\beta)h_c(\bar{\beta})h_d(\bar{\bar{\beta}})|\beta\in\Gamma^\times\}$.

(3) $J_2^u=J_5$, $J_3^u=J_4$, $J_1^v=J_3$, $J_5^v=J_6$.

(4) $J_0=J_2\cap J_5\cap J_6$ and J_0 has order q^2+q+1.

(5) J_2 acts regularly on $U_1^\sharp$, $U_3^\sharp$, and $U_4^\sharp$ and acts transitively on $U_5^\sharp$ and $U_6^\sharp$.

If $q\geq 4$, J_1 acts regularly on $U_2^\sharp$, $U_5^\sharp$, and $U_6^\sharp$ and acts semiregularly on $U_3^\sharp$ and $U_4^\sharp$.

(3.6) $\langle U_1, u\rangle=U_1J_6\cup U_1J_6uU_1\cong SL(2, q^3)$ and $\langle U_2, v\rangle=U_2J_4\cup U_2J_4vU_2\cong SL(2, q)$.

(3.7) $C_L(J_0)=C_L(J_0)'J$ and $C_L(J_0)'=\langle U_2, U_5, U_6, v, v^u, v^{uv}\rangle\cong SL(3, q)$. The element u acts on $C_L(J_0)'$ as a graph automorphism. Furthermore when $q\geq 4$, $C_L(J_1)=\langle U_1, u\rangle\times J_1$ and $C_L(J_2)=\langle U_2, v\rangle\times J_2$.

(3.8) We define $V=U_6$, $Y=U_5V$, $W=U_4Y$, $F=U_3W$, $D=U_2F$, and $M=U_1F$. Then

(1) $V=Z(S)=Z(D)=D'$, $Y=Z_2(S)=Z(M)=Z(F)$, $W=Z_3(S)=Z_2(M)=M'$, $F=Z_4(S)=S'$.

(2) $\mathscr{E}^*(S/W)=\{D/W, M/W\}$.

(3) $N_L(F)=N_L(S)$.

(4) $N_L(V)=N_L(D)=D(\langle U_1, u\rangle\times J_1)$.

(5) $N_L(Y)=N_L(W)=N_L(M)=M(\langle U_2, v\rangle\times J_2)$.

If Δ is a subset of Γ, define $x_i(\Delta)=\{x_i(\alpha)\mid\alpha\in\Delta\}$.

(3.9) The group L has exactly two conjugacy classes of involutions and we can choose $x_6(1)$ and $x_4(1)$ as their representatives. Moreover, $C_L(x_6(1))=C_L(V)=D\langle U_1, u\rangle$ and $C_L(x_4(\alpha))=x_1(\alpha^{-1}\Gamma_1)x_3(\alpha^{-1}\Gamma_1)W\langle U_2, v\rangle$ for $\alpha\in\Gamma^\times$. The group $C_L(x_6(1))$ is perfect and $O_2(C_L(x_4(1)))'$ contains $x_4(1)$. If $q\geq 4$, then $C_L(x_4(1))'$ is perfect.

(3.10) The group $\langle U_1, u\rangle$ acts irreducibly on D/V. Furthermore, Y is a natural module for $\langle U_2, v\rangle\cong SL(2, q)$, M/W is the direct sum of three natural modules for $\langle U_2, v\rangle$, and $\langle U_2, v\rangle J$ acts irreducibly on M/W.

(3.11) (1) Every abelian subgroup of D is of order at most q^5.

(2) Let B be a maximal elementary abelian subgroup of M. Then either $B=W$ or $N_L(B)$ is 2-closed. Moreover, if $B\not\leq F$ then $B^{xv}\leq F$ for some $x\in S$.

(3) $m(S)=5n$ and $J_r(S)=S$.

(4) If X is a subgroup of S such that $|\Omega_1(Z(X))|>q^2$ and $|X|>q^8$, then $X=C_S(x_4(\alpha)Y)=x_1(\alpha^{-1}\Gamma_1)x_3(\alpha^{-1}\Gamma_1)W$ for some $\alpha\in\Gamma^\times$ and so $X=O_2(C_L(x_4(\alpha)))$. In particular, $X^h=C_S(x_4(1)Y)$ for some $h\in J$. Note that $Z(X)=x_4(\alpha\Gamma_0)Y$ and $|X|=q^9$.

(3.12) Let $A=\mathrm{Aut}(L)$. Then A is a semidirect product of L and a cyclic group $\langle f\rangle$ of order $3n$, where f is the field automorphism of L induced by the automorphism: $\alpha\mapsto\alpha^2$ of Γ. Thus f maps $x_i(\alpha)$ to $x_i(\alpha^2)$, $1\leq i\leq 6$, and centralizes $\langle u, v\rangle$. If $h\in J$ and $g=h^f$, then $\psi_g(r)=\psi_h(r)^2$ for $r\in\Sigma$.

(3.13) If n is even, we denote by f_0 the involution of $\langle f\rangle$. Then $I(f_0S)=f_0^S$ and $C_L(f_0)\cong{}^3D_4(q^{3/2})$.

(3.14) $J_r(Q)=S$ for $S\leq Q\in\mathrm{Syl}_2(A)$.

(3.15) (1) $C_A(U_4)=W\langle U_2, v\rangle$ and $C_A(V)=C_L(V)\langle f^n\rangle$.

(2) $C_A(J_0)\leq L$. Moreover, if $q\geq 8$ then $C_A(J_4)=C_L(J_4)\langle f^n\rangle$ and if $q=4$ then $C_A(J_4)=C_L(J_4)\langle vf\rangle$.

(3) For any subgroup B of A with $B\geq L$ we have $O_2(N_B(S))=S$, $O_2(N_B(V))=D$, and $O_2(N_B(Y))=M$.

(4) $C_A(Y)=MJ_0\langle f^n\rangle$, $C_A(W/Y)=M\langle U_2, v\rangle$, $C_A(F/W)=SJ_3$, $C_A(D/V)=D$, and $C_A(S/F)=S$.

(5) $C_A(C_L(V)/D)=DJ_1$ and $C_A(\langle U_2, v\rangle M/M)=MJ_2\langle f^n\rangle$.

4. Structure of $C(t)$.

The symbols defined in Section 3 for various objects of the group ${}^3D_4(q^3)$ will retain their meaning for the rest of this paper.

In this section we assume

(4.1) *Hypothesis.* G is a group, t is an involution of G, and $C(t)\triangleright L\cong {}^3D_4(q^3)$ where $q=2^n\geq 2$. Furthermore, $C(L)$ has cyclic Sylow 2-subgroups.

Set $C=C(t)$. Let R be a Sylow 2-subgroup of $LC_C(L)$ with $S\leq R$ and set $T=C_R(L)$.

(4.2) (1) *If Q is a Sylow 2-subgroup of C containing R, then $J_r(Q)=R$.*

(2) $t^G\cap L=\emptyset$.

PROOF. (1) follows from (3.14). Let y be an involution of L. If $q\geq 4$, then $C_L(y)^{(\infty)}$ contains y. Since $C^{(\infty)}=L$ does not contain t, (2) holds in this case. If $q=2$, then $|C/LC_C(L)|=1$ or 3 and $E'\leq L$ for each 2-subgroup E of C. Since $(O_2(C_L(y)))'$ contains y, (2) holds.

(4.3) *Suppose t is a central involution of G. Then $t\in Z^*(G)$ and in particular $LO(G)\lhd G$.*

PROOF. Assume by way of contradiction that $t\notin Z^*(G)$. Let Q be a Sylow 2-subgroup of G such that $R\leq Q\leq C$. Then as in [16, (4.3)], $t^G\cap\langle t\rangle L>\{t\}$ and $t^G\cap\langle t\rangle Y=\{t\}$. Since every involution of L is conjugate to an element of W, $t^G\cap t(W-Y)\neq\emptyset$ by (4.2) (2).

We shall show that $t^{N(\langle t\rangle W)}\cap t(W-Y)\neq\emptyset$. As $N_L(S)$ acts transitively on $W-Y$, $t^{a^{-1}}=tx_4(1)$ for some $a\in G$ and $(\langle t\rangle K)^a\leq C$ where $K=C_L(x_4(1))$. By (4.2) (1), $(\langle t\rangle W)^a\leq\langle t\rangle L$. If $q\geq 4$, then $K=K'W$ and K' is perfect, so $(K')^a\leq C^{(\infty)}=L$ and $(\langle t\rangle K)^a<\langle t\rangle L$. If $q=2$, then $Q=R=S\times T$. Since $K=O^{2'}(K)$, $K^a\leq LC_C(L)$. Moreover, $t\notin O^2(G)$ by (4.2) (2) and [16, (2.3)], so $|O^2(G): L|$ is odd. This implies that $K^a\leq L$. Thus in either case $(\langle t\rangle K)^a\leq\langle t\rangle L$. Also, $t\in(\langle t\rangle W)^a$. Set $X=\langle t\rangle O_2(K)$, so that $X^a=\langle t\rangle(L\cap X^a)$. By (3.11) we can choose $b\in L$ such that $L\cap X^{ab}=C_S(x_4(1)Y)$. Set $E=L\cap(\langle t\rangle W)^{ab}$. Then $(\langle t\rangle W)^{ab}=\langle t\rangle E$ and E is an elementary abelian subgroup of M of order q^5. If $q\geq 4$, then $(K')^a<L$ and $N_L(E)$ contains $(K')^{ab}$. If $q=2$, then $K^a<L$ and $N_L(E)$ contains K^{ab}. Hence $N_L(E)$ is not 2-closed and by (3.11) we conclude that $E=W$ and $ab\in N(\langle t\rangle W)$. Thus $t^{N(\langle t\rangle W)}\cap t(W-Y)\neq\emptyset$. As in [16, (4.3)] this yields a contradiction.

(4.4) *Suppose t is a noncentral involution of G. Then*

(1) $t^{N(R)}=tV$.

(2) $\langle t\rangle\in\mathrm{Syl}_2(C(L))$. *In particular,* $C_C(L)=\langle t\rangle O(C)$.

PROOF. Same as for [16, (4.4)].

5. 2-local subgroups of G.

Henceforth we assume

(5.1) *Hypothesis.* G is a group which satisfies the hypotheses of the main theorem and $LO(G)\ntriangleleft G$.

Let t be an involution of $C(L)$ and set $C=C(t)$ and $R=\langle t\rangle S$. Thus $L\triangleleft C$. Furthermore, $C_C(L)=\langle t\rangle O(C)$ by (4.4).

(5.2) *Definition.* Let $C_1=O_2(N(\langle t\rangle V))$, $L_2=O_2(N(\langle t\rangle Y))$, $R_1=C_1S$, and $R_2=L_2S$.

(5.3) (1) $N(\langle t\rangle V)=C_1N_C(V)\leq N(D)$, $C_1\cap C=\langle t\rangle D$, $t^{C_1}=tV$, *and* $C_1/\langle t\rangle D\cong V$ *as* $N_C(V)$*-modules.*

(2) $N(\langle t\rangle Y)=N(\langle t\rangle M)=L_2N_C(Y)\leq N(Y)$, $L_2\cap C=\langle t\rangle M$, $t^{L_2}=tY$, *and* $L_2/\langle t\rangle M\cong Y$ *as* $N_C(Y)$*-modules.*

(3) $N(R)=C_1N_C(S)\leq N(S)$.

(4) $\mathscr{E}^*(R_2/\langle t\rangle M)=\{L_2/\langle t\rangle M, R_1/\langle t\rangle M\}$ *and* $R_2/R_1\cong Y/V$ *as* $N_C(S)$*-modules.*

(5) $\mho^1(C_1)=V$.

(6) $[O(C), L_2]=1$.

PROOF. When $q\geq 4$, the argument is similar to that of (5.3), (5.5), and (5.6) in [16]. Thus assume that $q=2$. Set $X=N(\langle t\rangle Y)$ and $K=\langle U_2, v\rangle$. As $Z_2(R)=\langle t\rangle Y$, it follows from (4.2) and (4.4) that $t^X=tY$ and $X\triangleright Y$. Put $\tilde{X}=X/C(\langle t\rangle Y)$. The permutation group $(\tilde{X}, tY)$ is transitive and the stabilizer of t is $\widetilde{N_C(Y)}=\tilde{K}$. Since K induces the full automorphism group of Y, $\widetilde{C_X(Y)}$ has order 4 and it is a regular normal subgroup of $(\tilde{X}, tY)$. By (3.15), $O_2(C_C(Y))=\langle t\rangle M$. Put $I=C_X(Y)$ and $\bar{X}=X/\langle t\rangle M$, so that $O(\bar{I})=\overline{C_C(Y)}$. Since K stabilizes the series $\overline{C_C(Y)}\geq\overline{C_L(Y)C_C(L)}\geq 1$ and $K=O^{2'}(K)$, K in fact centralizes $\overline{C_C(Y)}$. This implies that

$\bar{I}=O_2(\bar{I})\times O(\bar{I})$ and $X=N_C(Y)O_2(I)$. Since $O_2(N_C(Y))=\langle t\rangle M$ by (3.15), we obtain $L_2=O_2(I)$. Then as in [16, (5.5)], (2) and (6) follow.

Set $B=N(\langle t\rangle V)$. Then $t^B=tV$ by (4.2) and (4.4). Thus $|B: C_C(V)|=2$. Since $q=2$, R is a Sylow 2-subgroup of $C_C(V)$. Set $T=N_{R_2}(R)$. Then as $Z(R)=\langle t\rangle V$, T is a Sylow 2-subgroup of B. Note that $O^{2'}(C_C(V))=\langle t\rangle C_L(V)$ and $(\langle t\rangle C_L(V))'=C_L(V)$. Set $E=C_B(C_L(V)/D)$ and $\bar{B}=B/\langle t\rangle D$. As $C_L(V)/D\cong SL(2,8)$, we have $B=C_C(V)E$, $O(\bar{E})=\overline{O(C)}$, and $|\bar{E}: O(\bar{E})|=2$. Since $[O(C), T]=1$ by (6), $\bar{B}/C_{\bar{B}}(O(\bar{E}))$ has odd order and $\bar{E}=O(\bar{E})\times O_2(\bar{E})$. Thus we have $C_1=O_2(E)$, for $O_2(C_C(V))=\langle t\rangle D$ by (3.15). Hence (1) and (3) hold. For the proofs of (4) and (5), see [16, (5.6)].

(5.4) *$E(C(J_0))$ is semisimple of type $PSL(3,q)\times PSL(3,q)$ or $PSL(3,q^2)$ with center of odd order. In the former t interchanges the components of $E(C(J_0))$ and in the latter t acts on $E(C(J_0))$ as an involutive field automorphism. Furthermore, $E(C(J_0))$ is the normal closure of $C_L(J_0)'$ in $C(J_0)$ and $|C(J_0)|_2=2q^6$.*

PROOF. Set $H=C(J_0)$, $K=C_L(J_0)'$, and $E=E(H)$. Then $C_H(t)=C_L(J_0)C_C(L)\trianglerighteq K$ by (3.15). Moreover, $K\cong SL(3,q)$, $|C_H(t)|_2=2q^3$, and $\langle t\rangle$ is a Sylow 2-subgroup of $C_H(K)$. (5.3) (2) shows that $|H\cap R_2|=2q^5$, hence t is a noncentral involution of H. If $q\geq 4$, (5.4) follows from a result of Seitz [13] and [16, (2.10)]. If $q=2$, (2.2) implies that (5.4) holds or else $E/Z(E)\cong PSL(2,7^2)$. Assume that $E/Z(E)\cong PSL(2,7^2)$ and put $\bar{H}=H/O(H)$. Then $\bar{H}\leq \mathrm{Aut}(\bar{E})$. As $|H|_2\geq 2^6$, we have $\bar{H}=\mathrm{Aut}(\bar{E})$. But then $|C_{\bar{H}}(t)|_2=2^5$ since t acts on $\bar{E}$ as an involutive field automorphism, which contradicts the fact that $|C_H(t)|_2=2^4$. Finally, we remark that $|H|_2=2q^6$ is a consequence of $|C_H(t)|_2=2q^3$.

(5.5) *Definition.* Let $D_1=(C_1\cap E(C(J_0)))D$, $V_1=Z(D_1)$, $S_1=SD_1$, $M_1=L_2\cap S_1$, $F_1=M_1\cap D_1$, $W_1=WV_1$, and $Y_1=YV_1$.

(5.6) (1) *$C_1=\langle t\rangle D_1>D_1$ and $N_L(V)$ normalizes D_1.*

(2) *V_1 is elementary abelian of order q^2 with $V_1\cap C=V$ and $V_1\leq F_1$.*

(3) *For every $\langle t\rangle$-invariant Sylow 2-subgroup X of $E(C(J_0))$ such that $C_{D_1}(J_0)\leq X$, we have $V_1=Z(X)$ and X/V_1 is elementary abelian.*

PROOF. Set $H=C(J_0)$ and $E=E(H)$. Then $|H\cap C_1|=2q^4$ by (5.3) (1) and $|H: \langle t\rangle E|$ is odd by (5.4). Hence $|E\cap C_1|=q^4$. As $J_0^u=J_0$, $\langle u\rangle J$ normalizes D_1. Moreover, S normalizes D_1 since $R_1/D=S/D\times C_1/D$ by (5.3) (3). Thus (1) holds.

Set $Q=H\cap D_1$ and let X be a $\langle t\rangle$-invariant Sylow 2-subgroup of E containing Q. Then $N_X(C_X(t))=C_X(t)Z(X)$ has order q^4 by (5.4). As $C_X(t)=C_D(J_0)$ is a normal subgroup of Q and $|Q|=q^4$, we have $Q=C_D(J_0)Z(X)$ and $Z(Q)=Z(X)$. Set $B=Z(Q)$.

Then $[WB, J_0]=[W, J_0]=U_4$, for J_0 centralizes WB/W. Also, $C_{WB}(J_0)=YB\lhd WB$ by (5.3) (5). Thus $WB=YB\times U_4$ and so WB is abelian. Since u normalizes D_1 and J_0, B then centralizes $WW^u=D$. Since $D\cap B=V=Z(D)$ and $D_1=DB$, we have that $B=V_1$, and (3) holds.

Since $L_2\rhd Y$ and $N_L(Y)$ acts irreducibly on Y, $Z(L_2)\geq Y$. As $C_S(Y)=M$, this implies that $L_2=C_{R_2}(Y)$. Hence $V_1\leq F_1$.

(5.7) $Z(S_1)=V_1$.

PROOF. By (5.3) $Y\lhd R_2$, so $Y_1\lhd M_1$. As $C_{M_1}(J_0)=Y_1$, $[M_1, J_0]$ then centralizes Y_1 by [16, (2.4)] and we have $Y_1\leq Z(M_1)$. In particular, $V_1\leq Z(S_1)$ and as $S\cap V_1=Z(S)$, (5.7) holds.

(5.8) *Definition.* Let $Y_2=V_1V_1^v, W_2=WY_2$, $F_2=FY_2$, $M_2=MY_2$, and $S_2=SY_2$.

(5.9) (1) $L_2=\langle t\rangle M_2>M_2$, $R_2=\langle t\rangle S_2>S_2$.
(2) $Y_2=V_1\times V_1^v=Z(M_2)$ *and* $Y_2\cap C=Y$.
(3) $Z(S_2)=V_1$, $Z_2(S_2)=Y_2$.

PROOF. Set $H=C(J_0)$, $E=E(H)$, and $B=E\cap L_2$. Then $|H\cap L_2|=2q^4$ by (5.3) (2) and $|B|=q^4$ by (5.4). By (5.6) we have $V_1\leq Z(B)$. As v normalizes B, this implies that $Y_2\leq Z(B)$. Moreover, $V_1\cap V_1^v=1$ since $V_1\cap C=V$ and $V\cap V^v=1$. Thus $Y_2=V_1\times V_1^v=B$ and $Y_2\cap C=VV^v=Y$. As v normalizes M, (5.7) shows that $[M, Y_2]=1$. Then since $M\cap Y_2=Z(M)$, it follows that $Z(M_2)=Y_2$. Now $|L_2: M_2|=2$, so $C_{L_2}(Y_2)=M_2$ and $L_2=\langle t\rangle M_2$. Likewise $R_2=\langle t\rangle S_2>S_2=C_{R_2}(V_1)$.

By (5.3) (3) we have $N_{R_2}(R)=R_1\rhd S$. As $R_2\rhd\langle t\rangle Y$, this implies $N_{R_2}(S)=R_1$. Thus $Z(S_2)\leq R_1\cap S_2=S_1$, and hence $Z(S_2)=V_1$. Put $\bar{S}_2=S_2/V_1$. As $C_D(J_0)\leq E$, $C_D(J_0)$ centralizes $\bar{Y}_2$ by (5.6) (3). Since $S_2=M_2C_D(J_0)$, this shows that $\bar{S}_2$ is a central product of $\bar{S}_1$ and $\bar{Y}_2$. Hence $Z_2(S_2)=Y_2$.

(5.10) (1) $Z(R_2)=V$, $Z_2(R_2)=Y_1$, $Z_3(R_2)=\langle t\rangle W_2$, *and* $Z_4(R_4)=\langle t\rangle F_2$.
(2) $J(\langle t\rangle F_2)=F_2$.

PROOF. (1) follows from (3.8) (1) and (5.9). For the proof of (2), note that W and U_3Y are elements of $\mathscr{A}(F)$ and $F_2=F*Y_2$. Set $Q=\langle t\rangle F_2$. Then $Z(Q)=Y$ since $C_Q(t)=\langle t\rangle F$ implies $Z(Q)\leq\langle t\rangle Y$. Let B be an abelian subgroup of Q of order at least q^7 and suppose $B\not\leq F_2$. Then $Q=BF_2$ and $|B: B\cap F_2|=2$. Also, $|(B\cap F_2)Y_2|\leq q^7$ since $(B\cap F_2)Y_2$ is an abelian subgroup of F_2. But then $|B\cap F_2|\leq q^5$ since $B\cap Y_2\leq Z(Q)$. This contradiction proves (2).

(5.11) *Definition.* Let $C_5=O_2(N(C_1))$ and $R_5=C_5S$.

(5.12) (1) *$N(C_1)=C_5N_C(V)\leq N(D_1)$, $C_5\cap C=\langle t\rangle D$, and $C_5/C_1\cong D/V$ as $N_C(V)$-modules.*

(2) *C_5 acts transitively on $\langle t\rangle D_1/V_1-D_1/V_1$ and centralizes D_1/V_1.*

(3) *$C_5\geq Y_2$.*

(4) *$[O(C), C_5]=1$.*

PROOF. We first show that $N_C(V)\leq N(D_1)$. Note that $N_C(V)$ acts on $\tilde{C}_1=C_1/D$. If $q=2$, then $|N_C(V):\langle t\rangle N_L(V)|$ is odd and $\tilde{C}_1=\langle\tilde{t}\rangle\times\tilde{D}_1$ has order 4. Since $\langle t\rangle N_L(V)$ normalizes D_1, the assertion holds in this case. Recall that $W\in\mathscr{A}(D)$, $D=WW^u$, and $D_1=D*V_1$. Then we have $J(C_1)=D_1$ as in (5.10) when $q\geq 4$. Thus $N_C(V)$ normalizes D_1.

Set $\overline{N(V_1)}=N(V_1)/V_1$, $H=N(C_1)\cap N(D_1)$, and $Q=C_H(\bar{D}_1)$. Then $Q\geq Y_2$ by (5.9). Since $\mathscr{E}^*(\langle t\rangle V_1)=\{V_1, \langle t\rangle V\}$, we have $C_H(\bar{t})=N(\langle t\rangle V)$. In particular, $C_Q(\bar{t})=C_1C_C(L)$ by (3.15) and (5.3). Consider the $N_C(V)$-homomorphism of Q into $\bar{D}_1$ defined by $x\mapsto[\bar{t}, x]$. As $\bar{t}^{Y_2}=\bar{t}\bar{Y}_1$ and $N_L(V)$ acts irreducibly on $\bar{D}_1$ by (3.10), this map is surjective. Therefore, $Q/C_1C_C(L)\cong\bar{D}_1$ as $N_C(V)$-modules, $\bar{t}^Q=\bar{t}\bar{D}_1$, and $H=N_C(V)Q$.

Next we will show that $H=N(C_1)$. If $q\geq 4$, then $J(C_1)=D_1$ and the assertion holds. Thus we assume that $q=2$. Take $a\in V_1-V$, so that $D_1=\langle a\rangle\times D$. Set $V=\langle z\rangle$. Then $(ta)^2=z$. Since D is extra-special of order 2^9 with $Z(D)=\langle z\rangle$, we get $|I(tD_1)|=2^9$ where $I(tD_1)$ denotes the set of involutions in tD_1. Counting orders we have $t^H=I(tD_1)$. Suppose $t^G\cap D_1\neq\emptyset$. By [15, (3.3)] each involution of D is conjugate in $N_L(V)$ to an element of W. Hence $t^g\in W_1$ for some $g\in G$ and $C(t^g)$ contains W_2, so $m(C(t^g))>m(C)$ by (4.2), a contradiction. Thus $t^G\cap D_1=\emptyset$, and so $t^H=t^{N(C_1)}$. Since $N_C(C_1)\leq H$ by (5.3), we conclude that $H=N(C_1)$. Now (5.12) follows as in [16, (5.19)].

(5.13) *Definition.* Let $R_3=N_{R_5}(\langle t\rangle W_1)$ and $R_4=N_{R_5}(\langle t\rangle F_1)$.

(5.14) *R_3 acts transitively on $\langle t\rangle W_1/V_1-W_1/V_1$ and R_4 acts transitively on $\langle t\rangle F_1/V_1-F_1/V_1$. Furthermore, $R_2=N_{R_5}(\langle t\rangle Y_1)\lhd R_3\lhd R_4\lhd R_5$.*

PROOF. Since $R_2<R_5\leq N(D_1)$ and since $\mathscr{E}^*(\langle t\rangle Y_1)=\{Y_1, \langle t\rangle Y\}$, $\langle t\rangle Y$ is normal in $N_{R_5}(\langle t\rangle Y_1)$. Hence $R_2=N_{R_5}(\langle t\rangle Y_1)$ by (5.3). Now (5.14) follows from (5.12). (See [16, (5.21)]).

(5.15) $N_{R_5}(S_1)=N_{R_5}(M_1)=R_2$.

PROOF. As $Z_2(R_1)=\langle t\rangle Y_1$, we have $N_{R_5}(R_1)=R_2$ by (5.14). Since M_1 and S_1 are normal in R_2 and $M_1C_1=S_1C_1=R_1$, the assertion holds.

(5.16) *Definition.* Let $L_3=O_2(N(L_2))$.

(5.17) (1) $N(L_2)=N(\langle t\rangle W)=L_3N_C(Y)\leq N(M_2)$, $L_3\cap C=\langle t\rangle M$, $t^{L_3}=tW$, *and* $L_3/L_2\cong W/Y$ *as* $N_C(Y)$*-modules.*

(2) $N(R_2)=R_3N_C(S)$ *and* $R_3=L_3S$.

PROOF. Set $H=N(\langle t\rangle W_2)\cap N(Y_2)$ and $\overline{N(Y_2)}=N(Y_2)/Y_2$. As $Z(F_2)=Y_2$, it follows from (5.10) that $N(R_2)\leq H$. Moreover, $\mathscr{E}^*(\langle t\rangle W_2)=\{W_2, \langle t\rangle W\}$. Hence (5.14) shows that $R_3\leq H$ and $\bar{t}^{R_3}=\bar{t}^H=\bar{t}\,\bar{W}_2$. Now, $N_C(Y)=N_L(Y)N_C(S)$ by the Frattini argument and $N_L(Y)=\langle S, J, v\rangle$ normalizes Y_2. Then as $Z_2(L_2)=\langle t\rangle W_2$, (5.3) yields that $N(\langle t\rangle Y)\leq H$. In particular, $C_H(\bar{t})=N(\langle t\rangle Y)$ since $\mathscr{E}^*(\langle t\rangle Y_2)=\{Y_2, \langle t\rangle Y\}$. Let $B=C_H(\bar{t}\bar{W}_2)$, so that $B=\langle U_2, v\rangle L_2C_C(L)$ by (3.15) and we have $O^{2'}(B)=\langle U_2, v\rangle L_2$ and $O_2(B)=L_2$. Let $E=C_H(\bar{W}_2)$. Then $R_3\leq E\lhd H$ and the map defined by $Bx\mapsto[\bar{t}, x]$ for $x\in E$ is a $N_C(Y)$-isomorphism between E/B and $\bar{W}_2$. Put $\tilde{H}=H/L_2$ and $E_0=C_E(\widetilde{\langle U_2, v\rangle})$. Then since $\langle U_2, v\rangle L_2\lhd H$ and J_2 acts irreducibly on E/B, we have $\tilde{E}=\tilde{E}_0\times\widetilde{\langle U_2, v\rangle}$. Now $|\tilde{E}_0|=q^3|O(C)|$ and R_3 is a Sylow 2-subgroup of E. Hence $[O(C), R_3]=1$ implies $E_0=O_2(E_0)\times O(C)$. As $O_2(E)=O_2(E_0)$, we conclude that $H=O_2(H)C_H(\bar{t})=O_2(H)N_C(Y)$ with $O_2(H)=O_2(E)$.

Let $K=N(\langle t\rangle W_2)$, so that $K\leq N(\langle t\rangle W)$. Counting orders we obtain $t^K=t^H=tW$. Furthermore, $C_K(t)=N_C(W)=N_C(Y)\leq H$. Hence $K=H$. This in particular shows that $H=N(L_2)$ since $Z_2(L_2)=\langle t\rangle W_2$. Likewise $H=N(\langle t\rangle W)$ since $t^G\cap W=\emptyset$. As $M_2=C_{L_2}(Y_2)$, we have $H\leq N(M_2)$ and (1) holds. Finally, as $N(R_2)\leq H$ and $E\geq R_3\geq O_2(E)=L_3$, (2) holds.

(5.18) $L_3'=W_2$.

PROOF. By (5.17), $\mho^1(L_3)\leq L_2\cap C_5=\langle t\rangle F_2$ and hence $\mho^1(L_3)$ is contained in $\langle t\rangle F_2\cap(\langle t\rangle F_2)^v=\langle t\rangle W_2$. Put $\overline{N(Y_2)}=N(Y_2)/Y_2$. Then $\bar{t}^{L_3}=\bar{t}\,\bar{W}_2$ and $C_{L_3}(\bar{t})=L_2$. As $\bar{t}\notin\mho^1(\bar{L}_2)$, we have that $\mho^1(L_3)\leq W_2$. Now (5.18) follows from (5.15), since $W=M'\leq L_3'$ and $N_L(Y)$ acts irreducibly on W_2/W.

(5.19) *Definition.* Let $D_5=C_{C_5}(V_1)$, $S_3=C_{R_3}(V_1)$, $M_3=C_{L_3}(Y_2)$, $D_3=S_3\cap C_5$, $F_3=M_3\cap D_3$, and $W_3=C_{L_3}(W_2)$.

(5.20) $C_5=\langle t\rangle D_5>D_5=D_3D_3^u$, $D_3\cap D_3^u=D_1$, $L_3=\langle t\rangle M_3>M_3$, *and* $R_3=\langle t\rangle S_3>S_3$.

PROOF. Set $Q=[L_3, J]M_2$. Then $L_3/M_2=Q/M_2\times L_2/M_2$ by (5.17) and (5.18). We can verify that $C_Q(J_0)=Y_2$. Thus $Y_2\leq Z(Q)$ by [16, (2.4)], so $Q=M_3$ and $R_3=\langle t\rangle S_3>S_3=M_3S$ by (5.7). Moreover, $C_5/C_1=T/C_1\times T^u/C_1$ by (5.12) where T denotes $C_5\cap R_3$. Thus (5.20) holds.

(5.21) $W_3=F_3\cap F_3^v$ *and* W_3 *is elementary abelian of order* q^{10}.

PROOF. Let X be a cyclic subgroup of $\langle U_2, v\rangle$ of order $q+1$. Then X

centralizes U_4 and acts fixed-point-freely both on Y and on M/W by (3.10). Set $B=C_{M_3}(X)$ and $Q=BW_2$. We first show that $Q=W_3$. Since $M_2/W_2\cong M/W$ and $M_3/M_2\cong W/Y$ as X-modules, we have $M_3/W_2=Q/W_2\times M_2/W_2$ by (5.18). Likewise $W_2=C_{W_2}(X)\times[W_2, X]$ with $[W_2, X]=Y_2$ and $C_{W_2}(X)=C_W(X)=U_4$ since $W_2/Y_2\cong W/Y$ and $Y_2/Y\cong Y$ as X-modules. Thus $Q=B\times Y_2$ and $B\rhd U_4$. Note that J_2 acts irreducibly on U_4 and normalizes B since J_2 centralizes X. Hence $Z(Q)\geq U_4Y_2=W_2$. As $M_3=QM$ and $C_M(W)=W$, this implies that $Q=W_3$.

Then $\mho^1(W_3)\leq U_4$. Since $N_L(Y)$ normalizes W_3, we conclude that W_3 is elementary abelian. As $C_S(W/V)=D$, it follows from (5.12) (2) that $C_{R_3}(W_1/V_1)=\langle t\rangle D_3$. Hence $W_3\leq F_3$ and so we get $W_3=F_3\cap F_3^v$.

(5.22) (1) *$\langle t\rangle F_3$ and L_3 are characteristic subgroups of R_3 and $R_3'=F_2$.*
(2) $N(\langle t\rangle W_3)=L_3N_C(Y)$.

PROOF. As $F=S'\leq R_2'\leq F_1$ and J acts irreducibly on F_1/F, (5.3) (3) yields that $R_2'=F_1$ and $F_1\leq R_3'\leq C_5\cap L_3\cap R_2\cap S_3=F_2$. Thus $R_3'=F_2$ by (5.15) since J acts irreducibly on F_2/F_1. Using (5.10) and (5.15) we have $Z(R_3)=V$, $Z_2(R_3)=Y_1$, and $Z_3(R_3)\leq\langle t\rangle W_2$. Moreover, $Z_3(R_3)$ contains W_2, for $R_3=W_3R_2$ and W_3 is abelian. Therefore, $Z_3(R_3)=W_2$ since $R_3 \not\trianglerighteq \langle t\rangle Y_1$ by (5.14). Set $T=R_3\cap C_5$, so that $T=\langle t\rangle D_3$ and $L_3\cap T=\langle t\rangle F_3$. Note that $W_3\lhd R_3$ by (5.18), and $T/W_3\cong\langle t\rangle D/W$. Hence $\mho^1(T)\leq W_3\cap C_1=W_1$. In particular, $\bar{L}_3$ and $\bar{T}$ are elementary abelian where $\bar{R}_3=R_3/W_2$. Since $\mathscr{E}^*(S/W)=\{M/W, D/W\}$, we conclude that $\mathscr{E}^*(\bar{R}_3)=\{\bar{L}_3, \bar{T}\}$ and $Z(\bar{R}_3)=\overline{\langle t\rangle F_3}$. As $|L_3|\neq|T|$, (1) holds. Since $\mathscr{E}^*(\langle t\rangle W_3)=\{W_3, \langle t\rangle W\}$, (2) is a consequence of (5.17).

(5.23) *Definition.* Let $S_4=C_{R_4}(V_1)$, $L_4=C_{R_4}(Y)$, $M_4=C_{R_4}(Y_2)$, and $D_4=R_4\cap D_5$.

(5.24) $L_4=\langle t\rangle M_4>M_4$, $R_4/L_3=L_4/L_3\times R_3/L_3$, *and* $R_4'=F_3$.

PROOF. It follows from (5.14) that $R_4/R_3\cong F/W$ as J-modules. We first show that $\bar{R}_4=R_4/L_3$ is abelian. If $q\geq 4$, then J_3 centralizes R_4/R_3 and acts regularly on $\bar{R}_3^\sharp$ since $\bar{R}_3\cong S/M$. Thus $\bar{R}_4=C_{\bar{R}_4}(J_3)\times\bar{R}_3$ and so $\bar{R}_4$ is abelian. If $q=2$, then $|\bar{R}_3|=2$. Since J acts irreducibly on R_4/R_3, $\bar{R}_4$ is abelian or else extra-special. As $\bar{R}_4$ has order 2^4, it must be abelian.

Set $Q=[R_4, J_0]L_3$. Then $\bar{R}_4=\bar{R}_3\times\bar{Q}$ by the above. Set $B=C_Q(V_1)$. Thus $Q=\langle t\rangle B>B$ by (5.20) and $B/M_3\cong F/W$ as J-modules. We can now verify that $C_B(J_0)=Y_2$, which is normal in B by (5.22) since $Z(F_2)=Y_2$. So $Z(B)\geq Y_2$ by [16, (2.4)]. As $R_4=QS$ and $C_S(Y)=M$, this implies that $Q=L_4$ and $B=M_4$. Now we have that $F_2=R_3'\leq R_4'\leq S_4\cap L_3\cap C_5=F_3$. Since J acts irreducibly on F_3/F_2, it follows from (5.17) that $R_4'=F_3$.

(5.25) *$D_5'=V_1$, $C_5'=D_1$, and $N(C_5)=C_5N_C(V)$. Furthermore, $D_5/V_1=W_3/V_1\times W_3^u/V_1$.*

PROOF. Set $H=C(J_0)$, $E=E(H)$, and $X=Y_2Y_2^u$. Then $E\cap D_5$ contains X since $V_1\le E$ and $\langle u,v\rangle$ normalizes E. By (5.20) we have $Y_2\cap Y_2^u=Y_2\cap Y_2^u\cap D_1=Y_1\cap Y_1^u=V_1$. This implies that $|X|=q^6$ and $E\cap D_5=X\in \mathrm{Syl}_2(E)$, and thus X/V_1 is abelian by (5.6).

Note that $D_5/D_4\cong D/F$ and $D_4/D_3\cong F/W$ as J-modules and that F_3 is normal in R_5 by (5.24). Set $Q=[D_5, J_0]F_3$. Then $\tilde{Q}=[\tilde{D}_5, J_0, J_0]=[\tilde{D}_4, J_0]$ and $\tilde{D}_4=\tilde{D}_3\times\tilde{Q}$ where $\tilde{D}_5=D_5/F_3$. Moreover, since $H\cap D_5=X$ and since J_0 centralizes D_5/Q, D_5/Q is abelian by the above. Hence $D_5'\le Q\cap D_1=F_3\cap D_1=F_1$. Then $V\le D_5'\le V_1$, for $N_L(V)$ acts irreducibly on D_1/V_1. Since D is not normal in R_2 by (5.3) (3), we conclude that $D_5'=V_1$. Likewise $Y_1\le C_5'\le D_5\cap C_1=D_1$ and we have $C_5'=D_1$. Put $\bar{C}_5=C_5/V_1$. Then $\bar{C}_5\cap C(\bar{t})=\bar{C}_1$ since $\mathscr{E}^*(\langle t\rangle V_1)=\{V_1, \langle t\rangle V\}$, and so $\mathscr{E}^*(\bar{C}_5)=\{\bar{D}_5, \bar{C}_1\}$. This shows that $N(C_5)=N(C_1)$. Finally, as $D_3=W_3D_1$ and $W_3\cap D_1=W_1$, it follows from (5.20) that $D_5/V_1=W_3/V_1\times W_3^u/V_1$.

(5.26) *Definition.* Let $L_6=O_2(N(L_3)\cap N(M_3))$ and $R_6=L_6S$.

(5.27) (1) *$N(L_3)\cap N(M_3)=L_6N_C(Y)$ and $L_6/L_3=L_4/L_3\times L_4^v/L_3\cong M/W$ as $N_C(Y)$-modules.*

(2) $\mathscr{E}^*(R_6/L_3)=\{L_6/L_3, R_4/L_3\}$.

(3) $N(R_3)=R_4N_C(S)$.

PROOF. Set $H=N(L_3)\cap N(M_3)$ and $\overline{N(W_3)}=N(W_3)/W_3$. We have $N(L_3)\le N(W_3)$ by (5.18) and $C_H(\bar{t})=L_3N_C(Y)$ by (5.22). Set $T=C_H(Y)\cap C_H(W_2/Y_2)$. Then T contains L_4 and $C_T(\bar{t})=L_3C_C(L)$ by (3.15). Now $\bar{t}^{L_4}=\bar{t}\bar{F}_3$. As $\bar{F}_3\cap\bar{F}_3^v=1$, this implies that $L_4\cap L_4^v=L_3$. Counting orders we have $\bar{t}^T=\bar{t}^H=\bar{t}\bar{M}_3$ and $H=N_C(Y)T$. Moreover, $T=O(C)\times O_2(T)$ with $O_2(T)=L_4L_4^v$ since $[O(C), L_4]=1$ by (5.12) (4). As $O_2(N_C(Y))=\langle t\rangle M$, we get that $L_6=O_2(T)$.

As $N_L(Y)$ acts irreducibly on $\bar{M}_3$, L_6 centralizes $\bar{M}_3$. Thus the map defined by $L_3x\mapsto[\bar{t}, x]$ for $x\in L_6$ is a $N_C(Y)$-isomorphism of L_6/L_3 onto $\bar{M}_3$, which maps L_4/L_3 onto $\bar{F}_3$. Now $\mathscr{E}^*(S/W)=\{M/W, D/W\}$ implies (2) since $\bar{M}_3\cong M/W$. Set $K=N(R_3)$. Then as $Z(F_2)=Y_2$, K is contained in $N_H(\langle t\rangle F_3)$ by (5.22). Since $R_4\le K$, we have that $\bar{t}^K=\bar{t}\bar{F}_3$. Moreover, $C_K(\bar{t})=L_3N_C(S)$. Thus (3) holds.

(5.28) *Definition.* Let $M_6=C_{L_6}(Y_2)$ and $F_4=M_4\cap D_4$.

(5.29) (1) $L_6=\langle t\rangle M_6>M_6=M_4M_4^v$.

(2) *$Z(S_4)=V_1$, $Z_2(S_4)=Y_2$, $Z_3(S_4)=W_3$, and $Z_4(S_4)=F_4$.*

PROOF. (1) follows from (5.24) and (5.27). By (5.9) and (5.15) we have

$Z(S_4)=V_1$ and $Z_2(S_4)\leq Y_2$. As $S_4=M_4D_4$ and D_4/V_1 is abelian, it follows that $Z_2(S_4)=Y_2$. Let X be a cyclic subgroup of $\langle U_2, v\rangle$ of order $q+1$ and put $\widetilde{N(Y_2)}=N(Y_2)/Y_2$. Then, since X centralizes W/Y and $C_{M/W}(X)=1$, we get that $\tilde{M}_6\cap C(X)=\tilde{W}_3$, for $M_6/M_3\cong M_3/W_3\cong M/W$ as $N_L(Y)$-modules. Since $W_3\lhd M_6$, [16, (2.4)] yields that M_6 centralizes $\tilde{W}_3$. In particular, $Z(\tilde{S}_4)\geq\tilde{W}_3$. On the other hand, $N_{R_5}(S_2)=R_3$ by (5.17) since $R_2=C_1S_2$ and $C_1\lhd R_5$, so $Z_3(S_4)$ is contained in S_3. As $\tilde{S}_3=\tilde{W}_3*\tilde{S}$, we conclude that $Z_3(S_4)=W_3$. Put $\overline{N(W_3)}=N(W_3)/W_3$. Then $Z(\bar{M}_6)\geq\bar{M}_3$ since $N_L(Y)$ acts irreducibly on $\bar{M}_3$, so $\bar{M}_4=\bar{M}_3*\bar{F}_4$ and $\bar{S}_4=\bar{F}_4*\bar{S}$. Therefore, $Z_4(S_4)=F_4$.

(5.30) *$M_6'=W_3$, $L_6'=M_3$, and $N(L_6)=L_6N_C(Y)$. Furthermore, $M_6/W_3=F_4/W_3\times F_4^v/W_3$.*

PROOF. Since $R_4\lhd R_6$ by (5.27) (2), we have $S_4=C_{R_4}(V_1)\lhd R_6$ and thus F_4 is also normal in R_6 by (5.29). As $F_3F_3^v=M_3$ and $M_3F_4=M_4$, it then follows that $M_6/W_3=F_4/W_3\times F_4^v/W_3$. By (5.18), $W=M'\leq M_3'\leq W_2$. As $N_L(Y)$ acts irreducibly on W_2/W, (5.15) yields that $M_3'=W_2$, and so $W_2\leq M_6'\leq W_3$. Now, $N_{R_5}(M_2)=R_3$ by (5.17) since $R_2=C_1M_2$. Hence $M_6'=W_3$. Then $L_6'=M_3$ or W_3. From (5.22) (2) we have $L_6'=M_3$ and $\mathscr{E}^*(L_6/W_3)=\{M_6/W_3, L_3/W_3\}$ as well. Finally, $Z(M_3)=Y_2$ by (5.15) and $Z_2(M_3)=W_3$, so it follows from (5.27) that $N(L_6)=L_6N_C(Y)$.

(5.31) *Definition.* Let $P=O_2(N(R_4))\cap C(V_1)$.

(5.32) *$N(R_4)=N_C(S)P$ and $\langle t\rangle P/R_4=R_5/R_4\times R_6/R_4\cong S/F$ as $N_C(S)$-modules.*

PROOF. Set $K=N(R_4)$ and $T=\langle t\rangle F_4$. We first show that S_4 is characteristic in R_4 and that $N_K(T)=R_4N_C(S)$. Using (5.10) and (5.15) we can verify that $Z(R_4)=V$, $Z_2(R_4)=Y_1$, and $Z_3(R_4)=W_2$. Furthermore, $M_4=C_{R_4}(Y_1)$ and $Z(M_4)=Y_2$. Since $R_4'=F_3$ and $W_3=C_{F_3}(W_2)$, it then follows that $S_4=C_{R_4}(W_3/Y_2)$ is characteristic in R_4. Setting $\widetilde{N(W_3)}=N(W_3)/W_3$ we have $C_K(\tilde{t})=R_3N_C(S)$ and $\mathscr{E}^*(\tilde{T})=\{\tilde{F}_4, \widetilde{\langle t\rangle F_3}\}$ by (5.22). Thus $N_K(T)$ is contained in B where $B=N(\langle t\rangle F_3)\cap N(F_3)\cap N(W_3)$. Now $\tilde{t}^{R_4}=\tilde{t}\tilde{F}_3$ and $C_B(\tilde{t})=R_3N_C(S)$. Hence $\tilde{t}^B=\tilde{t}\tilde{F}_3$ and $N_K(T)=B=R_4N_C(S)$ as required.

By (3.8) (2) and (5.29) we have $\mathscr{E}^*(S_4/W_3)=\{M_4/W_3, D_4/W_3\}$, so both M_4 and D_4 are characteristic in S_4. Now, K contains R_5 and R_6 by (5.14) and (5.27) (2). Set $\overline{N(F_4)}=N(F_4)/F_4$ and $E=C_K(\bar{S}_4)$. Then E contains R_5 and R_6 since J acts irreducibly both on $\bar{M}_4$ and on $\bar{D}_4$. Moreover, $C_K(\bar{t})=R_4N_C(S)$ by the above and so $C_E(\bar{t})=R_4C_C(L)$ by (3.15). Here note that $\bar{t}^{R_5}=\bar{t}\bar{D}_4$ by (5.12) and $\bar{t}^{R_6}=\bar{t}\bar{M}_4$ (see the proof of (5.27)). Thus $R_5\cap R_6=R_4$. The map defined by $x\mapsto[\bar{t}, x]$ for $x\in E$ is a $N_C(S)$-homomorphism of E onto $\bar{S}_4$. In particular, $\bar{t}^E=\bar{t}\bar{S}_4$ and so $\bar{t}^E=\bar{t}^K$. Hence $K=N_C(S)E$. Moreover, $E=O(C)\times O_2(E)$ with $O_2(E)=R_5R_6$ since $O(C)$ centralizes

R_5 and R_6. As $O_2(N_C(S))=\langle t\rangle S$, we have $O_2(K)=O_2(E)$. Finally, counting orders we get that $O_2(K)=\langle t\rangle P>P=D_5M_6$.

(5.33) *$Z(P)=V_1$, $Z_2(P)=Y_2$, $Z_3(P)=W_3$, $Z_4(P)=F_4$, and $P/F_4=D_5/F_4\times M_6/F_4$.*

PROOF. As $L_3S_3=C_1S_3=R_3$, we have $N_{R_6}(S_3)=N_{R_5}(S_3)=R_4$ by (5.27) (3). Thus $N_P(S_3)=S_4$ by (5.32) since the only J-invariant proper subgroups of S/F are M/F and D/F. From this and (5.29) we can determine $Z(P)$ and $Z_i(P)$, $i=2,3,4$. Next set $S_5=D_5S$ and $S_6=M_6S$, so that $R_j=\langle t\rangle S_j>S_j$, $j=5,6$. By (5.27) (2) we have $\mathscr{E}^*(S_6/M_3)=\{M_6/M_3, S_4/M_3\}$, and so $\mathscr{E}^*(S_6/W_3)=\{M_6/W_3, D_4/W_3\}$ since $\mathscr{E}^*(S_4/W_3)=\{M_4/W_3, D_4/W_3\}$. Hence $M_6\lhd P$. Likewise (3.8) (2) yields that $\mathscr{E}^*(S_5/D_3)=\{S_4/D_3, D_5/D_3\}$ since $D_5/D_3\cong D/W$ as $N_C(S)$-modules (see the proofs of (5.12) and (5.14)). Therefore, $\mathscr{E}^*(S_5/W_3)=\{M_4/W_3, D_5/W_3\}$ and $D_5\lhd P$.

(5.34) *$(\langle t\rangle P)'=S_4$ and $N(\langle t\rangle P)=N_C(S)P$. Furthermore, if $q=2$ then $\langle t\rangle P$ is a Sylow 2-subgroup of G.*

PROOF. Set $Q=\langle t\rangle P$. Then by (5.25) and (5.30), Q' contains $D_1M_3=S_3$. As $Q'\leq P\cap R_4=S_4$ and J acts irreducibly on S_4/S_3, (5.27) (3) yields that $Q'=S_4$. Now $\mathscr{E}^*(Q/F_4)=\{P/F_4, R_4/F_4\}$ since $N_Q(\langle t\rangle F_4)=R_4$ (see the proof of (5.32)). Therefore, we have $N(Q)=N(R_4)$. If $q=2$, then Q is a Sylow 2-subgroup of $N(R_4)=N_C(S)P$.

(5.35) *Definition.* $N_1=E(N(D_5) \bmod D_5)$, $N_3=O^{2'}(E(C(J_0))\cap N(Y_2))$, and $N_2=N_3M_6$.

(5.36) *$N_1/D_5\cong SL(2,q^3)\times SL(2,q^3)$ or $SL(2,q^6)$ and in the former t interchanges the components of N_1/D_5 and in the latter t acts on N_1/D_5 as an involutive field automorphism. Moreover, $P\in \mathrm{Syl}_2(N_1)$ and $C_L(V)\leq N_1$.*

PROOF. Set $K=C_L(V)$, $H=N(D_5)$, and $\bar{H}=H/D_5$. Then $\bar{H}\cap C(\bar{t})=\overline{N_C(V)}$ by (5.25). Thus $\bar{K}$ is a standard subgroup of $\bar{H}$ isomorphic to $SL(2,q^3)$ and $\langle \bar{t}\rangle$ is a Sylow 2-subgroup of $\bar{H}\cap C(\bar{K})$. Moreover, $J_6\leq K$ and $\bar{P}=[\bar{P},\bar{J}_6]$. Since $\bar{P}$ is elementary abelian of order q^6, $\bar{N}_1$ is determined by [8] and [16, (2.10)].

(5.37) *$N_3/Y_2\cong SL(2,q)\times SL(2,q)$ or $SL(2,q^2)$ according as $E(C(J_0))$ is semisimple of type $PSL(3,q)\times PSL(3,q)$ or $PSL(3,q^2)$. Furthermore, $O_2(N_2)=M_6$ and $P\in\mathrm{Syl}_2(N_2)$.*

PROOF. Let $E=E(C(J_0))$ and $X=Y_2Y_2^u$. Then $X=C_{D_5}(J_0)$ and X is a Sylow 2-subgroup of E (see the proof of (5.25)). Now, Y_2 is an element of $\mathscr{E}^*(X)$ and hence N_3/Y_2 is determined. Notice that $N_3=\langle X, X^v\rangle\geq\langle U_2, v\rangle$. Hence N_3 normalizes M_6 and so $O_2(N_2)=M_6$ and $P=M_6X$ is a Sylow 2-subgroup of N_2.

(5.38) *If $q\geq 4$, then $N_2=E(N(M_6) \bmod M_6)$.*

PROOF. Assume that $q\geq 4$ and set $K=N_L(Y)'$, $H=N(M_6)$, and $\bar{H}=H/M_6$.

Then $\bar{H}\cap C(\bar{t})=\overline{N_C(Y)}$ by (5.30) and so $\bar{K}$ is a standard subgroup of $\bar{H}$ isomorphic to $SL(2, q)$ and $\langle\bar{t}\rangle$ is a Sylow 2-subgroup of $\bar{H}\cap C(\bar{K})$. Thus (5.38) follows from (5.37) and [8].

6. Case for ${}^3D_4(q^6)$.

Let $G_1=\langle N_1, N_2\rangle$. In this section we argue under Hypothesis (6.1) to show that $G_1=\langle L^G\rangle\cong{}^3D_4(q^6)$.

(6.1) *Hypothesis.* $N_1/D_5\cong SL(2, q^6)$ and $N_2/M_6\cong SL(2, q^2)$.

(6.2) $N(P)=N(D_5)\cap N(M_6)$.

PROOF. Since $N_{N_1}(P)$ acts irreducibly on P/D_5 by (6.1), we have $C_P(W_3/V_1)=D_5$. Likewise $C_P(Y_2)=M_6$, and so the assertion follows from (5.33).

(6.3) $G_1\cong{}^3D_4(q^6)$ *and* P *is a Sylow 2-subgroup of* G_1.

PROOF. By (6.1), $P\cap P^u=D_5$ and $P\cap P^v=M_6$. Then as in [16, (6.3)] we have

(a) $P=D_5(P\cap P^v\cap P^{uv}\cap P^{vuv}\cap P^{uvuv}\cap P^{vuvuv})$
$=M_6(P\cap P^u\cap P^{vu}\cap P^{uvu}\cap P^{vuvu}\cap P^{uvuvu})$,

(b) $P\cap P^{(uv)^3}=1$.

Next, since N_1 contains J_0, we can take a complement X of P in $N_{N_1}(P)$ such that $X\geq J_0$. Then $X\leq C(J_0)$ and so $N_{N_1}(P)$ normalizes N_2. Also, $N_{N_2}(P)$ normalizes N_1 by (6.2). Now apply (2.1) to obtain (6.3).

(6.4) $G_1=\langle L^G\rangle$.

PROOF. By (4.2) $m(C)=5n+1$, while in G_1 the centralizer of every involution has 2-rank $10n$. Thus $t^G\cap P=\varnothing$. Set $H=O^2(G)$. If $q=2$, then by (5.34) it follows that P is a Sylow 2-subgroup of H. If $q\geq 4$, then as in [16, (6.4)] we again have that $P\in \mathrm{Syl}_2(H)$. Since the group ${}^3D_4(2^{3n})$ is characterized by its Sylow 2-subgroups [11], (6.4) holds.

7. Case for ${}^3D_4(q^3)\times{}^3D_4(q^3)$.

In this section we use, unless otherwise specified, the following three bar conventions:

$$\widehat{N(W_3)}=N(W_3)/W_3\,, \qquad \widetilde{N(Y_2)}=N(Y_2)/Y_2\,, \qquad \text{and} \qquad \overline{N(V_1)}=N(V_1)/V_1\,.$$

(7.1) *Hypothesis.* $N_2/M_6\cong SL(2, q)\times SL(2, q)$.

In (7.2) through (7.7) we argue under Hypothesis (7.1).

(7.2) *Definition.* Let $K_3=O^{2'}(N_B(Y_2))$ where B denotes a component of $E(C(J_0))$.

Moreover, set $Q_3=P\cap K_3$ and $K_2=K_3M_6$.

(7.3) $N_3=K_3\times K_3^t$, $Y_2=O_2(K_3)O_2(K_3^t)$, *and* $Q_3=O_2(K_3)O_2(K_3^u)\in \mathrm{Syl}_2(K_3)$. *Furthermore, K_3 is isomorphic to* $N_3\cap C=\langle U_2, v\rangle Y$.

PROOF. Notice that $E=E(C(J_0))$ is semisimple of type $PSL(3, q)\times PSL(3, q)$ and $Z(E)$ has odd order. Let B be a component of E. As in (5.37), $Y_2Y_2^u$ is a Sylow 2-subgroup of E and Y_2 is a maximal elementary abelian subgroup of $Y_2Y_2^u$. Hence $Q_3\in \mathrm{Syl}_2(B)$ and $B\cap Y_2\in \mathscr{E}^*(Q_3)$. We have that $O_2(K_3)=B\cap Y_2$ and $N_3=K_3\times K_3^t$. Since N_3 contains $\langle U_2, v\rangle$ and $Y_2\cap C=Y$, it follows that $K_3\cong N_3\cap C=\langle U_2, v\rangle Y$. Now, u centralizes V_1 and normalizes E. Then since $1\neq Z(Q_3)=V_1\cap Q_3$, we get that $B=B^u$ and $Q_3=O_2(K_3)O_2(K_3^u)$.

(7.4) $N_2/M_6=K_2/M_6\times K_2^t/M_6$ *with* $K_2/M_6\cong SL(2, q)$ *and* $N(P)\leq N(M_6)$.

PROOF. Since $K_3=\langle Q_3, Q_3^v\rangle$ normalizes M_6, the first assertion holds. In particular, P/M_6 has no proper subgroup which is normal in $\langle t\rangle N_{N_2}(P)/M_6$ if $q\geq 4$ and the only $\langle t\rangle$-invariant proper subgroup of P/M_6 is SM_6/M_6 if $q=2$. As $C_S(Y)=M$, it follows that $C_P(Y_2)=M_6$. Hence $M_6\lhd N(P)$.

(7.5) *Definition.* Let $K_4=K_2^tK_3$.

(7.6) $\hat{N}_2=\hat{K}_4\times \hat{K}_4^t$ *and* $\hat{K}_4$ *is isomorphic to* $\langle U_2, v\rangle M/W$. *Furthermore,* $K_4=K_3O_2(K_4)\geq W_3$ *and* $M_6=O_2(K_4)O_2(K_4^t)$.

PROOF. We first show that $C_{N_2}(\hat{M}_6)=M_6$. If $q\geq 4$, then N_2/M_6 has no proper $\langle t\rangle$-invariant normal subgroups, so $C_{N_2}(\hat{M}_6)=M_6$. Suppose $q=2$. Arguing as in (7.4) we have $C_P(\hat{M}_6)=M_6$ since $C_S(M/W)=M$. Thus $C_{N_2}(\hat{M}_6)$ is a subgroup of $O(N_2 \bmod M_6)$. Now, the only $\langle t\rangle$-invariant normal subgroups of N_2/M_6 contained in $O(N_2/M_6)$ are 1 and $O(N_2/M_6)$. Since $O(\langle U_2, v\rangle)$ is contained in $O(N_2 \bmod M_6)$ and does not centralize M/W, we obtain $C_{N_2}(\hat{M}_6)=M_6$.

Next we show that $\hat{M}_6=Z(\hat{K}_2)\times Z(\hat{K}_2^t)$. Put $T=P\cap K_2$. Then $P=TT^t$ and $K_2=\langle T, T^v\rangle$. By (5.33) we have $\hat{F}_4=Z(\hat{P})=Z(\hat{T})\cap Z(\hat{T})^t$. Now $Z(\hat{K}_2)\leq \hat{N}_2\cap C(\hat{M}_6)=\hat{M}_6$, so $Z(\hat{K}_2)=Z(\hat{T})\cap Z(\hat{T})^v$. Likewise $Z(\hat{T}\hat{T}^{vt})=Z(\hat{T})\cap Z(\hat{T})^{vt}\neq 1$. Thus $\hat{F}_4\neq Z(\hat{T})$ and so $Z(\hat{K}_2)\neq 1$ by (5.30). Moreover, $Z(\hat{K}_2)\cap Z(\hat{K}_2^t)=1$ since $Z(\hat{N}_2)\leq \hat{F}_4\cap \hat{F}_4^v=1$. Set $\hat{I}=Z(\hat{K}_2)\times Z(\hat{K}_2^t)$. As J is a subgroup of $C(J_0)$ of odd order, J normalizes K_3. Hence $N_L(Y)$ normalizes $\hat{I}$. Since $\hat{M}_6\cap C(t)=\hat{M}_3$ by (5.22) and $N_L(Y)$ acts irreducibly on $\hat{M}_3\cong M/W$, it follows that $\hat{I}\cap C(t)=\hat{M}_3$. Therefore, $\hat{I}=\hat{M}_6$ as required.

By (7.3), $\hat{N}_3=\hat{K}_3\times \hat{K}_3^t$ with $\hat{K}_3\cong SL(2, q)$, so $\hat{N}_2=\hat{N}_3\hat{M}_6=\hat{H}\times \hat{H}^t$ where $\hat{H}=\hat{K}_3Z(\hat{K}_2^t)$. In particular, $\hat{H}$ is isomorphic to $\hat{N}_2\cap C(t)=\widehat{\langle U_2, v\rangle}\hat{M}_3\cong\langle U_2, v\rangle M/W$. As $(\langle U_2, v\rangle M)'$ contains M, this yields that $\hat{H}'\geq O_2(\hat{H})$. Since $\hat{K}_2=\hat{H}\times Z(\hat{K}_2)$ and

$O_2(\hat{H})=Z(\hat{K}_2^t)$, we conclude that $\hat{H}=\hat{K}_4$. Finally, $K_4 \geq M_6'=W_3$ by (5.30), and (7.6) holds.

(7.7) $N_1/D_5 \cong SL(2, q^3) \times SL(2, q^3)$.

PROOF. Suppose false. Then $N_1/D_5 \cong SL(2, q^6)$. Take a complement X of P in $N_{N_1}(P)$ such that $X \geq J_0$. Then X is a subgroup of $C(J_0)$ of odd order, so X normalizes K_3. Since $X \leq N(M_6)$ by (7.4), X normalizes K_4 as well. However, since $F_4 < F_4 O_2(K_4) < M_6$, X then can not act irreducibly on P/D_5, a contradiction.

(7.8) *Hypothesis.* $N_1/D_5 \cong SL(2, q^3) \times SL(2, q^3)$.

In (7.9) through (7.13) we argue under Hypothesis (7.8).

(7.9) *Definition.* Let $N_1/D_5 = K_1/D_5 \times K_1^t/D_5$ with $K_1/D_5 \cong SL(2, q^3)$.

(7.10) $N(P) \leq N(D_5)$.

PROOF. By (7.8), $\langle t \rangle N_{N_1}(P)$ acts irreducibly on P/D_5. Hence $C_P(W_3/V_1)=D_5$ and so D_5 is characteristic in P by (5.33).

(7.11) $\bar{N}_1 = \bar{K}_1' \times \bar{K}_1'^t$, $\bar{D}_5 = O_2(\bar{K}_1')O_2(\bar{K}_1')^t$, *and* $\bar{K}_1'$ *is isomorphic to* $C_L(V)/V$.

PROOF. Since N_1/D_5 has no proper $\langle t \rangle$-invariant normal subgroups, $C_{N_1}(\bar{D}_5)=D_5$. Set $A=P \cap K_1$ and $H=Z(A \bmod V_1)$. Then we have $1 \neq Z(\bar{A}\bar{A}^{ut})=\bar{H} \cap \bar{H}^{ut}$. As $P=AA^t$, $\bar{Y}_2=Z(\bar{P})=\bar{H} \cap \bar{H}^t$. We remark that J normalizes K_1 and A by (7.10) and that $\bar{D}_5 \cap C(\bar{t})=\bar{D}_1$ by (5.3) (1) since $\mathscr{E}^*(\langle t \rangle V_1)=\{V_1, \langle t \rangle V\}$.

We shall show that $Z(\bar{K}_1) \neq 1$. Suppose $Z(\bar{K}_1)=1$. Then $\bar{H} \cap \bar{H}^u=1$ since $K_1=\langle A, A^u \rangle$. Note that $\bar{D}_5 \cap C(J_0)=\bar{Y}_2 \times \bar{Y}_2^u$ and $[\bar{D}_5, J_0]=[\bar{F}_4, J_0]$. These imply that $H \leq F_4$, for $\bar{H} \cap \bar{Y}_2^u \leq \bar{H} \cap \bar{H}^u=1$. As $\bar{H} \cap \bar{H}^{ut} \neq 1$, we have $Y_2 < H$ and so $Y_2 < H \cap W_3$ by (5.33) since $H \lhd P$. Moreover, every J-invariant proper subgroup of $\tilde{W}_3$ has order q^3 since J acts transitively both on $\tilde{W}_2^\#$ and on $(W_3/W_2)^\#$. Then as $\bar{H} \cap \bar{H}^t = \bar{Y}_2$, it follows that $|H \cap W_3|=q^7$. If $H \leq W_3$, then $\bar{H} \cap \bar{H}^{ut} \leq \bar{W}_3 \cap \bar{W}_3^u=1$, a contradiction. If $HW_3=F_4$, then $|H|=q^{13}$, which is impossible since $HH^u \leq D_5$ and $\bar{H} \cap \bar{H}^u=1$. Now, every J-invariant proper subgroup of F_4/W_3 has order q^3 since J acts transitively both on $(F_3/W_3)^\#$ and on $(F_4/F_3)^\#$. Therefore, we conclude that $|H|=q^{10}$.

Put $\bar{Q}=\bar{H}^{ut} \cap \bar{W}_3$. If $\bar{Q}=1$, then $\bar{D}_5=\bar{H}^{ut} \times \bar{W}_3$. As A normalizes A^{ut} and $\bar{H}^{ut}$, this implies that $(D_5/W_3) \cap C(A)$ is isomorphic to $\bar{H}^{ut} \cap C(\bar{A})$. Now $\bar{H}^{ut} \cap C(\bar{A})=\bar{H} \cap \bar{H}^{ut}$, whose order is at most q^3 since $|H: H \cap W_3|=q^3$ and $\bar{Q}=1$. However, $(D_5/W_3) \cap C(A)$ contains F_4/W_3 by (5.33), a contradiction. Thus $\bar{Q} \neq 1$. Note that $\bar{W}_3 \cap C(J_0)=\bar{Y}_2$. Then, since $\bar{H}^{ut} \cap \bar{Y}_2 \leq (\bar{H}^u \cap \bar{H})^t=1$ and J_0 normalizes $\bar{H}^{ut}$, we have $\bar{Q} \leq [\bar{W}_3, J_0]$. If $\bar{Q}=[\bar{W}_3, J_0]$, then $\bar{H}^{ut} \cap \bar{D}_1 \geq [\bar{W}_1, J_0]$, which is impossible since $\bar{H}\cap\bar{H}^t=\bar{Y}_2$ and $\bar{D}_1=\bar{D}_5\cap C(\bar{t})$. Hence $\bar{Q}<[\bar{W}_3, J_0]$. Since $[\bar{W}_3, J_0]\cong\tilde{W}_3$ as J-modules and since every J-invariant proper subgroup of $\tilde{W}_3$ has order q^3, we must have $|\bar{Q}|=q^3$. Now $\bar{Q}\lhd\bar{A}$ implies $\bar{Q}\cap\bar{H}\neq 1$, so it follows that $\bar{Q}\leq\bar{H}$ by the action of J. By a property of $K_1^t/D_5\cong SL(2,q^3)$ we have $K_1^t=\langle A^{ut}, A^{utx}\rangle$ for each $x\in A^t-D_5$, and so $\bar{H}^{ut}\cap\bar{H}^{utx}=1$ since we are assuming that $Z(\bar{K}_1)=1$. Put

$$\Omega=\bigcup_x \bar{Q}^x$$

where x runs all over A^t. As $\bar{Q}$ is a subgroup of $\bar{H}^{ut}$ of order q^3 and $|A/D_5|=q^3$, $|\Omega|=q^3(q^3-1)+1$. On the other hand, $\bar{Q}\leq\bar{H}\cap\bar{W}_3$ by the above and hence $\Omega\leq\bar{H}\cap\bar{W}_3$. Since $|\bar{H}\cap\bar{W}_3|=q^5$, this is impossible. Therefore, $Z(\bar{K}_1)\neq 1$ as required.

Put $E=Z(K_1^t \bmod V_1)$. Since $N_1=\langle P, P^u\rangle$ and $C_{N_1}(\bar{D}_5)=D_5$, $Z(\bar{P})=\bar{Y}_2$ implies $Z(\bar{N}_1)=\bar{Y}_2\cap\bar{Y}_2^u=1$. Then $\bar{E}\cap\bar{E}^t=1$ and so $\bar{E}\cong(\bar{E}\bar{E}^t)\cap C(\bar{t})\leq\bar{D}_5\cap C(\bar{t})=\bar{D}_1$. Since $C_L(V)\leq N_1$ and $C_L(V)$ acts irreducibly on $\bar{D}_1$, it follows that $\bar{D}_5=\bar{E}\times\bar{E}^t$. This implies that $D_5/E=Z(K_1/E)$. Put $B=K_1'E$. Then, since the Schur multiplier of $SL(2,q^3)$ is trivial, $B/E\cong SL(2,q^3)$ and $\bar{K}_1=\bar{B}\times\bar{E}^t$. Hence $\bar{N}_1=\bar{B}\times\bar{B}^t$ and $\bar{B}$ is isomorphic to $\bar{N}_1\cap C(\bar{t})=\overline{C_L(V)}$. In particular, $\bar{B}$ is perfect and so $\bar{K}_1'=\bar{B}$. Thus $\bar{E}=O_2(\bar{K}_1')$ and $\bar{K}_1'\cong C_L(V)/V$, and (7.11) holds.

(7.12) $N_1=K_1'*K_1'^t=K_1''*K_1''^t$ *and* $K_1'\geq V_1$.

PROOF. By (5.25), $K_1'\geq V_1$. As $N_1=\langle P, P^u\rangle$, we have $V_1\leq Z(N_1)$. Set $Z=Z(K_1 \bmod V_1)$. Then $[K_1', Z]=1$ by the three-subgroup lemma. So $K_1=K_1'*Z$ and $K_1'=K_1''*V_1$. Moreover, $[K_1', K_1'^t]\leq V_1$ by (7.11), so $[K_1'', K_1'^t]=1$ and we have $N_1=K_1'*K_1'^t$.

(7.13) $N_2/M_6\cong SL(2,q)\times SL(2,q)$.

PROOF. It follows from (7.11) that $F_4<F_4O_2(K_1')<D_5$. Then, since $N_{N_2}(P)$ normalizes K_1 by (7.10), $N_{N_2}(P)$ does not act irreducibly on P/M_6. Hence (7.13) holds.

(7.14) *Hypothesis.* $N_1/D_5\cong SL(2,q^3)\times SL(2,q^3)$ and $N_2/M_6\cong SL(2,q)\times SL(2,q)$.

From now on we assume Hypothesis (7.14).

(7.15) *Definition.* Let $Q_1=P\cap K_1'$ and $Q_2=P\cap K_4$.

A complement of P in $N_{N_1}(P)$ containing J_0 normalizes K_3. Hence $N_{N_1}(P)$ normalizes K_2. On the other hand, $N_{N_2}(P)$ normalizes K_1 by (7.10). It follows from (7.6) that $F_4<O_2(K_4)F_4<M_6$. Since $Q_3\leq D_5$ and $Q_2=Q_3O_2(K_4)$, this implies that $D_5<Q_2D_5<P$. In view of the action of $N_{N_1}(P)$ on P/D_5, we have that $Q_2D_5=P\cap K_1$ or $P\cap K_1^t$. Changing notation if necessary, we may assume that

(7.16) $Q_2D_5=P\cap K_1$.

Next we will show

(7.17) $Q_2=Q_1W_3$.

PROOF. Put $A=P\cap K_1$, $B=P\cap K_2$, and $E=O_2(K_1')M_6$. By (7.11), $F_4<O_2(K_1')F_4<D_5$ and so $M_6<E<P$. When $q\geq 4$, the only proper subgroups of P/M_6 normalized by $N_{N_2}(P)$ are B/M_6 and B^t/M_6, so $E=B$ or B^t. When $q=2$, we again have that $E=B$ or B^t since $E^t\neq E$. If $E=B^t$, then $O_2(K_1')$ centralizes $Z(\hat{B})^t\geq O_2(\hat{K}_4)$. As $O_2(K_4)D_5=Q_2D_5=A$, $O_2(K_1')$ then centralizes $\hat{A}$. Since $P=AA^t$, this together with (5.33) and (7.11) forces $O_2(K_1')\leq F_4$, a contradiction. Hence $E=B$. Recall that $C_P(J_0)\leq D_5$ and $[P, J_0]\leq M_6$. As $Q_1\cap D_5=O_2(K_1')$, we have $Q_1\leq E$. Thus $Q_1M_6=B$.

Now, Q_1F_4 and Q_2F_4 are subgroups of $A\cap B$. Moreover, $|A\cap B|=q^4|F_4|$ since $P=D_5M_6=AB$. Therefore, comparing orders we get that $Q_1F_4=Q_2F_4=A\cap B$.

Let H be a complement of Q_1 in $K_1'\cap N(Q_1)$. Then $\bar{H}\bar{Q}_1\cong J_6S/V$ by (7.11). Hence we have $Z_2(\bar{Q}_1)=\bar{Q}_1\cap\bar{W}_3$, $Z_3(\bar{Q}_1)=\bar{Q}_1\cap\bar{F}_4$, and H acts fixed-point-freely on $Q_1\cap F_4/Q_1\cap W_3$. Now $Q_1M_6=B$ implies $|Q_1: Q_1\cap M_6|=q$, so $Q_1=(Q_1\cap M_6)(Q_1\cap D_5)$. Thus H acts fixed-point-freely on $Q_1/Q_1\cap D_5\cong Q_1\cap M_6/Q_1\cap F_4$ as well.

Set $U=A\cap B$. Then by the above $\hat{U}=\hat{Q}_1\hat{F}_4=\hat{Q}_1\times(\hat{Q}_1\cap\hat{F}_4)^t$. On the other hand, by (7.6) $\hat{U}=\hat{Q}_2\hat{F}_4=\hat{Q}_2\times(\hat{Q}_2\cap\hat{F}_4)^t$ and $\hat{Q}_2\cong S/W$. In particular, $\hat{U}'=\hat{Q}_2'$ has order q^3. As U/F_4 is abelian, we have $\hat{U}'=\hat{Q}_1\cap\hat{F}_4=\hat{Q}_2\cap\hat{F}_4$. Hence $\hat{U}\cap\hat{M}_6=(\hat{Q}_1\cap\hat{M}_6)\times\hat{U}'^t=(\hat{Q}_2\cap\hat{M}_6)\times\hat{U}'^t$. Now recall that $[\hat{Q}_1\cap\hat{M}_6, H]=\hat{Q}_1\cap\hat{M}_6$ and H centralizes $\bar{Q}_1^t$ and that H normalizes Q_2 since H lies in $N_{N_1}(P)$. Thus we have $\hat{Q}_1\cap\hat{M}_6=[\hat{U}\cap\hat{M}_6, H]=\hat{Q}_2\cap\hat{M}_6$. Moreover, as $\hat{M}_6=[\hat{P}, J_0]$ is abelian, it follows that $\hat{Q}_i\cap\hat{M}_6=[\hat{Q}_i, J_0]$ and $\hat{U}\cap C(J_0)=\hat{Q}_i\cap C(J_0)$ for $i=1, 2$. Therefore $\hat{Q}_1=\hat{Q}_2$.

(7.18) *Definition.* Let $K_5=K_4'K_3Y_2$.

(7.19) $\tilde{N}_2=\tilde{K}_5\times\tilde{K}_5^t$ *and* $\tilde{K}_5$ *is isomorphic to* $\langle U_2, v\rangle M/Y$. *Furthermore,* $K_5=K_3O_2(K_5)$, $M_6=O_2(K_5)O_2(K_5^t)$, *and* $P\cap K_5=Q_1Y_2$.

PROOF. Let H be a complement of Q_1 in $K_1'\cap N(Q_1)$, so that $\bar{H}\bar{Q}_1\cong J_6S/V$ by (7.11). As in the proof of (7.17) $Z(\bar{Q}_1)=\bar{Q}_1\cap\bar{Y}_2$, $Z_2(\bar{Q}_1)=\bar{Q}_1\cap\bar{W}_3$, and H acts fixed-point-freely on $Q_1\cap W_3/Q_1\cap Y_2$. Therefore, H acts fixed-point-freely on $Q_1\cap M_6/Q_1\cap Y_2$. Also, $\tilde{P}=\tilde{Q}_1\times\tilde{Q}_1^t$ and by (7.17) $\tilde{Q}_2=\tilde{Q}_1\times\tilde{T}$ where $\tilde{T}$ denotes $(\tilde{Q}_1\cap\tilde{W}_3)^t$. Now, $\mathscr{E}^*(\langle t\rangle Y_2)=\{Y_2, \langle t\rangle Y\}$ and (5.3) (2) imply $\tilde{N}_2\cap C(t)=\widetilde{\langle U_2, v\rangle M}$. Hence $\tilde{Q}_1\cong S/Y$ and $|\tilde{Q}_1'|=q^6$. Then it follows that $\tilde{Q}_2'=\tilde{Q}_1'=\tilde{Q}_1\cap\tilde{F}_4$.

We have $Z(\tilde{K}_4)\geq\tilde{W}_3$ since v normalizes $\tilde{W}_3=Z(\tilde{P})$. As $\tilde{Q}_2=\tilde{Q}_1\times\tilde{T}$, by Gaschütz's theorem $\tilde{T}$ has a complement $\tilde{A}$ in $\tilde{K}_4$. Then $\tilde{A}\geq\tilde{K}_4'=\tilde{Q}_2'$, so we have

$\tilde{K}_4'\tilde{W}_3=\tilde{K}_4'\times\tilde{T}$. Moreover, since $\hat{K}_4\cong\langle U_2, v\rangle M/W$, it follows that $\hat{K}_4'\geq O_2(\hat{K}_4)$ and hence $\tilde{K}_4'\tilde{W}_3$ contains $O_2(\tilde{K}_4)$. Thus $O_2(\tilde{K}_4)=O_2(\tilde{K}_4'\tilde{W}_3)=O_2(\tilde{K}_4')\times\tilde{T}$. Then by (7.6) we have $\tilde{K}_4=\tilde{K}_3O_2(\tilde{K}_4)=\tilde{K}_5\times\tilde{T}$ with $\tilde{K}_5=\tilde{K}_3O_2(\tilde{K}_4')$. In particular, $K_5=K_3O_2(K_5)$.

Set $\tilde{X}=\tilde{Q}_2\cap\tilde{M}_6$. Then $\tilde{X}=(\tilde{Q}_1\cap\tilde{M}_6)\times\tilde{T}$ and furthermore $\tilde{X}=O_2(\tilde{K}_4)=O_2(\tilde{K}_5)\times\tilde{T}$. As shown above H acts fixed-point-freely on $\tilde{Q}_1\cap\tilde{M}_6$, while H centralizes $\tilde{T}$ by (7.11). Since H normalizes K_5, it follows that $\tilde{Q}_1\cap\tilde{M}_6=[\tilde{X}, H]=O_2(\tilde{K}_5)$. Moreover, $\tilde{Q}_2\cap C(J_0)=\tilde{Q}_1\cap C(J_0)$ since $\tilde{W}_3\cap C(J_0)=1$. As J_0 centralizes Q_3, comparing orders we must obtain $P\cap K_5=Q_3O_2(K_5)=Q_1Y_2$. Now, $\tilde{M}_6=O_2(\tilde{K}_5)\times O_2(\tilde{K}_5^t)$ and $K_3=\langle Q_3, Q_3^v\rangle$ centralizes $O_2(\tilde{K}_5^t)$ since $[Q_1, Q_1^t]=1$. Thus (7.19) holds.

(7.20) $Q_1=\{Q_3(Q_1\cap Q_1^v)\}\times Z(Q_3^t)$ *and* $Q_1Y_2=\{Q_3(Q_1\cap Q_1^v)\}\times O_2(K_3^t)$.

PROOF. Put $E=O_2(K_3)$. By (7.3) $\bar{Y}_2=\bar{E}\times\bar{E}^t$ and when $q\geq 4$, the only proper subgroups of $\bar{Y}_2$ normalized by $N_{N_3}(Q_3Q_3^t)$ are $\bar{E}$ and $\bar{E}^t$. Also, $\bar{Y}_2=Z(\bar{Q}_1)\times Z(\bar{Q}_1^t)$ by (7.11). Moreover, $N_{N_3}(Q_3Q_3^t)$ lies in $N_{N_2}(P)$ and so it normalizes Q_1. Hence $Z(\bar{Q}_1)=\bar{E}$ or $\bar{E}^t$. If $E^t\leq Q_1$, then $[E, Q_1]=1$ by (7.12) and so E^t lies in the center of $Q_1^tQ_3M_6=P$, a contradiction. Thus $E\leq Q_1$. This implies that $Q_3\leq Q_1$ since u normalizes $Q_1\cap D_5=O_2(K_1')$ and $Q_3=EE^u$.

Put $X=Q_1\cap Q_1^v$, so that X is a subgroup of $P\cap P^v=M_6$. By the above $Q_1\cap Y_2=EV_1=E\times(E^t\cap V_1)$. Since $E\cap E^t=V_1\cap V_1^v=1$ and $E^v=E$, we get that $X\cap Y_2=E$. Now, $O_2(K_5)=(Q_1\cap M_6)Y_2$ by (7.19), so $|O_2(K_5): Q_1\cap M_6|=q$. Thus it follows that $O_2(K_5)=X\times E^t$. Furthermore, K_3^t centralizes X by (7.12) since $K_3=\langle Q_3, Q_3^v\rangle$ and $Q_3\leq Q_1$. Hence $X=O_2(K_5)\cap C(K_3^t)$. As $E\cap V_1=Z(Q_3)$ and as $Q_1Y_2=Q_3O_2(K_5)$ by (7.19), comparing orders we obtain (7.20).

(7.21) *Definition.* Let $A_1=K_1''$, $A_2=K_3O_2(K_5')$, and $P_0=Q_3(Q_1\cap Q_1^v)$.

(7.22) (1) $N_1=A_1\times A_1^t$, $D_5=O_2(A_1)\times O_2(A_1^t)$, *and* $A_1\cong C_L(V)$.

(2) $N_2=A_2\times A_2^t$, $M_6=O_2(A_2)\times O_2(A_2^t)$, *and* $A_2\cong\langle U_2, v\rangle M$.

(3) $P_0=P\cap A_i$, $i=1, 2$.

PROOF. Set $Z=Z(Q_3)$, $E=O_2(K_3)$, and $X=Q_1\cap Q_1^v$. Then as $Q_3'=Z$, K_5' contains $ZZ^v=E$. Thus $E\leq O_2(K_5')$ and $O_2(A_2)=O_2(K_5')$. Also, $O_2(K_5)=O_2(K_5'Y_2)$ since $\tilde{K}_5\cong\langle U_2, v\rangle M/Y$. As $Q_1Y_2=P_0\times E^t\in\mathrm{Syl}_2(K_5)$ by (7.20) and as $E^t\leq Z(K_5)$, E^t has a complement A in K_5 by Gaschütz's theorem and so $K_5=A\times E^t$. Then $K_5'\leq A$ and we have $K_5'Y_2=K_5'\times E^t$. Therefore, $O_2(K_5)=O_2(K_5')\times E^t$ and $K_5=K_3O_2(K_5)=A_2\times E^t$. In the proof of (7.20) we have shown that $O_2(K_5)=X\times E^t$ and $[X, K_3^t]=1$. Since $C_{M_6}(J_0)=Y_2=E\times E^t$, we get that $O_2(K_5')=[O_2(K_5), J_0]E=X$. Hence $N_2=A_2\times A_2^t$ and $P_0=P\cap A_2$. In particular, $A_2\cong N_2\cap C=\langle U_2, v\rangle M$.

Since $Z^t\leq V_1$ and Q_1 is a Sylow 2-subgroup of K_1', (7.20) shows that Z^t has a complement I in K_1' and so $K_1'=I\times Z^t$. This, together with (7.12), yields that

$K_1'=A_1\times Z^t$ and $N_1=A_1\times A_1^t$. Thus $A_1\cong N_1\cap C=C_L(V)$.

Set $B=P\cap A_1$. Then $Q_1=B\times Z^t$ by the above. On the other hand, $Q_1=P_0\times Z^t$ by (7.20). So $P_0'=B'$. Since $P=P_0\times P_0^t$ and P/F_4 is abelian, comparing orders we have that $P_0'=P_0\cap F_4$, and so $P_0'\geq E$. Since u normalizes $B\cap D_5=O_2(A_1)$, it follows from (7.3) that $Q_3=EE^u\leq B$. Now, $C_B(J_0)=C_{P_0}(J_0)=Q_3$ and $[B, J_0]=[Q_1, J_0]=[P_0, J_0]$. Therefore $B=P_0$. This completes the proof.

(7.23) *Definition.* Let $G_1=\langle N_1, N_2\rangle$ and $G_0=\langle A_1, A_2\rangle$.

(7.24) $C(G_1)$ *has odd order.*

PROOF. Since $C(G_1)$ is normal in $C(L)\cap N(G_1)$, this is a consequence of (4.4).

(7.25) $G_1=G_0\times G_0^t$ *and* $G_0\cong{}^3D_4(q^3)$.

PROOF. Proceed as in [16, (7.20)] and apply (2.1).

(7.26) *If* $q=2$, *then* $G=\langle t\rangle O^2(G)$ *and* $|O^2(G): G_1|$ *is odd.*

PROOF. In G_1 the centralizer of every involution has 2-rank $10n$, while $m(C)=5n+1$ by (4.2). Thus $t^G\cap P=\emptyset$ and so (5.34) and the Thompson fusion lemma yield the assertion.

(7.27) *For a Sylow* 2-*subgroup* Q *of* $N(G_1)$ *containing* P, *we have* $J_r(Q)=P$ *and* $J_r(C_Q(G_0^t))=P_0$.

PROOF. See (3.14) and [16, (7.21)].

(7.28) *If* $t\in N(G_1)^g$ *for an element* g *of* G, *then* $g\in N(G_1)$.

PROOF. Since $N_C(S)$ normalizes $\langle P, N_L(Y)'\rangle=N_2$, (5.3) (3) shows that $N(R)\leq N(G_1)$. Now proceed as in [16, (7.22)].

(7.29) (1) $|G: N(G_1)|$ *is odd.*

(2) $N(P_0)\leq N(G_1O(G))$.

PROOF. If $q\geq 4$, $N_2\lhd N(M_6)$ by (5.38). Then arguing as in (7.21) and (7.23) of [16] we have that $|G: N(G_1)|$ is odd and $N(P_0)\leq N(G_1)$. Next, suppose $q=2$. Let $\bar{G}=G/O(G)$, $H=N(P_0^t)$, and $H_1=H\cap O^2(G)$. Then $P_0\cong\bar{P}/\bar{P}_0^t$ is a Sylow 2-subgroup of $\bar{H}_1/\bar{P}_0^t$ by (7.26), so a result of Hughes [11] shows that $\bar{G}_0O(\bar{H}_1 \bmod \bar{P}_0^t)$ is a normal subgroup of $\bar{H}_1$ of odd index. Since $[G_0, P_0^t]=1$ and since $|H: H_1|\leq 2$, this implies that $L_{2'}(\bar{H})O(\bar{H})=\bar{G}_0O(\bar{H})$, where $L_{2'}(\bar{H})$ denotes the 2-layer of $\bar{H}$. Now $\bar{H}=N_{\bar{G}}(\bar{P}_0^t)$ is a 2-local subgroup of $\bar{G}$ and so $L_{2'}(\bar{H})=E(\bar{H})=\bar{G}_0$ by virtue of the condition (*) of the main theorem. As a consequence we have $E(C_{\bar{G}}(\bar{G}_0^t))=\bar{G}_0$, for $E(\bar{H})=E(C_{\bar{G}}(\bar{P}_0^t))$ and $P_0\leq G_0$. Hence $N_{\bar{G}}(\bar{G}_0^t)\rhd\bar{G}_0\bar{G}_0^t=\bar{G}_1$ and as $N_{\bar{G}}(\bar{P}_0)=\bar{H}^t\rhd\bar{G}_0^t$, it follows that $N(P_0)\leq N(G_1O(G))$. This completes the proof.

8. Conclusion of the proof.

Let G be a minimal counterexample to the main theorem. Then G satisfies Hypotheses (5.1) and (7.14). Suppose $O(G)\neq 1$ and let $\bar{G}=G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$ and $C_{\bar{G}}(\bar{L})$ has cyclic Sylow 2-subgroups and so the minimality of G implies that the conclusion of the main theorem holds for $\bar{G}$. But then by [16, (2.10)] the conclusion of the main theorem holds for G as well. Therefore we have $O(G)=1$. Now in view of (7.25), (7.27), (7.28), and (7.29) we can apply [14, Lemma (2.7)] to conclude that $E(G)\cong {}^3D_4(q^3)\times {}^3D_4(q^3)$, which is against the choice of G. The proof is complete.

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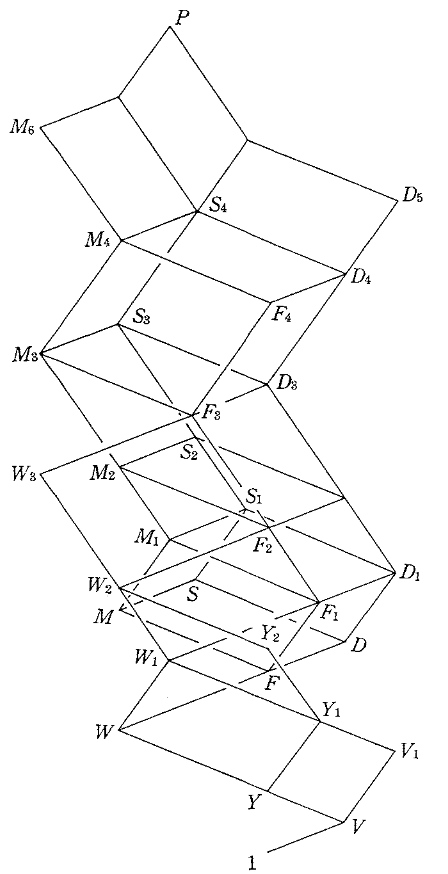
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Fig. 1.

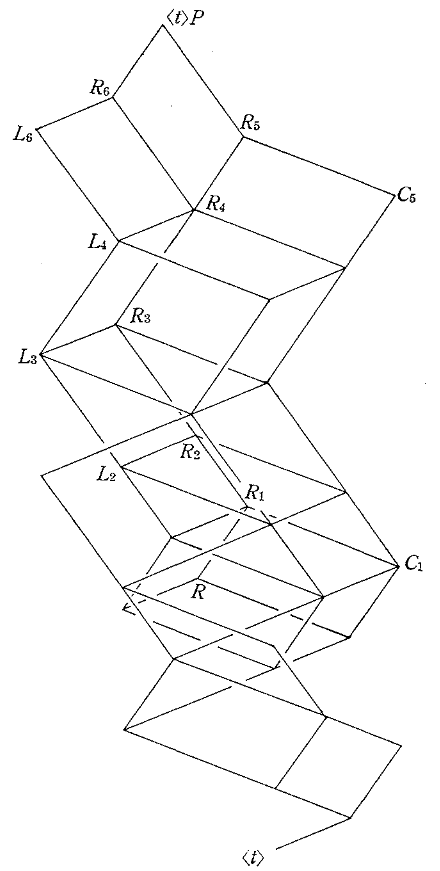
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Fig. 2.