Characteristic classes of $\mathscr{S}$-foliated vector bundles and Gel'fand-Fuks cohomology

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(Communicated by I. Tamura)

§ 0. Introduction

Let M be a C^∞-manifold and $\mathscr{S}$ be a geometric structure on M such as a Riemannian structure or a symplectic structure (see below for the precise definition of geometric structures considered in this paper). (In case there is no geometric structure on M we set $\mathscr{S}=\emptyset$.) Let $\mathfrak{A}(M,\mathscr{S})$ be the set of C^∞-vector fields on M which preserve the structure $\mathscr{S}$. (Such a vector field is called an $\mathscr{S}$-vector field.)

Suppose $\mathfrak{A}(M,\mathscr{S})$ is a Lie algebra with respect to the commutator [,]. With C^∞-topology $\mathfrak{A}(M,\mathscr{S})$ becomes a topological Lie algebra.

Let $C^p(\mathfrak{A}(M,\mathscr{S}))$ be the space of continuous skew symmetric p-forms on $\mathfrak{A}(M,\mathscr{S})$. Namely an element of $C^p(\mathfrak{A}(M,\mathscr{S}))$ is a continuous skew symmetric multi-linear form on $\mathfrak{A}(M,\mathscr{S})\times\cdots\times\mathfrak{A}(M,\mathscr{S})$ (p-times) with coefficients in $\boldsymbol{R}$.

The differential d: $C^p(\mathfrak{A}(M,\mathscr{S}))\to C^{p+1}(\mathfrak{A}(M,\mathscr{S}))$ is defined as

$$dc(\xi_1,\cdots,\xi_{p+1})\overset{\text{def}}{=}\sum_{i<j}(-1)^{i+j-1}c([\xi_i,\xi_j]\xi_1\cdots\hat{\xi}_i\cdots\hat{\xi}_j\cdots\xi_{p+1}).$$

The product $C^p(\mathfrak{A}(M,\mathscr{S}))\times C^q(\mathfrak{A}(M,\mathscr{S}))\to C^{p+q}(\mathfrak{A}(M,\mathscr{S}))$ is defined as

$$(c\cdot c')(\xi_1\cdots\xi_{p+q})\overset{\text{def}}{=}\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}\operatorname{sign}\begin{pmatrix}1&\cdots\cdots\cdots\cdots&p+q\\ i_1\cdots i_p&j_1\cdots&j_q\end{pmatrix}c(\xi_{i_1}\cdots\xi_{i_p})\cdot c'(\xi_{j_1}\cdots\xi_{j_q}).$$

By this product and the differential $C^*(\mathfrak{A}(M,\mathscr{S}))=\oplus C^p(\mathfrak{A}(M,\mathscr{S}))$ has the structure of a differential graded algebra (denoted shortly by D.G.A.).

The cohomology of this D.G.A. is a graded algebra and called Gel'fand-Fuks cohomology. This cohomology is denoted by $H^*_{GF}(\mathfrak{A}(M,\mathscr{S}))$ (cf. [6]).

Let K be a compact Lie group which acts effectively on M preserving the structure $\mathscr{S}$. K-basic elements of $C^*(\mathfrak{A}(M,\mathscr{S}))$ constitute a sub-algebra $C^*(\mathfrak{A}(M,\mathscr{S}),K)$. The cohomology of this algebra is denoted by $H^*_{GF}(\mathfrak{A}(M,\mathscr{S}),K)$.

Haefliger [5] has proved that this cohomology algebra gives characteristic classes for the category of Diff$(M,\mathscr{S})$-foliated K-bundles, where Diff$(M,\mathscr{S})$ is the subgroup of Diff M consisting of diffeomorphisms preserving the structure $\mathscr{S}$. This category is defined as follows.

Object: $(E, \mathcal{S})$

where E is a differentiable bundle $M \to E \to X$ with fiber M and structure group K, $\mathcal{S}$ is a Diff $(M, \mathscr{S})^{\delta}$-structure on E compatible with the bundle structure. (Diff $(M, \mathscr{S})^{\delta}$ is the group Diff $(M, \mathscr{S})$ equipped with the discrete topology.)

Morphism: f: $(E', \mathcal{S}') \to (E, \mathcal{S})$

where f is a bundle map $E' \to E$ which satisfies the condition $\mathcal{S}' = f^{-1}(\mathcal{S})$.

$\mathcal{S}$ induces a foliation $\mathscr{F}$ on E transversal to the fibers. $\mathscr{F}$ is called an $\mathscr{S}$-foliation and the pair $(E, \mathscr{F})$ is called an $\mathscr{S}$-foliated bundle.

In this paper we consider the case $M = \boldsymbol{R}^n$ (non-compact) and assume $\mathscr{S}$ satisfies the following condition. Let $(x_1, \cdots, x_n)$ be the canonical coordinate on $\boldsymbol{R}^n$ and $X = \sum_{i=1}^{n} f_i(\partial/\partial x_i)$ be a C^∞-vector field on $\boldsymbol{R}^n$. The necessary and sufficient condition for X to be an $\mathscr{S}$-vector field can be expressed by a system of linear-differential equations (not necessarily constant coefficient) w.r.t. $(f_1 \cdots f_n)$.

For example symplectic structure, volume-preserving structure and Riemannian structure satisfy this condition. We also consider $\mathscr{S}$-vector fields on $\boldsymbol{R}^n$ which vanish at 0. $\mathscr{S}$-vector fields on $\boldsymbol{R}^n$ which vanish at 0 constitute a topological Lie algebra $\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})$. $H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$ can be taken for the characteristic classes of $\mathscr{S}$-foliated vector bundles and $H^*_{GF}(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S}), K)$ can be taken for the characteristic classes of $\mathscr{S}$-foliated vector bundles whose base spaces are leaves.

THEOREM I. *Let*

$$\psi\colon\ C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})) \to \Omega^*_{DR}(X) \quad (\text{resp. } \psi\colon\ C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})) \to \Omega^*_{DR}(X))$$

be a continuous D.G.A. homomorphism with respect to the dual topology on $C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}))$ *and the* C^∞*-topology on* $\Omega^*_{DR}(X)$ *where* $\Omega^*_{DR}(X)$ *is the de Rham complex of* X. *Then, there is a unique foliated vector bundle structure* $\mathscr{F}$ *on* $X \times \boldsymbol{R}^n$ *(resp. with* $X \times \{0\}$ *a leaf) such that the characteristic homomorphism* $\lambda_{\mathscr{F}}$ *of* $\mathscr{F}$:

$$\lambda_{\mathscr{F}}\colon\ C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})) \to \Omega^*_{DR}(X) \quad (\text{resp. } C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})) \to \Omega^*_{DR}(X))$$

coincides with ψ.

This theorem shows that the existence of some kind of foliated vector bundle can be determined by an algebraic condition.

The author does not know, however, whether $\mathscr{F}$ may be an $\mathscr{S}$-foliation or not (see § 4).

Next, we consider deformations of $\mathscr{S}$-foliated vector bundles. Set $\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}) = \mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}) \otimes \boldsymbol{R}[t]/(t^2)$ (resp. $\hat{\mathfrak{A}}_0(\boldsymbol{R}^n, \mathscr{S}) = \mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S}) \times \boldsymbol{R}[t]/(t^2)$), where (t^2) stands for the ideal generated by t^2. Lie algebra structure is defined by

$$[\xi + t\eta, \xi' + t\eta'] = [\xi, \xi'] + t([\xi, \eta'] + [\eta, \xi']) .$$

It is shown that $H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K)$ (resp. $H^*_{GF}(\hat{\mathfrak{A}}_0(\boldsymbol{R}^n, \mathscr{S}), K)$) can be taken for the characteristic classes of deformations of $\mathscr{S}$-foliated vector bundles (resp. whose base spaces are leaves).

THEOREM II. *Assume $\mathscr{S}=\emptyset$, then the inclusions*

$$\begin{gathered} C^*_{pt}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)\subset C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K) \\ C^*_{pt}(\mathfrak{A}(\boldsymbol{R}^n), K)\subset C^*(\mathfrak{A}(\boldsymbol{R}^n), K) \\ \left(\begin{array}{l}\text{resp. } C^*_{pt}(\hat{\mathfrak{A}}_0(\boldsymbol{R}^n), K)\subset C^*(\hat{\mathfrak{A}}_0(\boldsymbol{R}^n), K) \\ \qquad\ C^*_{pt}(\mathfrak{A}_0(\boldsymbol{R}^n), K)\subset C^*(\mathfrak{A}_0(\boldsymbol{R}^n), K)\end{array}\right) \end{gathered}$$

induce isomorphisms on cohomologies, where C^*_{pt} stands for the sub D.G.A. consisting of elements supported at 0.

This theorem means that the above construction of characteristic classes is essentially equivalent to the construction given by Kamber-Tondeur (cf. [12]).

REMARK. It has been shown by Bott [2] that the inclusion $C^*_{pt}(\mathfrak{A}(\boldsymbol{R}^n))\subset C^*(\mathfrak{A}(\boldsymbol{R}^n))$ induces an isomorphism on cohomology. We prove Theorem II in similar way.

The author wishes to express his hearty thanks to Professor I. Tamura and Professor S. Morita for many valuable comments and suggestions.

§ 1. Construction of the characteristic homomorphism $H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)\to H^*_{DR}(X)$ (resp. $H^*_{GF}(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S}), K)\to H^*_{DR}(X)$)

We consider the case of $H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$. K acts on $\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$ as follows. For $g\in K$ and $\xi\in\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$ we define $g_*\xi\in\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$ as $(d/dt)(g\cdot \operatorname{Exp} t\xi\cdot g^{-1})|_{t=0}$, where $\operatorname{Exp} t\xi$ stands for the local 1-parameter transformation group of $\boldsymbol{R}^n$ generated by ξ.

Let k be the Lie algebra of K and $A\in k$, then $\exp tA\in K$ and acts on $\boldsymbol{R}^n$ preserving $\mathscr{S}$. Therefore $(d/dt)(\exp tA)|_{t=0}\in\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$ and k is a sub Lie algebra of $\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$. We define $C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$ as

$$C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)\overset{\text{def}}{=}\left\{\begin{array}{l} c\in C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}))\colon\ \iota(A)c=0 \text{ for } \forall A\in k \\ \qquad\qquad\qquad\qquad\quad g^*c=c \text{ for } \forall g\in K\end{array}\right\}.$$

Let $E\xrightarrow{\pi}X$ be an n-dim. vector bundle and $\mathscr{F}$ be an $\mathscr{S}$-foliation on E transversal to the fibers. Let $\{\varphi_U\}$ be the maximal atlas on E which represents $\mathscr{F}$. $\{U\}$ is an open covering of X and φ_U is an isomorphism $\pi^{-1}(U)\to U\times\boldsymbol{R}^n$ such that $p_1\circ\varphi_U=\pi$, where p_1 is the 1-st projection $U\times\boldsymbol{R}^n\to U$. We can write

$$\begin{array}{l} \varphi_U\circ\varphi_{U'}^{-1}\colon\ (U\cap U')\times\boldsymbol{R}^n\to(U\cap U')\times\boldsymbol{R}^n \\ \varphi_U\circ\varphi_{U'}^{-1}(x, z)=(x, r_{UU'}\circ z) \qquad x\in U\cap U' \quad z\in\boldsymbol{R}^n\ , \end{array}$$

here $r_{UU'}$: $U\cap U'\to \mathrm{Diff}(\boldsymbol{R}^n, \mathscr{S})$ is locally constant and $\{\pi^{-1}(U)\xrightarrow{\varphi_U} U\times \boldsymbol{R}^n \xrightarrow{p_2} \boldsymbol{R}^n\}$ determine leaves of $\mathscr{F}$. Let $\{\psi_V\}$ be the maximal atlas of E considered as a K-bunble,

$$\psi_V\circ\psi_{V'}^{-1}:\ (V\cap V')\times \boldsymbol{R}^n\to (V\cap V')\times \boldsymbol{R}^n$$
$$\psi_V\circ\psi_{V'}^{-1}(x, z)=(x, k_{VV'}(x)\circ z)\qquad x\in V\cap V'\quad z\in \boldsymbol{R}^n$$

where $k_{VV'}$: $V\cap V'\to K$ is a C^∞-map. $\{\psi_V\}$ naturally introduce a $\mathrm{Diff}(\boldsymbol{R}^n, \mathscr{S})$-structure on E. By compatibility of $\mathscr{F}$, $\{\varphi_U\}$ are $\mathrm{Diff}(\boldsymbol{R}^n, \mathscr{S})$-isomorphisms with respect to this $\mathrm{Diff}(\boldsymbol{R}^n, \mathscr{S})$-structure. Consider

$$\varphi_U\circ\psi_V^{-1}:\ (U\cap V)\times \boldsymbol{R}^n\to (U\cap V)\times \boldsymbol{R}^n$$
$$\varphi_U\circ\psi_V^{-1}(x, z)=(x, h_{UV}(x)\cdot z)\qquad x\in U\cap V\quad z\in \boldsymbol{R}^n$$

where h_{UV}: $U\cap V\to \mathrm{Diff}(\boldsymbol{R}^n, \mathscr{S})$ is a C^∞-map. The following equation holds

$$h_{U'V'}(x)=r_{U'U}\cdot h_{UV}(x)\cdot k_{VV'}(x)\ .$$

Now for h_{UV}: $U\cap V\to \mathrm{Diff}(\boldsymbol{R}^n, \mathscr{S})$ we define $(h'_{UV})_x$: $T_x(U\cap V)\to \mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$ as follows. Let

$$\xi\in T_x(U\cap V)\ ,\qquad \xi=\frac{d}{dt}x(t)\Big|_{t=0}\quad x(0)=x\ ,$$

then

$$(h'_{UV})_x(\xi)\overset{\text{def}}{=}\frac{d}{dt}(h_{UV}^{-1}(x(0))\circ h_{UV}(x(t)))\Big|_{t=0}\ .$$

For $c\in C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}))$ we define $h^*_{UV}c\in \Omega^*_{DR}(U\cap V)$ as

$$(h^*_{UV}c)(\xi_1\cdots\xi_q)(x)\overset{\text{def}}{=}c((h'_{UV})_x(\xi_1)\cdots(h'_{UV})_x(\xi_q))$$

where $\xi_i\in\mathfrak{A}(U\cap V)$, $x\in U\cap V$.

LEMMA. *If $c\in C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$, then $\{h^*_{UV}c\}$ determines a global form.*

PROOF. It suffices to show $h^*_{U'V'}c=h^*_{UV}c$. Let

$$\xi\in T_x(U\cap U'\cap V\cap V')\qquad \xi=\frac{d}{dt}x(t)\Big|_{t=0}\quad x(0)=x$$

and we compute

$$\begin{aligned}(h'_{U'V'})_x(\xi)&=\frac{d}{dt}(h_{U'V'}^{-1}(x(0))\cdot h_{U'V'}(x(t)))\Big|_{t=0}\\ &=\frac{d}{dt}[k_{VV'}^{-1}(x(0))\cdot h_{UV}^{-1}(x(0))\cdot r_{U'U}^{-1}][r_{U'U}\cdot h_{UV}(x(t))\cdot k_{VV'}(x(t))]\Big|_{t=0}\\ &=\frac{d}{dt}[k_{VV'}^{-1}(x(0))\cdot h_{UV}^{-1}(x(0))\cdot h_{UV}(x(t))\cdot k_{VV'}(x(0))][k_{VV'}^{-1}(x(0))k_{VV'}(x(t))]\Big|_{t=0}\\ &=(k_{VV'}^{-1}(x(0)))_*[(h'_{UV})_x(\xi)]+\frac{d}{dt}[k_{VV'}^{-1}(x(0))\cdot k_{VV'}(x(t))]\Big|_{t=0}\ .\end{aligned}$$

Since $(d/dt)[k_{V'V'}^{-1}(x(0))\cdot k_{VV'}(x(t))]|_{t=0}\in k$, we obtain

$$h_{U'V'}^{*}c=h_{UV}^{*}c\,. \qquad \text{q.e.d.}$$

The homomorphism $C^{*}(\mathfrak{A}(\boldsymbol{R}^{n},\mathscr{S}),K)\to\Omega_{DR}^{*}(X)$ constructed in this way is denoted by $\lambda_{\mathscr{S}}$.

LEMMA. $\lambda_{\mathscr{S}}\circ d=d\circ\lambda_{\mathscr{S}}$.

PROOF.

$$\begin{aligned}(dh^{*}c)(\xi_1,\cdots,\xi_{q+1})&=\sum_{i=1}^{q+1}(-1)^{i-1}\xi_i c(h'\xi_1,\cdots,\widehat{h'\xi_i},\cdots,h'\xi_{q+1})\\&\quad+\sum_{i<j}(-1)^{i+j}c(h'[\xi_i,\xi_j],h'\xi_1,\cdots,\widehat{h'\xi_i},\cdots,\widehat{h'\xi_j},\cdots,h'\xi_{q+1})\\(h^{*}dc)(\xi_1,\cdots,\xi_{q+1})&=\sum_{i<j}(-1)^{i+j-1}c([h'\xi_i,h'\xi_j],h'\xi_1,\cdots,\widehat{h'\xi_i},\cdots,\widehat{h'\xi_j},\cdots,h'\xi_{q+1})\,.\end{aligned}$$

It suffices to consider the case when $\deg c=1$, $h\colon \boldsymbol{R}^{2}\to\mathrm{Diff}(\boldsymbol{R}^{n},\mathscr{S})$, $\xi_1=\partial/\partial x$ and $\xi_2=\partial/\partial y$. Let $F^{\infty}(\boldsymbol{R}^{n})$ be the space of C^{∞}-functions on $\boldsymbol{R}^{n}$. For (x,y), $(\bar{x},\bar{y})\in\boldsymbol{R}^{2}$ we can define an automorphism of $F^{\infty}(\boldsymbol{R}^{n})$ by $[h_{(x,y)}^{-1}\circ h_{(\bar{x},\bar{y})}]^{*}\colon F^{\infty}(\boldsymbol{R}^{n})\to F^{\infty}(\boldsymbol{R}^{n})$. Fix (x,y). $p\in\boldsymbol{R}^{n}$, $f\in F^{\infty}(\boldsymbol{R}^{n})$. We have

$$\begin{cases}h'_{(x,y)}\left(\dfrac{\partial}{\partial x}\right)_{p}f=\dfrac{\partial}{\partial\bar{x}}([h_{(x,y)}^{-1}\circ h_{(\bar{x},y)}]^{*}f(p))\Big|_{\bar{x}=x}\\ h'_{(x,y)}\left(\dfrac{\partial}{\partial y}\right)_{p}f=\dfrac{\partial}{\partial\bar{y}}([h_{(x,y)}^{-1}\circ h_{(x,\bar{y})}]f(p))\Big|_{\bar{y}=y}\end{cases}.$$

If we describe $h'_{(x,y)}(\partial/\partial x)$ and $h'_{(x,y)}(\partial/\partial y)$ as

$$\begin{cases}h'_{(x,y)}\left(\dfrac{\partial}{\partial x}\right)=\displaystyle\sum_{i=1}^{n}a_i(x,y)\frac{\partial}{\partial z_i} & a_i(x,y)\in F^{\infty}(\boldsymbol{R}^{n})\\ h'_{(x,y)}\left(\dfrac{\partial}{\partial y}\right)=\displaystyle\sum_{i=1}^{n}b_i(x,y)\frac{\partial}{\partial z_i} & b_i(x,y)\in F^{\infty}(\boldsymbol{R}^{n})\end{cases}$$

then

$$\begin{aligned}&\left[h'_{(x,y)}\left(\frac{\partial}{\partial x}\right),h'_{(x,y)}\left(\frac{\partial}{\partial y}\right)\right]\\&\qquad=\sum_{k}\left(\sum_{i}a_i(x,y)\frac{\partial}{\partial z_i}b_k(x,y)\right)\frac{\partial}{\partial z_k}-\sum_{k}\left(\sum_{i}b_i(x,y)\frac{\partial}{\partial z_i}a_k(x,y)\right)\frac{\partial}{\partial z_k}\,.\end{aligned}$$

$$\begin{aligned}\sum_{i}a_i(x,y)\frac{\partial}{\partial z_i}b_k(x,y)&=h'_{(x,y)}\left(\frac{\partial}{\partial x}\right)b_k(x,y)\\&=\frac{\partial}{\partial x}([h_{(x,y)}^{-1}\circ h_{(\bar{x},y)}]^{*}b_k(x,y))\Big|_{\bar{x}=x}=\frac{\partial}{\partial\bar{x}}b_k(\bar{x},y)\Big|_{\bar{x}=x}\\&=\frac{\partial}{\partial x}b_k(x,y)\,.\end{aligned}$$

Similarly

$$\sum_i b_i(x, y)\frac{\partial}{\partial z_i} a_k(x, y)=\frac{\partial}{\partial y}a_k(x, y) .$$

Therefore

$$\left[h'\left(\frac{\partial}{\partial x}\right) h'\left(\frac{\partial}{\partial y}\right)\right]=\frac{\partial}{\partial x} h'\left(\frac{\partial}{\partial y}\right)-\frac{\partial}{\partial y} h'\left(\frac{\partial}{\partial x}\right) .$$

For $c \in C^1(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$

$$\begin{aligned}(dh^*c)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)&=\frac{\partial}{\partial x}c\left(h'\left(\frac{\partial}{\partial y}\right)\right)-\frac{\partial}{\partial y}c\left(h'\left(\frac{\partial}{\partial x}\right)\right)-c\left(h'\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]\right)\\ &=\frac{\partial}{\partial x}c\left(h'\left(\frac{\partial}{\partial y}\right)\right)-\frac{\partial}{\partial y}c\left(h'\left(\frac{\partial}{\partial x}\right)\right), \qquad \left(\because \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]=0\right)\end{aligned}$$

$$(h^*dc)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)= c\left(\left[h'\left(\frac{\partial}{\partial x}\right), h'\left(\frac{\partial}{\partial y}\right)\right]\right)$$

$$\begin{aligned}\frac{\partial}{\partial x}c\left(h'\left(\frac{\partial}{\partial y}\right)\right)&=\lim_{\bar{x}\to x}\frac{1}{\bar{x}-x}\left\{ c\left(h'_{(\bar{x},y)}\left(\frac{\partial}{\partial y}\right)\right)-c\left(h'_{(x,y)}\left(\frac{\partial}{\partial y}\right)\right)\right\}\\ &=c\left(\lim_{\bar{x}\to x}\frac{1}{\bar{x}-x}\left\{h'_{(\bar{x},y)}\left(\frac{\partial}{\partial y}\right)-h'_{(x,y)}\left(\frac{\partial}{\partial y}\right)\right\}\right)\\ &=c\left(\frac{\partial}{\partial x} h'\left(\frac{\partial}{\partial y}\right)\right) .\end{aligned}$$

Similarly

$$\frac{\partial}{\partial y}c\left(h'\left(\frac{\partial}{\partial x}\right)\right)=c\left(\frac{\partial}{\partial y} h'\left(\frac{\partial}{\partial x}\right)\right) .$$

Therefore the equation $[h'(\partial/\partial x), h'(\partial/\partial y)]=(\partial/\partial x)h'(\partial/\partial y)-(\partial/\partial y)h'(\partial/\partial x)$ implies $dh^*c=h^*dc$. q.e.d.

By this lemma, $\lambda_{\mathscr{F}}$ induces a homomorphism

$$H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)\to H^*_{DR}(X) .$$

This is also denoted by $\lambda_{\mathscr{F}}$.

DEFINITION. Let $(E_0, \mathscr{F}_0)$ and $(E_1, \mathscr{F}_1)$ be n-dim. $\mathscr{S}$-foliated vector bundles with common base space X. $(E_0, \mathscr{F}_0)$ and $(E_1, \mathscr{F}_1)$ are said to be integrable homotopic if and only if there exists an n-dim. $\mathscr{S}$-foliated vector bundle $(E, \mathscr{F})$ on $X\times I$ s.t. $(E, \mathscr{F})|X\times\{0\}=(E_0, \mathscr{F}_0)$ and $(E, \mathscr{F})|X\times\{1\}=(E_1, \mathscr{F}_1)$.

LEMMA. *$\lambda_{\mathscr{F}}$ depends only on the integrably homotopy class of $(E, \mathscr{F})$.*

PROOF. Let $c \in C^q(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$ be a closed form. We write $\lambda_{\mathscr{F}}(c) \in \Omega^*_{DR}(X\times I)$ as

$$\begin{aligned}\lambda_{\mathscr{F}}(c)=&\sum_{i_1<\cdots<i_q} f_{i_1\cdots i_q}(x_1 \cdots x_m, t)dx_{i_1}\wedge\cdots\wedge dx_{i_q}\\ &+\sum_{j_1<\cdots<j_{q-1}} g_{j_1\cdots j_{q-1}}(x_1 \cdots x_m, t)dx_{j_1}\wedge\cdots\wedge dx_{j_{q-1}}\wedge dt\end{aligned}$$

where $m=\dim X$ and t is the coordinate of $I=[0,1]$. By the naturality of the characteristic homomorphism $\lambda_{\mathscr{F}}$

$$\lambda_{\mathscr{F}_i}(c)=\lambda_{\mathscr{F}}(c)\,|\,X\times\{i\} \qquad (i=0,1)\ .$$

Therefore

$$\lambda_{\mathscr{F}_i}(c)=\sum_{i_1<\cdots<i_q} f_{i_1\cdots i_q}(x_1\cdots x_m, i)dx_{i_1}\wedge\cdots\wedge dx_{i_q} \qquad (i=0,1)\ .$$

The equation $d\lambda_{\mathscr{F}}(c)=\lambda_{\mathscr{F}}(dc)=0$ implies

$$\sum_{i_1<\cdots<i_q}(-1)^q\frac{\partial}{\partial t}f_{i_1\cdots i_q}(x_1\cdots x_m, t)dx_{i_1}\wedge\cdots\wedge dx_{i_q}\wedge dt$$
$$+\sum_{j_1<\cdots<j_{q-1}}\ \sum_{j\notin\{j_1\cdots j_{q-1}\}}\frac{\partial}{\partial x_j}g_{j_1\cdots j_{q-1}}(x_1\cdots x_m, t)dx_j\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{q-1}}\wedge dt=0\ .$$

We define φ: $\Omega^*_{DR}(X\times I)\to\Omega^{*-1}_{DR}(X)\hat{\otimes}F^\infty(I)$ as follows:

$$(\varphi\omega)(\xi_1\cdots\xi_{q-1})\overset{\text{def}}{=}\omega\left(\hat{\xi}_1\cdots\hat{\xi}_{q-1}\frac{\partial}{\partial t}\right) \qquad \text{for}\quad \omega\in\Omega^*_{DR}(X\times I)\ ,$$

where $\xi_i\in\mathfrak{A}(X)$ and $\hat{\xi}_i$ is the natural lift of ξ_i to $\mathfrak{A}(X\times I)$. Then

$$\varphi\lambda_{\mathscr{F}}(c)=\sum_{j_1<\cdots<j_{q-1}}g_{j_1\cdots j_{q-1}}(x_1\cdots x_m, t)dx_{j_1}\wedge\cdots\wedge dx_{j_{q-1}}$$
$$\int_0^1 dt:\quad \Omega^{*-1}_{DR}(X)\hat{\otimes}F^\infty(I)\to\Omega^{*-1}_{DR}(X)$$
$$\int_0^1\varphi\lambda_{\mathscr{F}}(c)dt=\sum_{j_1<\cdots<j_{q-1}}\left(\int_0^1 g_{j_1\cdots j_{q-1}}(x_1\cdots x_m)dt\right)dx_{j_1}\wedge\cdots\wedge dx_{j_{q-1}}\ .$$

The following equation holds.

$$d\left(\int_0^1\varphi\lambda_{\mathscr{F}}(c)dt\right)=(-1)^{q+1}\{\lambda_{\mathscr{F}_1}(c)-\lambda_{\mathscr{F}_0}(c)\}$$

$\because$) $d\left(\int_0^1\varphi\lambda_{\mathscr{F}}(c)dt\right)$

$$=\sum_{j_1<\cdots<j_{q-1}}\ \sum_{j\notin\{j_1\cdots j_{q-1}\}}\frac{\partial}{\partial x_j}\left(\int_0^1 g_{j_1\cdots j_{q-1}}(x_1\cdots x_m, t)dt\right)dx_j\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{q-1}}$$
$$=\int_0^1\left\{\sum_{j_1<\cdots<j_{q-1}}\ \sum_{j\notin\{j_1\cdots j_{q-1}\}}\frac{\partial}{\partial x_j}g_{j_1\cdots j_{q-1}}(x_1\cdots x_m, t)dx_j\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{q-1}}\right\}dt$$
$$=\int_0^1\left\{(-1)^{q+1}\sum_{i_1<\cdots<i_q}\frac{\partial}{\partial t}f_{i_1\cdots i_q}(x_1\cdots x_m, t)dx_{i_1}\wedge\cdots\wedge dx_{i_q}\right\}dt$$
$$=(-1)^{q+1}\sum_{i_1<\cdots<i_q}\left(\int_0^1\frac{\partial}{\partial t}f_{i_1\cdots i_q}(x_1\cdots x_m, t)dt\right)dx_{i_1}\wedge\cdots\wedge dx_{i_q}$$
$$=(-1)^{q+1}\{\lambda_{\mathscr{F}_1}(c)-\lambda_{\mathscr{F}_0}(c)\}\ .$$

Therefore the lemma results. q.e.d.

§2. Construction of the characteristic homomorphism $H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K) \to H^*_{DR}(X)$ (resp. $H^*_{GF}(\hat{\mathfrak{A}}_0(\boldsymbol{R}^n, \mathscr{S}), K) \to H^*_{DR}(X)$)

Let $\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}) = \mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}) \otimes \boldsymbol{R}[t]/(t^2)$. An element of $\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S})$ is expressed in the form of $\xi + t\eta$, $\xi, \eta \in \mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$. Lie algebra structure is defined as

$$[\xi + t\eta, \xi' + t\eta'] = [\xi, \xi'] + t([\xi, \eta'] + [\eta, \xi']) .$$

K acts on $\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S})$ by $g_*(\xi + t\eta) = g_*\xi + tg_*\eta$. We define $C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K)$ as

$$C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K) \overset{\text{def}}{=} \left\{ \begin{array}{l} c \in C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}))\colon\ \iota(A)c = 0 \text{ for } \forall A \in k \\ \qquad\qquad\qquad\qquad\quad g^*c = c \text{ for } \forall g \in K \end{array} \right\} .$$

Let $(E, \mathscr{F})$ be an n-dim. $\mathscr{S}$-foliated vector bundle on X and $\hat{\mathscr{F}} = \{\mathscr{F}_t\}$ be a deformation of $\mathscr{F}$. (We fix the bundle structure of E.) Then $\{\mathscr{F}_t\}$ is a foliation on $E \times \boldsymbol{R}$ and $(E \times \boldsymbol{R}, \{\mathscr{F}_t\})$ is a foliated vector bundle structure on X (not necessarily an $\mathscr{S}$-foliated structure). The structure group of this bundle is $K \times \mathrm{id}$. Let $\{h_{UV}\}$ be the transition functions associated with $(E \times \boldsymbol{R}, \{\mathscr{F}_t\})$:

$$h_{UV}\colon\ U \cap V \to \mathrm{Diff}\, \boldsymbol{R}^{n+1} .$$

Also let $\{h^t_{UV}\}$ be the transition functions associated with $(E, \mathscr{F}_t)$

$$h^t_{UV}\colon\ U \cap V \to \mathrm{Diff}\, (\boldsymbol{R}^n, \mathscr{S}) .$$

By definition of $\{\mathscr{F}_t\}$, there is no $(\partial/\partial t)$-component of $(\hat{h}'_{UV})_x \xi \in \mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})$ and the equation $(\hat{h}'_{UV})_x \xi|_{E_x \times \{t\}} = (h^{t\prime}_{UV})_x \xi$ holds for $\xi \in T_x(U \cap V)$, where E_x stands for the fiber over $x \in X$.

LEMMA. *If* $\sum_{i=1}^n f_i(x, t)(\partial/\partial x_i)_x$ *is an* $\mathscr{S}$*-vector field then* $\sum_{i=1}^n (\partial/\partial t) f_i(x, t)|_{t=0} (\partial/\partial x_i)_x$ *is also an* $\mathscr{S}$*-vector field.*

PROOF. Let $L_1(f_1(x), \cdots, f_n(x)) = 0, \cdots, L_m(f_1(x), \cdots, f_n(x)) = 0$ be the system of linear-differential equations which represent the necessary and sufficient condition for $X = \sum f_i(\partial/\partial x_i)$ to be an $\mathscr{S}$-vector field. Since each L_k is a linear differential equation,

$$L_k\left(\frac{\partial}{\partial t} f_1(x, t)\Big|_{t=0}, \cdots, \frac{\partial}{\partial t} f_n(x, t)\Big|_{t=0} \right) = \frac{\partial}{\partial t} L_k(f_1(x, t), \cdots, f_n(x, t))\Big|_{t=0} = 0 .$$

Therefore $\sum_{i=1}^n (\partial/\partial t) f_i(x, t)|_{t=0} (\partial/\partial x_i)_x$ is an $\mathscr{S}$-vector field. q.e.d.

By this lemma $J^1_{E \times 0}((\hat{h}'_{UV})_x \xi) = (h_{UV})_x \xi + t \cdot (d/dt)(h^{t\prime}_{UV})_x \xi|_{t=0}$ is contained in $\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S})$. This is denoted by $\hat{\xi}_x$. For $c \in C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K)$, $\hat{h}^*_{UV} c \in \Omega^*_{DR}(U \cap V)$ is defined as

$$(\hat{h}^*_{UV} c)(\xi_1 \cdots \xi_q)(x) \overset{\text{def}}{=} c(\hat{\xi}_{1,x} \cdots \hat{\xi}_{q,x})$$

where $\xi_i \in \mathfrak{A}(U \cap V)$, $x \in U \cap V$.

In the same way as $\lambda_{\mathscr{F}}$ we can construct the characteristic homomorphism $\lambda_{\hat{\mathscr{F}}}$: $H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K) \to H^*_{DR}(X)$.

Next we consider the relation to $H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$. Following [8], we define a map var: $C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K) \to C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K)$ as follows.

$$(\mathrm{var}\cdot c)(\xi_1 + t\eta_1 \cdots \xi_q + t\eta_q) \overset{\mathrm{def}}{=} \sum_{i=1}^{q} c(\xi_1 \cdots \xi_{i-1}\, \xi_i\, \xi_{i+1} \cdots \xi_q)\,.$$

Though var is not an multiplicative homomorphism, the relation

$$\mathrm{var}\,(c\cdot c') = \mathrm{var}\; c\cdot c' + c\cdot \mathrm{var}\; c'$$

holds. Since $d\cdot \mathrm{var} = \mathrm{var}\cdot d$ holds, var induces a cohomology homomorphism: $H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K) \to H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K)$. This homomorphism is also denoted by var.

PROPOSITION. *The following diagrams are commutative*

$$\begin{array}{ccc} H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K) & & \\ {\scriptstyle (d/dt)\lambda_{\mathscr{F}_t}|_{t=0}}\Big\downarrow & \searrow^{\mathrm{var}} & \\ & \supset & H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K) \\ & \swarrow_{\lambda_{\hat{\mathscr{F}}}} & \\ H^*_{DR}(X) & & \end{array} \qquad \begin{array}{ccc} H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K) & & \\ {\scriptstyle \lambda_{\mathscr{F}}}\Big\downarrow & \searrow^{p^*} & \\ & \supset & H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K) \\ & \swarrow_{\lambda_{\hat{\mathscr{F}}}} & \\ H^*_{DR}(X) & & \end{array}$$

where p is the natural projection.

PROOF.

$$\hat{\xi} = (h'_{UV})\xi + t\cdot\left(\frac{d}{dt} h^{t\prime}_{UV}\xi\Big|_{t=0}\right)$$

$$\begin{aligned}(\lambda_{\hat{\mathscr{F}}}\cdot \mathrm{var}\cdot c)(\xi_1 \cdots \xi_q) &= (\mathrm{var}\cdot c)(\hat{\xi}_1 \cdots \hat{\xi}_q) \\ &= \sum_{i=1}^{q} c\left(h'_{UV}\xi_1 \cdots h'_{UV}\xi_{i-1} \frac{d}{dt} h^{t\prime}_{UV}\xi_i\Big|_{t=0} h'_{UV}\xi_{i+1} \cdots h'_{UV}\xi_q\right) \\ &= \frac{d}{dt} c(h^{t\prime}_{UV}\xi_1 \cdots h^{t\prime}_{UV}\xi_q)\Big|_{t=0} = \frac{d}{dt}\lambda_{\mathscr{F}_t} c(\xi_1 \cdots \xi_q)\Big|_{t=0}\,. \end{aligned}$$

q.e.d.

DEFINITION. Elements of Ker{var: $H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K) \to H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n, \mathscr{S}), K)$} are called rigid classes.

If $\alpha \in H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}), K)$ is rigid, then by Prop. $\lambda_{\mathscr{F}_t}(\alpha)$ is constant for any deformation $\{\mathscr{F}_t\}$.

REMARK. In the case of $\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})$, it suffices to replace Diff $(\boldsymbol{R}^n, \mathscr{S})$ by $\mathrm{Diff}_0\,(\boldsymbol{R}^n, \mathscr{S})$ in § 1 and § 2. Where $\mathrm{Diff}_0\,(\boldsymbol{R}^n, \mathscr{S})$ is the group of $\mathscr{S}$-diffeomorphisms of $\boldsymbol{R}^n$ which fix the origin.

§ 3. Proof of Theorem I

THEOREM I. *Let ϕ: $C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S})) \to \Omega^*_{DR}(X)$ (resp. $C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})) \to \Omega^*_{DR}(X)$) be* *a continuous D.G.A. homomorphism with respect to the dual topology on* $C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}))$ *(resp.* $C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S}))$*) and the* C^∞*-topology on* $\Omega^*_{DR}(X)$*. Then there is a unique foliated vector bundle structure* $\mathscr{F}$ *on* $X\times\boldsymbol{R}^n$ *(resp. with* $X\times\{0\}$ *a leaf) such that the characteristic homomorphism* $\lambda_{\mathscr{F}}$ *of* $\mathscr{F}$ *coincides with* ψ (cf. [5] [11]).

It is not yet known whether $\mathscr{F}$ may be an $\mathscr{S}$-foliation or not.

PROOF. (i) The case when $\mathscr{S}=\emptyset$.

Let $\delta^i_{j_1\cdots j_s}(x)\in C^1(\mathfrak{A}(\boldsymbol{R}^n))$ be defined by

$$\delta^i_{j_1\cdots j_s}(x)\Big(\sum f_i\frac{\partial}{\partial x_i}\Big)=\frac{\partial^s f_i}{\partial x_{j_1}\cdots\partial x_{j_s}}(x)\,.$$

Let $U\subset X$ be a local coordinate neighborhood and $(\alpha_1\cdots\alpha_m)$ be the local coordinate on U. $\{\partial/\partial\alpha_1\cdots\partial/\partial\alpha_m\}$ forms a frame of $TX|U$. Consider $\langle\psi(\delta^i(x)), (\partial/\partial\alpha_j)(p)\rangle\in\boldsymbol{R}$ $(i=1,\cdots,n,\ j=1,\cdots,m)$ for $\psi(\delta^i(x))\in\Omega^1_{DR}(X)$ and $p\in U$. This is a C^∞-function w.r.t. p. We show that this is also a C^∞-function w.r.t. x. Set $t_s=(0\cdots\overset{s}{\check{t}}\,0\cdots 0)\in\boldsymbol{R}^n$ $(t\in\boldsymbol{R})$.

$$\lim_{t\to0}\frac{1}{t}\Big\{\Big\langle\psi(\delta^i(x+t_s)),\frac{\partial}{\partial\alpha_j}(p)\Big\rangle-\Big\langle\psi(\delta^i(x)),\frac{\partial}{\partial\alpha_j}(p)\Big\rangle\Big\}$$
$$=\Big\langle\psi\Big(\lim_{t\to0}\frac{\delta^i(x+t_s)-\delta^i(x)}{t}\Big),\frac{\partial}{\partial\alpha_j}(p)\Big\rangle\qquad(\because\ \psi\text{ is continuous}).$$

Since $\lim_{t\to0}(\delta^i(x+t_s)-\delta^i(x))/t=\delta^i_s(x)$, the limit exists. Therefore the limit

$$\lim_{t\to0}\frac{1}{t}\Big\{\Big\langle\psi(\delta^i(x+t_s)),\frac{\partial}{\partial\alpha_j}(p)\Big\rangle-\Big\langle\psi(\delta^i(x)),\frac{\partial}{\partial\alpha_j}(p)\Big\rangle\Big\}$$

exists. In the same way it is shown that $\langle\psi(\delta^i(x)),(\partial/\partial\alpha_j)(p)\rangle$ is C^∞-differentiable w.r.t. x. When $(\partial/\partial\alpha_j)(p)$ is fixed, $\sum_{i=1}^n\langle\psi(\delta^i(x)),(\partial/\partial\alpha_j)(p)\rangle(\partial/\partial x_i)_x$ is a C^∞-vector field on $\boldsymbol{R}^n$. We denote it by $\psi^*_p((\partial/\partial\alpha_j)(p))$. ψ^*_p is an $\boldsymbol{R}$-linear map $T_pU\to\mathfrak{A}(\boldsymbol{R}^n)$. Let π: $X\times\boldsymbol{R}^n\to X$ be a projection and π_x: $X\times\{x\}\to X$ $(x\in\boldsymbol{R}^n)$ be the restriction of π to $X\times\{x\}$. π_x is a diffeomorphism.

We define a codim n subspace $E_{(p,x)}$ of $T_{(p,x)}(X\times\boldsymbol{R}^n)$ as a space spanned by $\{\pi^*_x((\partial/\partial\alpha_j)(p))-\psi^*_p((\partial/\partial\alpha_j)(p))(x)\}$ $(j=1,\cdots,m)$. $E_{(p,x)}$ does not depend on the choice of local coordinates, and depends "smoothly" on (p,x). π: $E_{(p,x)}\to T_pX$ is a diffeomorphism. $\{E_{(p,x)}\}$ constitute a codim n-plane field on $X\times\boldsymbol{R}^n$. If this plane field is integrable, it determines a foliated vector bundle structure on $X\times\boldsymbol{R}^n$. Now we have

$$d\delta^i(x)=\sum_{s=1}^n\delta^s(x)\wedge\delta^i_s(x)$$

$$\left\langle \psi d\delta^i(x), \frac{\partial}{\partial\alpha_j}(p)\wedge\frac{\partial}{\partial\alpha_k}(p)\right\rangle$$

$$=\sum_{s=1}^{n}\left\langle \psi\delta^s(x)\wedge\psi\delta^i_s(x), \frac{\partial}{\partial\alpha_j}(p)\wedge\frac{\partial}{\partial\alpha_k}(p)\right\rangle$$

$$=\sum_{s=1}^{n}\left\langle \psi\delta^s(x), \frac{\partial}{\partial\alpha_j}(p)\right\rangle\left\langle \psi\delta^i_s(x), \frac{\partial}{\partial\alpha_k}(p)\right\rangle-\left\langle \psi\delta^s(x), \frac{\partial}{\partial\alpha_k}(p)\right\rangle\left\langle \psi\delta^i_s(x), \frac{\partial}{\partial\alpha_j}(p)\right\rangle$$

$$=\sum_{s=1}^{n}\left\langle \psi\delta^s(x), \frac{\partial}{\partial\alpha_j}(p)\right\rangle\frac{\partial}{\partial x^s}\left\langle \psi\delta^i(x), \frac{\partial}{\partial\alpha_k}(p)\right\rangle-\left\langle \psi\delta^s(x), \frac{\partial}{\partial\alpha_k}(p)\right\rangle\frac{\partial}{\partial x^s}\left\langle \psi\delta^i(x), \frac{\partial}{\partial\alpha_j}(p)\right\rangle .$$

Therefore

$$\sum_{i=1}^{n}\left\langle \psi d\delta^i(x), \frac{\partial}{\partial\alpha_j}(p)\wedge\frac{\partial}{\partial\alpha_k}(p)\right\rangle\left(\frac{\partial}{\partial x_i}\right)_x=\left[\psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right), \psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)\right]\in\mathfrak{A}(\boldsymbol{R}^n) .$$

On the other hand

$$\left\langle d\psi\delta^i(x), \frac{\partial}{\partial\alpha_j}(p)\wedge\frac{\partial}{\partial\alpha_k}(p)\right\rangle\left(\frac{\partial}{\partial x_i}\right)_x$$

$$=\frac{\partial}{\partial\alpha_j}\left\langle \psi\delta^i(x), \frac{\partial}{\partial\alpha_k}(p)\right\rangle-\frac{\partial}{\partial\alpha_k}\left\langle \psi\delta^i(x), \frac{\partial}{\partial\alpha_j}(p)\right\rangle-\left\langle \psi\delta^i(x), \left[\frac{\partial}{\partial\alpha_j}(p), \frac{\partial}{\partial\alpha_k}(p)\right]\right\rangle$$

$$=\frac{\partial}{\partial\alpha_j}\left\langle \psi\delta^i(x), \frac{\partial}{\partial\alpha_k}(p)\right\rangle-\frac{\partial}{\partial\alpha_k}\left\langle \psi\delta^i(x), \frac{\partial}{\partial\alpha_j}(p)\right\rangle \qquad \left(\because\ \left[\frac{\partial}{\partial\alpha_j}(p), \frac{\partial}{\partial\alpha_k}(p)\right]=0\right).$$

Therefore

$$\left[\psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right), \psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)\right]=\frac{\partial}{\partial\alpha_j}\psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)-\frac{\partial}{\partial\alpha_k}\psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right)$$

$$\left[\pi_x^*\left(\frac{\partial}{\partial\alpha_j}(p)\right)-\psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right)(x), \pi_x^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)-\psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)(x)\right]$$

$$=\left[\pi_x^*\left(\frac{\partial}{\partial\alpha_j}(p)\right), \pi_x^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)\right]-\left[\pi_x^*\left(\frac{\partial}{\partial\alpha_j}(p)\right), \psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)(x)\right]$$

$$+\left[\pi_x^*\left(\frac{\partial}{\partial\alpha_k}(p)\right), \psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right)(x)\right]+\left[\psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right), \psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)\right](x)$$

$$=-\frac{\partial}{\partial\alpha_j}\psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)(x)+\frac{\partial}{\partial\alpha_k}\psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right)(x)+\left[\psi_p^*\left(\frac{\partial}{\partial\alpha_j}(p)\right), \psi_p^*\left(\frac{\partial}{\partial\alpha_k}(p)\right)\right](x)$$

$$=0 .$$

Therefore $\{E_{(p,x)}\}$ is integrable.

Let $\mathscr{F}$ be a foliated vector bundle structure on $X\times\boldsymbol{R}^n$ induced from $\{E_{(p,x)}\}$. Let h be the transition function associated with $(X\times\boldsymbol{R}^n, \mathscr{F})$. By definition of $\mathscr{F}$, $h_p'((\partial/\partial\alpha_j)(p))=\psi_p^*((\partial/\partial\alpha_j)(p))$ holds. Let $\lambda_{\mathscr{F}}$: $C^*(\mathfrak{A}(\boldsymbol{R}^n))\to\Omega_{DR}^*(X)$ be the characteristic homomorphism of $\mathscr{F}$.

$$\left\langle \lambda_{\mathscr{F}}(\delta^i(x)), \frac{\partial}{\partial \alpha_j}(p)\right\rangle = \delta^i(x)\left(h'_p \frac{\partial}{\partial \alpha_j}(p)\right) = \delta^i(x)\left(\psi_p^* \frac{\partial}{\partial \alpha_j}(p)\right)$$

$$= \left\langle \psi(\delta^i(x)), \frac{\partial}{\partial \alpha_j}(p)\right\rangle \quad \text{for } \forall j=1, \cdots, m, \quad \forall p \in X$$

$$\lambda_{\mathscr{F}}(\delta^i(x)) = \psi(\delta^i(x)) \quad \text{for } \forall i=1, \cdots, n, \quad \forall x \in \boldsymbol{R}^n$$

$$\lambda_{\mathscr{F}}(\delta^i_s(x)) = \lim_{t \to 0} \lambda_{\mathscr{F}}\left(\frac{\delta^i(x+t_s) - \delta^i(x)}{t}\right)$$

$$= \lim_{t \to 0} \frac{1}{t}\{\lambda_{\mathscr{F}}(\delta^i(x+t_s)) - \lambda_{\mathscr{F}}(\delta^i(x))\}$$

$$= \lim_{t \to 0} \frac{1}{t}\{\psi(\delta^i(x+t_s)) - \psi(\delta^i(x))\}$$

$$\doteq \lim_{t \to 0}\left(\frac{\delta^i(x+t_s) - \delta^i(x)}{t}\right) = \psi(\delta^i_s(x)) .$$

Repeating this argument, we obtain

$$\lambda_{\mathscr{F}}(\delta^i_{j_1 \cdots j_s}(x)) = \psi(\delta^i_{j_1 \cdots j_s}(x)) .$$

Let $\mathscr{D}_\delta$ be the sub-algebra of $C^*(\mathfrak{A}(\boldsymbol{R}^n))$ generated by $\{\delta^i_{j_1 \cdots j_s}(x)\}$, then $\lambda_{\mathscr{F}} = \psi$ holds on $\mathscr{D}_\delta$.

Since $\mathscr{D}_\delta$ is dense in $C^*(\mathfrak{A}(\boldsymbol{R}^n))$, $\lambda_{\mathscr{F}} = \psi$ holds on $C^*(\mathfrak{A}(\boldsymbol{R}^n))$. Let $\mathscr{F}$ and $\mathscr{F}'$ be foliated vector bundle structures on $X \times \boldsymbol{R}^n$ s.t. $\lambda_{\mathscr{F}} = \lambda_{\mathscr{F}'}$, then the plane fields of $\mathscr{F}$ and $\mathscr{F}'$ coincide with each other. This implies the uniqueness.

In case of $C^*(\mathfrak{A}_0(\boldsymbol{R}^n))$ we can prove in the same way using $\delta^i(0)=0$ $\lim_{x \to 0} \delta^i(x)=0$.

(ii) The case when $\mathscr{S} \neq \emptyset$.

Let $L_1(f_1(x), \cdots, f_n(x))=0, \cdots, L_m(f_1(x), \cdots, f_n(x))=0$ be the system of linear-differential equations which represent the necessary and sufficient condition for $X = \sum f_i(\partial/\partial x_i)$ to be an $\mathscr{S}$-vector field.

Replacing terms $(\partial^s f_i/(\partial x_{j_1} \cdots \partial x_{j_s}))(x)$ in $L_k(f_1(x), \cdots, f_n(x))$ by $\delta^i_{j_1 \cdots j_s}(x)(X)$, we get the system of linear-differential equations $L_1(\delta^1(x), \cdots, \delta^n(x)), \cdots, L_m(\delta^1(x), \cdots, \delta^n(x))$. Since $L_k(\delta^1(x), \cdots, \delta^n(x))(X) = L_k(f_1(x), \cdots, f_n(x))$, the necessary and sufficient condition for X to be an $\mathscr{S}$-vector field is $L_1(\delta^1(x), \cdots, \delta^n(x))(X)=0, \cdots, L_m(\delta^1(x), \cdots, \delta^n(x))(X)=0$. We show that $\psi_p^*((\partial/\partial\alpha_j)(p)) = \sum_{i=1}^n \langle \psi\delta^i(x), (\partial/\partial\alpha_j)(p)\rangle (\partial/\partial x_i)_x$ is an $\mathscr{S}$-vector field. Since L_k is a linear equation and

$$\frac{\partial^s}{\partial x_{j_1} \cdots \partial x_{j_s}}\left\langle \psi\delta^i(x), \frac{\partial}{\partial \alpha_j}(p)\right\rangle = \left\langle \psi\delta^i_{j_1 \cdots j_s}(x), \frac{\partial}{\partial \alpha_j}(p)\right\rangle$$

holds,

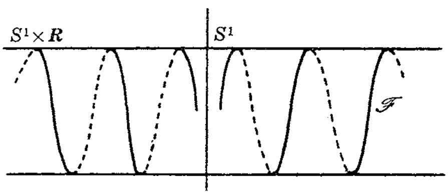
$$L_k\left(\left\langle \psi\delta^1(x), \frac{\partial}{\partial \alpha_j}(p)\right\rangle, \cdots, \left\langle \psi\delta^n(x), \frac{\partial}{\partial \alpha_j}(p)\right\rangle\right) = \left\langle \psi L_k(\delta^1(x), \cdots, \delta^n(x)), \frac{\partial}{\partial \alpha_j}(p)\right\rangle = 0 .$$

Therefore $\psi_p^*((\partial/\partial\alpha_j)(p))$ is an $\mathscr{S}$-vector field. Moreover, since $\mathfrak{A}(\boldsymbol{R}^n)$ is locally convex w.r.t. C^∞-topology, we can apply Hahn-Banach extension theorem, and conclude that $C^*(\mathfrak{A}(\boldsymbol{R}^n))\to C^*(\mathfrak{A}(\boldsymbol{R}^n, \mathscr{S}))$ is a subjection. Using these facts we can show that $\psi=\lambda_{\mathscr{F}}$. Uniqueness is proved in the same way as case (i). q.e.d.

Example 1).

We define a continuous D.G.A. homomorphism ψ: $C^*(\mathfrak{A}_0(\boldsymbol{R}^1))\to\Omega_{DR}^*(S^1)$ as $\psi(\delta(x))=x\cdot\mu$, $\psi(\delta'(x))=\mu$, $\psi(\delta''(x))=0, \cdots$, where μ is the volume element of S^1.

By Theorem I there exists a foliated vector bundle structure $\mathscr{F}$ on $S^1\times\boldsymbol{R}$ with $S^1\times\{0\}$ a leaf. Moreover, the characteristic homomorphism $\lambda_{\mathscr{F}}$ of $\mathscr{F}$ coincides with ψ and $\lambda_{\mathscr{F}}$: $H_{GF}^*(\mathfrak{A}_0(\boldsymbol{R}^1))\to H_{DR}^*(S^1)$ is a non-zero map.



Example 2).

Let $\mathscr{F}$ be a codim 1-oriented foliation on M^n. The normal bundle is trivial: $\nu(\mathscr{F})\cong M^n\times\boldsymbol{R}$. Let ω be a 1-form on M^n defining $\mathscr{F}$. Since $d\omega\wedge\omega=0$, there exists a 1-form θ s.t. $d\omega=\omega\wedge\theta$. The Godbillon-Vey form G.V. $(\mathscr{F})$ is represented by $\theta\wedge d\theta\in\Omega_{DR}^3(M^n)$. Since $\omega\wedge d\theta=0$, there exists a 1-form η s.t. $d\theta=\omega\wedge\eta$.

Let X be a vector field on M^n which is transversal to $\mathscr{F}$ and satisfies the condition $\omega(X)=1$. Let φ_t be a 1-parameter transformation group generated by X.

We define a continuous map ψ: $C^*(\mathfrak{A}(\boldsymbol{R}^1))\to\Omega_{DR}^*(M^n)$ as

$$\begin{cases} \psi(\delta(x))=\varphi_{-x}^*\omega \\ \psi(\delta'(x))=\mathscr{L}_X\varphi_{-x}^*\omega=(\iota(X)d+d\iota(X))\varphi_{-x}^*\omega \\ \qquad\qquad =\varphi_{-x}^*\theta-\theta(X)\varphi_{-x}^*\omega \\ \psi(\delta''(x))=\mathscr{L}_X\varphi_{-x}^*\theta-\theta(X)\mathscr{L}_X\varphi_{-x}^*\omega \\ \qquad\qquad =\varphi_{-x}^*\eta-\eta(X)\varphi_{-x}^*\omega+d\theta(X)-\theta(X)\varphi_{-x}^*\omega+\theta(X)^2\varphi_{-x}^*\omega\ . \end{cases}$$

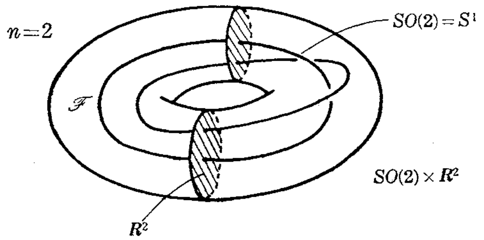
Then, $d\psi=\psi d$ holds, and ψ is a continuous D.G.A.-homomorphism. By Theorem I there exists a unique foliated vector bundle structure $\tilde{\mathscr{F}}$ on $\nu(\mathscr{F})\cong M^n\times\boldsymbol{R}$ and $\psi=\lambda_{\tilde{\mathscr{F}}}$ holds. Since

$$\lambda_{\tilde{\mathscr{F}}}(\delta(x)\wedge\delta'(x)\wedge\delta''(x))=-\varphi_{-x}^*\,\text{G.V.}\,(\mathscr{F})+\varphi_{-x}^*d(\theta(X)\cdot\omega\wedge\theta)\ ,$$

the homomorphism $\lambda_{\tilde{\mathscr{F}}}$: $H_{GF}^*(\mathfrak{A}(\boldsymbol{R}^1))\to H_{DR}^*(M^n)$ transforms a generator $\delta(0)\wedge\delta'(0)\wedge\delta''(0)$ into the Godbillon-Vey class of $\mathscr{F}$ (cf. Theorem II).

Example 3).

Let $\mathscr{S}$ be a Riemannian structure $\mathscr{R}$ on $\boldsymbol{R}^n$, then $\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{R}) \cong so(n)$ (cf. Matsushima "Introduction to differentiable manifolds"). Therefore $C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{R})) \cong \Lambda^* so(n)'$, where $so(n)'$ is the dual space of $so(n)$. Regarding $\Lambda^* so(n)'$ as a sub-algebra of $\Omega^*_{DR}(SO(n))$ consisting of left-invariant differential forms on $SO(n)$, a continuous D.G.A.-homomorphism ψ: $C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{R})) \to \Omega^*_{DR}(SO(n))$ can be constructed. By Theorem I there exists a foliated vector bundle structure $\mathscr{F}$ on $SO(n) \times \boldsymbol{R}^n$ s.t. $\psi = \lambda_{\mathscr{F}}$. Moreover, by propositions in § 4 $\mathscr{F}$ is an $\mathscr{R}$-foliated structure. The homomorphism $\lambda_{\mathscr{F}}$: $H^*_{GF}(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{R})) \to H^*_{DR}(SO(n))$ is an isomorphism.



REMARK. When $\mathscr{S} = \emptyset$, $\boldsymbol{R}^n$ can be replaced with a paracompact manifold M in Theorem I. In case of $\mathfrak{A}_0(M)$, some point on M must be fixed in advance.

PROOF. Let $U = (U, (x_1 \cdots x_n))$ be a coordinate neighborhood in M and $a \in U$. ${}^U\delta^i_{j_1 \cdots j_s}(a) \in C^1(\mathfrak{A}(M))$ is defined as follows. When $X \in \mathfrak{A}(M)$ and we denote $X|_U$ by $X|_U = \sum_{i=1}^n f_i(x)(\partial/\partial x_i)_x$,

$$ {}^U\delta^i_{j_1 \cdots j_s}(a)(X) \overset{\text{def}}{=} \frac{\partial^s}{\partial x_{j_1} \cdots \partial x_{j_s}} f_i(x(a)) . $$

Let $U' = (U', (x'_1 \cdots x'_n))$ be another coordinate neighborhood s.t. $a \in U'$. Similarly ${}^{U'}\delta^i_{j_1 \cdots j_s}(a) \in C^1(\mathfrak{A}(M))$ is defined. ${}^U\delta^i(a) = \sum_{k=1}^s (\partial(x_i \circ x'^{-1})/\partial x'_k)(x'(a))^{U'}\delta^k(a)$ holds. Let ψ: $C^*(\mathfrak{A}(M)) \to \Omega^*_{DR}(X)$ be a continuous D.G.A.-homomorphism.

$$ \begin{aligned} {}^U\psi^*_p\left(\frac{\partial}{\partial \alpha_j}(p)\right) &\overset{\text{def}}{=} \sum_{i=1}^n \left\langle \psi({}^U\delta^i(a)), \frac{\partial}{\partial \alpha_j}(p) \right\rangle \left(\frac{\partial}{\partial x_i}\right)_a \\ &= \sum_{i=1}^n \sum_{k=1}^n \frac{\partial(x_i \circ x'^{-1})}{\partial x'_k}(x'(a)) \left\langle \psi({}^{U'}\delta^k(a)), \frac{\partial}{\partial \alpha_j}(p) \right\rangle \left(\frac{\partial}{\partial x_i}\right)_a \\ &= \sum_{k=1}^n \left\langle \psi({}^{U'}\delta^k(a)), \frac{\partial}{\partial \alpha_j}(p) \right\rangle \sum_{i=1}^n \frac{\partial(x_i \circ x'^{-1})}{\partial x'_k}(x'(a)) \left(\frac{\partial}{\partial x_i}\right)_a \\ &= \sum_{k=1}^n \left\langle \psi({}^{U'}\delta^k(a)), \frac{\partial}{\partial \alpha_j}(p) \right\rangle \left(\frac{\partial}{\partial x'_k}\right)_a \\ &= {}^{U'}\psi^*_p\left(\frac{\partial}{\partial \alpha_j}(p)\right) . \end{aligned} $$

Therefore $\{{}^U\phi^*((\partial/\partial\alpha_j)(p))\}_U$ determine a C^∞-vector field on M. We denote it by $\phi_p^*((\partial/\partial\alpha_j)(p))$.

In the same way as $M=\boldsymbol{R}^n$, there exists a foliated bundle structure $\mathscr{F}$ on $X\times M$ and

$$\phi({}^U\delta^i_{j_1\cdots j_s}(a))=\lambda_{\mathscr{F}}({}^U\delta^i_{j_1\cdots j_s}(a))$$

holds on any coordinate neighborhood U.

Since M is paracompact, there exists a countable covering of M by coordinate neighborhoods and there exists a partition of unity subordinate to this covering.

Moreover, $c\in C^p(\mathfrak{A}(M))$ has a compact support and its local restriction by a partition of unity can be written as

$$c|_{U_1\times\cdots\times U_p}=\sum_{\text{finite sum}}\int_{U_1}\cdots\int_{U_p}\mu(x_1\cdots x_p)^{U_1}\delta^{i_1}_{j_1\cdots j_s}(x_1)\otimes\cdots\otimes{}^{U_p}\delta^{i_p}_{k_1\cdots k_{s_p}}(x_p)dx_1\cdots dx_p$$

(cf. [6]). Therefore the subalgebra $\mathscr{D}_\delta(M)$ of $C^*(\mathfrak{A}(M))$ generated by $\{{}^U\delta^i_{j_1\cdots j_s}(a)\}$ is dense in $C^*(\mathfrak{A}(M))$.

Since ϕ and $\lambda_{\mathscr{F}}$ is continuous, $\phi=\lambda_{\mathscr{F}}$ holds on $C^*(\mathfrak{A}(M))$. In case of $C^*(\mathfrak{A}_0(M))$ ${}^U\delta^i(a)=0$ and $\lim_{x\to a}{}^U\delta^i(x)=0$ holds for the base point a and any coordinate neighborhood U, and we can prove in the same way.

Uniqueness can be proved in the same way as the case of $M=\boldsymbol{R}^n$. q.e.d.

§4. Some condition for $\mathscr{F}$ to be an $\mathscr{S}$-foliation

PROPOSITION 1. *Let G be a finite dimensional Lie group and H be its closed subgroup. Let $g\cong T_eG$ and $h\cong T_eH$ be associated Lie algebras. Let $x(t)\in G$ $(t\in\boldsymbol{R})$ be a C^∞-path such that $x(0)=e$.*

If $(d/dt)x(t_0)^{-1}\cdot x(t)|_{t=t_0}$ is included in T_eH for $\forall t_0\in\boldsymbol{R}$, then $x(t)\in H$ $(t\in\boldsymbol{R})$ holds.

PROOF. Homogeneous space G/H is a C^∞-manifold. Let $\pi\colon G\to G/H$ be a natural projection and $\bar{e}=\pi(e)$. $\bar{x}(t)\cong\pi\circ x(t)$ is a C^∞-path in G/H s.t. $\bar{x}(e)=\bar{e}$. Let $\pi_*\colon T_xG\to T_{\pi(x)}(G/H)$ be a homomorphism induced from π and $L_{x(t_0)}\colon G\to G$ be a left-translation by $x(t_0)$. Then we get $\pi_*\circ L_{x(t_0)*}=L_{x(t_0)*}\circ\pi_*$ and

$$\begin{aligned}\frac{d}{dt}\bar{x}(t)\Big|_{t=t_0}&=\pi_*\left(\frac{d}{dt}x(t)\Big|_{t=t_0}\right)=\pi_*L_{x(t_0)*}\left(\frac{d}{dt}x(t_0)^{-1}\cdot x(t)\Big|_{t=t_0}\right)\\&=L_{x(t_0)*}\pi_*\left(\frac{d}{dt}x(t_0)^{-1}\cdot x(t)\Big|_{t=t_0}\right).\end{aligned}$$

Since $T_eH=\operatorname{Ker}\{\pi_*\colon T_eG\to T_{\bar{e}}(G/H)\}$, $(d/dt)x(t_0)^{-1}\cdot x(t)|_{t=t_0}\in T_eH$ implies $\pi_*((d/dt)x(t_0)^{-1}\cdot x(t)|_{t=t_0})=0$. Therefore $(d/dt)\bar{x}(t)|_{t=t_0}=0$ for $\forall t_0\in\boldsymbol{R}$ and $\bar{x}(0)=\bar{e}$. $\bar{x}(t)=\bar{e}$ (i.e. $x(t)\in H$) results from this. q.e.d.

We give some condition for $\mathscr{F}$ in Theorem I to be an $\mathscr{S}$-foliation in case of $\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})$.

The general linear group $GL(n)$ can be taken for a subgroup of $\mathrm{Diff}_0\,\boldsymbol{R}^n$. On the other hand, there exists a differential d: $\mathrm{Diff}_0\,\boldsymbol{R}^n \to GL(n)$. d is a homomorphism and identity on $GL(n)$. The Lie algebra $\mathfrak{gl}(n)$ of $GL(n)$ can be taken for a subalgebra of $\mathfrak{A}_0(\boldsymbol{R}^n)$ and the projection p: $\mathfrak{A}_0(\boldsymbol{R}^n) \to \mathfrak{gl}(n)$ is a Lie-homomorphism and identity on $\mathfrak{gl}(n)$.

Assume $\mathrm{Diff}_0\,(\boldsymbol{R}^n, \mathscr{S})$ is a closed subgroup of $\mathrm{Diff}_0\,\boldsymbol{R}^n$. We define $GL(n, \mathscr{S})$ and $\mathfrak{gl}(n, \mathscr{S})$ as follows.

$$GL(n, \mathscr{S}) \overset{\text{def}}{=} \mathrm{Diff}_0\,(\boldsymbol{R}^n, \mathscr{S}) \cap GL(n) = d(\mathrm{Diff}_0\,(\boldsymbol{R}^n, \mathscr{S}))$$
$$\mathfrak{gl}(n, \mathscr{S}) \overset{\text{def}}{=} \mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S}) \cap \mathfrak{gl}(n) = p(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S}))\,.$$

$GL(n, \mathscr{S})$ is a closed subgroup of $GL(n)$ and $\mathfrak{gl}(n, \mathscr{S})$ is a Lie algebra of $GL(n, \mathscr{S})$. There exist continuous D.G.A.-homomorphisms p^*, i^* such that

$$C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})) \underset{i^*}{\overset{p^*}{\leftrightarrows}} \Lambda^*\mathfrak{gl}(n, \mathscr{S})' \qquad i^* \circ p^* = \text{identity},$$

where i is an inclusion.

PROPOSITION 2. *Let ψ: $C^*(\mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})) \to \Omega^*_{DR}(X)$ be a continuous D.G.A.-homomorphism and $\mathscr{F}$ be a foliated vector bundle structure on $X \times \boldsymbol{R}^n$, such that $\psi = \lambda_{\mathscr{F}}$ (cf. Theorem* I). *Then, if there exists a D.G.A.-homomorphism ψ_1: $\Lambda^*\mathfrak{gl}(n, \mathscr{S})' \to \Omega^*_{DR}(X)$ such that $\psi = \psi_1 \circ i^*$, $\mathscr{F}$ is an $\mathscr{S}$-foliation.*

PROOF. Let $\{h_U$: $U \to \mathrm{Diff}_0\,\boldsymbol{R}^n\}$ be the transition functions associated with $\mathscr{F}$. Corresponding to $\mathscr{F}$ we define a foliated vector bundle structure $d\mathscr{F}$ on $X \times \boldsymbol{R}^n$ as a foliated structure whose transition functions are $\{d \circ h_U$: $U \to GL(n)\}$. Since $p \circ h'_U = (dh_U)'$: $TU \to \mathfrak{gl}(n)$, $\lambda_{\mathscr{F}} \circ p^* = \lambda_{d\mathscr{F}}$ holds. If $\psi = \psi_1 \circ i^*$, then $\psi_1 = \psi \circ p^* = \lambda_{\mathscr{F}} \circ p^* = \lambda_{d\mathscr{F}}$, $\psi = \psi_1 \circ i^* = \lambda_{d\mathscr{F}} \circ i^* = \lambda_{d\mathscr{F}}$. Therefore, by uniqueness, $\mathscr{F} = d\mathscr{F}$. This implies $(h_U)' = (dh_U)'$: $TU \to \mathfrak{gl}(n)$. On the other hand, $(h_U)'$: $TU \to \mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S})$. Therefore $(dh_U)'$: $TU \to \mathfrak{A}_0(\boldsymbol{R}^n, \mathscr{S}) \cap \mathfrak{gl}(n) = \mathfrak{gl}(n, \mathscr{S})$. Let $\alpha \in U$ be a fixed point and $x(t)$ be a path in U such that $x(0) = \alpha$. If we define $\varphi(t)$ as $\varphi(t) \overset{\text{def}}{=} dh_U(\alpha)^{-1} \circ dh_U(x(t))$, $\varphi(t)$ is a C^∞-path in $GL(n)$ and $(d/dt)\varphi(t_0)^{-1} \circ \varphi(t)|_{t=t_0} = (dh'_U)((d/dt)x(t)|_{t=t_0}) \in \mathfrak{gl}(n, \mathscr{S})$ holds for $\forall\, t_0 \in \boldsymbol{R}$.

By Prop. 1 $\varphi(t) = dh(\alpha)^{-1} \circ dh_U(x(t)) \in GL(n, \mathscr{S}) \subset \mathrm{Diff}_0\,(\boldsymbol{R}^n, \mathscr{S})$. If U is connected, $dh(\alpha)^{-1} \circ dh_U(x) \in \mathrm{Diff}_0\,(\boldsymbol{R}^n, \mathscr{S})$ for $\forall\, x \in U$. If we define $\tilde{h}_U$: $\pi^{-1}(U) \overset{\sim}{\to} U \times \boldsymbol{R}^n$ as $\tilde{h}_U = dh(\alpha)^{-1} \circ dh_U$, $\tilde{h}_U$ is a diffeomorphism preserving the structure $\mathscr{S}$. Moreover $\tilde{h}'_U = dh'_U$ holds. $\{\tilde{h}_U\}$ can be also taken for the maximal atlas on $X \times \boldsymbol{R}^n$ which defines $\mathscr{F} = d\mathscr{F}$. Therefore $\mathscr{F} = d\mathscr{F}$ is an $\mathscr{S}$-foliation. q.e.d.

By Prop. 2, in case of Riemannian structure, $\mathscr{F}$ in Theorem I is an $\mathscr{R}$-foliation (see § 3 example 3). Moreover, if $\mathscr{S}$ is a finite order G-structure, $\mathscr{F}$ in Theorem I is an $\mathscr{S}$-foliation.

§ 5. Proof of Theorem II

THEOREM II. *Assume $\mathscr{S}=\emptyset$, then the inclusions*

$$C^*_{pt}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)\subset C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)$$
$$C^*_{pt}(\mathfrak{A}(\boldsymbol{R}^n), K)\subset C^*(\mathfrak{A}(\boldsymbol{R}^n), K)$$
$$\left(\begin{array}{l}\text{resp. } C^*_{pt}(\hat{\mathfrak{A}}_0(\boldsymbol{R}^n), K)\subset C^*(\hat{\mathfrak{A}}_0(\boldsymbol{R}^n), K)\\ \qquad C^*_{pt}(\mathfrak{A}_0(\boldsymbol{R}^n), K)\subset C^*(\mathfrak{A}_0(\boldsymbol{R}^n), K)\end{array}\right)$$

induce isomorphisms on cohomologies, where C^*_{pt} stands for the sub D.G.A. consisting of elements supported at 0. (cf. [3])

PROOF. We consider the case of $C^*_{pt}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)\subset C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)$. Let

$$\hat{\mathfrak{A}}(\boldsymbol{R}^n)\ni\xi+t\cdot\eta \qquad \xi=\sum_{i=1}^n f_i\frac{\partial}{\partial x_i} \qquad \eta=\sum_{i=1}^n g_i\frac{\partial}{\partial x_i}\,.$$

Then $\delta^i_{j_1\cdots j_s}(x)$, $\rho^i_{j_1\cdots j_s}(x)\in C^1(\hat{\mathfrak{A}}(\boldsymbol{R}^n))$ are defined as

$$\begin{cases}\delta^i_{j_1\cdots j_s}(x)(\xi+t\eta)\overset{\text{def}}{=}\dfrac{\partial^s f_i}{\partial x_{j_1}\cdots\partial x_{j_s}}(x)\\[2mm] \rho^i_{j_1\cdots j_s}(x)(\xi+t\eta)\overset{\text{def}}{=}\dfrac{\partial^s g_i}{\partial x_{j_1}\cdots\partial x_{j_s}}(x)\,.\end{cases}$$

An element of $C^1(\hat{\mathfrak{A}}(\boldsymbol{R}^n))$ can be written as follows

$$\int_{-\infty}^{\infty}\mu(x)\delta^i_{j_1\cdots j_s}(x)dx \qquad \int_{-\infty}^{\infty}\nu(x)\rho^i_{j_1\cdots j_s}(x)dx$$

where μ, ν are continuous functions on $\boldsymbol{R}^n$ with compact supports.

We define an automorphism h_t $(t\neq 0)$ of $\boldsymbol{R}^n$ as $h_t(x)=t\cdot x$. h_t induces an automorphism h_t^* of $\hat{\mathfrak{A}}(\boldsymbol{R}^n)$, therefore induces an automorphism h_t^* of $C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n))$. Since $h_t\circ g=g\circ h_t$ holds for $\forall\, g\in K$, h_t^* induces an automorphism h^* of $C^*(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)$.

$$\begin{cases}h_t^*\delta^i_{j_1\cdots j_s}(x)=t^{1-s}\delta^i_{j_1\cdots j_s}(t^{-1}x)\\ h_t^*\rho^i_{j_1\cdots j_s}(x)=t^{1-s}\rho^i_{j_1\cdots j_s}(t^{-1}x)\end{cases}$$

holds.

LEMMA. $\lim_{t\to\infty}(1/t^{2n})h_t^*$ *exists.*

PROOF. If $s\geq 1$, there is no problem for $\delta^i_{j_1\cdots j_s}(x)$ and $\rho^i_{j_1\cdots j_s}(x)$. Suppose $s=0$. Moreover suppose $n=1$ in order to simplify the proof. The worst type is denoted as follows.

$$c=\int \mu(x_1 \cdots x_p\, y_1 \cdots y_q)\delta(x_1)\wedge\cdots\wedge\delta(x_p)\wedge\rho(y_1)\wedge\cdots\wedge\rho(y_q)dx_1\cdots dx_p\, dy_1\cdots dy_q$$

$$h_t^* c=t^{p+q}\int \mu(x_1 \cdots x_p\, y_1 \cdots y_q)\delta(x_1/t)\wedge\cdots\wedge\delta(x_p/t)\wedge\rho(y_1/t)\wedge\cdots\wedge\rho(y_q/t) \times dx_1\cdots dx_p\, dy_1\cdots dy_q\,.$$

We compute the order of

$$\begin{cases}\delta(x_1/t)\wedge\cdots\wedge\delta(x_p/t)\to 0 & (t\to\infty)\\ \rho(y_1/t)\wedge\cdots\wedge\rho(y_q/t)\to 0 & (t\to\infty)\end{cases}$$

$\xi_i \in \mathfrak{A}(\boldsymbol{R}^1)$ is denoted as $\xi_i=f_i(x)(\partial/\partial x)$ $(i=1\cdots p)$. We assume $f_i(x)=a_i+b_i x$ (terms over 2nd order can be ignored).

$$\delta(x_1/t)\wedge\cdots\wedge\delta(x_p/t)(\xi_1\cdots\xi_p)=\begin{vmatrix}\delta(x_1/t)(\xi_1) & \cdots & \delta(x_1/t)(\xi_p)\\ \vdots & & \vdots\\ \delta(x_p/t)(\xi_1) & \cdots & \delta(x_p/t)(\xi_p)\end{vmatrix}$$

$$=\begin{vmatrix}a_1+b_1(x_1/t) & \cdots & a_p+b_p(x_1/t)\\ \vdots & & \vdots\\ a_1+b_1(x_p/t) & \cdots & a_p+b_p(x_p/t)\end{vmatrix}$$

$$=\sum_{i=1}^{p}\begin{vmatrix}b_1(x_1/t) & \cdots & a_i & \cdots & b_p(x_1/t)\\ \vdots & & \vdots & & \vdots\\ b_1(x_p/t) & \cdots & \underset{\substack{\wedge\\ i}}{a_i} & \cdots & b_p(x_p/t)\end{vmatrix}=t^{1-p}\times\{\quad\}$$

Similarly

$$\rho(y_1/t)\wedge\cdots\wedge\rho(y_q/t)(\cdots)=t^{1-q}\times\{\quad\}$$
$$t^{p+q}\times t^{1-p}\times t^{1-q}=t^2\,.$$

Therefore $\lim_{t\to\infty}(1/t^2)h_t^*$ exists. q.e.d.

$\lim_{t\to\infty}(1/t^2)h_t^*$ is a homomorphism from $C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)$ to $C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)$.

For $R=\sum_{i=1}^{n} x_i(\partial/\partial x_i)$ we define θ_R: $C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n))\to C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n))$ as

$$(\theta_R c)(\xi_1+t\eta_1\cdots\xi_p+t\eta_p)\overset{\text{def}}{=}\sum_{i=1}^{p} c(\xi_1+t\eta_1\cdots[\xi_i+t\eta_i, R]\cdots\xi_p+t\eta_p)\,.$$

Since $g_*R=R$ and $[A,R]=0$ hold for $\forall g\in K$ and $\forall A\in k$, θ_R: $C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)\to C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)$ is induced. The equations

$$\begin{cases}\theta_R\delta^i_{j_1\cdots j_s}(0)=(1-s)\delta^i_{j_1\cdots j_s}(0)\\ \theta_R\rho^i_{j_1\cdots j_s}(0)=(1-s)\rho^i_{j_1\cdots j_s}(0)\end{cases}$$

hold.

We define the eigen-space $C^*_{pt}(\lambda)$ for θ_R as

$$C^*_{pt}(\lambda) \overset{\text{def}}{=} \{c \in C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \colon \theta_R c = \lambda c\} .$$

LEMMA. *$C^*_{pt}(\lambda)$ is a sub-complex of $C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)$.*

PROOF.

$$\text{(1)} \qquad \begin{cases} d\delta^i = \sum_k \delta^k \wedge \delta^i_k \\ d\rho^i = \sum_k \delta^k \wedge \rho^i_k + \sum_k \rho^k \wedge \delta^i_k \end{cases}$$

$$\text{(2)} \qquad \begin{cases} \theta_{\partial/\partial x_k} \delta^i_{j_1 \cdots j_s} = -\delta^i_{j_1 \cdots j_s, k} \\ \theta_{\partial/\partial x_k} \rho^i_{j_1 \cdots j_s} = -\rho^i_{j_1 \cdots j_s, k} \end{cases}$$

$$\text{(3)} \qquad d \circ \theta \frac{\partial}{\partial x_k} = \theta \frac{\partial}{\partial x_k} \circ d .$$

Using (1), (2) and (3), we can prove $dC^*_{pt}(\lambda) \subset C^*_{pt}(\lambda)$. q.e.d.

$C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)$ splits as follows.

$$C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) = \bigoplus_{\lambda \leq 2n} C^*_{pt}(\lambda) .$$

Since $\theta_R = d \circ \iota_R + \iota_R \circ d$, $H^*(C^*_{pt}(\lambda)) = 0$ for $\lambda \neq 0$. Therefore

$$H^*(C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)) \cong H^*(C^*_{pt}(0)) .$$

We define a filtration $\{F^\lambda C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)\}$ of $C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)$ as

$$F^\lambda C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \overset{\text{def}}{=} \left\{ c \in C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \colon \lim_{t \to \infty} \frac{1}{t^\lambda} h^*_t c \text{ exists} \right\} .$$

We get $C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) = F^{2n} C^* \supset F^{2n-1} C^* \supset \cdots$ and $0 \to F^{\lambda-1} C^* \to F^\lambda C^* \underset{\iota_\lambda}{\overset{\gamma_\lambda}{\rightleftarrows}} C^*_{pt}(\lambda) \to 0$ is a splitting exact sequence, where $\gamma_\lambda = \lim_{t \to \infty} (1/t^\lambda) h^*_t$ and ι_λ an inclusion.

$$\begin{array}{ccccccc} C^* = F^{2n} C^* & \supset F^{2n-1} C^* & \supset \cdots \supset & F^0 C^* & \supset F^{-1} C^* & \supset \cdots . \\ \downarrow{\scriptstyle \gamma_{2n}} & \downarrow{\scriptstyle \gamma_{2n-1}} & & \downarrow{\scriptstyle \gamma_0} & \downarrow{\scriptstyle \gamma_{-1}} & \\ C^*_{pt}(2n) & C^*_{pt}(2n-1) & & C^*_{pt}(0) & C^*_{pt}(-1) & \end{array}$$

We have

$$\begin{array}{ccc} C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) & \xleftarrow{\text{inclusion}} & C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \\ & {}_{\text{inclusion}} \nwarrow \quad \nearrow {}_{\text{inclusion}} & \\ & C^*_{pt}(0) & \end{array}$$

Since $H^*(C^*_{pt}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \cong H^*(C^*_{pt}(0))$ is already proved, we have only to prove $H^*_{GF}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \cong H^*(C^*_{pt}(0))$. We construct a map inverse to ι_0: $H^*(C^*_{pt}(0)) \to H^*_{GF}(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K)$. Define

$$Z^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \overset{\text{def}}{=} \{c \in C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \colon dc = 0\}$$
$$B^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \overset{\text{def}}{=} \{c \in C^*(\widehat{\mathfrak{A}}(\boldsymbol{R}^n), K) \colon c = d \ \exists\, c'\} .$$

Let c be an element of $Z(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)$. Since $H^*(C^*_{pt}(2n))=0$, $\gamma_{2n}c \in Z^*_{pt}(2n)=B^*_{pt}(2n)$. Therefore there exists $\eta_{2n} \in C^*_{pt}(2n)$ such that $\gamma_{2n}c=d\eta_{2n}$. Let c_{2n-1} be $c-d\iota_{2n}\eta_{2n}$, then $c_{2n-1} \in F^{2n-1}C^*$ and cohomologous to c. Since $H^*(C^*_{pt}(2n-1))=0$, $\gamma_{2n-1}c_{2n-1} \in Z^*_{pt}(2n)=B^*_{pt}(2n)$. Therefore there exists $\eta_{2n-1} \in C^*_{pt}(2n-1)$ such that $\gamma_{2n-1}c_{2n-1}=d\eta_{2n-1}$. Let c_{2n-2} be $c_{2n-1}-d\iota_{2n-1}\eta_{2n-1}$, then $c_{2n-2} \in F^{2n-2}C^*$ and cohomologous to c_{2n-1}. We proceed with this till we get $c_0 \in F^0C^*$.

We define a map π_0: $H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K) \to H^*(C^*_{pt}(0))$, as $\pi_0[c] \overset{\text{def}}{=} [\gamma_0 c_0]$, where [ ] denotes a cohomology class. Then π_0 is a desired inverse map to ι_0.

LEMMA. *This fact results from* $H^*(F^{-1}C^*)=0$.

PROOF. $\iota_0\gamma_0c_0 \simeq c_0 \bmod F^{-1}C^*$. Since $H^*(F^{-1}C^*)=0$, $\exists \rho \in F^{-1}C^*$ such that $c_0-\iota_0\gamma_0c_0=d\rho$. Therefore $[\iota_0\gamma_0c_0]=[c_0]=[c]$ and $\iota_0\cdot\pi_0=\text{id}$. $\pi_0\cdot\iota_0=\text{id}$ results from $\gamma_0\cdot\iota_0=\text{id}$. q.e.d.

Now we prove $H^*(F^{-1}C^*)=0$.

LEMMA.

$$\frac{d}{dt}h_t^* = \frac{1}{t}\{\iota_R\cdot d+d\cdot\iota_R\}h_t^* .$$

PROOF.

$$\frac{d}{dt}h_t^* = \lim_{\Delta\delta\to 0}\frac{h^*_{t+t\Delta\delta}-h_t^*}{t\Delta\delta} = \lim_{\Delta\delta\to 0}\frac{h_t^*t^*_{1+\Delta\delta}-h_t^*}{t\Delta\delta}$$
$$= \frac{1}{t}\lim_{\Delta\delta\to 0}\frac{h^*_{1+\Delta\delta}-h_1^*}{\Delta\delta}\cdot h_t^* = \frac{1}{t}\{\iota_R\cdot d+d\cdot\iota_R\}h_t^* . \qquad \text{q.e.d.}$$

$$K_Tc \overset{\text{def}}{=} \int_1^T \iota_R h_t^*(c)\frac{dt}{t}$$

$$\{dK_T+K_Td\}c = \int_1^T d\iota_R h_t^*(c)\frac{dt}{t} + \int_1^T \iota_R h_t^*(dc)\frac{dt}{t}$$
$$= \int_1^T \{d\iota_R+\iota_R d\}h_t^*(c)\frac{dt}{t} = \int_1^T \frac{d}{dt}h_t^*(c)dt$$
$$= h_T^*(c)-c .$$

If $c \in F^{-1}C^*$, then $K_\infty c \in F^{-1}C^*$ exists. Therefore $\{dK_\infty+K_\infty d\}c=h^*_\infty(c)-c=-c$ and $H^*(F^{-1}C^*)=0$. Now, $H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K) \cong H^*(C^*_{pt}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K))$ is proved. q.e.d.

Example. Determination of rigid classes.

Let $\mathfrak{A}(n)$ be the topological Lie algebra of n-dim. formal vector fields. An element of $\mathfrak{A}(n)$ is expressed in the form of $\sum_{i=1}^n f_i(\partial/\partial x_i)$, $f_i \in \boldsymbol{R}[[x_1 \cdots x_n]]$, where $\boldsymbol{R}[[x_1 \cdots x_n]]$ denotes the formal power series ring (cf. [7]). Let $\mathfrak{A}_0(n)$ be a sub-algebra consisting of elements such that each f_i has no constant term. Taking ∞-jets at 0 of elements of $\mathfrak{A}(\boldsymbol{R}^n)$ (resp. $\mathfrak{A}_0(\boldsymbol{R}^n)$), we can construct a Lie-algebra homomorphism: $\mathfrak{A}(\boldsymbol{R}^n)\to\mathfrak{A}(n)$ (resp. $\mathfrak{A}_0(\boldsymbol{R}^n)\to\mathfrak{A}_0(n)$). Similarly, we can construct a Lie-algebra homomorphism: $\hat{\mathfrak{A}}(\boldsymbol{R}^n)\to\hat{\mathfrak{A}}(n)$ (resp. $\hat{\mathfrak{A}}_0(\boldsymbol{R}^n)\to\hat{\mathfrak{A}}_0(n)$).

The following diagram is commutative and vertical maps are isomorphisms by Theorem II.

$$\begin{array}{ccc} H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n), K) & \xrightarrow[\text{var}]{} & H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K) \\ {\scriptstyle J_0^\infty}\uparrow \wr\Vert & \circlearrowleft & {\scriptstyle J_0^\infty}\uparrow \wr\Vert \\ H^*_{GF}(\mathfrak{A}(n), K) & \xrightarrow[\text{var}]{} & H^*_{GF}(\hat{\mathfrak{A}}(n), K)\ . \end{array}$$

Therefore

$$\operatorname{Ker}\{\text{var}: H^*_{GF}(\mathfrak{A}(\boldsymbol{R}^n), K)\to H^*_{GF}(\hat{\mathfrak{A}}(\boldsymbol{R}^n), K)\}$$
$$\cong\operatorname{Ker}\{\text{var}: H^*_{GF}(\mathfrak{A}(n), K)\to H^*_{GF}(\hat{\mathfrak{A}}(n), K)\}\ .$$

In case of $K=\{e\}$ and $SO(n)$, the basis of $H^*_{GF}(\mathfrak{A}(n), K)$ is expressed as follows (Vey, cf. [10]).

(i) $K=\{e\}$: $h_Ic_J=h_{i_1}\wedge\cdots\wedge h_{i_l}\otimes c_{j_1}\cdots c_{j_m}$
$I=\{1\leq i_1<\cdots<i_l\leq n\}$ $J=\{1\leq j_1\leq\cdots\leq j_m\leq n\}$
$i_1\leq j_1$ $i_1+|J|>n$

(ii) $K=SO(n)$

(a) $n=$odd: $h_Ic_J=h_{i_1}\wedge\cdots\wedge h_{i_l}\otimes c_{j_1}\cdots c_{j_m}$
$I=\{1\leq i_1<\cdots<i_l\leq n,\ i_k\text{: odd}\}$ $J=\{1\leq j_1\leq\cdots\leq j_m\leq n\}$
$i_1\leq\min(J\cap\text{odd integers})$ $i_1+|J|>n$

(b) $n=$even: h_Ic_J as above.

$$h_Ic_J\chi \quad\text{and}\quad \chi^2=c_n\ .$$

By Gel'fand-Feigin-Fuks ([8]),

$$\operatorname{Ker}\{\text{var}: H^*_{GF}(\mathfrak{A}(n))\to H^*_{GF}(\hat{\mathfrak{A}}(n))=\{h_Ic_J\in H^*_{GF}(\mathfrak{A}(n)):\ i_1+|J|>n+1\}$$
$$\operatorname{Ker}\{\text{var}: H^*_{GF}(\mathfrak{A}(n), SO(n))\to H^*_{GF}(\hat{\mathfrak{A}}(n), SO(n))\}$$
$$=\{h_Ic_J\in H^*_{GF}(\mathfrak{A}(n), SO(n)):\ i_1+|J|>n+1\}\ .$$

Next, we consider the case of $\mathfrak{A}_0(\boldsymbol{R}^n)$.

The following diagram is commutative.

$$\begin{array}{ccc} H^*_{GF}(\mathfrak{A}_0(\boldsymbol{R}^n), K) & \xrightarrow[\text{var}]{} & H^*_{GF}(\hat{\mathfrak{A}}_0(\boldsymbol{R}^n), K) \\ {\scriptstyle J_0^\infty}\uparrow \wr\Vert & \circlearrowleft & {\scriptstyle J_0^\infty}\uparrow \wr\Vert \\ H^*_{GF}(\mathfrak{A}_0(n), K) & \xrightarrow[\text{var}]{} & H^*_{GF}(\hat{\mathfrak{A}}_0(n), K) \\ \uparrow \wr\Vert & \circlearrowleft & \uparrow\cup \\ H^*(\mathfrak{gl}(n), K) & \xrightarrow[\text{var}]{} & H^*(\widehat{\mathfrak{gl}}(n), K)\ . \end{array}$$

Therefore

$$\operatorname{Ker}\{\text{var}: H^*_{GF}(\mathfrak{A}_0(\boldsymbol{R}^n), K)\to H^*_{GF}(\widehat{\mathfrak{A}}_0(\boldsymbol{R}^n), K)\}$$
$$\cong \operatorname{Ker}\{\text{var}: H^*(\mathfrak{gl}(n), K)\to H^*(\widehat{\mathfrak{gl}}(n), K)\}\ .$$

Assume $K=\{e\}$,

$$\Lambda^*\mathfrak{gl}(n)' \xrightarrow{\text{var}} \Lambda^*(\mathfrak{gl}(n)'\oplus t\mathfrak{gl}(n)')$$

where $\mathfrak{gl}(n)'$ stands for the dual space of $\mathfrak{gl}(n)$. Let $\{\delta^i_j\}$ be a basis of $\mathfrak{gl}(n)'$ and $\{\eta^i_j\}$ be a basis of $t\cdot gl(n)'$. $d\delta^i_j=-\delta^i_k\wedge\delta^k_j$ $d\eta^i_j=-\delta^i_k\wedge\eta^k_j-\eta^i_k\wedge\delta^k_j$ holds. Let W_r be a sub-space of $\Lambda^*(\mathfrak{gl}(n)'+t\mathfrak{gl}(n)')$ with r $t\mathfrak{gl}(n)'$ components. Since $dW_r\subset W_r$, W_r is a sub-complex.

$$H^*(\widehat{\mathfrak{gl}}(n))=\bigoplus_{r=0}^{n^2} H^*(W_r) \qquad H^*(\mathfrak{gl}(n))\xrightarrow{\text{var}} H^*(W_1)$$
$$H^*(\mathfrak{gl}(n))\cong\Lambda^*(h_1\cdots h_n) \qquad \deg h_i=2i-1$$
$$h_k=\delta^{i_1}_{i_2}\wedge\delta^{i_2}_{i_3}\wedge\cdots\wedge\delta^{i_{2k-1}}_{i_1}\ .$$

Let $\sigma=(\delta^i_j)$, $\tau=(\eta^i_j)$ be $n\times n$-matrices. $h_1=\operatorname{Tr}(\sigma)=\delta^i_i$ $\text{var}\,h_1=\eta^i_i$, therefore $\text{var}\,h_1\neq 0$ in $H^*(W_1)$. As for h_k with $k>1$,

$$\text{var}\,h_k=\text{var}(\operatorname{Tr}(\sigma^{2k-1}))=\sum_{i=1}^{2k-1}\operatorname{Tr}(\sigma\wedge\cdots\wedge\sigma\wedge\overset{\overset{i}{\vee}}{\tau}\wedge\sigma\wedge\cdots\wedge\sigma)=\text{constant}\times\operatorname{Tr}(\tau\wedge\sigma^{2k-2})\ .$$

On the other hand, $d\operatorname{Tr}(\tau\wedge\sigma^{2k-3})=-\operatorname{Tr}(\tau\wedge\sigma^{2k-2})$. Therefore $\text{var}\,h_k=\text{constant}\times d\operatorname{Tr}(\tau\wedge\sigma^{2k-3})$ and $\text{var}\,h_k=0$ in $H^*(W_1)$.

REMARK. Since

$$H^*(W(\mathfrak{gl}(n), K)_n)\cong H^*_{GF}(\mathfrak{A}(n), K)$$
$$H^*(\mathfrak{gl}(n), K)\cong H^*_{GF}(\mathfrak{A}_0(n), K)\ ,$$

Theorem II means that, in case of $M=\boldsymbol{R}^n$ and $\mathscr{S}=\emptyset$, the above construction of characteristic classes is essentially equivalent to the construction given by Kamber-Tondeur (cf. [12]). It is not yet known how other cases are.

References

[1] Bott, R., Lectures on characteristic classes and foliations. (Notes by Lawrence-Conlon) Springer Lecture Note #279, 1-94.
[2] Bott, R., On the Gelfand-Fuks cohomology, Proc. Symp. Math. A.M.S. vol. 27, 1975.
[3] Bott, R., Notes on Gel'fand-Fuks cohomology and characteristic classes. (Notes by Mostow-Perchik).
[4] Bott, R. and A. Haefliger, On characteristic classes of Γ-foliations. Bull. Amer. Math. Soc. 78 (1972), 1039-1044.
[5] Haefliger, A., Sur les classes caractéristiques de feuilletages. Sém. Bourbaki, #412.

[6] Gel'fand, I. M. and D. B. Fuks, Cohomology of the Lie algebra of tangent vector fields on a smooth manifold I, II. Functional Anal. Appl. **3** (1969), 194–210, **4** (1970), 110–116.
[7] Gel'fand, I. M. and D. B. Fuks, Cohomology of the Lie algebra of formal vector fields. Izvestija Akademii Nauk SSSR Serija Matematičeskaja? **34** (1970), 322–337.
[8] Gel'fand, I. M., B. L. Feigin and D. B. Fuks, Cohomologies of the Lie algebra of formal vector fields with coefficient in its adjoint space and variations of characteristic classes of foliations. Functional Anal. Appl. **8** (1974), 99–112.
[9] Heitsch, J. L., Deformations of secondary characteristic classes. Topology **12** (1973), 381–388.
[10] Godbillon, C., Cohomologie d'algèbres de Lie de champs de vecteurs formels. Sém. Bourbaki #412.
[11] Goldman, R., The holonomy ring on the leaves of foliated manifolds. J. Differential Geometry **11** (1976), 411–449.
[12] Kamber, F. and P. Tondeur, Foliated bundles and characteristic classes. Springer Lecture Notes #493.
[13] Kobayashi, S., Transformation groups in differential geometry. Springer Lecture Notes #70.

(Received November 15, 1978)

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