

# Finite groups with a standard subgroup isomorphic to $G_2(2^n)$

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## 1. Introduction.

A subgroup  $K$  of a finite group  $G$  is said to be *tightly embedded* in  $G$  if  $K$  has even order while  $K \cap K^g$  has odd order for each  $g \in G - N_G(K)$ . A *standard* subgroup of  $G$  is a quasisimple subgroup  $L$  such that  $K = C_G(L)$  is tightly embedded in  $G$ ,  $N_G(K) = N_G(L)$ , and  $[L, L^g] \neq 1$  for each  $g \in G$ . Let  $G_2(2^n)$  denote the Chevalley group of type  $(G_2)$  over the finite field  $GF(2^n)$  of  $2^n$  elements. The purpose of this paper is to prove the following theorem.

**THEOREM.** *Let  $G$  be a finite group and suppose  $L$  is a standard subgroup of  $G$  with  $L/Z(L) \cong G_2(q)$ , where  $q = 2^n \geq 4$ . Assume that  $Z(L)$  has odd order and  $C_G(L)$  has cyclic Sylow 2-subgroups. Assume furthermore that every section  $X$  of  $G$  satisfies the following condition:*

(\*) *In  $X/O(X)$ , the 2-layer of every 2-local subgroup is semisimple.*

*Then one of the following holds.*

- (1)  $LO(G) \triangleleft G$ .
- (2)  $\langle L^G \rangle \cong G_2(q^2)$ .
- (3)  $\langle L^G \rangle \cong G_2(q) \times G_2(q)$ .

We note that the Schur multiplier of  $G_2(q)$  is known (see [11]) and  $Z(L)$  is actually trivial.

The condition (\*) is a consequence of the  $B$ -conjecture of J. Thompson. In the proof of the theorem, we use an inductive argument due to Seitz [19] to show that a certain subgroup isomorphic to  $G_2(q) \times G_2(q)$  is in fact normal in  $G$ . Here we need to assume the condition (\*) for every section of  $G$ . Also, when  $q = 4$ , we use a result of Seitz [18] to determine the structure of  $C_G(E)$  for a certain subgroup  $E$  of order 3. Here again we require the condition (\*) for  $C_G(E)$ . All other parts of the proof are free from the condition.

The method used in this paper is similar to that used by K. Gomi in [9]. Let  $t$  be an involution of  $C_G(L)$ . Then  $L$  is normal in  $C_G(t)$  by our hypotheses. In Section 4 we shall show that Case (1) of the theorem holds if and only if  $t$  is a central involution of  $G$ , that is,  $t$  lies in the center of a Sylow 2-subgroup of  $G$ . Let  $P_i$ ,  $i=1, 2$ , denote the two maximal parabolic subgroups of  $L$  containing a fixed Sylow 2-subgroup of  $L$ . Then  $O_2(P_i)$  has order  $q^5$  and  $P_i'/O_2(P_i) \cong SL(2, q)$ . Under the assumption that  $LO(G) \not\trianglelefteq G$ , we construct in Section 5 two subgroups  $N_1$  and  $N_2$  with the property that  $O_2(N_i)$  has order  $q^{10}$ ,  $L \cap N_i = P_i'$ , and

$$N_i/O_2(N_i) \cong SL(2, q^2) \quad \text{or} \quad SL(2, q) \times SL(2, q).$$

The structure of  $N_i/O_2(N_i)$  is determined by a previous classification theorem of Griess, Mason, and Seitz [12]. The rest of the paper is devoted to proving that the subgroup  $G_1 = \langle N_1, N_2 \rangle$  is isomorphic to  $G_2(q^2)$  or  $G_2(q) \times G_2(q)$  and that  $G_1$  is a normal subgroup of  $G$ . If  $N_i/O_2(N_i) \cong SL(2, q^2)$ , then using Lemma (2.8) in Section 2 we can prove that  $G_1$  is isomorphic to  $G_2(q^2)$ . It then follows from [16] or [21] that  $G_1$  is normal in  $G$ . Thus  $\langle L^G \rangle \cong G_2(q^2)$  in this case. If  $N_i/O_2(N_i) \cong SL(2, q) \times SL(2, q)$ , then it will be shown that  $G_1$  is isomorphic to  $G_2(q) \times G_2(q)$ . Next, we apply [19, Lemma (2.7)] to conclude that  $G_1$  is normal in  $G$ . Thus  $\langle L^G \rangle \cong G_2(q) \times G_2(q)$  in this case.

Our notation is fairly standard. For a 2-group  $P$ ,  $\mathcal{A}(P)$  is the set of abelian subgroups of  $P$  of maximal order,  $J(P)$  is the Thompson subgroup generated by all members of  $\mathcal{A}(P)$ , and  $J_r(P)$  is the subgroup generated by abelian subgroups of  $P$  of maximal rank. Also,  $\mathcal{E}^*(P)$  is the set of maximal elementary abelian subgroups of  $P$ ,  $\mathcal{U}^1(P) = \langle x^2 \mid x \in P \rangle$ , and  $\Omega_1(P) = \langle x \in P \mid x^2 = 1 \rangle$ . For a subset  $U$  of a group,  $I(U)$  is the set of involutions in  $U$ . In addition,  $m(X)$  denotes the 2-rank of a group  $X$ ,  $X^\infty$  is the final term of the derived series of  $X$ , and  $X * Y$  is a central product of two groups  $X$  and  $Y$ .  $V(2, q)$  denotes the 2-dimensional vector space over  $GF(q)$ . An elementary abelian group  $E$  of order  $q^2$ , on which  $A \cong SL(2, q)$  acts, is said to be a *natural module* for  $A$  if  $E \cong V(2, q)$  as  $A$ -modules. We also use the bar convention for homomorphic images.

Two figures are attached at the end of this paper to illustrate the relations of subgroups which will appear in Section 5.

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## 2. Preliminaries.

In this section we collect together certain results to be used in later sections.

(2.1) Let  $S=AB$  be a nonabelian 2-group, where  $A$  and  $B$  are elementary abelian. Suppose that  $C_S(a)=A$  for all  $a \in A-B$ . Then

- (1)  $C_S(b)=B$  for all  $b \in B-A$ ,
- (2)  $\mathcal{E}^*(S)=\{A, B\}$ ,
- (3)  $Z(S)=A \cap B$ ,
- (4)  $|C_S(x)| \leq \min\{|A|, |B|\}$  for each  $x \in S-(A \cup B)$ .

PROOF. Clearly  $C_S(b)=B$  for all  $b \in B-A$ , so (1) and (3) hold. Let  $x=ab$  be an involution with  $a \in A$  and  $b \in B$ . Then  $x^2=[a, b]=1$ , so that  $a \in B$  or  $b \in A$ . Thus  $A^* \cup B^*$  is the set of involutions of  $S$  and (2) follows. For (4) we may assume that  $|A| \geq |B|$ . Take an element  $x \in S-(A \cup B)$ . If  $C_A(x) \not\leq B$ , then  $x \in C_S(a)$  for some  $a \in A-B$ , a contradiction. Hence  $C_A(x)=A \cap B$  and  $|S| \geq |C_S(x)A|=|C_S(x)| |A|/|A \cap B|$ . Thus  $|C_S(x)| \leq |B|$ .

The following two lemmas are well-known and their proofs are omitted.

(2.2) Let  $P$  be a Sylow  $p$ -subgroup of a group  $G$  for a prime  $p$  and  $W$  a subgroup of  $P$ . If  $W$  is weakly closed in  $P$  with respect to  $G$ , then  $N_G(W)$  controls the fusion in  $C_G(W)$ .

(2.3) Let  $P$  be a Sylow 2-subgroup of a group  $G$  and  $R$  a subgroup of  $P$  containing  $P'$ . Suppose that  $x$  is an involution in  $P-R$  with  $x^G \cap P \leq xR$ . Then  $x$  is not contained in the kernel of the transfer of  $G$  into  $P/R$ .

(2.4) Let  $X$  be a group acting on a group  $Y$  and  $z \in Y$ . If  $z^Y \leq C_Y(X)$ , then  $[X, Y] \leq C_Y(z)$ . In particular, if  $C_Y(X) \triangleleft Y$ , then  $[X, Y]$  centralizes  $C_Y(X)$ .

PROOF. For  $x \in X$  and  $y \in Y$  we have  $z^{x^{-1}y^{-1}xy} = z^{y^{-1}xy} = z^{y^{-1}y} = z$ .

(2.5) Suppose that a group  $G$  acts on an elementary abelian 2-group  $X$ ,  $|X| \geq 4$ , and normalizes a maximal subgroup  $E$  of  $X$ . Let  $t \in X-E$ . If  $C_G(E)$  has a subgroup  $H$  such that  $E = \langle [t, H]^{C_G(t)} \rangle$ , then the following conditions hold.

- (1)  $t^G = X-E$ .
- (2)  $C_G(E)/C_G(X)$  is a regular normal subgroup of the permutation group  $(G/C_G(X), t^G)$ .
- (3)  $C_G(E)/C_G(X) \cong E$  as  $C_G(t)$ -modules.

PROOF. We have  $t^G \leq X - E = tE$ . Define a map  $f$  from  $C_G(E)$  into  $E$  by  $y \mapsto [t, y] = tt^y$  for  $y \in C_G(E)$ . This is obviously a homomorphism commutable with the action of  $C_G(t)$ . The kernel of  $f$  is  $C_G(E) \cap C_G(t) = C_G(X)$ . The image of  $f$  is  $C_G(t)$ -invariant and contains  $[t, H]$ , so is equal to  $E$ . Hence (3) holds. Also,  $t^G = t^{C_G(E)} = tE$ . Thus  $\langle t^G \rangle = X$  and (2) holds.

The following three lemmas are due to K. Gomi and their proofs can be found in [9, Section 1].

(2.6) Let  $(G, X)$  be a 2-transitive permutation group and  $G_x$  the stabilizer of a point  $x \in X$ . If there exists  $N \triangleleft G_x$  such that  $(N, X) \cong (SL(2, q), V(2, q))$  as permutation groups, then  $G$  has a regular normal subgroup.

(2.7) Suppose that  $L \cong SL(2, 2^n)$  acts nontrivially on an elementary abelian 2-group  $V$  of order  $2^{2^n}$  and  $|C_V(P)| \geq 2^n$  for  $P \in \text{Syl}_2(L)$ . Then  $V$  is a natural module for  $L$ . In particular,  $|C_V(P)| = 2^n$ ,  $\mathcal{C}^*(PV) = \{V, PC_V(P)\}$ , and  $N_L(P)/P$  acts regularly both on  $C_V(P)^\#$  and on  $(V/C_V(P))^\#$ .

(2.8) Let  $S$  be a  $p$ -subgroup of a group  $G$  and let  $M_i$ ,  $i=1, 2$ , be normal subgroups of  $S$ . Assume that for each  $i$  there is a subgroup  $L_i$  of  $N_G(M_i)$  with  $S \leq L_i$  such that

(1)  $L_i/M_i$  is a perfect central extension of  $PSL(2, p^{n_i})$ ,  $PSU(3, p^{n_i})$ , or  $Sz(p^{n_i})$  for some  $n_i > 1$  with  $Z(L_i/M_i)$  a  $p'$ -group,

(2)  $S \leq \text{Syl}_p(L_i)$ , and

(3)  $N_{L_i}(S) \leq N_G(L_j)$  for  $\{i, j\} = \{1, 2\}$ .

Assume further that there is an involution  $r_i \in L_i - N_{L_i}(S)$  such that

(4)  $|r_1 r_2| = m$  is even  $> 2$ ,

(5)  $S = M_i(S \cap S^{r_j} \cap S^{r_i r_j} \cap \dots \cap S^{\overbrace{r_j r_i \dots r_j}^{m-1}})$  for  $\{i, j\} = \{1, 2\}$ , and

(6)  $S \cap S^{(r_1 r_2)^{m/2}} = 1$ .

Then  $H = \langle L_1, L_2 \rangle$  is a perfect central extension of

$$PSp(4, p^n), \quad PSU(4, p^n), \quad PSU(5, p^n),$$

$$G_2(p^n), \quad {}^3D_4(p^n), \quad \text{or} \quad {}^2F_4(p^n)$$

for some  $n$ , and  $Z(H)$  is a  $p'$ -group. Furthermore,  $S$  is a Sylow  $p$ -subgroup of  $H$  and the Weyl group of the natural  $BN$ -pair of  $H/Z(H)$  has order  $2m$ .

The next result is due to M. Aschbacher (see for example [1, Lemma 2.7]).

(2.9) Let  $A$  be a standard subgroup of a group  $G$  and  $X = \langle A^g \rangle$ . Assume that  $O(G) = 1$  and  $A \ntriangleleft G$ . Then  $F^*(G) = X$  and either

(1)  $X$  is simple, or

(2)  $X \cong A \times A$ ,  $A$  is simple,  $|C_G(A)| = 2$ , and the involution in  $C_G(A)$  interchanges the two components of  $X$ .

(2.10) Let  $A$  be a standard subgroup of a group  $G$ ,  $t$  an involution in  $C_G(A)$ ,  $X = \langle A^g \rangle$ , and assume that  $C_G(A)$  has cyclic Sylow 2-subgroups. Assume further that  $X$  has a  $\langle t \rangle$ -invariant 2-subgroup  $R$  such that  $1 \neq [R, t] \leq A$ . Then  $X$  is semisimple with  $|Z(X)|$  odd.

PROOF. We argue as in [9, Section 10]. Since  $C_G(A)$  is tightly embedded in  $G$  and  $N_G(C_G(A)) = N_G(A)$ , we have  $C_G(t) \leq N_G(A)$ . Let  $\bar{G} = G/O(G)$  and let  $D$  be the inverse image of  $C_{\bar{G}}(\bar{A})$  in  $G$ . Then  $C_{\bar{G}}(\bar{t}) = \overline{C_G(t)}$  and  $D \cap C_G(t)O(G) = C_D(t)O(G)$ . Now  $[A, C_D(t)] \leq A \cap O(G) \leq Z(A)$ , so that  $[A, C_D(t)] = 1$  by the three-subgroup lemma. Hence  $C_{\bar{G}}(\bar{t}) \cap C_{\bar{G}}(\bar{A}) = \overline{C_G(t) \cap C_G(A)}$ . In particular,  $C_{\bar{G}}(\bar{t}) \cap C_{\bar{G}}(\bar{A})$  has cyclic Sylow 2-subgroups. Since  $\bar{t}$  is an involution of  $C_{\bar{G}}(\bar{A})$ , this implies that a Sylow 2-subgroup of  $C_{\bar{G}}(\bar{A})$  is contained in  $C_{\bar{G}}(\bar{t})$  and is cyclic. Therefore, if  $\bar{A} \triangleleft \bar{G}$ , then  $\bar{t}$  is isolated in  $\bar{G}$  and so  $\bar{t} \in Z(\bar{G})$  by the  $Z^*$ -theorem [6]. This contradicts the assumption that  $[R, t] \neq 1$ . Hence  $\bar{A} \ntriangleleft \bar{G}$ . Also,  $\bar{A}$  is standard in  $\bar{G}$ . Now (2.9) shows that  $\bar{X}$  is semisimple with  $Z(\bar{X}) = 1$  and that  $\bar{X}$  has no proper  $\langle t \rangle$ -invariant normal subgroups. Since  $A \triangleleft C_G(t)$ , we have  $[C_{O(G)}(t), [R, t]] \leq [C_{O(G)}(t), A] = 1$ . Then by [9, Lemma (1J)],  $[O(G), [R, t]] = 1$ . So  $C_X(O(X)) \leq O(X)$  and  $X = C_X(O(X))O(X)$ . Since  $X$  is perfect and  $O(X)$  is solvable, we must have  $X = C_X(O(X))$ . This proves (2.10).

The next result is well-known (see [2]).

(2.11) (1) Each involution in  $\text{Aut}(\text{PSU}(3, 2^n)) - \text{PSU}(3, 2^n)$  is conjugate to an involutory field automorphism.

(2) Each involution in  $\text{Aut}(\text{PSL}(3, 2^n)) - \text{PSL}(3, 2^n)$  is conjugate to  $g$  if  $n$  is odd, and conjugate to  $f$ ,  $g$ , or  $fg$  if  $n$  is even, where  $g$  and  $f$  denote respectively a graph and an involutory field automorphism.

### 3. Properties of $G_2(q)$ .

In this section we will give a summary of some of the properties of  $G_2(q)$ , the Chevalley group of type  $(G_2)$  over  $GF(q)$ ,  $q = 2^n$ , and its automorphism group. A general discussion of Chevalley groups can be found in [3] and an excellent description of  $G_2(q)$  can be found in [20].

Let  $\mathcal{A}$  denote a simple Lie algebra of type  $(G_2)$  over the complex field and let  $\Sigma$  denote the set of roots of  $\mathcal{A}$ . Relative to some fixed ordering on  $\Sigma$ , the fundamental roots of  $\mathcal{A}$  may be denoted by  $a$  and  $b$  with  $a$  a short root and  $b$  a long root. Then the set of positive roots of  $\Sigma$  is

$$\Sigma^+ = \{a, b, a+b, 2a+b, 3a+b, 3a+2b\},$$

and  $\Sigma$  consists of the elements of  $\Sigma^+$  and their negatives. Let  $\mathcal{A}$  denote the additive group generated by the elements of  $\Sigma$ . Then  $\mathcal{A}$  is a free abelian group of rank 2 with generators  $a$  and  $b$ . For each  $r, s \in \Sigma$  we define the rational integer  $s(r)$  by  $s(r)=2$  if  $r=s$ , and  $s(r)=p_1-p_2$  if  $r \neq s$ , where

$$p_1 = \max\{i \mid s - ir \in \Sigma\} \quad \text{and} \quad p_2 = \max\{j \mid s + jr \in \Sigma\}.$$

Set  $\tilde{w}_r(s) = s - s(r)r$ . This defines for each  $r \in \Sigma$  a permutation  $\tilde{w}_r$  on  $\Sigma$  called the reflection with respect to  $r$ . It is of order 2 and additive on  $\Sigma$ .

(3.1) The group  $\langle \tilde{w}_r \mid r \in \Sigma \rangle$  is called the Weyl group on  $\Sigma$ , and is dihedral of order 12 generated by  $\{\tilde{w}_a, \tilde{w}_b\}$ . It acts transitively both on the set of short roots  $\{\pm a, \pm(a+b), \pm(2a+b)\}$  and on the set of long roots  $\{\pm b, \pm(3a+b), \pm(3a+2b)\}$ . Moreover,

$$\tilde{w}_a(a) = -a, \quad \tilde{w}_a(b) = 3a+b, \quad \tilde{w}_b(a) = a+b, \quad \tilde{w}_b(b) = -b.$$

(3.2) Let  $L = G_2(q)$ ,  $q = 2^n$ . Then  $L$  is simple except when  $q=2$ . In fact,  $|G_2(2) : G_2(2)'| = 2$  and  $G_2(2)' \cong PSU(3, 3^2)$ . The Schur multiplier of  $L$  is trivial if  $q \geq 8$  and is of order 2 if  $q=4$ .

(3.3) For each  $r \in \Sigma$  there is a nontrivial homomorphism  $\varphi_r : SL(2, q) \rightarrow L$ . For each  $\alpha \in GF(q)$  and  $\beta \in GF(q)^\times$ , set

$$\begin{aligned} x_r(\alpha) &= \varphi_r \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, & x_{-r}(\alpha) &= \varphi_r \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \\ w_r &= \varphi_r \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & h(r, \beta) &= \varphi_r \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}. \end{aligned}$$

(3.4) Let  $S_r = \{x_r(\alpha) \mid \alpha \in GF(q)\}$ . Then  $S_r$  is an elementary abelian group of order  $q$ . Let

$$S = S_a S_b S_{a+b} S_{2a+b} S_{3a+b} S_{3a+2b}.$$

Then  $S$  has order  $q^6$  and is a Sylow 2-subgroup of  $L$ . Every element  $x \in S$  has a unique expression:

$$x = x_a(\alpha_1) x_b(\alpha_2) x_{a+b}(\alpha_3) x_{2a+b}(\alpha_4) x_{3a+b}(\alpha_5) x_{3a+2b}(\alpha_6).$$

The structure of  $S$  is completely determined by the following commutator formulas:

$$\begin{aligned}
[x_a(\alpha), x_b(\beta)] &= x_{a+b}(\alpha\beta) x_{2a+b}(\alpha^2\beta) x_{3a+b}(\alpha^3\beta), \\
[x_a(\alpha), x_{a+b}(\beta)] &= x_{3a+b}(\alpha^2\beta) x_{3a+2b}(\alpha\beta^2), \\
[x_a(\alpha), x_{2a+b}(\beta)] &= x_{3a+b}(\alpha\beta), \\
[x_b(\alpha), x_{3a+b}(\beta)] &= x_{3a+2b}(\alpha\beta), \\
[x_{a+b}(\alpha), x_{2a+b}(\beta)] &= x_{3a+2b}(\alpha\beta), \\
[x_r(\alpha), x_s(\beta)] &= 1, \text{ for all other pairs of roots } r, s \in \Sigma^+.
\end{aligned}$$

(3.5) Let  $J = \langle h(r, \beta) \mid r \in \Sigma^+, \beta \in GF(q)^\times \rangle$ . There is an isomorphism  $\phi: J \rightarrow \text{Hom}(A, GF(q)^\times)$ . If  $\phi_h$  denotes the image of  $h \in J$  under  $\phi$ , then the isomorphism  $\phi$  is given by  $\phi_{h(r, \beta)}(s) = \beta^{s(r)}$ . The subgroup  $J$  is a direct product of two cyclic groups of order  $q-1$ .

(3.6)  $N_L(S) = SJ$ , and  $h \in J$  acts on  $S_r$  according to the formula  $hx_r(\alpha)h^{-1} = x_r(\phi_h(r)\alpha)$ .

(3.7)  $N_L(J) = \langle J, w_r \mid r \in \Sigma \rangle$  and  $N_L(J)/J \cong \langle \tilde{w}_a, \tilde{w}_b \rangle$ ; in fact, each  $w_r$  is a coset representative of  $\tilde{w}_r$ . If  $h \in J$ , then  $w_r h w_r = h'$  where  $h'$  is the element of  $J$  such that  $\phi_{h'}(s) = \phi_h(\tilde{w}_r(s))$ . Also,  $w_r x_s(\alpha) w_r = x_{\tilde{w}_r(s)}(\alpha)$ .

(3.8) For any two roots  $r, s \in \Sigma$ ,  $w_r w_s w_r = w_{r'}$  where  $r' = \tilde{w}_r(s)$ . The group  $\langle w_a, w_b \rangle$  is dihedral of order 12.

(3.9) The following subgroups of  $J$  play important roles in later sections.

$$\begin{aligned}
J_1 &= C_J(S_{3a+2b}), \quad J_2 = C_J(S_{2a+b}), \quad J_3 = C_J(S_a), \\
J_4 &= C_J(S_b), \quad J_5 = C_J(S_{3a+b}), \quad J_6 = C_J(S_{a+b}).
\end{aligned}$$

For convenience we set

$$r_1 = 3a+2b, \quad r_2 = 2a+b, \quad r_3 = a, \quad r_4 = b, \quad r_5 = 3a+b, \quad r_6 = a+b.$$

Then  $J_i = C_J(S_{r_i}) = \{h \in J \mid \phi_h(r_i) = 1\}$ ,  $1 \leq i \leq 6$ . Also,  $J_i = \{h(s, \beta) \mid \beta \in GF(q)^\times\}$  where  $s$  is the positive root orthogonal to  $r_i$ . (3.1) and (3.7) imply that  $J_1, J_4$ , and  $J_5$  are conjugate in  $N_L(J)$  and  $J_2, J_3$ , and  $J_6$  are conjugate in  $N_L(J)$ . Furthermore,  $J_i$  has the following properties.

(1)  $J_i$  is cyclic of order  $q-1$ .

(2) If  $q \not\equiv 1 \pmod{3}$ , then  $J_i \cap J_k = 1$  and  $J_i$  acts regularly on  $S_{r_k}^*$  for  $i, k$  with  $i \neq k$ .

(3) If  $q \equiv 1 \pmod{3}$ , then  $J_1 \cap J_4 = J_1 \cap J_5 = J_4 \cap J_5 = \{h \in J \mid \phi_h(a)^3 = 1, \phi_h(b) = 1\}$  has order 3 and  $J_i / \langle J_i \cap J_k \rangle$  acts semiregularly on  $S_{r_k}^\#$  for  $i, k$  with  $i \neq k$  and  $\{i, k\} \leq \{1, 4, 5\}$ , while if  $\{i, k\} \not\leq \{1, 4, 5\}$  then  $J_i \cap J_k = 1$  and  $J_i$  acts regularly on  $S_{r_k}^\#$ .

(4) If  $q \geq 8$ , then  $C_S(J_i) = S_{r_i}$  for  $1 \leq i \leq 6$  and  $J_i$  acts irreducibly on  $S_{r_k}$  for  $i \neq k$ . If  $q = 4$ , then  $J_1 = J_4 = J_5$ ,  $C_S(J_1) = S_b S_{3a+b} S_{3a+2b}$ , and  $C_S(J_i) = S_{r_i}$  for  $i = 2, 3, 6$ .

(3.10) The structure of  $C_L(J_i)$  is given in [2] or [4].

$$C_L(J_1) = \begin{cases} \langle S_{\pm(3a+2b)} \rangle \times J_1 & \text{if } q \geq 8, \\ \langle S_{\pm b}, S_{\pm(3a+b)}, S_{\pm(3a+2b)} \rangle \cong SL(3, 4) & \text{if } q = 4. \end{cases}$$

$$C_L(J_2) = \langle S_{\pm(2a+b)} \rangle \times J_2 \quad \text{for } q \geq 4.$$

(3.11)  $\langle S_{\pm a} \rangle = S_a J_1 \cup S_a J_1 w_a S_a$  and  $\langle S_{\pm b} \rangle = S_b J_2 \cup S_b J_2 w_b S_b$ .

(3.12) We define

$$\begin{aligned} U &= S_{3a+2b}, \quad Y = S_{3a+b} U, \quad X = S_{a+b} Y, \quad W = S_{2a+b} Y, \\ F &= XW, \quad D = S_b F, \quad \text{and} \quad M = S_a F. \end{aligned}$$

Then the following conditions hold.

(1)  $U = Z(S) = Z(D) = \mathfrak{U}^1(D) = D' = S''$ ,  $Y = Z_2(S) = Z(M) = \mathfrak{U}^1(M) = M'$ , and  $F = Z_3(S) = \mathfrak{U}^1(S) = S'$ .

(2)  $\mathcal{E}^*(F) = \{W, X\}$  and  $\mathcal{E}^*(S/Y) = \{M/Y, D/Y\}$ .

In particular,  $M$  and  $D$  are characteristic subgroups of  $S$ , while  $S$  has an involutory automorphism which interchanges  $W$  and  $X$  (see (6.24) of [20]). Also,

$$\mathcal{E}^*(S/W) = \{M/W, D/W\} \quad \text{and} \quad \mathcal{E}^*(S/X) = \{M/X, D/X\}.$$

(3.13) (1)  $N_L(X) = N_L(F) = N_L(S)$ .

(2)  $N_L(U) = N_L(D) = D \langle \langle S_{\pm a} \rangle \times J_3 \rangle$ .

(3)  $N_L(Y) = N_L(W) = N_L(M) = M \langle \langle S_{\pm b} \rangle \times J_4 \rangle$ .

(4) If  $q \geq 4$ , then both  $N_L(D)'$  and  $N_L(M)'$  are perfect.

REMARK. The groups  $N_L(D)$  and  $N_L(M)$  are the maximal parabolic subgroups containing  $N_L(S)$  and  $D \langle \langle S_{\pm a} \rangle \times J_3 \rangle$  and  $M \langle \langle S_{\pm b} \rangle \times J_4 \rangle$  are the Levi decompositions.

(3.14) (1)  $L$  has exactly two conjugate classes of involutions and we can choose  $x_{3a+2b}(1)$  and  $x_{2a+b}(1)$  as their representatives.



$$(2) \quad C_L(x_{3a+2b}(1))=C_L(U)=D\langle S_{\pm a} \rangle, \quad C_L(x_{2a+b}(1))=C_L(S_{2a+b})=W\langle S_{\pm b} \rangle.$$

$$(3.15) \quad N_L(S_{2a+b})=W(\langle S_{\pm b} \rangle \times J_4).$$

(3.16) (1)  $C_L(U)$  acts irreducibly on  $D/U$  if  $q \geq 8$  and  $N_L(U)$  acts irreducibly on  $D/U$  if  $q \geq 4$ .

(2) Both  $Y$  and  $M/W$  are natural modules for  $\langle S_{\pm b} \rangle \cong SL(2, q)$ .

PROOF. When  $q \geq 4$ , we can compute that  $\langle Y^{C_L(U)} \rangle = D$ . Let  $\overline{N_L(U)} = N_L(U)/U$ . Assume first that  $q \geq 8$  and let  $E$  be a normal subgroup of  $C_L(U)$  such that  $U < E \leq D$ . Then  $1 \neq C_E(S) \leq Z(\bar{S}) = \bar{Y}$ . Since  $J_1$  acts irreducibly on  $\bar{Y}$  by (3.9) (4), we have  $\bar{E} \geq \bar{Y}$ . Hence  $E = D$ . If  $q = 4$ , then we can verify that

$$|\bar{D} : \langle \overline{x_{3a+b}(1)}^{C_L(U)} \rangle| = 2^4.$$

In this case, however,  $J$  acts transitively on  $\bar{Y}^*$ , and by a similar argument as above we see that  $N_L(U)$  acts irreducibly on  $D/U$ . We remark that  $|D : \langle Y^{C_L(U)} \rangle| = 2$  if  $q = 2$ .

(2) follows immediately from (2.7). But we can describe this situation more explicitly. We have

$$(x_{3a+b}(\beta_1) x_{3a+2b}(\beta_2))^{x_b(\alpha)} = x_{3a+b}(\beta_1) x_{3a+2b}(\alpha\beta_1 + \beta_2).$$

As  $x_{-b}(\alpha) = w_b x_b(\alpha) w_b$ , we also have

$$(x_{3a+b}(\beta_1) x_{3a+2b}(\beta_2))^{x_{-b}(\alpha)} = x_{3a+b}(\beta_1 + \alpha\beta_2) x_{3a+2b}(\beta_2).$$

Thus, if we denote  $x_{3a+b}(\beta_1) x_{3a+2b}(\beta_2)$  by  $(\beta_1, \beta_2)$ , then

$$x_b(\alpha) : (\beta_1, \beta_2) \mapsto (\beta_1, \alpha\beta_1 + \beta_2) = (\beta_1, \beta_2) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

$$x_{-b}(\alpha) : (\beta_1, \beta_2) \mapsto (\beta_1 + \alpha\beta_2, \beta_2) = (\beta_1, \beta_2) \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

Next, let  $\overline{N_L(W)} = N_L(W)/W$ . Then  $\bar{M} = \bar{S}_a \bar{S}_{a+b}$ . We compute that

$$(\overline{x_a(\beta_1) x_{a+b}(\beta_2)})^{x_b(\alpha)} = \overline{x_a(\beta_1) x_{a+b}(\alpha\beta_1 + \beta_2)}$$

and

$$(\overline{x_a(\beta_1) x_{a+b}(\beta_2)})^{x_{-b}(\alpha)} = \overline{x_a(\beta_1 + \alpha\beta_2) x_{a+b}(\beta_2)}.$$

(3.17) (1)  $L$  has exactly two conjugate classes of maximal elementary abelian 2-subgroups and we can choose  $X$  and  $W$  as their representatives.

(2)  $C_L(X) = X$  and  $C_L(W) = W$ .

(3)  $m(S)=3n$  and  $J_r(S)=S$ .

(4) Any abelian subgroup contained in  $D$  or  $M$  has order at most  $q^3$ .

PROOF. (2) and (4) can be verified by direct computations. Since  $X^{wb}=S_a S_{3a+b} S_{3a+2b}$  and  $N_L(X)$  is not isomorphic to  $N_L(W)$ , (1) is a consequence of [20, (3.11)]. Then, it follows that  $m(S)=3n$ . Since  $D=WW^{w_a}$  and  $M=WX^{w_b}$ , we have  $J_r(S)=S$ . Thus (3) holds.

(3.18) Let  $A=\text{Aut}(L)$ . Then  $A=L\langle f \rangle$  is a semidirect product of  $L$  by a cyclic group  $\langle f \rangle$ , where  $f$  is the field automorphism defined by

$$f: x_r(\alpha) \mapsto x_r(\alpha^2)$$

for each  $r \in \Sigma$  and  $\alpha \in GF(q)$ . Thus  $f$  normalizes each root subgroup  $S_r$ . Also,  $f$  centralizes  $w_r$  and maps  $h(r, \beta)$  to  $h(r, \beta^2)$ .

(3.19) (1)  $C_A(S_r)=C_L(S_r)$  for all  $r \in \Sigma$ .

(2)  $C_{\langle f \rangle}(J_i)=1$  for  $1 \leq i \leq 6$ .

PROOF. Since each  $S_r$  is conjugate to  $U$  or  $S_{2a+b}$ , it suffices to show that  $C_A(U) \leq L$  and  $C_A(S_{2a+b}) \leq L$ . As  $f$  centralizes  $x_{3a+2b}(1)$ , (3.14) (2) yields that  $C_A(x_{3a+2b}(1))=C_L(U)\langle f \rangle$ . Certainly  $C_{\langle f \rangle}(U)=1$ , and hence  $C_A(U) \leq L$ . Likewise we have  $C_A(S_{2a+b}) \leq L$ . Since  $J_i=\{h(r, \beta) \mid \beta \in GF(q)^* \}$  for some  $r \in \Sigma$ , (2) is obvious.

(3.20) When  $n$  is even, we denote by  $f_0$  the involutory field automorphism. Then the following conditions hold.

(1)  $I(f_0 S)=f_0^S$  and  $I(A-L)=f_0^L$ .

(2)  $C_L(f_0) \cong G_2(q^{1/2})$ .

PROOF. Certainly (2) holds. We have  $|f_0^S|=|S:C_S(f_0)|=q^3$ . If  $|f_0 x|=2$  for  $x \in S$ , then  $f_0$  inverts  $x$ . By a direct computation we can verify that precisely  $q^3$  elements of  $S$  are inverted by  $f_0$ . Thus  $I(f_0 S)=f_0^S$ , and (1) holds.

(3.21)  $J_r(Q)=S$  for  $S \leq Q \in \text{Syl}_2(A)$ .

PROOF. We may assume that  $n$  is even. Let  $V$  be an abelian subgroup of  $S\langle f \rangle$  with  $m(V) \geq 3n$ . Then  $\Omega_1(V) \leq S\langle f_0 \rangle = S \cup Sf_0$ . As  $m(C_S(f_0)\langle f_0 \rangle) = 3n/2 + 1 < m(V)$ , (3.20) (1) shows that  $\Omega_1(V)$  is a subgroup of  $S$ . Then, by (3.17)  $\Omega_1(V)$  is conjugate to  $X$  or  $W$  in  $L$ , and so  $V \leq C_A(\Omega_1(V)) \leq L$  by (3.19). Let  $B$  be an abelian subgroup of  $Q$  such that  $m(B) \geq 3n$ . Then  $B^x \leq S\langle f \rangle$  for some  $x \in A$  and the above implies that  $B^x \leq L$ . Hence  $B \leq L \cap Q = S$ . This proves (3.21).

(3.22) If  $q \geq 8$ , then  $C_A(J_i) = C_L(J_i)$  for  $i=1, 2$ . If  $q=4$ , then  $C_A(J_1) = C_L(J_1)\langle w_a f \rangle$  and  $C_A(J_2) = C_L(J_2)\langle w_b f \rangle$ . Furthermore,  $C_A(C_L(J_i)) \leq C_L(J_i)$  for  $i=1, 2$ .

PROOF. Recall that  $J_1 = \{h(a, \beta) \mid \beta \in GF(q)^\times\}$  and  $J_2 = \{h(b, \beta) \mid \beta \in GF(q)^\times\}$ . Since  $h(a, \beta)^{w_a} = h(a, \beta^{-1})$  and  $h(a, \beta)^f = h(a, \beta^2)$ , it follows that  $\langle w_a, f \rangle \cap C(J_1)$  is trivial if  $q \geq 8$  and equal to  $\langle w_a f \rangle$  if  $q=4$ . Similarly  $\langle w_b, f \rangle \cap C(J_2)$  is trivial or equal to  $\langle w_b f \rangle$  according as  $q \geq 8$  or  $q=4$ .

If  $DN_L(J_1) \geq S$ , then  $S = S \cap DN_L(J_1) = DN_S(J_1)$ . But  $N_S(J_1) = C_S(J_1)$  is contained in  $D$  by (3.9), a contradiction. Thus  $\langle w_a \rangle JD \leq D(N_L(U) \cap N_L(J_1)) \neq N_L(U)$ . Since  $\langle w_a \rangle JD$  is a maximal subgroup of  $N_L(U)$ , we must have  $N_L(U) \cap N_L(J_1) = \langle w_a \rangle JC_D(J_1)$ , which is equal to  $\langle w_a \rangle JU$  or  $\langle w_a \rangle JS_b Y$  according as  $q \geq 8$  or  $q=4$ . Since  $N_A(U) \cap N_A(J_1) = (N_L(U) \cap N_L(J_1))\langle f \rangle$ , we have  $N_A(U) \cap C_A(J_1) = JU$  or  $\langle w_a f \rangle JS_b Y$  according as  $q \geq 8$  or  $q=4$ . Moreover, if  $q \geq 8$  then  $U \in \text{Syl}_2(C_L(J_1))$ , and if  $q=4$  then  $S_b Y \in \text{Syl}_2(C_L(J_1))$  and  $Z(S_b Y) = U$ . So we have  $C_A(J_1) = C_L(J_1)(N_A(U) \cap C_A(J_1))$  by a Frattini argument. Therefore,  $C_A(J_1)$  is equal to  $C_L(J_1)$  or  $C_L(J_1)\langle w_a f \rangle$  according as  $q \geq 8$  or  $q=4$ . Also, we have  $C_A(C_L(J_1)) \leq C_L(J_1)$  by (3.19) (1). As for  $J_2$ , it follows from (3.9) and (3.15) that  $N_L(S_{2a+b}) \cap N_L(J_2) = \langle w_b \rangle JS_{2a+b}$ . Moreover,  $N_A(S_{2a+b}) \cap C_A(J_2)$  is equal to  $JS_{2a+b}$  or  $\langle w_b f \rangle JS_{2a+b}$  according as  $q \geq 8$  or  $q=4$ . In addition,  $S_{2a+b}$  is a Sylow 2-subgroup of  $C_L(J_2)$ . Thus arguing as in the case  $J_1$ , we obtain the desired assertions.

(3.23) (1) For any subgroup  $B$  of  $A$  with  $B \geq L$  we have  $O_2(N_B(S)) = S$ ,  $O_2(N_B(U)) = D$ , and  $O_2(N_B(Y)) = M$ .

(2)  $C_A(X/Y) = SJ_6$ ,  $C_A(F/Y) = C_A(S/F) = S$ ,  $C_A(D/U) = D$ ,  $C_A(Y) = M(J_1 \cap J_5)$ ,  $C_A(W/Y) = M\langle S_{\pm b} \rangle$ , and  $C_A(M/W) = M$ .

(3) If  $q \geq 4$ , then  $C_A(N_L(U)/D) = DJ_3$  and  $C_A(N_L(Y)/M) = MJ_4$ .

PROOF. Set  $B = L\langle f_1 \rangle$  for some  $f_1 \in \langle f \rangle$ . Then  $N_B(S) = N_L(S)\langle f_1 \rangle$ . Let  $\overline{N_A(S)} = N_A(S)/S$ . Then  $\overline{N_B(S)} \triangleright \overline{N_L(S)} = \bar{J}$  and so  $\bar{J}$  centralizes  $\overline{O_2(N_B(S))}$ . Hence  $O_2(N_B(S)) = S$  by (3.19) (2). Likewise, the structure of  $N_L(U)$  given in (3.13) (2) shows that  $N_L(U)/D$  centralizes  $O_2(N_B(U))/D$ , and hence  $O_2(N_B(U)) = D$ . Similarly we have  $O_2(N_B(Y)) = M$ . (2) and (3) can be easily verified.

#### 4. Structure of $C(t)$ .

In this section we assume the following hypothesis.

(4.1) *Hypothesis.*  $G$  is a group,  $t$  is an involution of  $G$ , and  $C(t) > L \cong G_2(q)$  where  $q = 2^n \geq 4$ . Furthermore,  $C(L)$  has a cyclic Sylow 2-subgroup.

For subgroups of  $L$ , we use the symbols defined in Section 3. Thus  $S$  is a Sylow 2-subgroup of  $L$ . Set  $C=C(t)$ . Let  $R$  be a Sylow 2-subgroup of  $LC_C(L)$  with  $S \leq R$  and set  $T=C_R(L)$ .

(4.2) *The following conditions hold.*

- (1) *If  $Q$  is a Sylow 2-subgroup of  $C$  containing  $R$ , then  $J_r(Q)=R$ .*
- (2)  *$t^G \cap Y = \emptyset$ .*
- (3)  *$W \triangleleft N_C(S)$ .*

PROOF.  $R=S \times T$  has rank  $3n+1$  and  $J_r(R)=R$  by (3.17). Let  $R \leq Q \in \text{Syl}_2(C)$  and let  $V$  be an abelian subgroup of  $Q$  with  $m(V) \geq 3n+1$ . Then  $m(VT/T) \geq 3n$ . By (3.21) we have  $V \leq R$ . This implies (1). For an element  $y \in Y^\#$  we have  $C(y)^\infty \geq C_L(y)^\infty = C_L(y)$ . As  $C^\infty = L$  does not contain  $t$ , (2) follows. Since  $F=Z_3(S)$  and  $\mathcal{C}^*(F)=\{W, X\}$ ,  $N_C(S)$  acts on the set  $\{W, X\}$ . Furthermore,  $N_L(W) \neq N_L(X)$ . Thus (3) holds.

(4.3) *Suppose  $t$  is a central involution of  $G$ . Then  $t \in Z^*(G)$  and in particular  $LO(G) \triangleleft G$ .*

PROOF. Let  $Q$  be a Sylow 2-subgroup of  $C$  containing  $R$ . As  $t$  is a central involution of  $G$ ,  $Q \in \text{Syl}_2(G)$ . Assume by way of contradiction that  $t \notin Z^*(G)$ . Then  $t^G \cap C > \{t\}$  by the  $Z^*$ -theorem [6]. Take  $g \in G$  such that  $t \neq t^g \in C$ . So  $t \in C(t^g) \leq N(L^g)$ . If  $t \notin L^g C(L^g)$ , then we may assume that  $t$  acts on  $L^g$  as an involutory field automorphism. Thus  $I(tL^g) \leq t^G$  and  $C_{L^g}(t)^\infty$  has even order by (3.20). So  $\{t\} < I(tC_{L^g}(t)^\infty) \leq t^G \cap tL$  since  $C_{L^g}(t)^\infty \leq C^\infty = L$ . If  $t \in L^g C(L^g)$ , then  $t \neq t^{g^{-1}} \in LC_C(L)$ , and hence  $t^G \cap \langle t \rangle L > \{t\}$ . Therefore in any case  $t^G \cap \langle t \rangle L > \{t\}$ .

If  $t^G \cap \langle t \rangle Y > \{t\}$ , then the transitive action of  $N_L(Y)$  on  $Y^\#$  and (4.2) (2) yield  $t^G \cap \langle t \rangle Y = tY$ . By (4.2) (1)  $R$  is weakly closed in  $Q$  with respect to  $G$ , so  $N(R)$  controls the fusion of elements of  $Z(R)$  by (2.2). Thus  $t^{N(R)} = tU$ , since  $Z(R) = U \times T$  and  $\Omega_1(T) = \langle t \rangle$ . However,  $N_C(R)$  contains  $Q$  and so  $|t^{N(R)}|$  is odd, a contradiction. Therefore  $t^G \cap \langle t \rangle Y = \{t\}$ . In particular,  $N(\langle t \rangle Y) \leq C$ .

Since every involution of  $L$  is conjugate to an element of  $W$  in  $L$ , it follows that  $t^G \cap (\langle t \rangle W - \langle t \rangle Y) \neq \emptyset$ . As  $N_L(Y)$  acts transitively on  $W - Y$ , we have  $t \neq t^{x^{-1}} \in \langle t \rangle S_{2a+b}$  for some  $x \in G$ . Then  $C(t^{x^{-1}}) \geq \langle t \rangle W \langle S_{\pm b} \rangle$  by (3.14), and  $C \geq \langle \langle t \rangle W \langle S_{\pm b} \rangle \rangle^x$ . Take  $y \in C$  such that  $\langle \langle t \rangle W \rangle^{xy} \leq Q$ . As  $\langle t \rangle W$  is elementary abelian of order  $2q^3$ , (4.2) yields that  $\langle \langle t \rangle W \rangle^{xy} \leq \langle t \rangle S$ . Notice that  $t = t^y \in \langle \langle t \rangle S_{2a+b} \rangle^{xy} \leq \langle \langle t \rangle W \rangle^{xy}$ . Then  $\langle \langle t \rangle W \rangle^{xy} = \langle t \rangle (\langle \langle t \rangle W \rangle^{xy} \cap L)$ . Set  $K = \langle \langle t \rangle W \rangle^{xy} \cap L$ , so that  $K$  is an elementary abelian subgroup of order  $q^3$ . By (3.17) we can find an element  $z$  of  $L$  such that  $K^z = W$  or  $X$ . Now  $\langle S_{\pm b} \rangle^x \leq C^\infty = L$ , so  $N_L(K^z)$  contains  $\langle S_{\pm b} \rangle^{xy^z}$ . Hence we must have  $K^z = W$ , and  $\langle \langle t \rangle W \rangle^{xy^z} = \langle t \rangle W$ . Since

$t^{(xyz)^{-1}} = t^{z^{-1}}$  is contained in  $\langle t \rangle S_{2a+b}$ , we conclude that  $t^{N(\langle t \rangle W)} \cap (\langle t \rangle W - \langle t \rangle Y) \neq \emptyset$ .

There are five  $N_L(Y)$ -orbits in  $(\langle t \rangle W)^*$ , namely,  $\{t\}$ ,  $t(W-Y)$ ,  $tY^*$ ,  $W-Y$ , and  $Y^*$ . Put  $H = N(\langle t \rangle W)$ . Then  $H$  contains  $N_L(Y)$ , and so it follows that  $t^H$  is equal to  $\{t\} \cup t(W-Y)$ ,  $\{t\} \cup (W-Y)$ , or  $\{t\} \cup t(W-Y) \cup (W-Y)$ . If  $t^H = \{t\} \cup t(W-Y) \cup (W-Y)$ , then  $tY^* \cup Y^* = (\langle t \rangle W)^* - t^H$  is  $H$ -invariant. But then,  $H \leq N(\langle t \rangle Y) \leq C$ , a contradiction.

Suppose next  $t^H = \{t\} \cup t(W-Y)$ . As  $U = Z(S)$  is a normal subgroup of  $Q$ , we can take an element  $u$  of  $(Z(Q) \cap U)^*$ . Assume that  $H$  normalizes  $Y^*$  and set  $\overline{N(Y)} = N(Y)/Y$ . Then  $\bar{t}^H = \bar{t}\bar{W}$  has  $q$  elements and  $C_H(\bar{t}) = N_H(\langle t \rangle Y)$ . As  $Y = Z_2(S)$ ,  $\langle t \rangle Y$  is a normal subgroup of  $Q$ . Thus  $C_H(\bar{t})$  contains  $Q$  by (4.2) (3), and so  $|\bar{t}^H|$  is odd. This is a contradiction. Thus  $H$  does not normalize  $Y^*$ . Also, as  $\langle tY^* \rangle = \langle t \rangle Y$  and  $N(\langle t \rangle Y) \leq C$ ,  $H$  normalizes neither  $tY^*$  nor  $tY^* \cup Y^*$ . Therefore  $u^H = tY^* \cup (W-Y) \cup Y^*$ , which is equal to  $(\langle t \rangle W)^* - t^H$ , and  $|u^H| = q^3 + q^2 - 2$ . But  $|u^H|$  is odd since  $C_H(u)$  contains  $Q$ , a contradiction. Hence  $t^H \neq \{t\} \cup t(W-Y)$ . Similarly we can show that  $t^H \neq \{t\} \cup (W-Y)$ . The proof is complete.

(4.4) Suppose  $t$  is a noncentral involution of  $G$ . Then the following conditions hold.

$$(1) \quad t^{N(R)} = t^{N(\langle t \rangle U)} = tU.$$

$$(2) \quad \langle t \rangle \in \text{Syl}_2(C(L)). \text{ In particular, } C_C(L) = \langle t \rangle \times O(C).$$

PROOF. By (4.2) (1) we have  $\{t\} \neq t^{N(R)} \leq Z(R)$ . If  $|T| > 2$ , then  $t$  is the unique involution of  $\mathfrak{U}^1(Z(R)) = \mathfrak{U}^1(T)$  and hence  $N(R) \leq C$ , a contradiction. Thus (2) holds. Now  $Z(R) = \langle t \rangle U$ , so  $N(R) \leq N(\langle t \rangle U)$ . Since  $N_L(S)$  acts transitively on  $U^*$ , (1) follows from (4.2) (2).

## 5. 2-Local subgroups of $G$ .

From now on we assume the following hypothesis.

(5.1) *Hypothesis.*  $G$  is a group satisfying the hypotheses of the main theorem and  $LO(G) \ntriangleleft G$ .

The symbols defined in Section 3 will retain their meaning for the rest of this paper. In addition, let  $t$  be an involution of  $C(L)$  and set  $C = C(t)$  and  $R = \langle t \rangle S$ .

Then  $L \triangleleft C$  by (5.1), and so (4.1) is satisfied. As  $LO(G) \ntriangleleft G$ , it follows from (4.3) and (4.4) that  $\langle t \rangle \in \text{Syl}_2(C(L))$  and  $C_C(L) = \langle t \rangle \times O(C)$ .

(5.2) *Notation.*  $C_1 = O_2(N(\langle t \rangle U))$ ,  $R_1 = C_1 S$ .

- (5.3) (1)  $N(\langle t \rangle U) = N(\langle t \rangle D) = C_1 N_C(U) \leq N(D)$ ,  $C_1 \cap C = \langle t \rangle D$ .  
 (2)  $t^{C_1} = tU$ .  
 (3)  $N(R) = C_1 N_C(S) > R_1$ .  
 (4)  $C_1 / \langle t \rangle D \cong U$  as  $N_C(U)$ -modules.  
 (5)  $Z(C_1) = Z(R_1) = U$ .  
 (6)  $[O(C), C_1] = 1$ .

PROOF. We have  $C(\langle t \rangle U) = C_C(U) = C_L(U)C_C(L)$  by (3.19). Set  $\bar{C} = C/C_C(L)$ . Then  $O_2(N_{\bar{C}}(\bar{U})) = O_2(N_{\bar{L}}(\bar{U}))$  by (3.23) (1). Since  $N_{\bar{C}}(\bar{U}) = \bar{N}_C(\bar{U})$ , it follows that  $\overline{O_2(N_C(U))} \leq O_2(N_{\bar{C}}(\bar{U})) \leq \bar{L}$ , and thus  $O_2(N_C(U)) = O_2(N_L(U)C_C(L)) = \langle t \rangle D$ .

Set  $H = N(\langle t \rangle U)$  and  $K = C(\langle t \rangle U)$ . By (4.4)  $t^H = tU$ , so  $|H : N_C(U)| = q$ . As  $\langle t \rangle D = O_2(K)$  and  $O(C) = O(K)$ ,  $DC_C(L)$  is normal in  $H$ . Let  $\bar{H} = H/DC_C(L)$  and  $E = C_H(\bar{K})$ . Since  $\bar{K} \cong SL(2, q)$  and  $\text{Out}(SL(2, q))$  has order  $n$  where  $q = 2^n$ ,  $E \not\leq N_C(U)$ . This implies  $H = N_C(U)E$  since  $E \triangleleft H$  and  $H$  acts 2-transitively on  $tU$ . By (3.23) (3),  $C_E(t) = J_3 DC_C(L)$  and so  $E \cap K = DC_C(L)$ . Thus the permutation group  $(\bar{E}, tU)$  is 2-transitive and the stabilizer of  $t$  is  $\bar{J}_3$ . Since  $J_3$  acts regularly on  $U^*$ ,  $\bar{E}$  is a Frobenius group and  $\bar{E} = \bar{J}_3 O_2(\bar{E})$ . Since  $C_E(O(C))O(C)$  contains  $J_3 DC_C(L)$  and is normal in  $E$ , it follows that  $E = C_E(O(C))O(C)$  and so  $V = C_V(O(C))O(C)$  where  $V$  denotes  $O_2(E \text{ mod } DC_C(L))$ . Thus  $V = O(C) \times O_2(V)$ ,  $H = N_C(U)O_2(V)$ , and  $N_C(U) \cap O_2(V) = \langle t \rangle D$ . As  $O_2(N_C(U)) = \langle t \rangle D$ , we have  $O_2(V) = C_1$ . Hence (2) and (6) hold. Now  $O^*(K) = \langle t \rangle C_L(U)$  and  $(\langle t \rangle C_L(U))^* = C_L(U)$ , so  $D = O_2(C_L(U))$  is normal in  $H$  and (1) holds. Also,  $[R, C_1] \leq \langle t \rangle C_L(U) \cap C_1 = \langle t \rangle D$ , hence  $N(R) \geq C_1$ . As  $Z(R) = \langle t \rangle U$ , (3) follows from (1). Now  $Z(C_1) \leq C_{C_1}(t) = \langle t \rangle D$ , and so  $Z(C_1) \leq \langle t \rangle U$ . Then, since  $U \triangleleft H$ , we have  $Z(C_1) = U$ . Similarly  $Z(R_1) = U$  and (5) holds. Finally the map defined by  $x \mapsto [t, x]$  for  $x \in C_1$  is a  $N_C(U)$ -homomorphism from  $C_1$  onto  $U$  whose kernel is  $\langle t \rangle D$ . Thus (4) holds.

(5.4) Notation.  $L_2 = O_2(N(\langle t \rangle Y))$ ,  $R_2 = L_2 S$ .

- (5.5) (1)  $N(\langle t \rangle Y) = N(\langle t \rangle M) = L_2 N_C(Y)$ ,  $L_2 \cap C = \langle t \rangle M$ .  
 (2)  $t^{L_2} = tY$ .  
 (3)  $L_2 / \langle t \rangle M \cong Y$  as  $N_C(Y)$ -modules.  
 (4)  $Z(L_2) = Y$ ,  $Z(R_2) = U$ .  
 (5)  $[O(C), L_2] = 1$ .

PROOF. Arguing as in the first paragraph of the proof of (5.3), we have  $C(\langle t \rangle Y) = C_L(Y)C_C(L)$  and  $O_2(N_C(Y)) = \langle t \rangle M$ . As  $Z_2(R) = \langle t \rangle Y$ , (5.3) (3) implies  $R_1 \leq N(\langle t \rangle Y)$ . Since  $N_L(Y)$  acts transitively on  $Y^*$ , it follows from (5.3) (2) and (4.2) (2) that  $t^{N(\langle t \rangle Y)} = tY$ . In particular  $N(\langle t \rangle Y) \leq N(Y)$ .

Let  $\widetilde{N(\langle t \rangle Y)} = N(\langle t \rangle Y)/C(\langle t \rangle Y)$ . Then the permutation group  $(\widetilde{N(\langle t \rangle Y)}, tY)$  is 2-transitive and the stabilizer of  $t$  is  $\widetilde{N_C(Y)}$ . Since  $\widetilde{N_C(Y)} \triangleright \widetilde{N_L(Y)} \cong N_L(Y)/M$ , and since  $(N_L(Y)/M, Y)$  is isomorphic to  $(SL(2, q), V(2, q))$  as permutation groups, it follows from (2.6) that  $\widetilde{N(\langle t \rangle Y)}$  has a regular normal subgroup  $V/C(\langle t \rangle Y)$ . Then  $N_L(Y)$  acts transitively on  $\tilde{V}^*$ , and  $\tilde{V}$  is elementary abelian of order  $q^2$ .

Since  $C(\langle t \rangle Y) = C_L(Y)C_C(L)$ ,  $O_2(C(\langle t \rangle Y)) = \langle t \rangle M$  is a Sylow 2-subgroup of  $C(\langle t \rangle Y)$ . Set  $\widetilde{N(\langle t \rangle Y)} = N(\langle t \rangle Y)/\langle t \rangle M$ . Then  $O(\tilde{V}) = \overline{C(\langle t \rangle Y)}$ . Since  $C_L(Y) \cap N_L(Y) = M$ , we have that  $[O(\tilde{V}), \overline{N_L(Y)}] = 1$ . As  $N_L(Y)$  acts irreducibly on  $\tilde{V}$ , it follows that  $\tilde{V} = O(\tilde{V})C_{\tilde{V}}(O(\tilde{V}))$ . Thus we have  $\tilde{V} = O(\tilde{V}) \times O_2(\tilde{V})$ ,  $\widetilde{N(\langle t \rangle Y)} = O_2(\tilde{V})\overline{N_C(Y)}$ , and  $O_2(\tilde{V}) \cap \overline{N_C(Y)} = 1$ . Then,  $O_2(N_C(Y)) = \langle t \rangle M$  implies that  $O_2(V) = L_2$ . Consequently,  $N(\langle t \rangle Y) = L_2 N_C(Y)$ ,  $L_2 \cap C = \langle t \rangle M$ , and  $t^{L_2} = tY$ . As  $Z(\langle t \rangle M) = \langle t \rangle Y$ , (1) and (2) hold. Moreover,  $Z(L_2) \leq L_2 \cap C = \langle t \rangle M$  and so  $Z(L_2) \leq \langle t \rangle Y$ . Recall that  $N(\langle t \rangle Y) \leq N(Y)$ . Thus  $Y \cap Z(L_2) \neq 1$ , and hence  $Y = Z(L_2)$ . Similarly  $Z(R_2) = U$ . Now the map defined by  $\langle t \rangle Mx \mapsto [t, x]$  for  $x \in L_2$  is an  $N_C(Y)$ -isomorphism from  $L_2/\langle t \rangle M$  onto  $Y$ . Thus (3) holds. Finally, since  $O(C)$  stabilizes the series  $L_2 > \langle t \rangle M > 1$ , (5) holds.

$$(5.6) \quad (1) \quad R_1 = N_{R_2}(R) \triangleleft R_2 = L_2 C_1.$$

$$(2) \quad \mathcal{E}^*(R_2/\langle t \rangle M) = \{L_2/\langle t \rangle M, R_1/\langle t \rangle M\}.$$

$$(3) \quad R_2/R_1 \cong Y/U \text{ as } N_C(S)\text{-modules.}$$

$$(4) \quad \mathfrak{U}^1(C_1) = U.$$

PROOF. Since  $Z_2(R) = \langle t \rangle Y$  and  $R_2 \cap C = R$ , it follows from (5.3) (3) and (5.5) (1) that  $R_1 \leq N(R) \leq N(R_2) \cap N(\langle t \rangle Y) = L_2 N_C(R)$ . Note that  $L_2 N_C(R)/R_2 \cong N_C(R)/R$  has cyclic Sylow 2-subgroups, since  $\text{Out}(L)$  is cyclic. As  $R_1/R$  is elementary abelian of order  $q$ , we must have  $R_1 \cap R_2 > R$ . Moreover  $J$  acts irreducibly on  $R_1/R$  by (5.3) (4). Thus  $R_1 \leq R_2$ . Then, by (5.3) (3)  $N_{R_2}(R) = C_1 R = R_1$ . Certainly  $R_2 = L_2 C_1$ .

Set  $\bar{R}_2 = R_2/\langle t \rangle M$ . Then  $\bar{R}_2$  is a semidirect product of  $\bar{L}_2$  by  $\bar{R}$ . In particular  $\overline{R_1 \cap L_2} = N_{\bar{L}_2}(\bar{R}) \leq Z(\bar{R}_2)$ . Recall that  $S$  is a semidirect product of  $M$  by  $S_b$ , and so  $\bar{R} = \bar{S}_b$ . Therefore (5.5) (3) shows that there exists an isomorphism between two semidirect products  $\bar{R}_2$  and  $S_b Y$ , under which  $\bar{L}_2$  and  $\bar{R}$  are mapped onto  $Y$  and  $S_b$  respectively. Now the commutator formula  $[x_b(\alpha), x_{3a+b}(\beta)] = x_{3a+2b}(\alpha\beta)$  shows that  $\mathcal{E}^*(S_b Y) = \{Y, S_b U\}$  and  $Z(S_b Y) = U$ . Since  $\overline{R_1 \cap L_2}$  has order  $q$ , it follows that  $Z(\bar{R}_2) = \overline{R_1 \cap L_2}$  is mapped onto  $U$  and  $\bar{R}_1 = \bar{R}(\overline{R_1 \cap L_2})$  is mapped onto  $S_b U$ . Thus (2) holds. Since  $L_2$  and  $\langle t \rangle M$  are normal in  $R_2$ ,  $R_1$  is also normal in  $R_2$  by (2), and thus all parts of (1) hold.

As was shown in the proof of (5.5) (3), the map defined by  $\langle t \rangle Mx \mapsto [t, x]$  for  $x \in L_2$  is an  $N_C(Y)$ -isomorphism from  $L_2/\langle t \rangle M$  onto  $Y$ . Under this isomorphism  $(L_2 \cap R_1)/\langle t \rangle M \cong U$ , and so  $L_2/(L_2 \cap R_1) \cong Y/U$ . Thus (3) holds. As  $C_1/(C_1 \cap L_2)$  and  $C_1/\langle t \rangle D$  are elementary abelian,  $\mathfrak{U}^1(C_1) \leq L_2 \cap \langle t \rangle D = \langle t \rangle F$ . Then by (3.16)  $\mathfrak{U}^1(C_1) \leq \langle t \rangle U$ , and so  $\mathfrak{U}^1(C_1) = \langle t \rangle U$  or  $U$  since  $\mathfrak{U}^1(D) = U$ . If  $\mathfrak{U}^1(C_1) = \langle t \rangle U$ , then  $x^2 \in \langle t \rangle U - U = tU$  for some  $x \in C_1$ , and as  $tU = t^{C_1}$ , we can take an element  $y$  of  $C_1$  such that  $y^2 = t$ . But then  $y \in C_1 \cap C = \langle t \rangle D$ , and  $y^2 \in U$ . This contradiction proves (4).

(5.7) *Notation.* Let  $L_1 = L_2 \cap R_1$ ,  $E_1 = L_2 \cap C_1$ ,  $D_1 = [C_1, J_3]D$ ,  $S_1 = SD_1$ ,  $M_1 = L_1 \cap S_1$ , and  $F_1 = E_1 \cap D_1$ .

(5.8) (1)  $D_1$  is an  $N_L(U)$ -invariant subgroup with  $C_1 = \langle t \rangle D_1 \triangleright D_1$ .

(2)  $\mathfrak{U}^1(R_1) = \mathfrak{U}^1(S_1) = F$ .

(3)  $W$  and  $X$  are normal in  $R_2$ .

(4)  $L_2 = L_1 L_1^{w_b}$  and  $L_1 \cap L_1^{w_b} = \langle t \rangle M$ .

(5)  $M \triangleleft N(\langle t \rangle Y)$ .

(6)  $N_{R_2}(S) = N_{R_2}(D) = R_1$ .

PROOF. Recall that  $N_L(U) = D(\langle S_{\pm a} \rangle \times J_3)$ . Then certainly  $D_1/D = [C_1/D, J_3]$  is  $N_L(U)$ -invariant. By (5.3) (4)  $J_3$  acts transitively on  $(C_1/\langle t \rangle D)^*$ , and so  $C_{C_1/D}(J_3) = \langle t \rangle D/D$ . Hence (5.6) (4) implies that  $C_1/D = \langle t \rangle D/D \times D_1/D$ , and thus (1) holds. In particular,  $S_1$  is an  $N_L(S)$ -invariant subgroup such that  $R_1 = \langle t \rangle S_1 \triangleright S_1$ . Thus  $\mathfrak{U}^1(R_1) \leq C_1 \cap L_1 \cap R \cap S_1 = F$ . Then, since  $\mathfrak{U}^1(S) = F$ , (2) holds. (2) and (5.6) (1) show that  $R_2$  acts on  $\mathcal{E}^*(F) = \{W, X\}$ , hence  $|R_2 : N_{R_2}(W)| \leq 2$ . Also  $R_1 \triangleright W$  by (5.6) (4). Since  $J$  acts irreducibly on  $R_2/R_1$  by (5.6) (3), we obtain (3). The fact that  $Y = U \times U^{w_b}$  yields (4) as in the proof of (5.6) (3). It follows from (2) that  $M \triangleleft L_1$ , and so  $M \triangleleft L_1 L_1^{w_b} = L_2$ . Since  $S \in \text{Syl}_2(L)$ ,  $C = N_C(S)L$  by a Frattini argument. As described in Section 3,  $Y = Z(M)$ ,  $M$  is a characteristic subgroup of  $S$ , and  $N_L(Y) = N_L(M)$ . Hence  $N_C(Y) = N_C(M)$ . Now (5) is a consequence of (5.5) (1). By (2),  $S$  and  $D$  are normal in  $R_1$ . Since  $\langle t \rangle M \triangleleft R_2$ , and since  $\langle t \rangle MS = \langle t \rangle MD = R \not\triangleleft R_2$  by (5.6) (1), we have  $S \not\triangleleft R_2 \not\triangleright D$ . Then the irreducible action of  $J$  on  $R_2/R_1$  yields (6).

(5.9) *Notation.* Set  $U_1 = C_{D_1}(J_1)$  if  $q \geq 8$ , and  $U_1 = Z(C_{D_1}(J_1))$  if  $q = 4$ . Furthermore, set  $Y_1 = YU_1$ ,  $W_1 = WU_1$ , and  $X_1 = XU_1$ .

(5.10) (1)  $D_1 = DU_1$ ,  $U_1 \cap C = U$ , and  $|U_1| = q^2$ .

(2)  $U_1$  is elementary abelian or homocyclic abelian of exponent 4 if  $q \geq 8$ , and elementary abelian if  $q = 4$ .



(3)  $U_1 = Z(D_1) \leq F_1$ .

(4) If  $q=4$ , then for any  $\langle t \rangle$ -invariant Sylow 2-subgroup  $V$  of  $\langle C_L(J_1)^{C(J_1)} \rangle$  such that  $C_{D_1}(J_1) \leq V$ , we have  $U_1 = Z(V)$  and  $V/U_1$  is elementary abelian.

PROOF. Set  $H = C(J_1)$ ,  $K = C_L(J_1)'$ , and  $B = \langle K^H \rangle$ . We distinguish two cases:  $q \geq 8$  and  $q=4$ . First we assume that  $q \geq 8$ . So  $U_1 = C_{D_1}(J_1)$ . Since  $J_1$  centralizes  $D_1/D$  by (5.3) (4) and  $C_D(J_1) = U$ , (1) holds.

It follows from (3.22) that  $C_H(t) = C_L(J_1)C_C(L)$ . Thus  $K = \langle S_{\pm(3a+2b)} \rangle$  is a standard subgroup of  $H$  isomorphic to  $SL(2, q)$  and  $\langle t \rangle$  is a Sylow 2-subgroup of  $C_H(K)$ . Moreover,  $t$  is a noncentral involution of  $H$  since  $|U_1| = q^2$ . Thus  $KO(H) \not\triangleleft H$ . Now a result of Griess, Mason, and Seitz [12], together with (2.10), shows that  $Z(B)$  has odd order and  $B/Z(B)$  is isomorphic to one of the following groups:

$SL(2, q) \times SL(2, q)$ ,  $SL(2, q^2)$ ,  $PSU(3, q^2)$ ,  $PSL(3, q)$ , or  $G_2(3)$ .

Considering the conjugacy classes of involutions of the automorphism groups of these groups and using the fact that  $K \triangleleft C_B(t)$ , we have that  $t$  acts on  $B/Z(B)$  as an involutory field automorphism if  $B/Z(B) \cong SL(2, q^2)$  or  $PSU(3, q^2)$ ,  $t$  acts on  $B/Z(B)$  as a graph automorphism if  $B/Z(B) \cong PSL(3, q)$ , and  $t$  interchanges the two components of  $B/Z(B)$  if  $B/Z(B) \cong SL(2, q) \times SL(2, q)$ . Recall that  $J_3 = K \cap J$  and  $J_3$  acts on  $U^*$  transitively, so  $U_1 = [U_1, J_3] \leq B$  by (1) and (5.3) (4). Thus  $B/Z(B)$  is not isomorphic to  $G_2(3)$ . Also,  $U_1$  is elementary abelian if  $B/Z(B) \cong SL(2, q) \times SL(2, q)$  or  $SL(2, q^2)$ . When  $B/Z(B) \cong PSU(3, q^2)$  or  $PSL(3, q)$ , we use one more condition that  $[U_1, t, t] = 1$ , which is a consequence of (5.6) (4). By an easy calculation we have that  $U_1$  is homocyclic abelian of exponent 4 in these cases.

By (5.6) (4),  $U_1$  and  $W$  are normal in  $C_1$ . In particular  $U_1$  normalizes  $[W, J_1]$ . Note that  $C_W(J_1) = U$  and  $[W, J_1] \cap U_1 = 1$ . Thus  $[[W, J_1], U_1] = 1$ . Then, since  $U_1$  is abelian, we obtain  $[W, U_1] = 1$ . Moreover,  $w_a$  normalizes  $J_1$  and  $D_1$  by (5.8) (1). Hence  $U_1 = U_1^{w_a}$  centralizes  $W^{w_a}$  as well. Since  $D = WW^{w_a}$  and  $Z(D) = U$ , (1) implies that  $Z(D_1) = U_1$ . As  $C_D(Y) = F$ , it follows from (5.5) (4) that  $C_{D_1}(Y) = L_2 \cap D_1 = F_1$ , and hence  $Z(D_1) \leq F_1$ . Thus in the case where  $q \geq 8$ , the proof of (5.10) is complete.

Assume next  $q=4$ . Then  $U_1 = Z(C_{D_1}(J_1))$  by the definition. In this case  $|C_H(t) : C_L(J_1)C_C(L)| \leq 2$ ,  $C_D(J_1) = S_b Y \in \text{Syl}_2(C_L(J_1))$ ,  $K = C_L(J_1) = \langle S_{\pm b}, S_{\pm(3a+b)}, S_{\pm(3a+2b)} \rangle \cong SL(3, q)$ , and  $C_H(t) \cap C(K) = J_1 C_C(L)$ . Thus  $K$  is a standard subgroup of  $H$  and  $C_H(K)$  has a cyclic Sylow 2-subgroup  $\langle t \rangle$ . Since  $J_1$  centralizes  $D_1/D$ , we have  $D_1 = C_{D_1}(J_1)D$ , and so  $C_{D_1}(J_1)$  is of order  $q^4$ . Hence  $C_1 \cap H$  has order  $2q^4$ , so that  $t$  is a noncentral involution of  $H$ . This time a result of Seitz [18], together with (2.10), shows that  $|Z(B)|$  is odd and  $B/Z(B)$  is isomorphic to either

$$PSL(3, q) \times PSL(3, q) \quad \text{or} \quad PSL(3, q^2).$$

Moreover,  $t$  interchanges the two components of  $B/Z(B)$  in the former case and  $t$  acts on  $B/Z(B)$  as an involutory field automorphism in the latter case. In either case we can easily verify that  $N_V(C_V(t)) = C_V(t)Z(V)$  and  $|N_V(C_V(t))| = q^4$  for any  $\langle t \rangle$ -invariant Sylow 2-subgroup  $V$  of  $B$ .

Now  $J_3$  acts transitively on  $U^*$ , so also on  $(C_{D_1}(J_1)/C_D(J_1))^*$ . Thus  $J_3 \leq K$  implies  $C_{D_1}(J_1) = [C_{D_1}(J_1), J_3] \leq B$ . Let  $V$  be a  $\langle t \rangle$ -invariant Sylow 2-subgroup of  $B$  containing  $C_{D_1}(J_1)$ . Then  $N_V(C_D(J_1))$  is equal to  $C_D(J_1)Z(V)$  and so it is of order  $q^4$ . Since  $Z(V)$  is elementary abelian of order  $q^2$ , the order consideration shows  $C_{D_1}(J_1) = N_V(C_D(J_1))$ ,  $U_1 = Z(V)$ , and  $C_D(J_1) \cap U_1 = U$ . Hence (1), (2), and (4) hold.

By (5.6) (4),  $Y_1$  is normal in  $W_1$ . Since  $C_{W_1}(J_1) = Y_1$  and since  $[W_1, J_1] = [W, J_1] = S_{2a+b}$ , we have that  $W_1 = Y_1 \times S_{2a+b}$ . In particular,  $[W, U_1] = 1$ . Then (3) follows as in the case  $q \geq 8$ . The proof is complete.

$$(5.11) \quad (1) \quad Z(S_1) = U_1.$$

$$(2) \quad J(C_1) = D_1.$$

$$(3) \quad \mathcal{E}^*(R_1/Y) = \{L_1/Y, C_1/Y\}.$$

$$(4) \quad L_1 \text{ and } C_1 \text{ are characteristic subgroups of } R_1.$$

PROOF. By (5.10) (3),  $U_1$  is normal in  $S_1$ . If  $q \geq 8$ , then  $C_{S_1}(J_1) = U_1$  so that  $[S_1, J_1]$  centralizes  $U_1$  by (2.4), and hence  $U_1$  is contained in  $Z(S_1)$ . If  $q = 4$ , then  $C_{M_1}(J_1) = Y_1$  since  $M_1 = U_1 M$ . Now,  $Y$  is normal in  $R_2$ , so  $Y_1$  is normal in  $M_1$ . Thus  $[M_1, J_1]$  centralizes  $Y_1$  and we have  $Y_1 \leq Z(M_1)$ . As  $S_1 = M_1 D_1$ , this implies that  $U_1$  is contained in  $Z(S_1)$ . Hence in either case  $U_1 \leq Z(S_1)$ . As  $U_1 \cap S = Z(S)$ , we obtain (1).

For the proof of (2), recall that  $W \in \mathcal{A}(D)$  and  $D = WW^{w_a}$ . Thus  $W_1 \in \mathcal{A}(D_1)$  by (5.10). Suppose that there exists an abelian subgroup  $B$  of  $C_1$  not contained in  $D_1$  and  $|B| \geq q^4$ . Then (5.8) (1) yields  $C_1 = D_1 B$  and  $B \cap D_1$  has index 2 in  $B$ . Since  $(B \cap D_1)U_1$  is an abelian subgroup of  $D_1$ , we have that the order of  $(B \cap D_1)U_1$  is at most  $q^4$ . Moreover,  $B \cap U_1 \leq Z(C_1) = U$  by (5.3). Thus  $|B \cap D_1| \leq q^3$ , and so  $|B| \leq 2q^3$ , which is a contradiction. Therefore,  $\mathcal{A}(D_1) = \mathcal{A}(C_1)$  and  $J(C_1) = W_1 W_1^{w_a} = D_1$ .

Since  $R_1/Y_1$  is isomorphic to  $R/Y$ ,  $\mathcal{E}^*(R_1/Y_1) = \{L_1/Y_1, C_1/Y_1\}$  by (3.12). Now  $C_1/Y$  is elementary abelian by (5.6) (4). Furthermore, by (5.5)  $\langle t \rangle M$  is normal in  $L_1$ , and thus  $L_1/Y = \langle t \rangle M/Y \times Y_1/Y$  is also elementary abelian. Thus (3) holds. Since  $Z(F) = Y$ , (5.8) (2) implies that  $Y$  is a characteristic subgroup of  $R_1$ . Since  $\mathfrak{U}^1(C_1) = U$  and  $\mathfrak{U}^1(L_1) \geq \mathfrak{U}^1(M) = Y$ ,  $C_1$  is not isomorphic to  $L_1$ . Hence (4) is a consequence of (3).

(5.12) *Notation.*  $Y_2 = U_1 U_1^{wb}$ ,  $M_2 = MY_2$ , and  $S_2 = SY_2$ .

(5.13) (1)  $Y_2 = U_1 \times U_1^{wb} = Z(M_2)$ ,  $Y_2 \cap C = Y$ ,  $L_2 = \langle t \rangle M_2 > M_2$ .

(2)  $J(L_2) = M_2$ .

(3)  $S_2$  is a subgroup of  $R_2$  with  $R_2 = \langle t \rangle S_2 > S_2$ .

(4)  $\mathfrak{U}^1(L_2) = \mathfrak{U}^1(\langle t \rangle Y_2) = Y$ .

(5)  $Z(S_2) = U_1$ ,  $Z_2(S_2) = Y_2$ .

PROOF. Since  $U_1 \cap U_1^{wb}$  is a  $\langle t \rangle$ -invariant 2-group and since  $C \cap U_1 \cap U_1^{wb} = U \cap U^{wb} = 1$  by (5.10) (1), we have  $U_1 \cap U_1^{wb} = 1$ . It follows from (5.10) (3) and (5.11) that  $U_1$  is a characteristic subgroup of  $R_1$ , hence  $U_1 \triangleleft R_2$  by (5.6) (1). In particular,  $U_1$  and  $U_1^{wb}$  are normal in  $L_2 = L_2^{wb}$ . Thus  $Y_2$  is a direct product of  $U_1$  and  $U_1^{wb}$  and  $Y_2 \cap C = UU^{wb} = Y$ . Furthermore (5.11) (1) implies that  $M$  centralizes  $U_1$  and  $U_1^{wb}$ . Then, since  $Z(M) = Y$ , we have  $Z(M_2) = Y_2$ . Comparing orders we obtain  $L_2 = \langle t \rangle M_2 > M_2$ , and (1) holds.

Recall that  $W$  and  $X$  are elements of  $\mathcal{A}(M)$  and  $M = WXX^{wb}$ . Then  $W_2$  and  $X_2$  are elements of  $\mathcal{A}(M_2)$  and  $|W_2| = |X_2| = q^5$ , where we put  $W_2 = WY_2$  and  $X_2 = XY_2$ . Suppose that there exists an abelian subgroup  $B$  of  $L_2$  such that  $B \not\leq M_2$  and  $|B| \geq q^5$ . Then  $L_2 = BM_2$  and  $B \cap M_2$  is of index 2 in  $B$ . As  $(B \cap M_2)Y_2$  is an abelian subgroup of  $M_2$ , its order is at most  $q^5$ . Moreover,  $B \cap Y_2 \leq Z(L_2) = Y$  by (5.5) (4). Hence we have  $|B \cap M_2| \leq q^3$  and so  $|B| \leq 2q^3$ , which is a contradiction. Therefore  $\mathcal{A}(L_2) = \mathcal{A}(M_2)$  and  $J(L_2) = W_2 X_2 X_2^{wb} = M_2$ . Hence (2) holds. Now (3) follows easily.

(1) and (2) in particular show that  $Y_2$  is a characteristic subgroup of  $L_2$ . Hence  $L_2/Y$  is a direct product of  $\langle t \rangle M/Y$  and  $Y_2/Y$  by (5.5) and so is elementary abelian. Since  $\mathfrak{U}^1(M) = Y$ , we obtain  $\mathfrak{U}^1(L_2) = Y$ . Then  $1 \neq \mathfrak{U}^1(\langle t \rangle Y_2) \leq Y$ . Thus the irreducible action of  $N_L(Y)$  on  $Y$  yields  $\mathfrak{U}^1(\langle t \rangle Y_2) = Y$ , proving (4).

Since  $Z(S_1) = U_1$ , it follows that  $Z(S_2)$  contains  $U_1$ . On the other hand (5.8) (6) shows that  $Z(S_2) \leq S_1$ . Thus  $Z(S_2) = U_1$ . For  $Z_2(S_2)$ , we distinguish two cases:  $q \geq 8$  and  $q = 4$ . Note that  $Y_2/Y$  is  $N_L(Y)$ -isomorphic to  $Y$  and  $Y_2/Y_1$  is  $N_L(S)$ -isomorphic to  $Y/U$  (see (5.5) (3) and (5.6) (3)). Assume first  $q \geq 8$ . Then  $C_S(J_5) = S_{3a+b}$  by (3.9), and we can check that  $C_{S_2/U_1}(J_5) = Y_2/U_1$ . Then, since  $Y_2$  is normal in  $S_2$ ,  $[S_2/U_1, J_5]$  centralizes  $Y_2/U_1$  by (2.4). Hence  $Y_2/U_1$  is contained in  $Z(S_2/U_1)$ . On the other hand,  $Z(S/U) = Y/U$  implies  $Z(S_1/U_1) = Y_1/U_1$ . Thus, comparing orders we have  $Y_2/U_1 = Z(S_2/U_1)$ . Assume next  $q = 4$ , and set  $H = C(J_1)$  and  $K = C_L(J_1)$ . In this case we need (5.10). Recall that  $C_S(J_1) = S_b Y$  and  $J_3$  acts fixed-point-freely on  $C_S(J_1)$ . Thus  $C_{S_2}(J_1) = S_b Y_2$ , on which  $J_3$  acts fixed-point-freely. Hence  $J_3 \leq K$  yields  $S_b Y_2 = [S_b Y_2, J_3] \leq \langle K^H \rangle$ .

Then it follows from (5.10) that  $S_b Y_2 / U_1$  is abelian. In particular, (1) implies  $Y_2 / U_1 \leq Z(S_2 / U_1)$ . Thus we have  $Y_2 = Z_2(S_2)$ . The proof is complete.

(5.14) *Notation.*  $W_2 = WY_2$ ,  $X_2 = XY_2$ ,  $F_2 = FY_2$ ,  $D_2 = DY_2$ ,  $V_2 = \langle t \rangle W_2$ ,  $U_2 = \langle t \rangle X_2$ ,  $E_2 = \langle t \rangle F_2$ , and  $C_2 = \langle t \rangle D_2$ .

(5.15) (1)  $Z(R_1) = U$ ,  $Z_2(R_1) = \langle t \rangle Y_1$ ,  $Z_3(R_1) = E_1$ .

(2)  $Z(R_2) = U$ ,  $Z_2(R_2) = Y_1$ ,  $Z_3(R_2) = E_2$ .

(3)  $\mathfrak{U}^1(R_2) = \mathfrak{U}^1(S_2) = F_1$ ,  $\mathfrak{U}^1(C_2) = Y_1$ .

(4)  $J(E_2) = F_2$ .

(5)  $\mathcal{E}^*(R_2/Y_1) = \{L_2/Y_1, C_2/Y_1\}$ ,  $\mathcal{E}^*(C_2/Y) = \{E_2/Y, C_1/Y\}$ .

(6)  $L_2$ ,  $C_2$ ,  $C_1$ , and  $S_2$  are characteristic subgroups of  $R_2$ .

PROOF. Recall that  $Z(S/U) = Y/U$  and  $Z(S/Y) = F/Y$ . In (5.3) and (5.5) we have already shown that  $Z(R_1) = Z(R_2) = U$ . Let  $\overline{N(U)} = N(U)/U$ . Then, since  $\langle t \rangle U$ ,  $S$ , and  $U_1$  are normal in  $R_1$ , we have  $\overline{R_1} = \overline{S} \times \overline{U_1} \times \langle \bar{i} \rangle$ . Hence  $Z_2(R_1) = \langle t \rangle Y_1$  and  $Z_3(R_1) = \langle t \rangle F U_1 = E_1$ . It follows from (5.6) (1) and (5.3) (1) that  $R_2 \cap N(\langle t \rangle U) = R_1$ . Thus  $Z(\overline{R_2})$  is contained in  $C_{\overline{R_2}}(\bar{i}) = \overline{R_1}$ , and so  $Z(\overline{R_2}) \leq Z(\overline{R_1}) = \langle \bar{i} \rangle \overline{Y_1}$ . On the other hand, since  $R_2 = Y_2 R_1$ ,  $Z(\overline{R_2})$  contains  $\overline{Y_1}$ . Thus we have  $Z(\overline{R_2}) = \overline{Y_1}$ . Recall that  $R_1$  and  $Y_2$  are normal in  $R_2$ . Then, since  $R_1 \cap Y_2 = Y_1$ ,  $R_2/Y_1$  is a direct product of  $R_1/Y_1$  and  $Y_2/Y_1$ . Thus we get (1) and (2).

Now  $R_2/F_2 \cong R/F$  is elementary abelian. Then by (5.8) (2), we have  $F \leq \mathfrak{U}^1(S_2) \leq \mathfrak{U}^1(R_2) \leq F_2 \cap R_1 = F_1$ . Since  $J$  acts transitively on  $(F_1/F)^*$  by (5.3) (4), this implies  $\mathfrak{U}^1(S_2)$  is equal to  $F$  or  $F_1$ . Then (5.8) (6) forces  $\mathfrak{U}^1(S_2) = \mathfrak{U}^1(R_2) = F_1$ . Let  $\widetilde{N(Y_1)} = N(Y_1)/Y_1$ . Then  $\widetilde{C_2}$  is a direct product of  $\widetilde{Y_2}$  and  $\widetilde{C_1}$  by (2), so that  $\widetilde{C_2}$  is elementary abelian. This, together with (5.13) (4), shows that  $Y \leq \mathfrak{U}^1(C_2) \leq Y_1$ . Thus again (5.8) (6) forces  $\mathfrak{U}^1(C_2) = Y_1$ , and (3) holds.

Since  $\mathcal{E}^*(F) = \{W, X\}$ , (5.13) (1) shows that  $Z(F_2) = Y_2$  and  $\mathcal{A}(F_2)$  contains  $W_2$  and  $X_2$ . Moreover,  $Z(E_2) \leq E_2 \cap C = \langle t \rangle F$  and so  $Z(E_2)$  is contained in  $\langle t \rangle Y$ . Hence  $Z(E_2) = Y$ . Now as in the proof of (5.11) (2) we can show that  $\mathcal{A}(E_2) = \mathcal{A}(F_2)$  and  $J(E_2) = W_2 X_2 = F_2$ . Thus (4) holds.

It follows from (3.12) (2) that  $\mathcal{E}^*(R_2/Y_2) = \{L_2/Y_2, C_2/Y_2\}$ . This, together with (3) and (5.13) (4), yields that  $\mathcal{E}^*(R_2/Y_1) = \{L_2/Y_1, C_2/Y_1\}$ . In particular,  $L_2$  and  $C_2$  are characteristic subgroups of  $R_2$ . Similarly, as  $\mathfrak{U}^1(R_1) = F$ , (5.6) (2) yields that  $\mathcal{E}^*(R_2/F) = \{L_2/F, R_1/F\}$ . Then, since  $L_2 \cap C_2 = E_2$  and  $R_1 \cap C_2 = C_1$ , and since  $C_1 E_2 = C_2$ , we have that  $\mathcal{E}^*(C_2/F) = \{E_2/F, C_1/F\}$ . Now, (5.13) (4) implies that  $\mathfrak{U}^1(E_2) = Y$ . Also,  $C_1/Y$  is elementary abelian by (5.6) (4). Thus we conclude that  $\mathcal{E}^*(C_2/Y) = \{E_2/Y, C_1/Y\}$ . As shown above  $L_2$  and  $C_2$ , and hence  $Y = \mathfrak{U}^1(L_2)$ , are characteristic in  $R_2$ . Thus  $E_2$  and  $C_1$  are also character-

istic in  $R_2$ . By (5.13) (2) and (5.11) (2),  $S_2 = M_2 D_1$  is characteristic in  $R_2$  as well. This completes the proof of (5.15).

(5.16)  $U_1$  is elementary abelian.

PROOF. Suppose false. Then by (5.10),  $q \geq 8$ ,  $U_1$  is homocyclic abelian of exponent 4, and  $\langle K^H \rangle / Z(\langle K^H \rangle)$  is isomorphic to  $PSU(3, q^2)$  or  $PSL(3, q)$ , where  $H = C(J_1)$  and  $K = C_L(J_1)'$ . By (3.22),  $C_C(J_1) = C_L(J_1) C_C(L)$  and so (5.3) shows that  $N_H(\langle t \rangle U) = U_1 J C_C(L)$ . In particular,  $N_H(\langle t \rangle U)$  is a proper subgroup of  $N_H(\langle t \rangle U_1)$ . Since  $U_1$  is characteristic in  $C_1$  by (5.10) (3) and (5.11),  $\langle t \rangle U_1$  is normal in  $N(\langle t \rangle U)$ . Thus  $N(\langle t \rangle U)$  is a proper subgroup of  $N(\langle t \rangle U_1)$ .

Next we set  $H_0 = N(\langle t \rangle U_1)$  and  $C_0 = O_2(H_0)$  and prove the following assertion.

$$(*) \quad H_0 = C_0 N(\langle t \rangle U) \triangleright C_1 \quad \text{and} \quad C_0 \cap N(\langle t \rangle U) = C_1.$$

Since  $\mathcal{A}(\langle t \rangle U_1) = \{U_1\}$  and  $\Omega_1(U_1) = U$ , we have  $H_0 \leq N(U_1) \leq N(U)$ . Let  $\overline{N(U)} = N(U)/U$ . Then  $C_{N(U)}(t) = N(\langle t \rangle U) < H_0$  by the above, and so  $H_0$  acts transitively on  $\overline{U_1}$ . Moreover, since  $C_C(U) = C_L(U) C_C(L)$  by (3.19), it follows from (5.3) (4) that  $C_{H_0}(\overline{U_1}) = N(\langle t \rangle U) \cap C(\overline{U_1}) = C_1 C_L(U) C_C(L)$ , which is normal in  $H_0$ . In particular,  $C_1$  and  $O(C)$  are normal in  $H_0$ . Now arguing as in (5.3) we obtain (\*).

Finally we estimate the order of  $N(C_1)$  and derive a contradiction. Set  $B = N(C_1)$  and  $\widetilde{N(U_1)} = N(U_1)/U_1$ . Then  $B \leq N(D_1) \leq N(U_1)$  by (5.10) and (5.11), and  $C_B(\tilde{t}) = H_0$  by (\*). Set  $V = C_B(\tilde{D}_1)$  and consider the map  $h$  defined by  $x \mapsto [\tilde{t}, x]$  for  $x \in V$ . This is an  $N_L(U)$ -homomorphism of  $V$  into  $\tilde{D}_1$  since  $\tilde{t}^B \leq \tilde{C}_1 - \tilde{D}_1 = \tilde{t} \tilde{D}_1$ . By (5.15),  $B$  contains  $R_2$ . Hence (5.13) (5) shows  $Y_2 \leq V$ . Since  $Y_2$  acts transitively on  $tY$  by (5.5) (2), we have that the image of  $h$  contains  $\tilde{Y}_1$ . Then, since  $N_L(U)$  acts irreducibly on  $\tilde{D}_1$  by (3.16), the image of  $h$  is equal to  $\tilde{D}_1$ . This implies  $\tilde{t}^V = \tilde{t}^B = \tilde{t} \tilde{D}_1$  and  $|B : H_0| = q^4$ . Also,  $C_B(t) = C_{H_0}(t) = N_C(U)$  and  $|H_0 : N_C(U)| = q^2$  by (\*) and (5.3). Thus  $|t^B| = q^6$ . As  $B$  normalizes  $D_1$ , it follows that  $t^B = t D_1$ . But  $D \leq D_1$  and  $tD$  contains an element of order 4, a contradiction.

$$(5.17) \quad N(\langle t \rangle U_1) = N(\langle t \rangle U), \quad N(\langle t \rangle Y_2) = N(\langle t \rangle Y), \quad \text{and} \quad N(R_1) = N(\langle t \rangle Y_1) = R_2 N_C(S).$$

PROOF. As  $C_{U_1}(t) = U$ , it follows from (5.16) that  $\mathcal{E}^*(\langle t \rangle U_1) = \{\langle t \rangle U, U_1\}$ . In particular,  $N(\langle t \rangle U_1) \leq N(\langle t \rangle U)$  since  $|\langle t \rangle U| \neq |U_1|$ . On the other hand,  $U_1$  is normal in  $N(\langle t \rangle U)$  by (5.10) (3) and (5.11) (2). Thus we have  $N(\langle t \rangle U_1) = N(\langle t \rangle U)$ . Similarly, (5.13) (1) shows that  $\mathcal{E}^*(\langle t \rangle Y_2) = \{\langle t \rangle Y, Y_2\}$ , and hence  $N(\langle t \rangle Y_2) \leq N(\langle t \rangle Y)$ . Also, (1) and (2) of (5.13) show that  $\langle t \rangle Y_2$  is normal in  $N(\langle t \rangle Y)$ . Thus  $N(\langle t \rangle Y_2) = N(\langle t \rangle Y)$ . Since  $Z_2(R_1) = \langle t \rangle Y_1$  and since  $\mathcal{E}^*(\langle t \rangle Y_1)$

$=\{\langle t \rangle Y, Y_1\}$ , both  $\langle t \rangle Y$  and  $Y_1$  are normal in  $N(R_1)$ . Now  $N_C(S)$  is a maximal subgroup of  $N_C(Y)$  by (3.13). Moreover,  $w_b$  does not normalize  $Y_1$ . Therefore, it follows from (5.6) (1) and (5.5) that  $N(R_1)=N(\langle t \rangle Y) \cap N(Y_1)=R_2 N_C(S)$ .

(5.18) *Notation.*  $C^*=O_2(N(C_1))$ ,  $R^*=C^*S$ .

REMARK. This notation will be renewed in (5.48).

(5.19) (1)  $N(C_1)=C^*N(\langle t \rangle U)$ ,  $C^* \cap N(\langle t \rangle U)=C_1$ .

(2)  $\bar{i}^{C^*}=\bar{i}\bar{D}_1$  where  $\bar{N}(\bar{U}_1)=N(U_1)/U_1$ .

(3)  $C^*$  centralizes  $D_1/U_1$ .

(4)  $C^*/C_1 \cong D/U$  as  $N_C(U)$ -modules.

(5)  $C_2=C^* \cap R_2$ .

(6)  $[O(C), C^*]=1$ .

PROOF. We have  $R_2 \leq N(C_1) \leq N(D_1) \leq N(U_1)$  by (5.10), (5.11), and (5.15). Let  $\bar{N}(\bar{U}_1)=N(U_1)/U_1$ . Then it follows from (5.17) that  $C_{N(C_1)}(\bar{i})=N(\langle t \rangle U)$ . Now  $N_C(U) \cap C(D/U)=DC_C(L)$  by (3.23). Thus (5.3) and (5.6) (4) show that  $C_{N(C_1)}(\bar{i}\bar{D}_1)=C_{N(\langle t \rangle U)}(\bar{D}_1)=C_1 C_C(L)$ . Set  $V=C_{N(C_1)}(\bar{D}_1)$  and consider the map  $h$  defined by  $x \mapsto [\bar{i}, x]$  for  $x \in V$ . This is an  $N_C(U)$ -homomorphism from  $V$  into  $\bar{D}_1$  and the kernel is  $C_1 C_C(L)$ . By (5.13) (5),  $V$  contains  $Y_2$ . Moreover, (5.5) (2) shows that  $Y_2$  acts transitively on  $\bar{i}\bar{Y}_1$ . Hence the image of  $h$  contains  $\bar{Y}_1$ . Since  $N_L(U)$  acts irreducibly on  $D/U$ , we conclude that the image of  $h$  is equal to  $\bar{D}_1$ . Thus  $h$  induces an  $N_C(U)$ -isomorphism between  $V/C_1 C_C(L)$  and  $\bar{D}_1$ . Also, we have  $\bar{i}^V=\bar{i}^{N(C_1)}=\bar{i}\bar{D}_1$ . This implies  $N(C_1)=N(\langle t \rangle U)V$ .

Let  $\widetilde{N(C_1)}=N(C_1)/C_1$ . Then  $O(\tilde{V})=\widetilde{C_C(L)}$  and so  $O(\tilde{V})$  centralizes  $\widetilde{N_L(U)}$ . Moreover, since  $V/C_1 C_C(L)$  is  $N_C(U)$ -isomorphic to  $\bar{D}_1$ , we have that  $N_L(U)$  acts irreducibly on  $\tilde{V}/O(\tilde{V})$ . Thus  $\tilde{V}=O(\tilde{V})C_{\tilde{V}}(O(\tilde{V}))$ . Hence  $\tilde{V}$  is a direct product of  $O_2(\tilde{V})$  and  $O(\tilde{V})$ . Thus  $V=O_2(V)C_C(L)$ . Since  $N(C_1)=N(\langle t \rangle U)V$ , we must have  $O_2(V)=C^*$ . Now (1), (2), (3), and (4) hold. As was shown above  $Y_2$  is contained in  $V$ , so  $C_2=Y_2 C_1 \leq V$ . Hence (5) holds. Finally,  $O(C)$  stabilizes the series  $C^* > C_1 > 1$ , and (6) holds.

(5.20) *Notation.*  $R_3^*=R^* \cap N(\langle t \rangle W_1)$ ,  $R_4^*=R^* \cap N(\langle t \rangle X_1)$ , and  $R_5^*=R^* \cap N(E_1)$ .

(5.21) (1)  $\bar{i}^{R_3^*}=\bar{i}\bar{W}_1$ ,  $\bar{i}^{R_4^*}=\bar{i}\bar{X}_1$ , and  $\bar{i}^{R_5^*}=\bar{i}\bar{F}_1$ , where  $\bar{N}(\bar{U}_1)=N(U_1)/U_1$ .

(2)  $R_2=R_3^* \cap R_4^* \triangleleft R_5^*=R_3^* R_4^* \triangleleft R^*$ .

(3)  $\bar{i}^{N(R_2)}=\bar{i}\bar{F}_2$  and  $C_{N(R_2)}(\bar{i})=R_2 N_C(S)$ , where  $\widetilde{N(Y_2)}=N(Y_2)/Y_2$ .

(4)  $N(R_2)=N(E_2) \leq N(W_2) \cap N(X_2)$ .

PROOF. Let  $\overline{N(U_1)} = N(U_1)/U_1$ . Then we have

$$R_3^* = \{x \in R^* \mid \bar{i}^x \in \overline{tW_1}\},$$

$$R_4^* = \{x \in R^* \mid \bar{i}^x \in \overline{tX_1}\},$$

$$R_5^* = \{x \in R^* \mid \bar{i}^x \in \overline{tF_1}\}.$$

Indeed, by (5.19) (3)  $W_1$  is normal in  $R^*$ , so that  $R_3^*$  contains the set  $\{x \in R^* \mid \bar{i}^x \in \overline{tW_1}\}$ . On the other hand,  $R^*$  acts transitively on  $\overline{tD_1}$  by (5.19) (2), and hence  $\bar{i}^x \in \overline{tD_1} \cap \langle \bar{i} \rangle \overline{W_1} = \overline{tW_1}$  for all  $x \in R_3^*$ . Thus the above holds for  $R_3^*$ . The others are similarly verified. Then (1) follows.

Since  $R_2$  is contained in  $R^*$ ,  $R_2 = R^* \cap N(\langle t \rangle Y_1)$  by (5.17). Thus, arguing as above we have that  $R_2 = \{x \in R^* \mid \bar{i}^x \in \overline{tY_1}\}$ . Since  $\overline{tW_1} \cap \overline{tX_1} = \overline{tY_1}$ , it then follows that  $R_3^* \cap R_4^* = R_2$ . Now, the order consideration shows  $R_5^* = R_3^* R_4^*$ . Set  $C_i^* = R_i^* \cap C^*$  for  $i=3, 4, 5$ , and consider the  $N_C(U)$ -isomorphism of  $C^*/C_1$  onto  $\overline{D_1}$  defined by  $C_1 x \mapsto [\bar{i}, x]$  for  $x \in C^*$ . Obviously, this isomorphism maps  $C_2/C_1$ ,  $C_3^*/C_1$ ,  $C_4^*/C_1$ , and  $C_5^*/C_1$  onto  $\overline{Y_1}$ ,  $\overline{W_1}$ ,  $\overline{X_1}$ , and  $\overline{F_1}$  respectively. Thus we obtain the following three  $J$ -isomorphisms:

$$C_3^*/C_2 \cong W/Y, \quad C_4^*/C_2 \cong X/Y, \quad \text{and} \quad C^*/C_5^* \cong D/F.$$

In particular,  $J$  acts transitively on  $(C_3^*/C_2)^*$ ,  $(C_4^*/C_2)^*$ , and  $(C^*/C_5^*)^*$ . Now, we have  $R_3^* = R_2 C_3^*$  and  $R_2 \cap C_3^* = C_2$ , so that  $C_3^* \cap N(R_2)$  properly contains  $C_2$ . Hence the transitive action of  $J$  on  $(C_3^*/C_2)^*$  implies that  $R_2$  is normal in  $R_3^*$ . Likewise we have  $R_2 \triangleleft R_4^*$  and  $R_5^* \triangleleft R^*$ . Thus (2) holds.

Let  $\widetilde{N(Y_2)} = N(Y_2)/Y_2$ . As  $Z(F_2) = Y_2$ , it follows from (5.15) that  $N(R_2) \leq N(E_2) \leq N(F_2) \leq N(Y_2)$ . Then (1) and (2) imply that both  $N(R_2)$  and  $N(E_2)$  act transitively on  $\bar{i}\tilde{F}_2$ . Moreover, (5.17) and (5.5) show that  $C_{N(R_2)}(\bar{i}) = N(R_2) \cap N(\langle t \rangle Y_2) = R_2 N_C(S)$ . Thus (3) holds. Furthermore, since  $N(E_2) \cap N_C(Y) = N_C(S)$ , we have  $C_{N(E_2)}(\bar{i}) = C_{N(R_2)}(\bar{i})$  and thus  $N(R_2) = N(E_2)$ . As  $\mathcal{E}^*(F_2) = \{W_2, X_2\}$ , it follows that  $|N:H| \leq 2$ , where we put  $N = N(R_2)$  and  $H = N_N(W_2)$ . Hence  $\bar{i}^H = \bar{i}^N$  or  $|\bar{i}^H| = |\bar{i}^N|/2$ . Now,  $\bar{i}\tilde{F}_2$  decomposes into four orbits under the action of  $J$ , and these orbits have respectively 1,  $q-1$ ,  $q-1$ , and  $q^2-2q+1$  elements. Thus we must have  $\bar{i}^H = \bar{i}^N$ . This implies that  $N(R_2) \leq N(W_2) \cap N(X_2)$ . The proof is complete.

$$(5.22) \quad \text{Notation. } L_3 = O_2(N(V_2)), \quad R_3 = L_3 S.$$

$$(5.23) \quad (1) \quad N(V_2) = N(\langle t \rangle W) = L_3 N(\langle t \rangle Y) \leq N(Y_2) \text{ and } L_3 \cap N(\langle t \rangle Y) = L_2.$$

$$(2) \quad t^{L_3} = tW.$$

$$(3) \quad L_3/L_2 \cong W/Y \text{ as } N_C(Y)\text{-modules.}$$

$$(4) \quad R_3 = R_3^*.$$

PROOF. Let  $H = N(V_2) \cap N(Y_2)$  and  $\overline{N(Y_2)} = N(Y_2)/Y_2$ . As  $\mathcal{C}^*(V_2) = \{\langle t \rangle W, W_2\}$ , it follows that  $N(V_2) \leq N(\langle t \rangle W) \cap N(W_2) \leq N(W)$ . By (5.21),  $R_2$  and  $\langle t \rangle W_1$  are normal in  $R_3^*$ . Since  $Y_2$  is a characteristic subgroup of  $R_2$  (see (5.13) and (5.15)), and since  $\langle t \rangle W_1 Y_2 = V_2$ , we have  $R_3^* \leq H$ . Now, (5.21) (1) implies that  $R_3^*$  acts transitively on  $\overline{tW_2}$ , whence  $t^H = \overline{tW_2}$ . Furthermore,  $C_H(\bar{t}) = N_H(\langle t \rangle Y_2) = N(\langle t \rangle Y)$  by (5.17). Set  $K = C_H(\overline{tW_2})$ . Then  $K \triangleleft H$ . As  $\overline{W_2} \cong W/Y$ , we have  $K = C_{N(\langle t \rangle Y)}(\overline{W_2}) = L_2 N_L(Y) C_C(L)$  by (3.23) and (5.5). In particular,  $L_2 = O_2(K)$  and  $O(C) = O(K)$  are normal in  $H$ . Let  $\tilde{H} = H/L_2 C_C(L)$  and  $E = C_H(\tilde{K})$ . We now proceed as in (5.3). Since  $\tilde{K} \cong SL(2, q)$  and  $t^H = \overline{tW_2}$  with  $C_H(\bar{t}) = N(\langle t \rangle Y)$ , it follows that  $H = N(\langle t \rangle Y)E$ . Note that  $C_E(\bar{t}) = J_4 L_2 C_C(L)$  by (3.23) and that  $J_4$  acts regularly on  $\overline{W_2}^*$ . Thus the permutation group  $(\tilde{E}, \overline{tW_2})$  is a Frobenius group and  $\tilde{E} = \tilde{J}_4 O_2(\tilde{E})$ . Since  $C_E(O(C))O(C)$  contains  $J_4 L_2 C_C(L)$ ,  $E = C_E(O(C))O(C)$ . So if we set  $V = O_2(E \bmod L_2 C_C(L))$ , then  $V$  is a direct product of  $O(C)$  and  $O_2(V)$ ,  $H = N(\langle t \rangle Y)O_2(V)$ , and  $N(\langle t \rangle Y) \cap O_2(V) = L_2$ . In particular, we have  $O_2(H) = O_2(V)$ .

Since  $|t^H| = q^3$  and  $H \leq N(V_2) \leq N(\langle t \rangle W) \cap N(W)$ , it follows that  $t^H = t^{N(V_2)} = tW$ . Moreover  $H$  contains  $N_C(V_2)$ , for  $N_C(Y) = N_C(W)$  by (3.13). Thus we have  $H = N(V_2)$ . As was shown in (4.2),  $C$  has 2-rank  $3n+1$ , while  $W_2$  is elementary abelian of order  $q^5$ . Hence  $t^G \cap W_2 = \emptyset$ . This implies  $t^{N(\langle t \rangle W)} = tW$  and  $N(V_2) = N(\langle t \rangle W)$ . By a similar argument as in (5.3) (4), we have  $L_3/L_2 \cong W_2/Y_2$  as  $N_C(Y)$ -modules. Thus (1), (2), and (3) hold. Since  $[R_2, L_3] \leq K \cap L_3 = L_2$ ,  $L_3$  normalizes  $R_2$ . Thus  $N(R_2) \cap N(V_2) = R_3 N_C(S)$ , which contains  $R_3^*$ . As  $R_3 N_C(S)/R_3 \cong N_C(S)/R$  has cyclic Sylow 2-subgroups and  $R_3^*/R_2$  is elementary abelian of order  $q$ , it follows that  $R_2 \neq R_3 \cap R_3^*$ . This implies  $R_3 = R_3^*$ , since  $J$  acts irreducibly on  $R_3/R_2$ .

(5.24) Notation.  $R_4 = O_2(N(U_2))$ .

(5.25) (1)  $N(U_2) = N(\langle t \rangle X) = R_4 N_C(S) \leq N(R_2)$ .

(2)  $t^{R_4} = tX$ .

(3)  $R_4/R_2 \cong X/Y$  as  $N_C(S)$ -modules.

(4)  $R_4 = R_4^*$ .

PROOF. Let  $H = N(U_2) \cap N(Y_2)$  and  $\overline{N(Y_2)} = N(Y_2)/Y_2$ . Note that  $\mathcal{C}^*(U_2) = \{\langle t \rangle X, X_2\}$  and so  $N(U_2) \leq N(\langle t \rangle X) \cap N(X_2) \leq N(X)$ . Since  $Y_2$  is characteristic in  $R_2$  and  $\langle t \rangle X_1 Y_2 = U_2$ , it follows from (5.21) that  $R_4^* \leq N(R_2) \cap N(\langle t \rangle X_1) \leq H$  and both  $R_4^*$  and  $H$  act transitively on  $\overline{tX_2}$ . Furthermore, (5.5) and (5.17) show that  $C_H(\bar{t}) = N_H(\langle t \rangle Y) = R_2 N_C(S)$ , since  $N_C(X) = N_C(S)$ . Thus  $H = R_4^* N_C(S)$  and  $R_2$  is normal in  $H$ . Moreover,  $N_C(S)$  normalizes  $R_4^*$  by the definition of  $R_4^*$ . Hence  $O_2(H) = R_4^*$ . Since  $X$  and  $\langle t \rangle X$  are normal in  $H$ , we have  $t^H = tX$ .



Since  $t^g \cap W_2 = \emptyset$  and since every involution of  $L$  is conjugate to an element of  $W$ ,  $t^g \cap L = \emptyset$ . In particular,  $t^{N(\langle t \rangle X)} = tX$ . Thus  $H = N(U_2) = N(\langle t \rangle X)$ . Hence (1), (2), and (4) hold. The map defined by  $x \mapsto [\bar{i}, x]$  for  $x \in R_4$  is a  $N_C(S)$ -homomorphism of  $R_4$  onto  $\bar{X}_2$  whose kernel is  $R_2$ . As  $\bar{X}_2 \cong X/Y$ , (3) holds.

(5.26) Notation.  $R_5 = O_2(N(R_2))$ .

(5.27) (1)  $N(R_2) = R_5 N_C(S)$ ,  $R_5 \cap C = R$ .

(2)  $R_5/R_2 = R_3/R_2 \times R_4/R_2 \cong F/Y$  as  $N_C(S)$ -modules.

(3)  $R_5 = R_5^*$ .

PROOF. Recall that  $N(R_2) \leq N(F_2) \leq N(Y_2)$ . Let  $\bar{N}(\bar{Y}_2) = N(Y_2)/Y_2$  and  $V = C_{N(R_2)}(\bar{F}_2)$ . Then  $V$  is normal in  $N(R_2)$ . As  $\bar{F}_2 \cong F/Y$ , we have  $V \cap R_2 N_C(S) = R_2 C_C(L)$  by (3.23). In particular,  $|V : R_2 C_C(L)| \leq q^2$  since  $|N(R_2) : R_2 N_C(S)| = q^2$  by (5.21) (3). Now, (5.21) (4) shows that  $W_2$  and  $X_2$  are normal subgroups of  $R_i$  for  $i=3, 4$ . Then, since  $J$  acts irreducibly both on  $\bar{W}_2$  and on  $\bar{X}_2$ ,  $Z(\bar{R}_i)$  contains  $\bar{W}_2 \bar{X}_2 = \bar{F}_2$ . Hence  $R_i C_C(L) \leq V$ . Moreover,  $R_3 C_C(L) \cap R_4 C_C(L) = R_2 C_C(L)$  by (5.21) (2). Thus comparing orders we have  $V = R_3 R_4 C_C(L)$ ,  $|V : R_2 C_C(L)| = q^2$ , and  $N(R_2) = N_C(S)V$ . In particular,  $V = O_2(V) \times O(C)$  with  $O_2(V) = R_3 R_4$ . Thus  $R_3 R_4 = R_5$  and (1) and (3) hold. The map defined by  $x \mapsto [\bar{i}, x]$  for  $x \in R_5$  is a  $N_C(S)$ -homomorphism of  $R_5$  onto  $\bar{F}_2$ , which induces an  $N_C(S)$ -isomorphism between  $R_5/R_2$  and  $\bar{F}_2$ . Thus (2) holds.

(5.28)  $N_{R_5}(R_1) = N_{R_5}(S_1) = N_{R_5}(M_1) = R_2$ .

PROOF. (5.17) yields  $N_{R_5}(R_1) = R_2$ . Now,  $S_1$  and  $M_1$  are normal in  $R_2$  by (5.15) (3) and  $C_1$  is normal in  $R_5$ . As  $S_1 C_1 = M_1 C_1 = R_1$ , the assertion holds.

(5.29) Notation.  $C_3 = R_3 \cap O_2(N(C_1))$ ,  $E_3 = L_3 \cap C_3$ ,  $V_3 = C_{R_3}(J_2)V_2$ ,  $W_3 = [V_3, J]$ ,  $F_3 = W_3 F_2$ ,  $M_3 = W_3 M_2$ ,  $S_3 = W_3 S_2$ , and  $D_3 = S_3 \cap C_3$ .

(5.30) (1)  $R_3/C_2 = C_3/C_2 \times R_2/C_2$  and  $E_3/V_2 = V_3/V_2 \times E_2/V_2$ , and both are elementary abelian of order  $q^2$ .

(2)  $W_3$ ,  $F_3$ ,  $M_3$ ,  $S_3$ , and  $D_3$  are groups such that  $V_3 = \langle t \rangle W_3 > W_3 > W_2$ ,  $E_3 = \langle t \rangle F_3 > F_3$ ,  $L_3 = \langle t \rangle M_3 > M_3$ ,  $R_3 = \langle t \rangle S_3 > S_3$ , and  $C_3 = \langle t \rangle D_3 > D_3$ .

(3)  $M_3$  is  $N_L(Y)$ -invariant.

PROOF. (5.19) and (5.23) show that  $C_2 \triangleleft R_3 = C_3 R_2$ . Thus  $R_3/C_2$  is a direct product of  $C_3/C_2$  and  $R_2/C_2$  and so it is elementary abelian of order  $q^2$ . Since  $R_3/E_3 \cong S/F$  and since  $J_2$  acts fixed-point-freely on  $S/F$ , we have  $C_{R_3}(J_2) = C_{E_3}(J_2)$ . Recall that  $E_2$  and  $V_2$  are normal subgroups of  $R_3$ . As  $L_3 = L_2 E_3$  and  $L_2 \cap E_3 = E_2$ , (5.23) (3) implies that  $J_2$  centralizes  $E_3/E_2$ . On the other hand,  $J_2$  acts

regularly on  $(E_2/V_2)^*$  since  $E_2/V_2 \cong X/Y$ . Thus  $E_2/V_2$  is contained in  $Z(E_3/V_2)$ , and  $E_3/V_2$  is a direct product of  $V_3/V_2$  and  $E_2/V_2$ . This proves (1).

Next, we will show that  $V_3/W_2$  is elementary abelian. Let  $\overline{N(\overline{Y_2})} = N(Y_2)/Y_2$ . Then  $C_{R_3}(\bar{i}) = R_2$  and  $V_3$  acts transitively on  $\bar{i}\overline{W_2}$  by (5.21) and (5.23). If  $U^1(V_3)$  is not contained in  $W_2$ , then there exists an element  $x$  of  $V_3$  such that  $\bar{x}^2 \in \bar{i}\overline{W_2}$ . So we can choose  $y \in V_3$  such that  $\bar{y}^2 = \bar{i}$ . But then  $\bar{y} \in C_{\overline{V_3}}(\bar{i}) = \overline{V_2}$ , and  $\bar{y}^2 = 1$ , a contradiction. Thus  $V_3/W_2$  is elementary abelian.

We can easily check that  $C_{V_3/W_2}(J) = V_2/W_2$  and  $W_2 = [W_2, J]$ . Hence  $V_3 = \langle t \rangle W_3 > W_3 > W_2$ . Notice that  $F_2, M_2$ , and  $S_2$  are characteristic in  $R_2$  (see (5.13) and (5.15)). Thus  $F_3, M_3$ , and  $S_3$  are groups and (2) follows. In particular,  $L_3/M_2$  is a direct product of  $M_3/M_2$  and  $L_2/M_2$ . Then, it follows from (5.23) (3) that  $[L_3/M_2, J_4] = M_3/M_2$ . This implies (3), since  $N_L(Y) = M(\langle S_{\pm b} \rangle \times J_4)$  normalizes  $L_3$  and  $M_2$ .

(5.31) *Notation.*  $C_4 = R_4 \cap O_2(N(C_1))$ ,  $L_4 = C_{R_4}(J_6)L_2$ ,  $E_4 = L_4 \cap C_4$ ,  $U_4 = C_{R_4}(J_6)U_2$ ,  $X_4 = [U_4, J]$ ,  $F_4 = X_4F_2$ ,  $M_4 = X_4M_2$ ,  $S_4 = X_4S_2$ , and  $D_4 = S_4 \cap C_4$ .

The proof of the next result closely resembles that of (5.30) and is omitted.

(5.32) (1)  $R_4/C_2 = C_4/C_2 \times R_2/C_2$ ,  $R_4/L_2 = L_4/L_2 \times R_2/L_2$ , and  $E_4/U_2 = U_4/U_2 \times E_2/U_2$ . These three groups are elementary abelian of order  $q^2$ .

(2)  $X_4, F_4, M_4, S_4$ , and  $D_4$  are groups such that  $U_4 = \langle t \rangle X_4 > X_4 > X_2$ ,  $E_4 = \langle t \rangle F_4 > F_4$ ,  $L_4 = \langle t \rangle M_4 > M_4$ ,  $R_4 = \langle t \rangle S_4 > S_4$ , and  $C_4 = \langle t \rangle D_4 > D_4$ .

(5.33) If  $q \geq 8$ , then the following conditions hold.

(1)  $Z(S_3) = Z(S_4) = U_1$ ,  $Z_2(S_3) = Z_2(S_4) = Y_2$ .

(2)  $Z(M_3) = Y_2$ .

(3)  $W_3$  is elementary abelian of order  $q^6$  and normalized by  $N_C(Y)$ .

PROOF. Assume that  $q \geq 8$ . It follows from (5.23) and (5.25) that  $S_3/S_2$  and  $S_4/S_2$  are  $J$ -isomorphic to  $W/Y$  and  $X/Y$  respectively. Thus using (3.9) we can verify that  $C_{S_i}(J_1) = C_{S_2}(J_1) = U_1$  for  $i=3, 4$ . Now  $U_1 = Z(S_2) \triangleleft S_i$  by (5.13). So  $[S_i, J_1]$  centralizes  $U_1$  by (2.4), and hence  $U_1 \leq Z(S_i)$ . On the other hand, (5.28) yields that  $Z(S_i) \leq S_2$ . Thus  $Z(S_3) = Z(S_4) = U_1$ . Let  $\overline{N(\overline{U_1})} = N(U_1)/U_1$ . Then  $Z(\overline{S_2}) = \overline{Y_2}$  by (5.13). Now we can verify that  $C_{\overline{S_i}}(J_5) = \overline{Y_2}$ . Hence  $Z(\overline{S_i})$  contains  $\overline{Y_2}$  by (2.4). Then, (5.28) again yields that  $Z(\overline{S_3}) = Z(\overline{S_4}) = \overline{Y_2}$ . Thus (1) holds. Moreover, (5.30) (3) implies that  $Z(M_3)$  contains  $U_1 U_1^{wb} = Y_2$ . Then (2) follows immediately from (5.13) (1) and (5.28).

Now  $J_2$  centralizes  $W_3/Y_2$  since  $W_3/W_2 \cong W_2/Y_2 \cong W/Y$  as  $J$ -modules. Also  $C_{Y_2}(J_2) = 1$ . Then, as  $Y_2$  lies in  $Z(W_3)$  by (2), we have  $W_3 = C_{W_3}(J_2) \times Y_2$ .

Since  $C_{W_3}(J_2) \supset C_{W_2}(J_2) = C_W(J_2) = S_{2a+b}$ , the irreducible action of  $J$  on  $S_{2a+b}$  implies that  $S_{2a+b} \leq Z(C_{W_3}(J_2))$ . Hence  $Z(W_3)$  contains  $W_2$ . Then, as  $L_3 = \langle t \rangle W_3 M$  and  $C_M(W) = W$ , it follows that  $C_{L_3}(W_2) = W_3$ . In particular,  $N_C(Y)$  normalizes  $W_3$ . As  $W_3 = C_{W_3}(J_2) \times Y_2$ , we have  $\mathfrak{U}^1(W_3) \leq S_{2a+b}$ . Then, since  $S_{2a+b}$  has no nontrivial  $N_L(S)$ -invariant subgroups,  $W_3$  is elementary abelian of order  $q^6$ .

(5.34) *If  $q=4$ , then the following conditions hold.*

- (1)  $Z(S_3) = Z(S_4) = U_1$ ,  $Z_2(S_3) = Z_2(S_4) = Y_2$ .
- (2)  $Z(M_3) = Y_2$ .
- (3)  $W_3$  is elementary abelian of order  $q^6$  and normalized by  $N_C(Y)$ .
- (4)  $X_4$  is elementary abelian of order  $q^6$  and normalized by  $N_C(S)$ .

PROOF. Assume that  $q=4$ . We have four  $J$ -isomorphisms:  $M_3/M_2 \cong S_{2a+b}$ ,  $M_2/F_2 \cong S_a$ ,  $F_2/W_2 \cong S_{a+b}$ , and  $W_2/Y_2 \cong S_{2a+b}$ . So  $C_{M_3}(J_1)$  is equal to  $Y_2$ , which is normal in  $M_3$ . Hence  $[M_3, J_1]$  centralizes  $Y_2$ , and  $Z(M_3)$  contains  $Y_2$ . On the other hand, (5.28) yields that  $Z(M_3)$  is a subgroup of  $M_2$ . Thus (2) is a consequence of (5.13) (1). In particular,  $Z(W_3)$  contains  $Y_2$ . Furthermore,  $J_2$  centralizes  $W_3/Y_2$  and acts fixed-point-freely on  $Y_2$ . Hence arguing as in (5.33), we get (3).

Since  $X_4/X_2 \cong X_2/Y_2 \cong S_{a+b}$  as  $J$ -modules, we have  $C_{X_4}(J_1) = Y_2$ . Since  $Y_2$  is normal in  $X_4$ , this implies  $Y_2 \leq Z(X_4)$ . Moreover, we can check that  $J_6$  centralizes  $X_4/Y_2$  and acts fixed-point-freely on  $Y_2$ . (Remark. These two conditions are valid for  $q \geq 4$ . But when  $q \geq 8$  we can not yet show that  $Y_2 \leq Z(X_4)$ , which will be done in (5.45)). Then (4) follows in a similar manner as (5.33) (3). Finally, (1) follows from (5.28) and (5.13) (5), since  $S_3 = W_3 S_2$  and  $S_4 = X_4 S_2$ .

(5.35) (1)  $Z(R_3) = U$ ,  $Z_2(R_3) = Y_1$ ,  $Z_3(R_3) = F_2$ .

- (2)  $Z(L_3) = Y$ ,  $Z_2(L_3) = W_2$ .
- (3)  $\mathfrak{U}^1(V_3) = W$ ,  $\mathfrak{U}^1(C_3) = W_1$ ,  $\mathfrak{U}^1(L_3) = W_2$ ,  $\mathfrak{U}^1(M_3) = Y_2$ .
- (4)  $W_2$ ,  $X_2$ ,  $E_3$ ,  $L_3$ , and  $C_3$  are characteristic subgroups of  $R_3$ .
- (5)  $W_3$  is a characteristic subgroup of  $L_3$ .

PROOF. Since  $R_3 \cap C = R$ ,  $Z(R_3)$  is contained in  $Z(R_2) = U$ . Hence we have  $Z(R_3) = U$ . Also,  $Z(R_3 \text{ mod } U)$  is contained in  $N_{R_3}(\langle t \rangle U) = R_1$  by (5.3) and so  $Z(R_3/U) \leq Z(R_2/U) = Y_1/U$  by (5.15). Moreover  $R_3 = W_3 R_2$  and  $W_3 \geq Y_1$ . As  $W_3$  is abelian, this implies  $Z_2(R_3) = Y_1$ . Similarly, (5.28) and (5.15) show that  $Z(R_3 \text{ mod } Y_1)$  is a subgroup of  $E_2$ . On the other hand,  $W_1 = W_2 \cap F_1$  and  $X_1 = X_2 \cap F_1$  are normal in  $R_3$  by (5.15) (3) and (5.21) (4). Thus  $Z(R_3 \text{ mod } Y_1)$

contains  $Y_2 W_1 X_1 = F_2$ , since  $J$  acts irreducibly on  $W_1/Y_1$ ,  $X_1/Y_1$ , and  $Y_2/Y_1$ . Hence  $Z_3(R_3) = F_2$ , proving (1).

Now  $Z(L_3)$  contains  $UU^{wb} = Y$  and  $Z(L_3) \leq L_3 \cap C = \langle t \rangle M$ . Hence we have  $Z(L_3) = Y$ . As  $L_3 = W_3 L_2$  and  $W_3$  is abelian, (5.13) (4) shows that  $Z_2(L_3) \geq W_2$ . On the other hand, (5.5) (1) yields that  $Z_2(L_3)$  is a subgroup of  $L_2$ . Furthermore, by (3.16) (2)  $N_L(Y)$  acts irreducibly on  $M_2/W_2$ . Hence  $Z_2(L_3)$  is equal to  $W_2$  or  $M_2$ . Now (5.28) forces that  $Z_2(L_3) = W_2$ , and we get (2).

By (5.8) (3),  $N_{R_3}(X)$  is equal to  $R_2$  or  $R_3$ . (2) shows that  $Z(L_3/Y)$  and  $X/Y$  intersect trivially, and hence  $X$  is not normal in  $L_3$ . So we must have  $N_{R_3}(X) = R_2$ . Since  $\mathcal{E}^*(F) = \{W, X\}$  and  $W \triangleleft R_3$ , we have  $N_{R_3}(F) = R_2$  as well. Note that  $V_3 = \langle t \rangle W_3$  is  $N_C(Y)$ -invariant and so  $V_3$  is normal in  $R_3$ . Now  $C_3 = DV_3$  and  $D \cap V_3 = W$ , so  $C_3/V_3 \cong D/W$ . Also,  $C_3/C_1$  is elementary abelian by (5.19). Hence  $\mathfrak{U}^1(C_3)$  is contained in  $V_3 \cap C_1 \cap D_3 = W_1$ , and so  $\mathfrak{U}^1(V_3)$  lies in  $W_1 \cap W_1^{wb} = W$ . Then (5.13) (4) implies that  $\mathfrak{U}^1(V_3)$  is equal to  $W$  or  $Y$ . As  $V_3$  does not normalize  $\langle t \rangle Y$ , we conclude that  $\mathfrak{U}^1(V_3) = W$ . Then  $\mathfrak{U}^1(C_3) = W_1$ , since  $\mathfrak{U}^1(C_2) = Y_1$  by (5.15) (3). Thus  $\mathfrak{U}^1(E_3) = W$  or  $W_1$ . As was shown above  $N_{R_3}(F) = R_2$ , so we have  $\mathfrak{U}^1(E_3) = W_1$ . In particular,  $\mathfrak{U}^1(L_3)$  contains  $W_1 W_1^{wb} = W_2$ . We also have  $\mathfrak{U}^1(L_3) \leq V_3 \cap M_2 = W_2$ . Hence  $\mathfrak{U}^1(L_3) = W_2$ . Let  $\overline{N(Y_2)} = N(Y_2)/Y_2$ . Then  $\overline{W_3}$  centralizes  $\overline{F_2}$  by (1). As  $M_2$  is a product of  $F_2$  and  $F_2^{wb}$ , it follows that  $\overline{M_2}$  is a central product of  $\overline{W_3}$  and  $\overline{M_2}$ . Hence  $Y_2 \geq \mathfrak{U}^1(M_3) \geq \mathfrak{U}^1(M) = Y$ . Since  $\mathfrak{U}^1(M_3) \neq Y$  by (5.28) and since  $N_L(Y)$  acts irreducibly on  $Y_2/Y$ , we conclude that  $\mathfrak{U}^1(M_3) = Y_2$ . Therefore (3) holds.

As  $R_3 = W_3 R$  and  $W_3 \cap R = W$ , we have  $\mathcal{E}^*(R_3/W_3) = \{L_3/W_3, C_3/W_3\}$  by (3.12). Thus (3) yields that  $\mathcal{E}^*(R_3/W_2) = \{L_3/W_2, C_3/W_2\}$  and  $Z(R_3/W_2) = E_3/W_2$ . On the other hand,  $Z(R_3 \text{ mod } X_2) \leq R_3 \cap N(U_2) = R_2$  by (5.25) and thus  $Z(R_3/X_2)$  is contained in  $Z(R_2/X_2) = E_2/X_2$ . Therefore,  $|Z(R_3/W_2)| \neq |Z(R_3/X_2)|$ . As  $Z_3(R_3) = F_2$  and  $\mathcal{E}^*(F_2) = \{W_2, X_2\}$ , this implies that  $W_2$  and  $X_2$  are characteristic subgroups of  $R_3$ . Hence (4) holds. Finally, (5) is a consequence of (2), since  $W_3 = C_{L_3}(W_2)$ .

(5.36) *Notation.*  $C_5 = R_5 \cap O_2(N(C_1))$ ,  $L_5 = L_3 L_4$ ,  $E_5 = L_5 \cap C_5$ ,  $S_5 = S_3 S_4$ ,  $M_5 = M_3 M_4$ ,  $D_5 = D_3 D_4$ , and  $F_5 = F_3 F_4$ .

(5.37) (1)  $R_5/L_3 = L_5/L_3 \times R_3/L_3$  is elementary abelian of order  $q^2$ .

(2)  $S_5$ ,  $M_5$ ,  $D_5$ , and  $F_5$  are groups such that  $R_5 = \langle t \rangle S_5 > S_5$ ,  $L_5 = \langle t \rangle M_5 > M_5$ ,  $C_5 = \langle t \rangle D_5 > D_5$ , and  $E_5 = \langle t \rangle F_5 > F_5$ .

(3)  $\mathfrak{U}^1(S_5) = \mathfrak{U}^1(S_4) = \mathfrak{U}^1(R_5) = F_2$ .

PROOF. By (5.35) (4),  $L_3$  is normal in  $R_5$ . Since  $R_5 = L_5 R_3$  and  $L_5 \cap R_3 = L_3$ , (5.25) (3) shows that  $L_5/L_3 \cong R_5/R_3 \cong X/Y$  as  $J$ -modules. In particular,  $J_6$  centralizes  $L_5/L_3$  and  $R_5/R_3$ . Moreover, since  $R_3/L_3$  is isomorphic to  $S/M$  as

$J$ -modules,  $J_6$  acts fixed-point-freely on  $R_3/L_3$ . Then, since  $R_3/L_3 \leq Z(R_5/L_3)$ , we obtain

$$R_5/L_3 = C_{R_5/L_3}(J_6)[R_5/L_3, J_6] = L_5/L_3 \times R_3/L_3.$$

Thus (1) holds. By (5.19),  $R_5/C_4$  is a direct product of  $C_5/C_4$  and  $R_4/C_4$  and so is elementary abelian. Thus  $\mathcal{U}^1(R_5)$  is contained in  $L_3 \cap C_4 = E_2$  and  $\mathcal{U}^1(S_i)$  is contained in  $F_2$ ,  $i=3, 4$ . This, together with (5.15) (3) and (5.28), shows that  $\mathcal{U}^1(S_i) = F_2$ . Then, arguing as in (5.30) we obtain  $\mathcal{U}^1(R_5) = F_2$ . Now (2) is easily verified.

$$(5.38) \text{ Notation. } L_6 = O_2(N(L_2)), \quad R_6 = L_6 S.$$

$$(5.39) \quad (1) \quad N(L_2) = L_6 N(V_2) \leq N(L_3), \quad L_6 \cap N(V_2) = L_3.$$

$$(2) \quad L_6/L_3 \cong M/W \text{ as } N_C(Y)\text{-modules.}$$

$$(3) \quad [O(C), L_6] = 1.$$

PROOF. We examine first the structure of  $H = N(L_2) \cap N(W_2)$ , and later on we shall show that  $N(L_2) \leq N(W_2)$ . Notice that  $N(L_3) \leq N(W_2)$  by (5.35) (2) and that  $M_2/W_2$  is isomorphic to  $M/W$ . Let  $\overline{N(W_2)} = N(W_2)/W_2$ . We proceed as in (5.5). It follows from (5.13) (2), (5.15) (6), and (5.21) (4) that  $N(R_2) \leq H \leq N(M_2)$ . Moreover, (5.21) (3) indicates that  $N(R_2)$  acts transitively on  $\overline{tF_2}$ . Thus  $\overline{tF_2} \leq \overline{t}^H \leq \overline{L_2} - \overline{M_2} = \overline{tM_2}$ . As  $H$  contains  $N_L(Y)$  and  $\overline{M_2} \cong M/W$ , we conclude that  $\overline{t}^H = \overline{tM_2}$ . (5.23) implies that  $N(V_2) = L_3 N_C(Y) \leq H$ , and so  $C_H(\overline{t}) = N(V_2)$  since  $V_2 = \langle t \rangle W_2$ . Now  $N_C(Y) \cap C(M/W) = MC_C(L)$  by (3.23). As  $L_3$  centralizes  $\overline{M_2}$  by (5.35) (3), we have that  $C_H(\overline{tM_2}) = L_3 C_C(L)$ . Let  $\tilde{H} = H/L_3 C_C(L)$ . Then the permutation group  $(\tilde{H}, \overline{tM_2})$  is 2-transitive and the stabilizer of the point  $\overline{t}$  is  $\widetilde{N(V_2)}$ . Furthermore,  $\widetilde{N(V_2)} \triangleright \widetilde{N_L(Y)}$  and  $(\widetilde{N_L(Y)}, \overline{tM_2})$  is isomorphic to  $(SL(2, q), V(2, q))$  as permutation groups. Thus by (2.6),  $\tilde{H}$  has a regular normal subgroup  $E/L_3 C_C(L)$ . Arguing as in (5.5) we have that  $E/L_3$  is a direct product of  $O_2(E/L_3)$  and  $L_3 C_C(L)/L_3$ . Thus  $H = O_2(E)N(V_2)$ ,  $O_2(E) \cap N(V_2) = L_3$ , and  $O_2(E) = O_2(H)$ . Moreover,  $O(C)$  stabilizes the series  $O_2(H) > L_3 > 1$  and so centralizes  $O_2(H)$ . As  $M_2$  is a normal subgroup of  $H$ ,  $Z(\overline{O_2(H)})$  and  $\overline{M_2}$  intersect nontrivially, so that  $Z(\overline{O_2(H)})$  actually contains  $\overline{M_2}$ . Thus the map defined by  $L_3 x \mapsto [\overline{t}, x]$  for  $x \in O_2(H)$  is an  $N_C(Y)$ -isomorphism of  $O_2(H)/L_3$  onto  $\overline{M_2}$ .

It remains to show that  $H = N(L_2)$ . Note that  $N(L_2) \leq N(M_2) \leq N(Y_2)$  by (5.13). Let  $\widehat{N(Y_2)} = N(Y_2)/Y_2$ . Then from (5.17) we see that  $C_H(\overline{t}) = N(\langle t \rangle Y) = C_{N(L_2)}(\overline{t})$ . In particular,  $O_2(H) \cap C(\overline{t}) = L_2$ . Then since  $|O_2(H) : L_2| = q^3$ , both  $H$  and  $N(L_2)$  act transitively on  $\overline{tM_2}$ . This implies  $H = N(L_2)$ .

- (5.40) (1)  $R_5 = N_{R_6}(R_3) \triangleleft R_6$ .  
 (2)  $\mathcal{E}^*(R_6/L_3) = \{L_6/L_3, R_5/L_3\}$ .  
 (3)  $R_6/R_5 \cong M/F$  as  $N_C(S)$ -modules.  
 (4)  $L_5 = L_6 \cap R_5$ .

PROOF. As  $L_2$  and  $R_3$  are normal subgroups of  $R_5$ , we see from (5.38) that  $R_5$  normalizes  $R_6$ . (5.39) (1) implies that  $N(R_6) \cap N(L_2) = R_6 N_C(S)$ . Now  $R_6 N_C(S)/R_6 \cong N_C(S)/R$  has cyclic Sylow 2-subgroups. Since  $R_5/R_3$  is elementary abelian of order  $q$ , the irreducible action of  $J$  on  $R_5/R_3$  implies that  $R_5$  is a subgroup of  $R_6$ . Moreover,  $N(R_3) \leq N(E_3)$  by (5.35) (4) and  $L_2 \cap E_3 = E_2$ . Then, it follows from (5.21) (4) and (5.27) that  $N_{R_6}(R_3) \leq N(L_2) \cap N(E_3) \leq R_5 N_C(S)$ . Thus we have  $R_5 = N_{R_6}(R_3)$ . The remaining part of the proof is similar to that of (5.6) and is omitted. Notice that (2) corresponds to the fact that  $\mathcal{E}^*(S/W) = \{M/W, D/W\}$  under the  $N_C(Y)$ -isomorphism between  $L_6/L_3$  and  $M/W$  defined in (5.39) (2).

- (5.41) (1)  $L_6/L_2 \cong M/Y$  as  $N_C(Y)$ -modules.  
 (2)  $\mathfrak{U}^1(L_6) = M_2$ .  
 (3)  $N(V_2) = N(V_3)$ .  
 (4)  $N(L_2) = N(L_3) \leq N(M_3)$ .

PROOF. Let  $\overline{N(Y_2)} = N(Y_2)/Y_2$ . (5.13) shows that  $N(L_2) \leq N(M_2) \leq N(Y_2)$ . Now  $W_2$  is normal in  $R_6$  by (5.35) (2) and (5.39). Moreover, since  $F_2 = \mathfrak{U}^1(R_6) \triangleleft R_6$  by (5.37) and  $\mathcal{E}^*(F_2) = \{W_2, X_2\}$ ,  $X_2$  is also normal in  $R_6$ . Hence  $Z(\overline{R_6})$  contains  $\overline{W_2} \overline{X_2} = \overline{F_2}$ , and so  $Z(\overline{L_6})$  contains  $\overline{F_2} \overline{F_2}^{wb} = \overline{M_2}$ . Thus the map defined by  $L_2 x \mapsto [\tilde{t}, x]$  for  $x \in L_6$  is an  $N_C(Y)$ -isomorphism of  $L_6/L_2$  onto  $\overline{M_2}$  (see the proof of (5.39)). Thus (1) holds.

In particular,  $L_6/L_2$  is elementary abelian. Arguing as in (5.6) (4) we have that  $L_6/M_2$  is elementary abelian, while  $\mathfrak{U}^1(L_6)$  contains  $W_2$  by (5.35) (3). Since  $N_L(Y)$  acts irreducibly on  $M_2/W_2$  and since  $L_6$  is not contained in  $N(V_2)$ , we obtain (2). As  $\mathcal{E}^*(V_3) = \{\langle t \rangle W, W_3\}$ , (3) is a consequence of (5.23).

Finally we prove (4). (5.35) and (5.39) show that  $N(L_2) \leq N(L_3) \leq N(W_3)$ . Let  $\widetilde{N(W_3)} = N(W_3)/W_3$ . Then  $N(L_2)$  acts transitively on  $\tilde{t}\tilde{M}_3$ . Moreover,  $C_{N(L_3)}(\tilde{t}) = N(V_2)$  by (3). Since  $|N(L_2) : N(V_2)| = q^2$  by (5.39), the length of the  $N(L_3)$ -orbit on  $\tilde{L}_3^\#$ , in which  $\tilde{t}$  lies, is even. Now  $\tilde{L}_3^\#$  decomposes into three orbits  $\{\tilde{t}\}$ ,  $\tilde{M}_3^\#$ , and  $\tilde{t}\tilde{M}_3^\#$  under the action of  $N_L(Y)$ . Thus  $\tilde{t}^{N(L_3)} = \tilde{t}\tilde{M}_3^\#$ . This proves (4).

- (5.42)  $N_{R_6}(R_2) = N_{R_6}(S_2) = N_{R_6}(C_1) = R_5$ .

PROOF. It follows from (5.27) (1) that  $N_{R_6}(R_2)=R_5$ . Moreover, (5.15) (6) shows that  $S_2$  and  $C_1$  are normal in  $R_5$ . Since  $L_2$  is normal in  $R_6$  and  $R_2=L_2S_2=L_2C_1$ , the assertion holds.

$$(5.43) \text{ Notation. } M_6=M_5M_5^{wb}, \quad S_6=M_6S.$$

$$(5.44) \quad (1) \quad L_6=\langle t \rangle M_6 > M_6, \quad R_6=\langle t \rangle S_6 > S_6.$$

$$(2) \quad M_6=F_5F_5^{wb}, \quad F_5 \cap F_5^{wb}=W_3.$$

PROOF. By (5.37) and (5.41),  $M_5$  is a normal subgroup of  $R_6$ . (5.40) shows that  $R_6/L_5$  is a direct product of  $L_6/L_5$  and  $R_5/L_5$  and  $R_6/R_5 \cong M/F$  as  $J$ -modules. Moreover,  $R_5/L_5 \cong S/M$  as  $J$ -modules. Thus  $J_4$  centralizes both  $R_5/L_5$  and  $L_5/M_5$  and acts fixed-point-freely on  $R_6/R_5$ . Hence  $C_{R_6}(J_4)M_5=R_5$  and  $[R_6, J_4]$  centralizes  $R_5/M_5$  by (2.4). Thus  $R_6/M_5$  is a direct product of  $[R_6/M_5, J_4]$  and  $R_5/M_5$ , so it is elementary abelian since  $[R_6, J_4]$  is contained in  $L_6$ . The  $N_C(Y)$ -isomorphism between  $L_6/L_3$  and  $M/W$  mentioned in (5.39) maps  $L_5/L_3$  onto  $F/W$ . Hence  $M/W=F/W \times F^{wb}/W$  implies that  $L_6/L_3$  is a direct product of  $L_5/L_3$  and  $L_6^{wb}/L_3$ . In particular,  $M_5 \cap M_5^{wb}=M_3$ . Then, counting orders we obtain  $L_6=\langle t \rangle M_6 > M_6$ . Thus (1) holds. Since  $L_5/E_3$  is a direct product of  $L_3/E_3$  and  $E_5/E_3$  and since  $L_3/V_3$  is a direct product of  $E_3/V_3$  and  $E_3^{wb}/V_3$ , (2) follows easily.

$$(5.45) \quad (1) \quad Z(S_6)=Z(S_6)=U_1, \quad Z_2(S_6)=Z_2(S_6)=Y_2.$$

$$(2) \quad Z(M_6) \cong Y_2.$$

$$(3) \quad X_4 \text{ is elementary abelian of order } q^6 \text{ and normalized by } N_C(S).$$

PROOF. Assume first  $q \geq 8$ . As  $S_5=S_3S_4$ , (5.33) (1) implies  $U_1 \leq Z(S_5)$ . Thus  $Z(S_5)=U_1$  by (5.13) (5) and (5.28). Now  $S_6/S_5 \cong M/F$  and  $S_5/S_3 \cong F/W$  as  $J$ -modules by (5.40), and we have  $C_{S_6}(J_1)=U_1$ . As  $U_1$  is normal in  $S_6$ , this implies that  $[S_6, J_1]$  centralizes  $U_1$ , and hence  $U_1 \leq Z(S_6)$ . Moreover, (5.42) forces  $Z(S_6) \leq S_5$ . Thus  $Z(S_6)=U_1$ . In particular  $Z(M_6)$  contains  $U_1U_1^{wb}=Y_2$ , proving (2). Let  $\overline{N(U_1)}=N(U_1)/U_1$ . Then (5.13) (5), (5.28), and (5.33) (1) show that  $Z(\tilde{S}_5)=\tilde{Y}_2$ . This, together with (2), implies  $\tilde{Y}_2 \leq Z(\tilde{S}_6)$ . Hence (5.42) again yields  $Z(\tilde{S}_6)=\tilde{Y}_2$ . Now  $X_4/X_2 \cong X_2/Y_2 \cong X/Y$  as  $J$ -modules, and so  $J_6$  centralizes  $X_4/Y_2$ . Since  $J_6$  acts fixed-point-freely on  $Y_2$  and since  $X_4$  centralizes  $Y_2$  by (2),  $X_4$  is a direct product of  $C_{X_4}(J_6)$  and  $Y_2$ . Then as in (5.33) we obtain (3).

Assume next  $q=4$ . In this case we have already proved (3) in (5.34). Since  $M_5=M_3X_4$ , it follows from (5.34) (2) that  $M_5$  centralizes  $Y_2$ . Then as  $M_6=M_5M_5^{wb}$ , (2) holds. For (1), the argument is similar to that of the case  $q \geq 8$  and is omitted.

- (5.46) (1)  $F_5/Y_2 = W_3/Y_2 \times X_4/Y_2$  is elementary abelian of order  $q^4$ .  
 (2)  $\mathcal{E}^*(S_5/Y_2) = \{M_5/Y_2, D_5/Y_2\}$ .  
 (3)  $J(R_5/Y_2) = S_5/Y_2$ .  
 (4)  $F_5, M_5, D_5$ , and  $S_5$  are characteristic subgroups of  $R_5$ .

PROOF. By (5.21) and (5.35),  $X_2$  and  $W_3$  are normal in  $R_5$ . Moreover,  $X$  is a self-centralizing subgroup of  $S$  and  $S_4$  is normal in  $R_5$  by (5.37), so that  $X_4 = C_{S_4}(X_2)$  is also normal in  $R_5$ . Thus (1) holds.

Next we will show that  $\mathcal{U}^1(M_5) = Y_2$  and  $\mathcal{U}^1(D_5) \leq Y_1$ . Recall that  $M_5/M_3 \cong X/Y \cong F_3/W_3$  and  $M_3/F_3 \cong M/F$  as  $J$ -modules and that  $M_5/F_3$  is a direct product of  $M_3/F_3$  and  $F_5/F_3$ . Then  $J_6$  centralizes both  $M_5/M_3$  and  $F_3/W_3$  and acts regularly on  $(M_3/F_3)^*$ . So  $[M_5/W_3, J_6] = [M_5/W_3, J_6, J_6] = [M_3/W_3, J_6]$  and  $M_3/W_3 = [M_3/W_3, J_6] \times F_3/W_3$ . Hence  $M_5/W_3$  is a direct product of  $[M_5/W_3, J_6]$  and  $F_5/W_3$ , so it is elementary abelian. Now  $M_5 = M_3 X_4$  and  $M_3 \cap X_4 = X_2$ , so that  $M_5/X_4$  is isomorphic to  $M_3/X_2$ . Then, since  $\mathcal{U}^1(M_3) = Y_2$  by (5.35) (3), it follows that  $\mathcal{U}^1(M_5) = W_3 \cap X_4 = Y_2$ . Since  $D_5/D_4 \cong W/Y \cong F_4/X_4$  and  $D_4/F_4 \cong D/F$  as  $J$ -modules,  $J_2$  centralizes both  $D_5/D_4$  and  $F_4/X_4$  and acts regularly on  $(D_4/F_4)^*$ . Hence  $D_5/X_4$  is a direct product of  $[D_5/X_4, J_2]$  and  $F_5/X_4$  and so is elementary abelian. Also, since  $D_5/D_3 \cong X/Y \cong F_3/W_3$  and  $D_3/F_3 \cong D/F$  as  $J$ -modules,  $J_6$  centralizes both  $D_5/D_3$  and  $F_3/W_3$  and acts regularly on  $(D_3/F_3)^*$ . Hence  $D_5/W_3 = [D_5/W_3, J_6] \times F_5/W_3$ , which is elementary abelian. Furthermore,  $C_5/C_1$  is elementary abelian by (5.19). Therefore,  $\mathcal{U}^1(D_5)$  is contained in  $X_4 \cap W_3 \cap C_1 = Y_1$  as required.

Let  $\overline{N(Y_2)} = N(Y_2)/Y_2$ . Then  $\mathcal{E}^*(\bar{S}_2) = \{\bar{M}_2, \bar{D}_2\}$ . Thus the above proves (2).

(5.5) and (5.17) imply that  $Z(R_5 \text{ mod } Y_2)$  is a subgroup of  $N_{R_5}(\langle t \rangle Y_2) = R_2$ , and hence  $Z(\bar{R}_5) \leq Z(\bar{R}_2) = \bar{E}_2$ . Thus  $Z(\bar{R}_5) = \bar{F}_2$ , which has order  $q^2$ . Moreover, as  $|\bar{M}_5| = |\bar{D}_5| = q^5$ , it follows from (2) and (2.1) that every abelian subgroup of  $\bar{S}_5$  has order at most  $q^5$ . Assume that  $\bar{R}_5$  has an abelian subgroup  $\bar{A}$  such that  $\bar{A} \not\leq \bar{S}_5$  and  $|\bar{A}| \geq q^5$ . Then  $\bar{R}_5 = \bar{A} \bar{S}_5$ . As  $Z(\bar{S}_5) = \bar{F}_5$ ,  $\bar{A} \cap \bar{F}_5$  is contained in  $Z(\bar{R}_5)$ . Since  $(\bar{A} \cap \bar{S}_5) \bar{F}_5$  is an abelian subgroup of  $\bar{S}_5$ , its order is at most  $q^5$ . But then  $|\bar{A} \cap \bar{S}_5| \leq q^3$  and  $|\bar{A}| \leq 2q^3$ , a contradiction. So  $\mathcal{A}(\bar{R}_5) = \mathcal{A}(\bar{S}_5)$  and  $J(\bar{R}_5) = \bar{M}_5 \bar{D}_5 = \bar{S}_5$ . Thus (3) holds.

Now, since  $\mathcal{U}^1(R_5) = F_2$  by (5.37) and  $Z(F_2) = Y_2$ ,  $S_5$  is a characteristic subgroup of  $R_5$ . Moreover,  $|\mathcal{U}^1(M_5)| \neq |\mathcal{U}^1(D_5)|$  so that (2) implies (4).

$$(5.47) \quad N(R_5) \cap N(E_5) = N(R_2).$$

PROOF. Recall that  $N(R_2) = N(E_2) = R_5 N_C(S)$ . As  $E_5 = \langle t \rangle F_5$ , (5.46) (4) shows that  $N(R_2)$  normalizes  $E_5$ . Now,  $N(R_5)$  acts on  $\mathcal{E}^*(F_2) = \{W_2, X_2\}$  by (5.37) (3). Moreover,  $R_5 \cap N(V_2) = R_3$  and  $R_5 \cap N(U_2) = R_4$  by (5.23) and (5.25).



So it follows from (5.46) (1) that  $\mathcal{E}^*(E_5/W_2) = \{F_5/W_2, E_2/W_2\}$  and  $\mathcal{E}^*(E_5/X_2) = \{F_5/X_2, E_4/X_2\}$ . Now  $|F_5/W_2| = |F_5/X_2| = q^3 > |E_2/W_2| = |E_4/X_2| = 2q^2$ . Thus for any element  $x$  of  $N(R_5) \cap N(E_5)$ , either  $E_2^x = E_2$  and  $E_4^x = E_4$ , or  $E_3^x = E_4$  and  $E_4^x = E_3$ . In any case  $x$  normalizes  $E_3 \cap E_4 = E_2$ .

(5.48) *Notation.*  $C_7 = O_2(N(C_1))$ ,  $R_7 = C_7 S$ , and  $R_8 = O_2(N(R_5))$ .

REMARK.  $C_7$  and  $R_7$  were previously denoted by  $C^*$  and  $R^*$  respectively in (5.18).

(5.49)  $N(R_5) = R_8 N_C(S)$  and  $R_8 \cap C = R$ . Moreover,  $R_8/R_5 = R_6/R_5 \times R_7/R_5 \cong S/F$  as  $J$ -modules.

PROOF. Let  $\overline{N(F_5)} = N(F_5)/F_5$  and  $H = N(R_5)$ . Then  $H$  contains  $R_6$  and  $R_7$  by (5.40) and (5.21) (2) and  $F_5$ ,  $M_5$ ,  $D_5$ , and  $S_5$  are normal subgroups of  $H$  by (5.46). In particular,  $\bar{i}^H \leq \bar{R}_5 - \bar{S}_5 = \bar{i}\bar{S}_5$ . Now,  $C_H(\bar{i}) = R_5 N_C(S)$  by (5.47) and  $C_H(\bar{i}\bar{S}_5) = R_5 C_C(L)$  since  $C_C(S/F) = SC_C(L)$  by (3.23). Set  $V = C_H(\bar{S}_5)$  and consider the  $N_C(S)$ -homomorphism  $h$  of  $V$  into  $\bar{S}_5$  defined by  $x \mapsto [\bar{i}, x]$ . Since  $M_5$  and  $D_5$  are normal in  $R_i$  for  $i=6, 7$ ,  $Z(\bar{R}_i)$  contains  $\bar{M}_5 \bar{D}_5 = \bar{S}_5$ . Hence  $R_i \leq V$ . Now,  $R_6$  and  $R_7$  act transitively on  $\bar{i}\bar{M}_5$  and  $\bar{i}\bar{D}_5$  respectively (see the proof of (5.39) and (5.19)). This implies that the image of  $h$  contains  $\bar{M}_5$  and  $\bar{D}_5$ , and so  $h$  is surjective. The kernel of  $h$  is  $R_5 C_C(L)$ . Therefore,  $h$  induces an  $N_C(S)$ -isomorphism of  $V/R_5 C_C(L)$  onto  $\bar{S}_5$ . Since  $N_V(R_2) = R_5 C_C(L)$ , we have  $N(R_5) = N(R_2)V$ . Now  $R_i C_C(L) = R_i \times O(C)$  for  $i=5, 6, 7$ . Moreover,  $R_6 \cap R_7 = R_5$  by (5.42). Thus  $V = R_6 R_7 O(C)$  and  $O_2(V) = R_6 R_7$ . Since  $O_2(N(R_2)) = R_5$ , the assertion holds.

(5.50) *Notation.*  $P = [R_8, J]$ ,  $S_7 = P \cap R_7$ , and  $D_7 = P \cap C_7$ .

(5.51)  $R_8 = \langle t \rangle P > P$ ,  $R_7 = \langle t \rangle S_7 > S_7$ , and  $C_7 = \langle t \rangle D_7 > D_7$ .

PROOF. Arguing as in (5.6) (4) we have that  $R_8/S_5$  is elementary abelian. Moreover,  $J$  acts fixed-point-freely on  $S_5$  and so  $S_5$  is contained in  $P$ . Then (5.49) implies that  $C_{R_8/S_5}(J) = R_5/S_5$  and  $R_8/S_5$  is a direct product of  $R_5/S_5$  and  $P/S_5$ . Thus the assertion holds.

(5.52)  $Z(P) = U_1$  and  $Z_2(P) = Y_2$ .

PROOF. (5.27) and (5.42) show that  $N_{R_8}(R_2) = N_{R_6}(S_2) = R_5$ . Since  $C_2$  is normal in  $C_7 R_2 = R_7$  by (5.19) and  $S_2 C_2 = R_2$ , we also have  $N_{R_7}(S_2) = R_5$ . Furthermore, it follows from (5.49) that the only nontrivial  $J$ -invariant subgroups of  $R_8/R_5$  are  $R_6/R_5$ ,  $R_7/R_5$ , and  $R_8/R_5$ . Thus  $N_{R_8}(S_2) = R_5$ .

Assume  $q \geq 8$ . Notice that  $S_7/S_5 \cong D/F$  and  $S_6/S_5 \cong M/F$  as  $J$ -modules. Now  $J_1$  acts fixed-point-freely on  $D/F$  and  $M/F$ . Thus  $C_P(J_1) = C_{S_5}(J_1) = U_1$  (see the proof of (5.45) (1)). Since  $U_1 = Z(S_5)$  is normal in  $P$ ,  $Z(P)$  contains  $U_1$  by (2.4). On the other hand,  $Z(P) \leq R_5 \cap P = S_5$  by the above. Hence  $Z(P) = U_1$ . Likewise we can verify that  $C_{P/U_1}(J_5) = C_{S_5/U_1}(J_5) = Y_2/U_1$ . Since  $Z_2(S_5) = Y_2$ , the above yields that  $Z_2(P) = Y_2$ .

In the case where  $q=4$  we appeal to the structure of  $C(J_1)$ . Set  $A = C_P(J_1)$ . In this case  $J_1 = J_4 = J_5$ , so that  $D_7 = AF_5$  and  $A \cap F_5 = Y_2$ . Thus  $A$  has order  $q^6$ . As  $\langle J_1, J_3 \rangle = J$ ,  $J_3$  acts fixed-point-freely on  $A$ . Hence  $A$  is equal to  $[A, J_3]$  and so is contained in  $\langle C_L(J_1)^{C(J_3)} \rangle$ . Therefore, according to (5.10) we have that  $Z(A) = U_1$  and  $A/U_1$  is elementary abelian. In particular,  $Z(P)$  contains  $U_1$  since  $P = AS_5$  and  $Z(S_5) = U_1$  by (5.45). Then as in the case  $q \geq 8$  we have  $Z(P) = U_1$ . Since  $Z_2(S_5) = Y_2$  is a subgroup of  $A$ , it follows that  $Z_2(P) = Y_2$ . This completes the proof of (5.52).

$$(5.53) \quad \mathfrak{U}^1(C_7) = D_1.$$

PROOF. By (5.19) and (5.51) we have  $\mathfrak{U}^1(C_7) \leq C_1 \cap D_7 = D_1$ . Also,  $Y_1 \leq \mathfrak{U}^1(C_7)$  by (5.15) (3). Since  $N_L(U)$  acts irreducibly on  $D_1/U_1 \cong D/U$ , the assertion holds.

$$(5.54) \quad D_7 = D_3 D_3^{w_a} = W_3 W_3^{w_a}, \quad D_3 \cap D_3^{w_a} = D_1, \text{ and } W_3 \cap W_3^{w_a} = U_1.$$

PROOF. Since  $D/U = W/U \times W^{w_a}/U$ , we have that  $C_7/C_1$  is a direct product of  $C_3/C_1$  and  $C_3^{w_a}/C_1$  (see the proof of (5.19) and (5.23)). Set  $H = D_3 D_3^{w_a}$ . Then  $H$  is a  $J$ -invariant subgroup of  $C_7$ . Moreover,  $D_3 \cap D_3^{w_a} = D_1$  and  $|C_7 : H| = 2$ . As  $D_5/D_3$  has order  $q$ ,  $D_5 \cap H$  properly contains  $D_3$ . Hence the irreducible action of  $J$  on  $D_5/D_3$  yields that  $H \geq D_5$ . Arguing similarly we obtain  $H = D_7$ . Since  $D_3/W_1$  is a direct product of  $W_3/W_1$  and  $D_1/W_1$  and since  $D_1/U_1$  is a direct product of  $W_1/U_1$  and  $W_1^{w_a}/U_1$ , (5.54) follows.

$$(5.55) \quad (1) \quad \mathfrak{U}^1(D_7) = D'_7 = U_1, \quad \mathfrak{U}^1(M_6) = M'_6 = Y_2.$$

$$(2) \quad \mathcal{E}^*(C_7/U_1) = \{D_7/U_1, C_1/U_1\}, \quad \mathcal{E}^*(L_6/Y_2) = \{M_6/Y_2, L_2/Y_2\}.$$

$$(3) \quad N(C_7) = N(C_1) \leq N(D_7), \quad N(L_6) = N(L_2) \leq N(M_6).$$

PROOF. Let  $\overline{N(U_1)} = N(U_1)/U_1$ . By (5.19) (3),  $C_7$  centralizes  $\overline{D_1}$ . Hence (5.52) implies that  $Z(\overline{D_7})$  contains  $\overline{D_1} \overline{Y_2}$ . Now,  $D_7$  is  $N_L(U)$ -invariant and so  $N_L(U)$  acts irreducibly on  $D_7/D_1$  by (5.19) (4). Therefore,  $\overline{D_7}$  is elementary abelian. Then  $\mathfrak{U}^1(D_7) = U_1$  or  $U$  and since (5.8) (6) indicates that  $\mathfrak{U}^1(D_7)$  is not contained in  $D$ , we have  $\mathfrak{U}^1(D_7) = U_1$ . Similarly  $D'_7 = U_1$ . From (5.35) (2) and (5.39) we know that  $W_2$  is normal in  $R_6$ . As  $\mathfrak{U}^1(R_5) = F_2$  and  $\mathcal{E}^*(F_2) = \{W_2, X_2\}$ ,  $X_2$  is also normal in  $R_6$ . Moreover, (5.41) (2) shows that  $M_4$  is normal in  $R_6$ .

Since  $M_4 = X_4 M$  and  $C_M(X) = X$ , we must have  $R_6 \supset C_{M_4}(X_2) = X_4$ . Hence (5.44) and (5.46) (1) show that  $M_6/Y_2$  is a direct product of  $F_5^{wb}/Y_2$  and  $X_4/Y_2$  and so it is elementary abelian. Thus  $\mathfrak{U}^1(M_6) = Y_2$  by (5.35) (3). Furthermore,  $Y = M' \leq M'_6$  and  $M'_6 \leq M_1$  by (5.28). Thus  $M'_6 = Y_2$ , since  $N_L(Y)$  acts irreducibly on  $Y_2/Y$ .

By (5.10) and (5.53),  $U_1$  is normal in  $N(C_7)$ . Since  $C_{C_7}(\bar{i}) = N_{C_7}(\langle t \rangle U_1) = C_1$  by (5.3) and (5.17), it follows that  $\mathcal{E}^*(\bar{C}_7) = \{\bar{D}_7, \bar{C}_1\}$ . Hence  $D_7$  and  $C_1$  are normal subgroups of  $N(C_7)$ . Likewise  $N(L_6) \leq N(Y_2)$  by (5.13) (1) and (5.41) (2). Also,  $C_{L_6}(\bar{i}) = N_{L_6}(\langle t \rangle Y_2) = L_2$  by (5.5) and (5.17), where  $\widetilde{N(Y_2)} = N(Y_2)/Y_2$ . Hence  $\mathcal{E}^*(\bar{L}_6) = \{\bar{M}_6, \bar{L}_2\}$  and both  $M_6$  and  $L_2$  are normal in  $N(L_6)$ . The proof is complete.

(5.56)  $D_7$  and  $M_6$  are normal subgroups of  $R_8$ .

PROOF. We have that  $C_7/C_4$  is  $N_L(S)$ -isomorphic to  $D/X$  (see the proof of (5.19) and (5.21)). Then  $\mathcal{E}^*(S/X) = \{M/X, D/X\}$  implies  $\mathcal{E}^*(R_7/C_4) = \{R_5/C_4, C_7/C_4\}$ . Hence  $\mathcal{E}^*(S_7/D_4) = \{S_5/D_4, D_7/D_4\}$ . This, together with (5.46) (2), yields that  $\mathcal{E}^*(S_7/Y_2) = \{M_5/Y_2, D_7/Y_2\}$ . In particular,  $D_7 \triangleleft R_8$ . Similarly (5.40) (2) shows that  $\mathcal{E}^*(S_6/M_3) = \{S_5/M_3, M_6/M_3\}$ , and hence  $\mathcal{E}^*(S_6/Y_2) = \{D_5/Y_2, M_6/Y_2\}$ . Thus  $M_6 \triangleleft R_8$  as desired.

(5.57)  $Z(P) = U_1$ ,  $Z_2(P) = Y_2$ , and  $Z_3(P) = F_5$ . Furthermore,  $P/F_5 = M_6/F_5 \times D_7/F_5$  is elementary abelian.

PROOF. We have already proved the first two assertions in (5.52). By (5.55),  $Z_3(P)$  contains  $M_6 \cap D_7 = F_5$ . Now as was shown in the proof of (5.52),  $N_{R_8}(S_2) = R_5$ . This yields that  $Z_3(P) = F_5$  since  $Z(S_5/Y_2) = F_5/Y_2$ . The last assertion is an immediate consequence of (5.56).

(5.58) *Notation.* Set  $N_1$  to be the normal closure of  $N_L(U)D_7$  in  $N(D_7)$  and  $N_2$  to be the normal closure of  $N_L(Y)M_6$  in  $N(M_6)$ .

(5.59)  $N_1/D_7$  (resp.  $N_2/M_6$ ) is isomorphic to  $SL(2, q) \times SL(2, q)$  or  $SL(2, q^2)$ . In the former case  $t$  interchanges the components of  $N_1/D_7$  (resp.  $N_2/M_6$ ). Furthermore,  $C_{N(D_7)}(N_1/D_7) = O(N(D_7) \bmod D_7)$ ,  $C_{N(M_6)}(N_2/M_6) = O(N(M_6) \bmod M_6)$ , and  $P \in \text{Syl}_2(N_i)$  for  $i=1, 2$ .

PROOF. Let  $\overline{N(D_7)} = N(D_7)/D_7$  and  $K_1 = N_L(U)$ . Then by (5.55), (5.19), (5.3), and (3.13) we have

$$C_{\overline{N(D_7)}}(\bar{i}) = \overline{N(C_7)} = \overline{N_C(U)} \supset \bar{K}_1 \cong SL(2, q).$$

Note that  $D_\tau \cap C = D$ . Since  $N_C(U) \cap C(K_1/D) = DJ_3 C_C(L)$  by (3.23), we also have

$$C_{\overline{N(D_\tau)}}(\bar{i}) \cap C(\bar{K}_1) = \langle \bar{i} \rangle \bar{J}_3 \overline{O(C)}.$$

Thus  $\bar{K}_1$  is a standard subgroup of  $\overline{N(D_\tau)}$  isomorphic to  $SL(2, q)$ . Notice that  $D_\tau \triangleleft P$  by (5.56) and  $J_1$  is a subgroup of  $K_1$  by (3.11). Now,  $J_1$  acts regularly both on  $(P/S_\tau)^*$  and on  $(S_\tau/D_\tau)^*$  since  $P/S_\tau \cong S_\tau/D_\tau \cong S/D$  as  $J$ -modules. Thus  $P = [P, J_1] D_\tau \leq N_1$ . Moreover,  $[\bar{P}, t, t] = 1$  since  $t$  centralizes both  $\bar{S}_\tau$  and  $P/S_\tau$ . As  $\bar{P}$  is elementary abelian of order  $q^2$  and  $P = P^t$ , it follows from [12] and (2.10) that (5.59) holds for  $N_1$ . Indeed, as in (5.10)  $\bar{N}_1/O(\bar{N}_1)$  is isomorphic to neither  $PSU(3, q^2)$  nor  $PSL(3, q)$ , since  $[\bar{P}, t, t] = 1$ . Thus the only possibility to be considered is that  $q=4$  and  $\bar{N}_1/O(\bar{N}_1)$  is isomorphic to one of the following groups:

$$A_7, A_9, M_{12}, J_1, J_2, PSL(2, 5^2), PSL(3, 5), \text{ or } PSU(3, 5^2).$$

Since  $\overline{N(D_\tau)} \cap C(\bar{K}_1)$  has cyclic Sylow 2-subgroups, neither  $A_9$  nor  $J_2$  occurs. Each of the other groups has 2-rank at most 3. Therefore we have (5.59) for  $N_1$ .

As to  $N_2$ , we argue in  $\widetilde{N(M_6)} = N(M_6)/M_6$ . Set  $K_2 = N_L(Y)$ . By (5.55), (5.39), (5.23), (5.5), and (3.13) we have

$$C_{\widetilde{N(M_6)}}(\bar{i}) = \widetilde{N(L_6)} = \widetilde{N_C(Y)} \triangleright \tilde{K}_2 \cong SL(2, q).$$

Note that  $M_6 \cap C = M$ . We also have

$$C_{\widetilde{N(M_6)}}(\bar{i}) \cap C(\tilde{K}_2) = \langle \bar{i} \rangle \tilde{J}_4 \widetilde{O(C)}$$

by (3.23). Moreover,  $J_2$  is a subgroup of  $K_2$  by (3.11) and  $J_2$  acts regularly both on  $(P/S_6)^*$  and on  $(S_6/M_6)^*$  since  $P/S_6 \cong S_6/M_6 \cong S/M$  as  $J$ -modules. Thus  $P \leq N_2$ . Hence (5.59) holds for  $N_2$  as well.

(5.60) (1)  $N_1 = C_{N_1}(J_3) D_\tau$ ,  $C_{N_1}(J_3) \cap D_\tau = 1$ , and  $C_{N_1}(J_3) \cong \langle S_{\pm a} \rangle$ .

(2)  $N_2 = C_{N_2}(J_4) M_6$  and  $C_{N_2}(J_4) \cong \langle S_{\pm b} \rangle$ .

(3) If  $q \geq 8$ , then  $C_{N_2}(J_4) \cap M_6 = 1$ . If  $q=4$ , then  $C_{N_2}(J_4) \cap M_6 = Y_2$  and  $C_{N_2}(J_4)$  splits over  $Y_2$ .

PROOF. Let  $\overline{N(D_\tau)} = N(D_\tau)/D_\tau$ . Recall that  $N_L(U) = D(\langle S_{\pm a} \rangle \times J_3)$  and  $\langle S_{\pm a} \rangle = \langle S_a, w_a \rangle$ . We can easily verify that  $P$  is a semidirect product of  $D_\tau$  by  $C_P(J_3)$ . Now if  $\bar{N}_1 \cong SL(2, q^2)$ , then certainly  $\bar{N}_1 = \langle \bar{P}, \bar{w}_a \rangle$ . If  $\bar{N}_1 \cong SL(2, q) \times SL(2, q)$ , then  $C_{\bar{N}_1}(t) = \langle \bar{S}_{\pm a} \rangle$  and  $\bar{N}_1 = \langle \bar{P}, C_{\bar{N}_1}(t) \rangle = \langle \bar{P}, \bar{w}_a \rangle$  by [1, Lemma 2.5]. Thus  $J_3$  centralizes  $\bar{N}_1$  and (1) holds. Next recall that  $N_L(Y) = M(\langle S_{\pm b} \rangle \times J_4)$  and  $\langle S_{\pm b} \rangle = \langle S_b, w_b \rangle$ . By a similar argument as above we have that  $J_4$  centralizes  $N_2/M_6 = \langle P, w_b \rangle/M_6$ . When  $q \geq 8$ ,  $P$  is a semidirect product of  $M_6$

by  $C_P(J_4)$  and the required assertions hold. If  $q=4$ , then  $P=C_P(J_1)M_6$  and  $C_{M_6}(J_1)=Y_2$ . Moreover,  $C_P(J_1)$  is equal to  $[C_P(J_1), J_3]$  and so is contained in  $\langle C_L(J_1)^{C \cup J_1} \rangle$  (see the proof of (5.52)). Thus we know from (5.10) that  $C_P(J_1)$  is isomorphic to a Sylow 2-subgroup of  $SL(3, q) \times SL(3, q)$  or  $SL(3, q^2)$ . Now, a Sylow 2-subgroup of  $SL(3, q)$  is a product of the two maximal elementary abelian subgroups whose orders are  $q^2$ . Since  $Y_2$  is elementary abelian of order  $q^4$ ,  $Y_2$  has a complement in  $C_P(J_1)$ . So  $Y_2$  has a complement in  $C_{N_2}(J_1)$  by Gaschütz's theorem [15, p.121].

## 6. The case $N_2/M_6 \cong SL(2, q^2)$ .

We retain the notation of Section 5. In addition, we set  $G_1 = \langle N_1, N_2 \rangle$ ,  $r=w_a$ , and  $s=w_b$ . In this section we argue under Hypothesis (6.1) below and will show that  $G_1 = \langle L^G \rangle \cong G_2(q^2)$ .

(6.1) *Hypothesis.*  $N_2/M_6 \cong SL(2, q^2)$ .

(6.2) (1)  $P \cap P^s = M_6$ ,  $P \cap P^r = D_7$ .

(2)  $N(P) = N(M_6) \cap N(D_7)$ .

(3)  $N_1/D_7 \cong SL(2, q^2)$ .

PROOF. It follows from (6.1) that  $P \cap P^s = M_6$  and  $N_2 = \langle P, P^s \rangle$ . Let  $\overline{N(Y_2)} = N(Y_2)/Y_2$ . Then  $Z(\bar{P}) = \bar{F}_6$  by (5.57) and  $\bar{M}_6$  is a self-centralizing subgroup of  $\bar{N}_2$ . Hence  $Z(\bar{N}_2) = Z(\bar{P}) \cap Z(\bar{P})^s = \bar{W}_3$  by (5.44). Likewise,  $\bar{Z} = Z(\bar{N}_2 \text{ mod } \bar{W}_3)$  is contained in  $\bar{M}_6$ . As  $[\bar{N}_2, \bar{Z}] \leq \bar{W}_3$ , we have  $[\bar{N}_2', \bar{Z}] = 1$  by the three-subgroup lemma. So  $\bar{Z} = Z(\bar{N}_2)$  and  $Z(N_2/W_3) = 1$ , since  $\bar{N}_2 = \bar{N}_2' \bar{M}_6$ . Let  $\widetilde{N(W_3)} = N(W_3)/W_3$ . Then  $\tilde{F}_5 \leq Z(\tilde{P}) \leq C_{\tilde{N}_2}(\tilde{M}_6)$  and  $Z(\tilde{P}) \cap Z(\tilde{P})^s = Z(\tilde{N}_2) = 1$ . Since  $\tilde{M}_6$  is a direct product of  $\tilde{F}_5$  and  $\tilde{F}_5^s$  by (5.44), we have  $Z(\tilde{P}) = \tilde{F}_5$ . Recall that  $\tilde{P} = \tilde{M}_6 \tilde{D}_7$  where  $\tilde{M}_6$  and  $\tilde{D}_7$  are elementary abelian of order  $q^4$ . Then  $\tilde{M}_6$  has a complement in  $\tilde{P}$ , so that  $\tilde{M}_6$  has a complement  $\tilde{B}$  in  $\tilde{N}_2$  by Gaschütz's theorem. Now (2.7) shows that  $\tilde{M}_6$  is a natural module for  $\tilde{B} \cong SL(2, q^2)$  and in particular  $\mathcal{E}^*(\tilde{P}) = \{\tilde{M}_6, \tilde{D}_7\}$  and  $N_{N_2}(P)$  acts transitively on  $(M_6/F_6)^*$ . Since  $\bar{M}_6$  and  $\bar{D}_7$  are elementary abelian, it follows that  $\mathcal{E}^*(\bar{P}) = \{\bar{M}_6, \bar{D}_7\}$ . Thus  $M_6$  and  $D_7$  are characteristic subgroups of  $P$ , and (2) holds.

Suppose (3) is false. Then in view of (5.59) we may write  $N_1/D_7 = K/D_7 \times K^t/D_7$ , where  $K/D_7$  is isomorphic to  $SL(2, q)$ . As  $P$  is a Sylow 2-subgroup of  $N_2$ , there exists a complement  $H$  of  $P$  in  $N_{N_2}(P)$ . By (2)  $H$  acts on  $\{K/D_7, K^t/D_7\}$ ; the set of components of  $N_1/D_7$ , hence  $H$  in fact normalizes  $K$  and  $K^t$ . So  $H$  normalizes  $P \cap K$ . However, as shown above,  $H$  acts transitively on  $(M_6/F_6)^*$ , so also on  $(P/D_7)^*$ , a contradiction. Thus (3) holds.

(6.3)  $G_1 \cong G_2(q^2)$  and  $P \in \text{Syl}_2(G_1)$ .

PROOF. We appeal to (2.8). To do so, it suffices to show

- (1)  $P = D_7(P \cap P^s \cap P^{rs} \cap P^{sr} \cap P^{rst} \cap P^{rst})$ ,
- (2)  $P = M_6(P \cap P^r \cap P^{sr} \cap P^{rst} \cap P^{rst} \cap P^{rst})$ ,
- (3)  $P \cap P^{(rs)^3} = 1$ .

The other conditions required in (2.8) are easily verified (see (5.59) and (6.2)). Note that  $\langle r, s \rangle$  is dihedral of order 12.

Since  $P \cap C = S$ , the fact that  $S \cap S^{(rs)^3} = 1$  shows  $P \cap P^{(rs)^3} \cap C = 1$ . As  $P \cap P^{(rs)^3}$  is a  $\langle t \rangle$ -invariant 2-group, this proves (3).

By (6.2) (1) we have

$$P^r \cap P \cap P^s = D_7 \cap M_6 = F_5.$$

Conjugating by  $r$  and  $s$  respectively, we have

$$P \cap P^r \cap P^{sr} = F_5^r \quad \text{and} \quad P \cap P^s \cap P^{rs} = F_5^s.$$

Then by (5.44)

$$\begin{aligned} P^r \cap P \cap P^s \cap P^{rs} &= F_5 \cap F_5^s = W_3, \\ P \cap P^r \cap P^{sr} \cap P^{rst} &= W_3^r. \end{aligned}$$

Similarly we have

$$\begin{aligned} P^s \cap P \cap P^r \cap P^{sr} &= F_5 \cap F_5^r, \\ P \cap P^s \cap P^{rs} \cap P^{rst} &= F_5^s \cap F_5^r. \end{aligned}$$

Then

$$\begin{aligned} P^s \cap P \cap P^r \cap P^{sr} \cap P^{rst} &= F_5 \cap W_3^r, \\ P \cap P^s \cap P^{rs} \cap P^{rst} \cap P^{rst} &= F_5^s \cap W_3^{rs}. \end{aligned}$$

Furthermore

$$\begin{aligned} P^r \cap P \cap P^s \cap P^{rs} \cap P^{rst} &= F_5 \cap F_5^s \cap F_5^{rs} = W_3 \cap F_5^{rs}, \\ P \cap P^r \cap P^{sr} \cap P^{rst} \cap P^{rst} &= W_3^r \cap F_5^{rst}. \end{aligned}$$

Now by (5.54), we have

$$\begin{aligned} P^r \cap P \cap P^s \cap P^{rs} \cap P^{rst} \cap P^{rst} &= F_5 \cap F_5^s \cap W_3^{rs} \\ &= W_3 \cap W_3^{rs} = (W_3 \cap W_3^r)^s = U_1^s, \end{aligned}$$

and so

$$P \cap P^r \cap P^{sr} \cap P^{rst} \cap P^{rst} \cap P^{rst} = U_1^{sr}.$$

Finally

$$P^s \cap P \cap P^r \cap P^{sr} \cap P^{r sr} \cap P^{sr sr} = F_5 \cap W_3^r \cap F_5^{r sr},$$

and so

$$P \cap P^s \cap P^{rs} \cap P^{sr s} \cap P^{r sr s} \cap P^{sr sr s} = F_5^s \cap W_3^s \cap F_5^{r sr s}.$$

As  $M_6 \cap C = M$  and  $U_1 \cap C = U = S_{3a+2b}$ , we have that  $M_6 \cap U_1^{sr} \cap C = M \cap U^{sr} = M \cap S_b = 1$ . Thus  $M_6 \cap U_1^{sr} = 1$ . Since  $|P| = |M_6| |U_1|$ , this implies  $P = M_6 U_1^{sr}$ . Hence (2) holds.

Next set

$$V = P \cap P^s \cap P^{rs} \cap P^{sr s} \cap P^{r sr s} \cap P^{sr sr s}.$$

Then  $V = F_5^s \cap W_3^s \cap F_5^{r sr s}$  by the above, and so

$$V^{sr} = F_5^r \cap W_3 \cap F_5^{rs} = (W_3 \cap F_5^r) \cap (W_3 \cap F_5^r)^s.$$

By (5.54)  $D_7 = W_3 W_3^r > F_5 > W_3$ , so we have  $F_5 = W_3 (F_5 \cap W_3^r)$ . Moreover,  $W_3 \cap F_5 \cap W_3^r = W_3 \cap W_3^r = U_1$ . Now  $|W_3| = q^6$ ,  $|F_5| = q^8$ , and  $|U_1| = q^2$ , so that  $|F_5^r \cap W_3| = |F_5 \cap W_3^r| = q^4$ . Since  $(W_3 \cap F_5^r)(W_3 \cap F_5^r)^s$  is a subgroup of  $W_3$ , it follows that the order of  $V$  is at least  $q^2$ . Thus, as for the proof of (1), it is sufficient to show that  $D_7 \cap V = 1$ . Note that  $D_7 \cap C = D$ ,  $F_5 \cap C = F$ , and  $W_3 \cap C = W$ . Then

$$\begin{aligned} V^{sr} \cap C &= (W \cap F^r) \cap (W \cap F^r)^s \\ &= S_{2a+b} S_{3a+2b} \cap (S_{2a+b} S_{3a+2b})^s \\ &= S_{2a+b}. \end{aligned}$$

So  $V \cap C = (S_{2a+b})^{rs} = S_a$  and  $D_7 \cap V \cap C = D \cap S_a = 1$ . Hence  $D_7 \cap V = 1$  as required. Now (6.3) follows from (2.8) and the structure of  $P$  given in (5.57).

$$(6.4) \quad G_1 = \langle L^G \rangle.$$

PROOF. Since  $G_1$  contains  $L = \langle N_L(U)', N_L(Y)' \rangle$ , we have  $C(G_1) \triangleleft C(L) \cap N(G_1)$ . As  $\langle t \rangle$  is a Sylow 2-subgroup of  $C(L) \cap N(G_1)$  by (4.4) and  $t \notin C(G_1)$ ,  $C(G_1)$  has odd order. Let  $B$  be a Sylow 2-subgroup of  $N(G_1)$  containing  $\langle t \rangle P$ . Then since  $P \in \text{Syl}_2(G_1)$  and  $C_B(G_1) = 1$ , it follows from (3.18) and (3.21) that  $B/P$  is cyclic and  $J_r(B) = P$ . In particular,  $N(B) \leq N(P) \leq N(G_1)$  by (5.58) and (6.2) (2), and hence  $B$  is a Sylow 2-subgroup of  $G$ . Moreover, as was shown in (4.2)  $m(C) = 3n+1$ , while in  $G_1 \cong G_2(q^2)$  the centralizer of any involution has 2-rank  $6n$ , where  $q = 2^n$ . Thus  $t^G \cap P = \emptyset$ . Now we can apply (2.3) to conclude that  $P$  is a Sylow 2-subgroup of  $O^2(G)$ . Set  $H = O^2(G)$  and  $\bar{H} = H/O(H)$ . Then, since  $G_2(2^n)$  is characterized by its Sylow 2-subgroups ([16], [21]), it follows that  $\bar{G}_1 = O^2(\bar{H})$ . Hence  $G_1 O(G) \triangleleft G$ . This, together with (2.10), completes the proof of (6.4).

**7. The case  $N_2/M_6 \cong SL(2, q) \times SL(2, q)$ .**

In this section we assume the following hypothesis.

(7.1) *Hypothesis.*  $N_2/M_6 \cong SL(2, q) \times SL(2, q)$ .

For brevity we set  $r=w_a$  and  $s=w_b$ . We also use, unless otherwise specified, the following three bar conventions:

$$\widehat{N(W_3)} = N(W_3)/W_3, \quad \widetilde{N(Y_2)} = N(Y_2)/Y_2, \quad \text{and} \quad \overline{N(U_1)} = N(U_1)/U_1.$$

(7.2) *Notation.* Let  $N_2/M_6 = K_2/M_6 \times K_2^t/M_6$  with  $K_2/M_6 \cong SL(2, q)$ . Let  $A^*$  be a complement of  $Y_2$  in  $C_{N_2}(J_4)$  if  $q=4$  and  $A^* = C_{N_2}(J_4)$  if  $q \geq 8$ . Thus  $A^*$  is a complement of  $M_6$  in  $N_2$  by (5.60). Moreover, we set  $A_2 = K_2 \cap A^*$ ,  $P_2 = P \cap K_2$ , and  $T_2 = P \cap A_2$ .

(7.3) (1)  $P \cap P^s = M_6$ ,  $K_2 = \langle P_2, P_2^s \rangle$ , and  $J$  normalizes  $P_2$ .

(2)  $Z(N_2) = 1$ ,  $Z(N_2/Y_2) = W_3/Y_2$ .

(3)  $Z(\hat{N}_2) = Z(\hat{K}_2) \cap Z(\hat{K}_2^t) = 1$ ,  $Z(\hat{K}_2) = Z(\hat{P}_2) \cap Z(\hat{P}_2^s)$ ,  $Z(\hat{P}) = Z(\hat{P}_2) \cap Z(\hat{P}_2^t) = \hat{F}_5$ .

(4)  $\hat{N}_2 = \hat{K}_2^t \times \hat{K}_2'^t$ ,  $\hat{K}_2' = \hat{A}_2 Z(\hat{K}_2^t)$  is perfect,  $Z(\hat{K}_2^t) = O_2(\hat{K}_2^t)$ ,  $\hat{M}_6 = Z(\hat{K}_2^t) \times Z(\hat{K}_2)$ ,  $Z(\hat{P}_2) = \hat{F}_5 Z(\hat{K}_2)$ .

(5)  $O_2(\hat{K}_2^t)$  is a natural module for  $\hat{K}_2'/O_2(\hat{K}_2^t) \cong SL(2, q)$ .

(6)  $\mathcal{C}^*(\hat{P}) = \{\hat{M}_6, \hat{D}_7, Z(\hat{K}_2)(\hat{D}_7 \cap \hat{P}_2), Z(\hat{K}_2^t)(\hat{D}_7 \cap \hat{P}_2^t)\}$ .

(7)  $M_6$  and  $D_7$  are characteristic subgroups of  $P$ . In particular,  $N(P) = N(M_6) \cap N(D_7)$ .

(8)  $N_1/D_7 \cong SL(2, q) \times SL(2, q)$ .

PROOF. Since  $J$  is a subgroup of  $N(M_6)$  of odd order,  $J$  normalizes  $K_2$  and  $P_2$ . Other parts of (1) follow immediately from (7.2). We have  $N(M_6) \leq N(Y_2)$  by (5.55). Since the only  $\langle t \rangle$ -invariant normal subgroups of  $N_2/M_6$  are 1 and  $N_2/M_6$ , and since  $Z(\tilde{P}) = \tilde{F}_5$  by (5.57),  $\tilde{M}_6$  is a self-centralizing subgroup of  $\tilde{N}_2$ . Thus arguing as in (6.2), we have  $Z(\tilde{N}_2) = \tilde{W}_3$ ,  $Z(\hat{N}_2) = 1$ , and  $Z(\hat{P}) = \hat{F}_5$ . Also,  $Z(N_2) = U_1 \cap U_1^s = 1$  by (5.13) and (5.57). Then (2) and (3) follow. As  $P_2 P_2^{st}$  is a 2-group,  $Z(\hat{P}_2 \hat{P}_2^{st}) = Z(\hat{P}_2) \cap Z(\hat{P}_2^{st}) \neq 1$ . Since  $\hat{M}_6$  is a direct product of  $\hat{F}_5$  and  $\hat{F}_5^s$  by (5.44), this and (3) imply that  $Z(\hat{K}_2) \neq 1$ . As  $K_2^t/M_6$  acts nontrivially on  $Z(\hat{K}_2)$ , Lemma (4B) of [5] implies that  $Z(\hat{K}_2)$  has order  $q^2$  and  $\hat{M}_6$  is a direct product of  $Z(\hat{K}_2^t)$  and  $Z(\hat{K}_2)$ . Now by (2.7),  $Z(\hat{K}_2)$  is a natural module for  $K_2^t/M_6 \cong SL(2, q)$ . As  $\hat{K}_2$  is a direct product of  $\hat{A}_2 Z(\hat{K}_2^t)$  and  $Z(\hat{K}_2)$ , it follows that  $\hat{K}_2' = \hat{A}_2 Z(\hat{K}_2^t)$  is perfect. Thus (4) and (5) hold.



(2.7) also shows that  $\mathcal{E}^*(\hat{P})$  has exactly four elements, each of which has order  $q^4$ . Certainly  $\hat{D}_7$  and  $\hat{M}_6$  are elements of  $\mathcal{E}^*(\hat{P})$ . (3) and (4) show that  $Z(\hat{K}_2) \cap \hat{P}_2 \cap \hat{D}_7 = Z(\hat{K}_2) \cap \hat{F}_5 = Z(\hat{K}_2) \cap Z(\hat{P}_2)$  has order  $q$ , hence  $Z(\hat{K}_2)(\hat{P}_2 \cap \hat{D}_7)$  is also elementary abelian of order  $q^4$ . Furthermore,

$$Z(\hat{K}_2)(\hat{P}_2 \cap \hat{D}_7) \cap \hat{D}_7 = \hat{P}_2 \cap \hat{D}_7$$

and

$$Z(\hat{K}_2)(\hat{P}_2 \cap \hat{D}_7) \cap \hat{M}_6 = Z(\hat{K}_2) \hat{F}_5 = Z(\hat{P}_2).$$

Thus (6) holds. In particular,  $\tilde{D}_7$  is the unique elementary abelian subgroup of  $\tilde{P}$  of order  $q^6$  whose intersection with  $\tilde{M}_6$  is equal to  $\tilde{F}_5$ . Now  $Z_2(P) = Y_2$ ,  $Z_3(P) = F_5$ , and  $Y_2 \leq Z(M_6)$ . Hence  $C_{N_2}(Y_2) = M_6$  as in the first paragraph, and so  $M_6 = C_P(Y_2)$  is characteristic in  $P$ . This proves (7).

Finally, suppose (8) is false. Then  $N_1/D_7$  is isomorphic to  $SL(2, q^2)$  by (5.59). Let  $H$  be a complement of  $P$  in  $N_{N_1}(P)$ . Then  $H$  is cyclic of order  $q^2 - 1$  and acts irreducibly on  $P/D_7$ . On the other hand, (7) implies that  $H$  acts on  $\{K_2/M_6, K_2^t/M_6\}$ ; the set of components of  $N_2/M_6$ , so that  $H$  in fact normalizes  $K_2$ . Hence  $P_2$  is  $H$ -invariant. Also, (2) implies that  $W_3$  is  $H$ -invariant. But  $F_5 < Z(P_2 \text{ mod } W_3) < M_6$ , contrary to the irreducible action of  $H$  on  $P/D_7 \cong M_6/F_5$ . Thus (8) holds.

(7.4) (1)  $Z(K_2 \text{ mod } Y_2) = Z(K_2 \text{ mod } W_3)$  and  $Z(P_2 \text{ mod } Y_2) = Z(P_2 \text{ mod } W_3)$ .

(2)  $\tilde{K}_2'$  is perfect and  $[\tilde{K}_2', \tilde{K}_2''] = 1$ .

PROOF. Set  $H = Z(K_2 \text{ mod } W_3)$ . Then  $[\tilde{K}_2, \tilde{H}]$  is contained in  $\tilde{W}_3 = Z(\tilde{N}_2)$ , so that  $[\tilde{K}_2', \tilde{H}] = 1$  by the three-subgroup lemma. Now,  $\tilde{K}_2 = \tilde{K}_2' Z(\tilde{K}_2)$  and  $\tilde{K}_2$  is a central product of  $\tilde{K}_2'$  and  $\tilde{H}$ . Thus  $Z(\tilde{K}_2) = \tilde{H}$ , and (1) holds. Moreover,  $\tilde{K}_2' = (\tilde{K}_2' * \tilde{H})' = \tilde{K}_2''$  is perfect. Now (7.3) (4) shows that  $[\tilde{K}_2', \tilde{K}_2''] \leq \tilde{W}_3$ . Hence using again the three-subgroup lemma we obtain (2).

(7.5) Notation. Let  $N_1/D_7 = K_1/D_7 \times K_1^t/D_7$  with  $K_1/D_7 \cong SL(2, q)$ . Set  $A_1 = C_{K_1}(J_3)$ ,  $P_1 = P \cap K_1$ , and  $T_1 = P \cap A_1$ . Thus  $C_{N_1}(J_3) = A_1 \times A_1^t$  is a complement of  $D_7$  in  $N_1$  by (5.60),  $T_1 \cap T_1^t = 1$ , and  $A_1 = \langle T_1, T_1^t \rangle \cong SL(2, q)$ . Furthermore,  $A_1 A_1^t \cap C = \langle S_{\pm a} \rangle$ .

(7.6) (1)  $P \cap P^r = D_7$ ,  $K_1 = \langle P_1, P_1^r \rangle$ , and  $J$  normalizes  $P_1$ .

(2)  $Z(N_1) = Z_2(N_1) = U_1$ ,  $C_{\bar{N}_1}(\bar{D}_7) = \bar{D}_7$ .

(3)  $Z(\bar{P}) = Z(\bar{P}_1) \cap Z(\bar{P}_1^t) = \bar{Y}_2$ ,  $Z(\bar{K}_1) = Z(\bar{P}_1) \cap Z(\bar{P}_1^t)$ .

(4)  $Z(\bar{P}_1) \cap \bar{D}_1 = \bar{Y}_1$ ,  $Z(\bar{K}_1) \cap Z(\bar{K}_1^t) = Z(\bar{K}_1) \cap \bar{D}_1 = 1$ .

PROOF. Arguing as in (7.3) we obtain (1), (2), and (3). Notice that  $C_{\bar{D}_7}(t) = \bar{D}_1$  (see (5.19)). Then (3) shows that  $Z(\bar{P}_1) \cap C(t) = Z(\bar{P}_1) \cap \bar{D}_1 = \bar{P}_1$ . As  $N_1 = K_1 K_1^t$ , (2) yields that  $Z(\bar{K}_1) \cap Z(\bar{K}_1^t) = Z(\bar{N}_1) = 1$ , and (4) holds.

Now let  $H$  be a complement of  $P$  in  $N_{N_1}(P)$ . Then  $H$  has odd order and so normalizes  $K_2$  by (7.3) (7), hence  $P_2$  is  $H$ -invariant. Thus  $Z(\bar{P}_2)\bar{D}_7$  is also  $H$ -invariant and has order  $q|\bar{D}_7|$  by (7.4). Since the only  $H$ -invariant proper subgroups of  $P/D_7$  are  $P_1/D_7$  and  $P_1^t/D_7$ , changing notation if necessary, we may assume that

$$(7.7) \quad \bar{P}_1^t = Z(\bar{P}_2)\bar{D}_7.$$

Next we shall prove

$$(7.8) \quad Z(\bar{K}_1) \neq 1.$$

PROOF. First of all we observe some properties of the action of  $J$  on  $P$ , most of which have been already mentioned in (5.45) or (5.52).  $\bar{W}_3$  is a direct product of  $C_P(J_2)$  and  $Y_2$  and  $X_4$  is a direct product of  $C_P(J_6)$  and  $Y_2$ . When  $q \geq 8$ , we have  $Y_2 = C_P(J_5) \times U_1$ ,  $D_7 = C_P(J_4)F_5$ , and  $C_{F_5}(J_4) = 1$ . When  $q = 4$ , we have  $J_1 = J_4 = J_5$ ,  $D_7 = C_P(J_5)F_5$ , and  $C_{F_5}(J_5) = Y_2$ . For  $i = 2, 6$ ,  $J$  acts transitively both on  $(C_P(J_i)/C_S(J_i))^{\#}$  and on  $C_S(J_i)^{\#}$ , so that each  $J$ -invariant subgroup of  $C_P(J_i)$  has order 1,  $q$ , or  $q^2$ . Also recall that  $\bar{D}_7 = \bar{W}_3 \times \bar{W}_3^r$  is elementary abelian by (5.54) and that  $P_1$  and  $P_2$  are normalized by  $J$ .

Henceforth we assume by way of contradiction that  $Z(\bar{K}_1) = 1$ . Let  $\bar{Z} = Z(\bar{P}_1)$ . Then  $\bar{Z} \cap \bar{Z}^r = 1$  by (7.6) (3). We will show that  $\bar{F}_5 \geq \bar{Z}$  and  $\bar{D}_7 = \bar{Z} \times \bar{Z}^r$ . For this we distinguish two cases:  $q = 4$  and  $q \geq 8$ . Suppose first  $q = 4$ . Then  $J_4 = J_5$ ,  $\bar{D}_7 = C_{\bar{P}}(J_5) \times C_{\bar{P}}(J_2) \times C_{\bar{P}}(J_6)$ ,  $|C_{\bar{P}}(J_5)| = q^4$ , and  $|C_{\bar{P}}(J_2)| = |C_{\bar{P}}(J_6)| = q^2$ . As  $J = J^r$  and  $(S_5)^r = S_{3a+b}$ , certainly we have  $J_5^r = J_4 = J_5$ , and hence  $C_{\bar{P}}(J_5) = C_{\bar{D}_7}(J_5)$  is normalized by  $r$ . Now the order consideration shows that  $C_{\bar{P}}(J_5)$  is a direct product of  $\bar{Y}_2$  and  $\bar{Y}_2^r$ . Thus  $\bar{Z} \cap \bar{Z}^r = 1$  implies that  $C_{\bar{Z}}(J_5) = \bar{Y}_2$ . Hence  $\bar{Z} = \bar{Y}_2 \times C_{\bar{Z}}(J_2) \times C_{\bar{Z}}(J_6) \leq \bar{F}_5$ . If  $|C_{\bar{Z}}(J_2)| = q^2$ , then  $C_{\bar{Z}}(J_2)$  is equal to  $C_{\bar{P}}(J_2)$  and so contains  $C_{\bar{D}_1}(J_2)$ . But since  $\bar{W}_1$  is a direct product of  $C_{\bar{D}_1}(J_2)$  and  $\bar{Y}_1$ , this contradicts (7.6) (4). Thus we have  $|C_{\bar{Z}}(J_2)| \neq q^2$ . Likewise  $|C_{\bar{Z}}(J_6)| \neq q^2$ . If  $C_{\bar{Z}}(J_2) = 1$ , then  $\bar{Z} = \bar{Y}_2 \times C_{\bar{Z}}(J_6) \leq \bar{Y}_2 \times C_{\bar{D}_7}(J_6)$ . Since  $J_2^r = J_6$ , it follows that  $\bar{Z}^{rt} \leq \bar{Y}_2^r \times C_{\bar{D}_7}(J_2)$  and hence  $\bar{Z} \cap \bar{Z}^{rt} = 1$ . However,  $P_1 P_1^{rt}$  is a 2-group so that  $Z(\bar{P}_1 \bar{P}_1^{rt}) = \bar{Z} \cap \bar{Z}^{rt} \neq 1$ , a contradiction. Thus  $C_{\bar{Z}}(J_2) \neq 1$ . Similarly  $C_{\bar{Z}}(J_6) \neq 1$ . Therefore we have  $|C_{\bar{Z}}(J_2)| = |C_{\bar{Z}}(J_6)| = q$  (see the first paragraph), and  $\bar{Z}$  is of order  $q^4$ . Hence  $\bar{D}_7 = \bar{Z} \times \bar{Z}^r$  as desired. Suppose next  $q \geq 8$ . Then  $\bar{D}_7 = C_{\bar{P}}(J_5) \times C_{\bar{P}}(J_2) \times C_{\bar{P}}(J_6) \times C_{\bar{P}}(J_4)$ ,  $\bar{Y}_2 = C_{\bar{P}}(J_5) = C_{\bar{D}_7}(J_5)$ , and  $\bar{Y}_2^r = C_{\bar{P}}(J_4)$ . So  $C_{\bar{Z}}(J_4) \leq \bar{Z} \cap \bar{Z}^r = 1$ , and thus  $\bar{Z} = [\bar{Z}, J_4] \leq \bar{F}_5$ . Now arguing as in the case  $q = 4$ , we obtain  $|C_{\bar{Z}}(J_2)| = |C_{\bar{Z}}(J_6)| = q$  and  $\bar{D}_7 = \bar{Z} \times \bar{Z}^r$ .

Next we prove that  $C_{\bar{P}}(T_1) = \tilde{M}_6$ . Notice that  $\bar{Z} = C_{\bar{D}_7}(T_1)$ . Since  $Z_2(P) = Y_2$  and  $Z_3(P) = F_5$ , certainly  $(\bar{D}_7/\bar{Y}_2) \cap C(T_1)$  contains  $\bar{F}_5/\bar{Y}_2$ . By the above  $\bar{D}_7$  is a direct product of  $\bar{Z}$  and  $\bar{Z}^r$  and  $\bar{Y}_2 \leq \bar{Z}^t \leq \bar{F}_5$ , so that  $\bar{D}_7/\bar{Y}_2$  is a direct product of  $\bar{Z}^t/\bar{Y}_2$  and  $\bar{Z}^{rt}\bar{Y}_2/\bar{Y}_2$ . Moreover,  $\bar{Z}^{rt} \cap C(T_1)$  is contained in  $\bar{D}_7 \cap C(T_1) = \bar{Z}$ . Therefore  $(\bar{D}_7/\bar{Y}_2) \cap C(T_1)$  is equal to  $\bar{F}_5/\bar{Y}_2$ , and hence  $C_{\bar{D}_7}(T_1) = \bar{F}_5$ . Now  $T_1 T_1^t = C_P(J_3)$  is abelian and  $M_6 = C_P(J_3)F_5$ . Thus  $C_{\bar{P}}(T_1) = \bar{F}_5 \bar{T}_1 \bar{T}_1^t = \tilde{M}_6$  as desired.

Since  $J_3$  acts fixed-point-freely on  $D_7$ , (7.7) implies that  $\bar{T}_1^t = \bar{P}_1^t \cap C(J_3) \leq Z(\bar{P}_2)$ . Thus  $C_{\bar{P}}(T_1)$  contains  $\bar{P}_2^t$ . However,  $P_2^t > M_6$  and we have arrived at the final contradiction.

(7.9) (1)  $\bar{N}_1 = \bar{K}_1' \times \bar{K}_1'^t$ ,  $\bar{K}_1' = \bar{A}_1 Z(\bar{K}_1^t)$  is perfect,  $Z(\bar{K}_1^t) = O_2(\bar{K}_1^t)$ , and  $\bar{D}_7 = Z(\bar{K}_1^t) \times Z(\bar{K}_1)$ .

(2)  $N_1 = K_1' * K_1'^t = K_1'' * K_1''^t$ ,  $K_1''$  is perfect, and  $K_1' \cap K_1'^t = U_1$ .

(3)  $Z(\bar{P}_1) = \bar{Y}_2 Z(\bar{K}_1)$  has order  $q^5$ .

(4)  $Z(\bar{P}_1) \cap \bar{F}_5 = \bar{Y}_2 \times C_{Z(\bar{P}_1)}(J_2) \times C_{Z(\bar{P}_1)}(J_6)$  and  $|C_{Z(\bar{P}_1)}(J_2)| = |C_{Z(\bar{P}_1)}(J_6)| = q$ .

PROOF. Let  $\bar{V} = Z(\bar{K}_1)$ . As was shown in (7.6) (4),  $\bar{V} \cap \bar{V}^t = 1$ . Now  $A_1 A_1^t J$  normalizes  $\bar{V} \times \bar{V}^t$ . Moreover,  $A_1 A_1^t$  contains  $\langle S_{\pm a} \rangle$ ,  $N_L(U) = \langle S_{\pm a} \rangle JD$ , and  $C_{\bar{D}_7}(t) = \bar{D}_1$  by (5.19). Thus it follows from (7.8) that  $\bar{V} \bar{V}^t \cap \bar{D}_1$  is a nontrivial  $N_L(U)$ -invariant subgroup. Furthermore,  $N_L(U)$  acts irreducibly on  $\bar{D}_1 \cong D/U$  by (3.16). Hence  $\bar{V} \bar{V}^t$  contains  $\bar{D}_1$ , so that  $\bar{D}_7 = \bar{V} \times \bar{V}^t$ . Now  $\bar{N}_1 = \bar{A}_1 \bar{A}_1^t \bar{D}_7$  is a direct product of  $\bar{A}_1 \bar{V}^t$  and  $\bar{A}_1^t \bar{V}$ , and so  $\bar{A}_1 \bar{V}^t$  is isomorphic to  $C_{\bar{N}_1}(t) = \overline{\langle S_{\pm a} \rangle \bar{D}_1} \cong C_L(U)/U$ . Since  $C_L(U)$  is perfect, so also is  $\bar{A}_1 \bar{V}^t$ . Since  $\bar{K}_1 = \bar{A}_1 \bar{D}_7$ , we have  $\bar{K}_1' = \bar{A}_1 \bar{V}^t$  and (1) holds. Since  $D_1' = U_1$  by (5.55), arguing as in (7.4) we obtain (2).

Now  $\bar{D}_7 = Z(\bar{P}_1) Z(\bar{P}_1^t)$  and (7.6) (3) shows that  $Z(\bar{P}_1)$  has order  $q^5$ . Thus the order of  $\bar{Y}_2 \cap \bar{V}$  is at least  $q$ . Since  $J$  acts transitively both on  $(Y_2/Y_1)^*$  and on  $\bar{Y}_1^*$  and since  $\bar{Y}_2$  is not contained in  $\bar{V}$ , we must have  $|\bar{Y}_2 \cap \bar{V}| = q$  and  $Z(\bar{P}_1) = \bar{Y}_2 \bar{V}$ , proving (3). Finally, arguing as in (7.8) we get (4).

(7.10) Notation. Let  $B_1 = O_2(K_1')$  and  $B_2 = O_2(K_2')$ . Thus  $B_1 \geq U_1$  and  $B_2 \geq Y_2$  by (5.55).

(7.11) (1)  $\tilde{N}_2 = \tilde{K}_2' \times \tilde{K}_2'^t$ ,  $\tilde{K}_2' = \tilde{A}_2 \tilde{B}_2$ , and  $\tilde{M}_6 = \tilde{B}_2 \times \tilde{B}_2^t$ .

(2)  $T_1 B_1 Y_2 = T_2 B_2$ .

PROOF. We first show that  $P_2 = B_1 M_6$ . As  $N_{N_2}(P)/P$  has odd order,  $N_{N_2}(P)$  normalizes  $K_1$  and  $B_1 M_6$ . Since the only  $N_{N_2}(P)$ -invariant proper subgroups of  $P/M_6$  are  $P_2/M_6$  and  $P_2^t/M_6$ ,  $B_1 M_6$  is equal to  $P_2$  or  $P_2^t$ . Now it follows from (7.4) and (7.7) that  $\tilde{P} = Z(\tilde{P}_2^t) \tilde{P}_1^t$ . Moreover,  $\tilde{B}_1 = Z(\tilde{K}_1)^t$  by (7.9). Thus, if  $B_1 M_6 = P_2^t$  then  $\tilde{B}_1 \leq Z(\tilde{P}) = \tilde{F}_5$ , a contradiction. Hence  $P_2 = B_1 M_6$  as asserted.

Set  $Q = P \cap K_2'$  and  $\tilde{E} = \cup^1(\tilde{Q})$ . (7.4) shows that  $\tilde{N}_2 = \tilde{K}_2' * \tilde{K}_2'^t * \tilde{W}_3$ . In particular

$\tilde{P} = \tilde{Q} * \tilde{Q}^t * \tilde{W}_3$ , so  $\mathfrak{U}^1(\tilde{P}) = \tilde{E} \tilde{E}^t$ . Recall that  $S_2 = SY_2$  and  $\mathfrak{U}^1(\tilde{S}_2) = \tilde{S}_2' = \tilde{F}_2 \geq \tilde{W}_2$ . As was shown above  $P_2 = B_1 M_6$ , so that  $\tilde{E} \leq \mathfrak{U}^1(\tilde{P}_2) \leq \tilde{B}_1$  since  $\tilde{M}_6$  is elementary abelian. Now from (7.9) it follows that  $\tilde{D}_7 = \tilde{B}_1 \times \tilde{B}_1^t$  and  $\tilde{W}_3 = (\tilde{W}_3 \cap \tilde{B}_1) \times (\tilde{W}_3 \cap \tilde{B}_1^t)$ . Notice that  $\tilde{W}_3 = C_{\tilde{P}}(J_2)$ . Hence  $\tilde{W}_3 \cap \mathfrak{U}^1(\tilde{P})$  is a direct product of  $\tilde{W}_3 \cap \tilde{E}$  and  $\tilde{W}_3 \cap \tilde{E}^t$  and contains  $\tilde{W}_2 = C_{\tilde{W}_3}(t)$ . Thus we conclude that  $\tilde{W}_3 \cap \tilde{E} = \tilde{W}_3 \cap \tilde{B}_1$ . Since  $\tilde{Q} \tilde{W}_3 / \tilde{E}$  is elementary abelian, there exists a complement of  $\tilde{W}_3 \tilde{E} / \tilde{E}$  in  $\tilde{Q} \tilde{W}_3 / \tilde{E}$ . Hence  $\tilde{W}_3 / (\tilde{W}_3 \cap \tilde{E})$  has a complement in  $\tilde{Q} \tilde{W}_3 / (\tilde{W}_3 \cap \tilde{E})$ . Certainly  $\tilde{Q} \tilde{W}_3$  is a Sylow 2-subgroup of  $\tilde{K}_2' \tilde{W}_3$ . Thus by Gaschütz's theorem [15, p. 121],  $\tilde{W}_3 / (\tilde{W}_3 \cap \tilde{E})$  has a complement in  $\tilde{K}_2' \tilde{W}_3 / (\tilde{W}_3 \cap \tilde{E})$ . Since  $\tilde{N}_2 = \tilde{K}_2' \times \tilde{K}_2'^t$  and  $\tilde{K}_2'$  is perfect, this yields that  $\tilde{K}_2' \cap \tilde{K}_2'^t = 1$ . As  $A_2$  is perfect, (1) then follows from (7.3).

In particular,  $\tilde{P} = \tilde{Q} \times \tilde{Q}^t$  and  $\tilde{Q}$  is isomorphic to  $C_{\tilde{P}}(t)$ . Now  $C_{\tilde{P}}(t) = \widetilde{N_P(\langle t \rangle Y_2)} = \tilde{S}_2$  (see (5.17) and (5.5)). Thus  $\tilde{Q}'$  is equal to  $\tilde{E}$  and isomorphic to  $\tilde{F}_2$ . By (7.9) we have  $\tilde{D}_7 = \tilde{B}_1 \times \tilde{B}_1^t$ , so that  $\tilde{P} = \tilde{Q} \times \tilde{Q}^t = (\tilde{T}_1 \tilde{B}_1) \times (\tilde{T}_1 \tilde{B}_1)^t$ . Then  $(\tilde{T}_1 \tilde{B}_1)'$  is equal to  $\tilde{Q}'$  or  $\tilde{Q}'^t$  by the Krull-Schmidt theorem. Hence  $\tilde{E} \leq \tilde{B}_1$  implies  $\tilde{Q}' = (\tilde{T}_1 \tilde{B}_1)'$ . Moreover,  $\tilde{F}_5 = \tilde{Q}' \times \tilde{Q}'^t$ ,  $\tilde{Q}' = \tilde{Q} \cap \tilde{F}_5$ , and  $(\tilde{T}_1 \tilde{B}_1)' = \tilde{T}_1 \tilde{B}_1 \cap \tilde{F}_5$ . In the first paragraph we have shown that  $P_2 = B_1 M_6$ , which implies that  $\tilde{T}_2 = C_{\tilde{P}_2}(J_4)$  is contained in  $\tilde{B}_1$  since  $J_4$  acts fixed-point-freely on  $\tilde{M}_6$ . Likewise (1) and (7.3) (4) show that  $\tilde{B}_2 \tilde{F}_5 = Z(\tilde{P}_2')$ , and hence (7.7) yields that  $P_1 = B_2 D_7$ . Thus  $T_1 = C_{P_1}(J_3)$  is contained in  $B_2$ , since  $J_3$  acts fixed-point-freely on  $D_7$ . As  $Q = T_2 B_2$ , comparing orders, we now conclude that  $\tilde{T}_1 \tilde{B}_1 = \tilde{T}_2 \tilde{B}_2$ . Thus (2) holds.

(7.12) *Notation.*  $N_3 = C_{N_2}(J_4) Y_2$ ,  $K_3 = K_2 \cap N_3$ , and  $P_3 = P \cap K_3$ .

(7.13) (1)  $P_3 \cap P_3^s = Y_2$ ,  $K_3 = \langle P_3, P_3^s \rangle$ , and  $P_3 P_3^t = P \cap N_3 \in \text{Syl}_2(N_3)$ .

(2)  $C_{N_3}(Y_2) = Y_2$ ,  $Z(N_3) = 1$ .

(3)  $N_3 = K_3' \times K_3'^t$ ,  $K_3' = A_2 Z(K_3')$  is perfect,  $Z(K_3') = O_2(K_3')$ ,  $Y_2 = Z(K_3') \times Z(K_3)$ ,  $Z(P_3) = Z(K_3) U_1$ , and  $U_1 = (Z(K_3) \cap U_1) \times (Z(K_3') \cap U_1)$ .

(4)  $O_2(K_3')$  is a natural module for  $K_3' / O_2(K_3') \cong SL(2, q)$ .

(5) The only  $N_{N_2}(P)$ -invariant proper subgroups of  $\bar{Y}_2$  are  $\overline{Z(P_3)}$  and  $\overline{Z(P_3)}^t$ .

(6) The only  $N_{N_2}(P)$ -invariant proper subgroups of  $Z(P_3) / (Z(K_3) \cap U_1)$  are  $Z(K_3) / (Z(K_3) \cap U_1)$  and  $U_1 / (Z(K_3) \cap U_1)$ .

PROOF.  $N_3$  is a semidirect product of  $Y_2$  by  $A^*$  containing  $\langle S_{\pm b} \rangle$ ,  $K_3 = A_2 Y_2$ , and  $N_3 / Y_2$  is a direct product of  $K_3 / Y_2$  and  $K_3' / Y_2$ . Furthermore,  $P_3 = T_2 Y_2$  is a Sylow 2-subgroup of  $K_3$ . As  $\tilde{K}_3$  is isomorphic to  $SL(2, q)$ , (1) is easily verified. Now recall that both  $Y$  and  $Y_2 / Y$  are natural modules for  $\langle S_{\pm b} \rangle \cong SL(2, q)$  (see (5.5)). In particular,  $C_{Y_2}(\langle S_{\pm b} \rangle) = 1$ . Since  $\tilde{N}_3$  has no proper  $\langle t \rangle$ -invariant normal subgroups, we have  $C_{N_3}(Y_2) = Y_2$ . Hence  $Z(N_3) = 1$ , proving (2).

Recall that  $Z(P)=U_1$  and  $Y_2=U_1 \times U_1^s$ . Then arguing as in (7.3) we obtain (3) and (4).

Set  $B=O_2(K'_3)$ . Then  $C_B(T_2)=B \cap Z(P_3)=B \cap U_1$ . From (4) we find out that  $N_{A_2}(T_2)$  acts transitively both on  $(B/C_B(T_2))^{\#}$  and on  $C_B(T_2)^{\#}$ . As  $K_2=A_2M_6$  and  $P_2=T_2M_6$ , we have  $N_{K_2}(P_2)=N_{A_2}(T_2)M_6$ . Also,  $N_{N_2}(P)=N_{K_2}(P_2)N_{K_2}(P_2)^t$  and by (5.45)  $Y_2 \leq Z(M_6)$ . Let  $Z=Z(P_3)$ . Then  $\bar{Y}_2=\bar{Z} \times \bar{Z}^t$  and so  $\bar{Y}_2^{\#}$  decomposes, under the action of  $N_{N_2}(P)$ , into three orbits  $\bar{Z}^{\#}$ ,  $(\bar{Z}^t)^{\#}$ , and  $\bar{Z}^{\#}(\bar{Z}^t)^{\#}$ . Thus (5) holds. Likewise (6) holds.

$$(7.14) \quad B_1 \cap Y_2 = Z(P_3^t).$$

PROOF. By (7.3) (7),  $N_{N_2}(P)$  normalizes  $K_1$ . Thus  $B_1 \cap Y_2$  is a  $N_{N_2}(P)$ -invariant subgroup of order  $q^8$  containing  $U_1$ . Then (7.13) (5) implies that  $B_1 \cap Y_2$  is equal to  $Z(P_3)$  or  $Z(P_3^t)$ . Now  $Z(P_3)$  centralizes  $T_2M_6$ , hence (7.11) (2) shows that  $[Z(P_3), T_1B_1]=1$ . Moreover, it follows from (7.9) (2) that  $P$  is a central product of  $T_1B_1$  and  $(T_1B_1)^t$ . Thus, if  $B_1 \cap Y_2=Z(P_3)$  then  $Z(P_3) \leq Z(P)=U_1$ , a contradiction. So the assertion holds.

$$(7.15) \quad (1) \quad N_2=K_2'' \times K_2''^t, K_2''=A_2O_2(K_2'') \text{ is perfect, and } M_6=O_2(K_2'') \times O_2(K_2'')^t.$$

$$(2) \quad O_2(K_2'')=T_1B_1 \cap (T_1B_1)^s.$$

PROOF. Set  $H=T_1B_1 \cap (T_1B_1)^s$ . Then  $H \leq T_2B_2 \cap (T_2B_2)^s \leq B_2$  by (7.11) (2). As  $|T_2B_2 : T_1B_1|=q$ , we have  $|B_2 : H| \leq q^2$ . Moreover, from (7.13) and (7.14) it follows that  $H \cap Y_2=Z(P_3^t) \cap Z(P_3^t)^s=Z(K_3^t)$ . This implies that  $B_2=H \times Z(K_3)$ , since  $Y_2$  is a direct product of  $Z(K_3)$  and  $Z(K_3^t)$ .

Set  $V=\langle T_1B_1, (T_1B_1)^s \rangle$ . Then by (7.9) (2),  $H$  centralizes  $V^t$ . Now  $V$  is  $\langle s \rangle$ -invariant and contains  $U_1U_1^s=Y_2$ , so that  $T_2B_2 \leq V$  by (7.11) (2). As  $P_3=T_2Y_2$ ,  $V$  in fact contains  $K_3$ . Thus we have  $[H, K_3^t]=1$ . Then the above yields that  $C_{B_2}(K_3^t)=H$ , which is normal in  $K_2'$ . Therefore,  $K_2'$  is a direct product of  $A_2H$  and  $Z(K_3)$ . In particular,  $A_2H/Z(K_3^t)$  is isomorphic to  $\tilde{K}_2'$  and so is perfect by (7.4). Also,  $A_2Z(K_3^t)$  is perfect by (7.13). Hence  $K_2''=A_2H$ . Now (7.15) follows immediately from (7.11) (1).

$$(7.16) \quad (1) \quad N_1=K_1'' \times K_1''^t, K_1''=A_1O_2(K_1'') \text{ is perfect, and } D_7=O_2(K_1'') \times O_2(K_1'')^t.$$

$$(2) \quad T_1O_2(K_1'')=T_2O_2(K_2'').$$

PROOF. Set  $H=O_2(K_2'') \cap U_1$ . We first show that  $T_1B_1=T_2O_2(K_2'') \times H^t$ . As  $N_{A_2}(T_2) \leq N_{N_2}(P) \leq N(K_1)$ ,  $N_{A_2}(T_2)$  normalizes  $P \cap K_1'=T_1B_1$ . Also,  $T_2=[T_2, N_{A_2}(T_2)]$ . Since  $T_1B_1Z(K_3)=T_2B_2$  by (7.11) and (7.14) and since  $N_{A_2}(T_2)$  centralizes  $Z(K_3)$ , we obtain that  $T_2 \leq T_1B_1$  and so  $T_2 \leq C_P(J_4) \cap T_1B_1 \leq D_7 \cap T_1B_1=B_1$ . Thus by (7.15),  $T_1B_1$  is a direct product of  $T_2O_2(K_2'')$  and  $H^t$  as asserted.

As  $U_1=H \times H^t$ , it then follows that  $T_1B_1/H$  is a direct product of  $T_2O_2(K_2'')/H$

and  $U_1/H$ . Since  $T_1B_1$  is a Sylow 2-subgroup of  $K'_1$ ,  $U_1/H$  has a complement  $E/H$  in  $K'_1/H$  by Gaschütz's theorem. Then  $K'_1 = E \times H^t$  and  $E = A_1(B_1 \cap E)$ . Moreover,  $E = K''_1 H$  since  $\bar{K}'_1$  is perfect. The order of any nontrivial  $J$ -invariant subgroup of  $U_1$  is at least  $q$ . Hence if  $K''_1 \not\cong H$ , then  $K''_1 \cap U_1 = K''_1 \cap H = 1$  and  $K'_1 = K''_1 \times U_1$ , contrary to (7.9) (2). Thus  $E = K''_1$  and (1) holds.

Now  $T_1B_1 = T_1O_2(K''_1) \times H^t$ . Also,  $T_1B_1 = T_2O_2(K''_2) \times H^t$  by the above. Set  $I_i = T_iO_2(K''_i)$  for  $i=1, 2$ . Then  $I'_1 = I'_2$ . Moreover,  $T_1 \leq O_2(K''_2)$  since  $T_1 \leq C_P(J_3) \leq M_6$  and  $C_{U_1}(J_3) = 1$ . On the other hand, we have shown in the first paragraph that  $T_2 \leq B_1$ , hence  $T_2 \leq I_1 \times H^t$ . Now  $N_{A_2}(T_2)$  centralizes  $H^t$  and normalizes  $P \cap K''_1 = I_1$ , so that  $T_2 = [T_2, N_{A_2}(T_2)] \leq I_1$ . From (1) and (7.15) it follows that  $P = I_i \times I'_i$  for  $i=1, 2$ . Then  $I_i$  is isomorphic to  $C_P(t) = S$  and as  $S' = F$ , we must have  $F_5 = I'_i \times I'_i$  and  $|I'_i| = q^4$ . Thus counting orders we obtain (2).

(7.17) *Notation.* Let  $P_0 = P \cap K''_i$ , so that  $P_0 = T_iO_2(K''_i) \in \text{Syl}_2(K''_i)$  for  $i=1, 2$ . Furthermore, set  $D_0 = D_7 \cap P_0$ ,  $M_0 = M_6 \cap P_0$ ,  $F_0 = F_5 \cap P_0$ ,  $W_0 = W_3 \cap P_0$ , and  $U_0 = U_1 \cap P_0$ .

(7.18) (1)  $K''_1 = \langle P_0, P_0^r \rangle$ ,  $K''_2 = \langle P_0, P_0^s \rangle$ .

(2)  $P_0 \cap P_0^r = D_0$ ,  $P_0 \cap P_0^s = M_0$ ,  $P_0 = D_0 M_0$ ,  $D_0 \cap M_0 = F_0$ ,  $M_0 = F_0 F_0^s$ ,  $F_0 \cap F_0^s = W_0$ ,  $D_0 = W_0 W_0^r$ ,  $W_0 \cap W_0^r = U_0$ .

(3)  $C(N_1) = U_1 O(C(N_1))$  and  $C(N_2)$  has odd order.

PROOF. (1) is obvious. As  $D_0 = O_2(K''_1)$  and  $M_0 = O_2(K''_2)$ , certainly  $P_0 \cap P_0^r = D_0$  and  $P_0 \cap P_0^s = M_0$ . Also,  $P = P_0 \times P_0^t$ . Since  $T_1 \leq C_P(J_3) \leq M_6$ , we have  $T_1 \leq M_0$  so that  $D_0 M_0 = P_0$ . As  $|P_0| = q^6$  and  $|D_0| = |M_0| = q^5$ ,  $D_0 \cap M_0$  has order  $q^4$ . Since  $F_5 = M_6 \cap D_7$  has order  $q^8$ , we must have  $F_5 = (D_0 \cap M_0)(D_0 \cap M_0)^t$  and  $D_0 \cap M_0 = F_0$ . Remaining parts of (2) can be verified similarly. As  $D_7 = D_0 \times D_0^r$ ,  $D_0$  is isomorphic to  $C_{D_7}(t) = D$ . This in particular shows that  $Z(D_7) = U_1$ . Likewise  $Z(M_6) = Y_2$ . Now  $C_{N_1}(D_7) = U_1$  by (7.6) (2), so it follows from (5.59) that  $C(D_7) = U_1 O(C(D_7))$ . Similarly  $C(M_6) = Y_2 O(C(M_6))$ . Thus (3) holds.

(7.19) *Notation.* Let  $G_1 = \langle N_1, N_2 \rangle$  and  $G_0 = \langle K''_1, K''_2 \rangle$ .

(7.20)  $G_1 = G_0 \times G_0^t$  and  $G_0 \cong G_2(q)$  with  $P_0 \in \text{Syl}_2(G_0)$ .

PROOF. Set

$$V_0 = P_0 \cap P_0^s \cap P_0^{rs} \cap P_0^{srs} \cap P_0^{rsrs} \cap P_0^{srsrs}.$$

Then, arguing as in (6.3) we obtain the following three equations:

$$(1) \quad P_0 \cap P_0^r \cap P_0^{sr} \cap P_0^{rsr} \cap P_0^{srsr} \cap P_0^{rsrsr} = U_0^{sr}.$$

$$(2) \quad P_0 = D_0 V_0 = M_0 U_0^{sr}.$$

$$(3) \quad P_0 \cap P_0^{(rs)^3} = 1.$$

Next, take an involution  $u \in K_1''$  (resp.  $v \in K_2''$ ) such that  $r = uu^t$  (resp.  $s = vv^t$ ). We also set

$$V = P \cap P^s \cap P^{rs} \cap P^{sr} \cap P^{rsts} \cap P^{srsts}.$$

Then  $V = V_0 \times V_0^t$ . As a property of  $G_2(q)$  we have

$$S \cap S^s \cap S^{rs} \cap S^{sr} \cap S^{rsts} \cap S^{srsts} = S_a$$

and there exists an element  $x$  of  $S_a$  such that  $r \in (S_a)^x = (S_a)^{rx}$ . Thus  $r \in V^{rx} = V_0^{rx} \times V_0^{xt}$ . Since  $V_0^{rx}$  is a subgroup of  $K_1''$ , we must have  $u \in V_0^{rx}$ . Now

$$V_0 \leq P_0^{rsts} \cap P_0^{srsts} = (P_0^s \cap P_0)^{srsts} = M_0^{srsts},$$

and  $V_0 \leq M_0^{(sr)^3} = M_0^{(rs)^3}$  since  $|rs| = 6$ . Moreover,  $U_0 < W_0^r < F_0^{sr} < D_0^{sr} = D_0^{rsr}$  and hence  $U_0^{sr} < D_0^{(rs)^3}$ . Thus  $[V_0^r, U_0^{srst}] = 1$ . In addition, as  $x \in P^{sr}$  and  $U_0^t \leq U_1 = Z(P)$ , we have  $[x, U_0^{srst}] = 1$ . Hence  $[V_0^{rx}, U_0^{srst}] = 1$ . Since  $[V_0^{rx}, M_0^t] \leq [K_1'', K_1''^t] = 1$  and since  $P_0^s = M_0 U_0^{sr}$  by (2), this implies that  $[u, P_0^s] \leq [V_0^{rx}, P_0^s] = 1$ . Then, as  $P_0^r = P_0^u$ , we obtain

$$[P_0^r, P_0^{st}] = [P_0^u, P_0^{st}] = [P_0, P_0^{st}]^u \leq [K_2'', K_2''^t]^u = 1.$$

Furthermore,  $[P_0, P_0^t] = [P_0, P_0^{st}] = 1$  and  $[P_0^r, P_0^t] \leq [K_1'', K_1''^t] = 1$ . Thus  $[K_1'', K_2''^t] = 1$ , and so  $[G_0, G_0^t] = 1$ . Consequently,  $G_1$  is a central product of  $G_0$  and  $G_0^t$ . This, together with (2) and (3), shows that

$$(4) \quad P_0 = D_0(P_0 \cap P_0^v \cap P_0^{uv} \cap P_0^{vu} \cap P_0^{uvuv} \cap P_0^{uvvu}) \\ = M_0(P_0 \cap P_0^u \cap P_0^{vu} \cap P_0^{uv} \cap P_0^{vuuv} \cap P_0^{uvvu}),$$

$$(5) \quad P_0 \cap P_0^{(uv)^3} = 1.$$

Moreover,  $1 = (rs)^6 = (uv)^6(u^t v^t)^6$  and so  $(uv)^6 \in G_0 \cap G_0^t \leq Z(G_1)$ . Since  $Z(G_1)$  has odd order by (7.18) (3) and since  $\langle u, v \rangle$  is dihedral, we have  $|uv| = 6$ . Now  $P_0 = T_i O_2(K_i'')$  is a Sylow 2-subgroup of  $K_i''$  for  $i = 1, 2$ . Furthermore,  $K_i'' \cap N(P_0) \leq N(P) = N(M_6) \cap N(D_7)$  by (7.3), so that  $K_1'' \cap N(P_0) \leq N(K_2'')$  and  $K_2'' \cap N(P_0) \leq N(K_1'')$ . Hereupon we can apply (2.8) to conclude that  $G_0$  is isomorphic to  $G_2(q)$ . This completes the proof of (7.20).

(7.21) (1)  $C(G_1)$  has odd order.

$$(2) \quad \langle N(M_6), N(D_7) \rangle \leq N(G_1).$$

(3) If  $Q$  is a Sylow 2-subgroup of  $N(G_1)$  containing  $P$ , then  $J_r(Q)=P$  and  $J_r(C_Q(G_0))=P_0$ .

(4)  $|G : N(G_1)|$  is odd.

PROOF. (1) is a consequence of (7.18) (3). Since  $P \in \text{Syl}_2(N_i)$  for  $i=1, 2$  and since  $N(P)=N(M_6) \cap N(D_7)$ , we have  $N(M_6)=N_2N(P)$  and  $N(D_7)=N_1N(P)$  by a Frattini argument. Hence (2) holds. Take a Sylow 2-subgroup  $Q$  of  $N(G_1)$  with  $Q \geq P$ . We have  $J_r(P)=P$  and  $m(P)=6n$  by (7.20). Let  $B$  be an abelian subgroup of  $Q$  of rank at least  $6n$ . If  $B \not\leq N(G_0)$ , then  $|B : N_B(G_0)|=2$  and  $B \cap P_0=1$ . Now  $m(N_Q(G_0)/P) \leq 2$  since  $\text{Out}(G_0)$  is cyclic, and  $m(P/P_0)=3n$ . Hence  $m(N_B(G_0)) \leq 3n+2$  and  $m(B) \leq 3n+3$ , a contradiction. So  $B \leq N(G_0)$ . Then (3.21) yields that  $B \leq P$ , and so  $J_r(Q)=P$ . Furthermore,  $J_r(C_Q(G_0))=P_0$ . Now  $N(Q) \leq N(P) \leq N(G_1)$ , and (4) holds.

(7.22) If  $t \in N(G_1)^g$  for an element  $g$  of  $G$ , then  $g \in N(G_1)$ .

PROOF. Suppose that  $t \in N(G_0^g) \cap N(G_0^{tg})$ . If  $t$  centralizes  $G_0^g$  or acts on  $G_0^g$  as an inner automorphism, we have  $m(G_0^g \cap C)=3n$ . Otherwise, by (3.20) we may assume that  $t$  acts on  $G_0^g$  as an involutory field automorphism and  $m(G_0^g \cap C)=3n/2$ . Similarly  $m(G_0^{tg} \cap C)$  is equal to  $3n$  or  $3n/2$ . Now,  $G_1^g \cap C$  is a direct product of  $G_0^g \cap C$  and  $G_0^{tg} \cap C$  and  $m(C)=3n+1$  by (4.2). Thus  $t$  acts both on  $G_0^g$  and on  $G_0^{tg}$  as an involutory field automorphism. Hence  $(G_1^g \cap C)^\infty$  is isomorphic to  $G_2(q^{1/2}) \times G_2(q^{1/2})$  or  $\text{PSU}(3, 3^2) \times \text{PSU}(3, 3^2)$  when  $q=4$ . However,  $L=C^\infty$  does not contain such subgroups, a contradiction.

Therefore,  $G_0^{gt}=G_0^{tg}$  and so  $L=G_1^g \cap C=\{xx^t \mid x \in G_0^g\}$ . Take  $E \in \text{Syl}_2(G_0^g)$  and set  $P^*=\{ee^t \mid e \in E\}$ . Then  $\langle t \rangle P^*$  and  $R=\langle t \rangle S$  are Sylow 2-subgroups of  $\langle t \rangle L$ , so that  $R^x=\langle t \rangle P^*$  for some  $x \in L$ . Let  $y$  be an element of  $Z(E)^\#$ . Then  $[y, P^*]=1$  and  $t^y \in \langle t \rangle P^*$ , so  $y \in N(\langle t \rangle P^*)$ . Here notice that  $N(R)=R_1N_C(S) \leq N(P) \leq N(G_1)$  (see (5.3) (3)). Then  $y \in N(R)^x \leq N(G_1)$ . Hence  $G_1^g=\langle y, L \rangle \leq N(G_1)$  by Lemma 2.5 of [1]. Since  $N(G_1)/G_1C(G_1)$  is solvable, from (7.21) (1) we conclude that  $G_1^g=G_1$ .

(7.23)  $N(P_0) \leq N(G_1)$ .

PROOF. Suppose that  $N(P_0) \not\leq N(G_1)$  and take an element  $g \in N(P_0) - N(G_1)$ . In view of (7.22) we can apply Lemma (2.2) of [19] and find  $a \in t^G$  such that  $G_1^g=G_1^a$ . Then  $G_0^g$  is equal to  $G_0^a$  or  $G_0^{ta}$ . Assume that  $G_0^g=G_0^a$ . Then  $P_0^g=P_0^a \leq G_0$ . Take  $B \in \text{Syl}_2(\langle P_0, P_0^g \rangle)$  such that  $B^a=B$ . Then, as  $P_0$  and  $B$  are Sylow 2-subgroups of  $G_0$ ,  $B^h=P_0$  for some  $h \in G_0$ . Choose a nontrivial element  $x$  in  $Z(B) \cap C(a)$ . Then  $x^h \in Z(P_0) \leq Z(P)$  and  $[x^h, a^h]=1$ . Put  $H=C(x^h)$ . For a Sylow 2-subgroup  $T$  of  $N_H(G_1)$  containing  $P$  we have  $J_r(T)=P$  by (7.21) (3).



Hence  $N_H(T) \leq N_H(P) \leq N_H(G_1)$ , so that  $T \in \text{Syl}_2(H)$ . Thus  $a^h$  is conjugate to an element of  $N_H(G_1)$ . But since  $N_H(G_1) \leq N(G_0) \cap N(G_0')$  and  $a \in t^a$ , this contradicts (7.22). So we must have  $G_0^g = G_0^{ga}$ . In particular,  $P_0^a = P_0^{ga} \in \text{Syl}_2(G_0')$  and  $P_0 P_0^a \in \text{Syl}_2(G_1)$ , so  $P_0 P_0^a = P^y$  for some  $y \in G_1$ . Then  $a \in N(P_0 P_0^a) = N(P)^y \leq N(G_1)$ , contrary to the choice of  $g$ . The proof is complete.

### 8. Conclusion of the proof.

Let  $G$  be a minimal counterexample to the main theorem. Then (4.3) implies that  $G$  satisfies Hypothesis (5.1). Moreover, (5.59) and (6.4) imply that  $G$  satisfies Hypothesis (7.1). Now, in view of (7.20), (7.21), (7.22), and (7.23) we can apply Lemma (2.7) of [19] to conclude that  $\langle L^G \rangle \cong G_2(q) \times G_2(q)$ , which is against the choice of  $G$ . This completes the proof of the main theorem.

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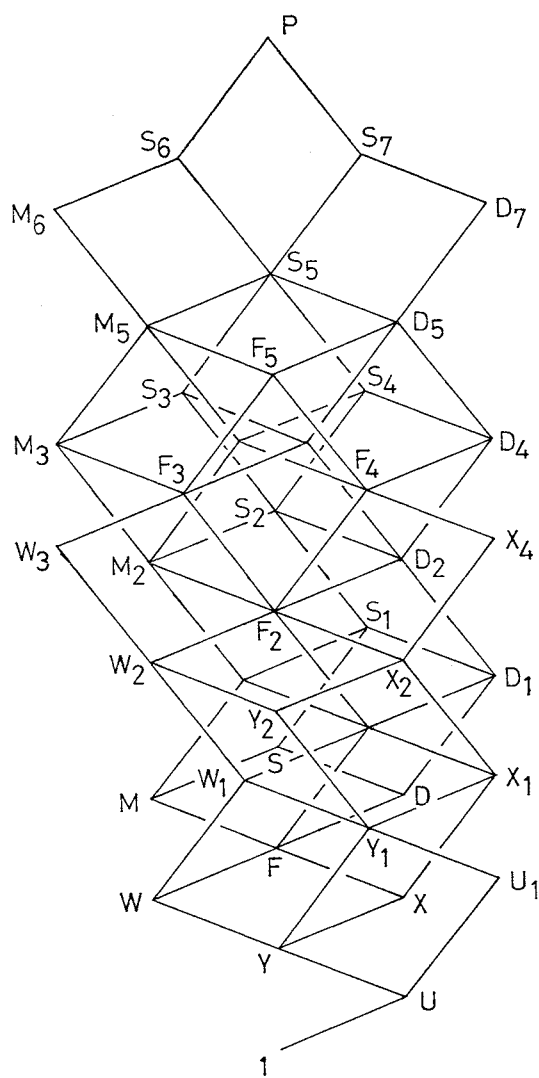


Fig. 1

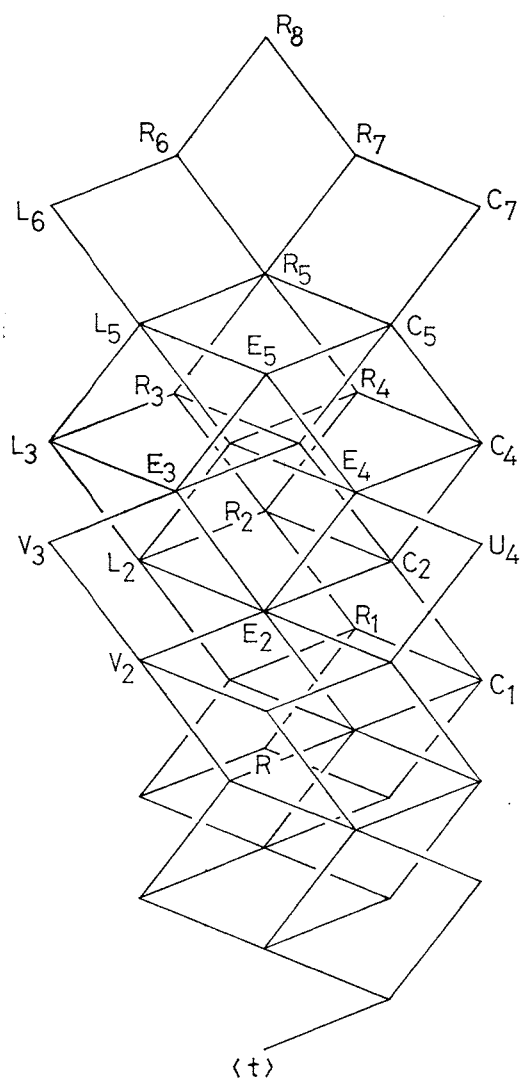


Fig. 2