Gaussian Markov random fields

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(Communicated by S. Itô)

0. Introduction.

Random fields are stochastic processes with a multidimensional parameter. They will be needed to describe phenomena which vary randomly in space, or in both space and time. We suppose here that the parameter set T is an open subset of $\boldsymbol{Z}^{\nu}$ or $\boldsymbol{R}^{\nu}$, $\nu=1, 2, \cdots$; but many of the results are also true in more general parameter spaces. We restrict ourselves also to real valued fields.

One of the main points of interest in the theory of random fields has been the Markov property of Gaussian fields: see [11], [12], [13], [4] for a continuous parameter; [17], [20] for a discrete parameter. In our paper we develop the general theory of discrete parameter Gaussian Markov fields with the help of a generalized potential theory (section 4), and we give simplified proofs for existence theorems of homogeneous Gaussian Markov fields (sections 2 and 5.2). More precisely, our paper contains the following results:

It is well known that for $\nu \geqq 2$, it is not reasonable to define the Markov property by asking the values on the boundary of a domain to be sufficient for best prediction of the inside; for instance in the Gaussian, translation-and rotation-invariant case this would give the constant as the only Markov field. In section 2 we give a new short proof of this fact, rectifying an error in the papers [21], [22].

It is important to distinguish clearly between Markovity for compact and for arbitrary domains, the latter we call G-Markovity. We show in section 3 that contrary to an assertion of Molchan [12], even a purely non deterministic Markov field need not to be G-Markov.

Spitzer [19] and Williams [20] investigated those discrete parameter Markov fields whose covariance is the Green function of a random walk on the parameter space, using identities of discrete potential theory. We show in section 4 that in the theory of an arbitrary Gaussian field having the Markov property for all one point sets there arise functions which satisfy identities analogous to the ones in potential theory. We then calculate the conditional distributions

*) This research was partially supported by a scholarship of the Japanese Ministry of Education.

of the field given the values outside a finite set and prove a structure theorem (see the beginning of section 4 for a more detailed summary).

In section 5 we deal with continuous parameter Gaussian fields. In 5.1 we characterize the Hilbert space with reproducing kernel corresponding to a Markov field, improving a result of Pitt [13]. As an application of this result, we give in 5.2 conditions for Markovity and G-Markovity of homogeneous fields. They are due to Kotani-Okabe [7], Kotani [6] and Pitt [14]. Our proofs are shorter and more straightforward since they do not rely on the characterization of subspaces of the Hilbert space with reproducing kernel corresponding to the field.

1. Definitions and preparations.

1.1. Markov property.

Intuitively speaking, a Markov process is the simplest model of dependence; for the prediction of the future we do not need the whole past, but only the present. As is easily proved (see Chay [1], Proposition 5), this implies that for the prediction of the interior of an interval we need only the two values at the boundary points, not the whole exterior.

Since this last property is independent of the time direction, one can use it for a definition of Markovity of random fields, as Lévy [8] first did. But this definition turns out to be too narrow: Lévy's multidimensional parameter Brownian motion has not this simple Markov property, and the only Gaussian homogeneous, isotropic Markov random field would be the constant (see section 2).

So we have to look for a broader definition, i.e. to allow some more information for the best prediction. McKean [11] proposed to take everything what can be observed in an arbitrary small neighbourhood of the boundary. Then the Lévy Brownian motion becomes Markov for ν odd.

Following McKean [11], we give now the exact definition of such a Markov property.

In the case of a random field $X(t)$ with continuous parameter define

$$\mathcal{B}(O)=\sigma(X(t),\ t\in O) \qquad \text{for an open set } O\subset T,$$

$$\mathcal{B}(D)=\bigcap_{\varepsilon>0}\mathcal{B}(D_\varepsilon) \qquad \text{for a closed set } D\subset T,$$

$$\text{where } D_\varepsilon=\{t\in T\ ;\ \inf_{s\in D}|t-s|<\varepsilon\}.$$

In the case of a random field with discrete parameter we take any set to be open and closed at the same time, so $\mathcal{B}(D)=\sigma(X(t),\ t\in D)$ for any $D\subset T$. Instead, we generalize the notion of the boundary: We suppose that we have a fixed neighbourhood relation $\gamma: T\times T\rightarrow\{0, 1\}$ satisfying the following conditions:

i) for any t $\gamma(t, s)=1$ only for finitely many s;

ii) for any t and s $\gamma(t, s)=\gamma(s, t)$.

∂D is then defined by $\partial D=\{t \notin D ; \gamma(t, s)=1 \text{ for some } s \in D\}$.

DEFINITION: A random field has the *Markov property with respect to an open O* if for any $\mathscr{B}(\bar{O})$-summable U

$$E(U|\mathscr{B}(O^c))=E(U|\mathscr{B}(\partial O)). \tag{1.1}$$

A random field is *Markov* if it has the Markov property with respect to all open, precompact O (in the relative topology).

A random field is *G-Markov* if it has the Markov property with respect to all open O.

Next we show that this definition is symmetric with respect to the inside and outside.

LEMMA 1.1. *The following statements are equivalent for any open set $O \subset T$:*

i) *X has the Markov property with respect to O.*

ii) *$P(B_1 \cap B_2|\mathscr{B}(\partial O))=P(B_1|\mathscr{B}(\partial O)) \cdot P(B_2|\mathscr{B}(\partial O))$ for any $B_1 \in \mathscr{B}(\bar{O})$ and $B_2 \in \mathscr{B}(O^c)$.*

iii) *X has the Markov property with respect to $\bar{O}^c$.*

The proof is the same as in the case of usual Markov processes. For a proof in the context of random fields see Pitt [13], Lemma 2.1.

REMARK 1.1. For a discussion of equivalence of different definitions see Mandrekar [10].

1.2. Gaussian random fields.

Let $X(t)$ be a Gaussian random field. If not stated otherwise, we suppose $EX(t)=0$. The advantage of Gaussian fields lies in the fact that the field is completely characterized by its covariance. Markovity and other properties can all be expressed in the following subspaces of $L_2(dP)$:

$H(O)=\text{lin}(X(t), t \in O)$ (completion in $L_2(dP)$) for an open set O,

$H(D)=\bigcap_{\varepsilon>0} H(D_\varepsilon)$ for a closed set D.

For short we write H instead of $H(T)$. Furthermore

$H_\infty=\bigcap_{C \text{ compact}} H(C^c)$ and $H_0=H_\infty^\perp$ (orthogonal complement).

It follows easily that $H_0=\overline{(\bigcup_C H(C^c)^\perp)}$.

REMARK 1.2. Lemma 3.3 of Mandrekar [10] shows that for closed D $\mathscr{B}(D)=\sigma(Y \in H(D))$.

The following definitions are standard:

X is *regular* (purely non deterministic) if $H_\infty=\{0\}$,
X is *singular* (purely deterministic) if $H_\infty=H$,
X is *non singular with respect to a set* C if $H(C^c)\neq H$.
It follows that X is not singular iff it is non singular with respect to some C. It is well known that every Gaussian random field X has a decomposition $X=Y+Z$ with Y and Z independent, Y singular, Z regular: Y is simply the projection of X onto H_∞.

The following definitions are generalizations of the concept of stationarity. Let $T=\boldsymbol{Z}^\nu$ or $T=\boldsymbol{R}^\nu$. Then a Gaussian field is called *homogeneous* if the mean and the covariance are translation-invariant. It is called *isotropic* if the mean and the covariance are rotation-invariant. We have the following multidimensional version of Bochners theorem.

PROPOSITION 1.1. *$R(t)$ is the covariance of a mean continuous homogeneous Gaussian random field iff $R(t)=\int e^{itx}dF(x)$, where $dF(x)=dF(-x)$ is a finite measure on $\boldsymbol{R}^\nu$ if $T=\boldsymbol{R}^\nu$, on $[-\pi, \pi]^\nu$ if $T=\boldsymbol{Z}^\nu$. Then there is a unitary operator A from H onto $L_2^+(dF)$ with $A(X(t))=e^{itx}$, where $L_2^+(dF)=\{f\in L_2(dF);\ f(-x)=\overline{f(x)}\}$.*

For the proof see Gikhman-Skorokhod [3], chap. IV, § 2 and § 5. There is a similar theorem for homogeneous and isotropic fields, see Gikhman-Skorokhod [3], chap. IV, § 2.

Regular homogeneous fields are characterized by

LEMMA 1.2. *A homogeneous Gaussian field is regular iff the spectral measure dF is absolutely continuous and the density f satisfies*
i) *in the case $T=\boldsymbol{Z}^\nu$: There is a trigonometric polynomial P,*

$$P(x)=\sum_{t\in C} c(t)e^{itx}\not\equiv 0 \text{ such that } \int |P(x)|^2 f(x)^{-1}dx<\infty . \tag{1.2}$$

ii) *in the case $T=\boldsymbol{R}^\nu$: There is a function $u\not\equiv 0$ with compact support such that*

$$\int |\hat{u}(x)|^2 f(x)^{-1}dx<\infty . \tag{1.3}$$

For the proof see Rozanov [17], Theorem 1, in the case i) and Pitt [13], Proposition 8.2, in the case ii).

REMARK 1.3. It follows from the proof that, if $\sum_{t\in C} c(t)e^{itx}$ satisfies (1.2) or if u with supp $u\subset C$ satisfies (1.3), then X is non singular with respect to this set C.

Finally, the Markov property can be expressed in a simpler form.

LEMMA 1.3. *For a Gaussian field X and for any open set O the following statements are equivalent:*

i) *X has the Markov property with respect to O.*

ii) *The orthogonal projection of* $H(\bar{O})$ *on* $H(O^c)$ *is* $H(\partial O)$.

iii) $H=H(\bar{O})\oplus[H(O^c)\ominus H(\partial O)]=H(O^c)\oplus[H(\bar{O})\ominus H(\partial O)]$.

i)⇒ii) and ii)⇔iii) are not difficult to prove. For ii)⇒i) see Lemma 2.3 of Pitt [13], remembering Remark 1.2.

2. Homogeneous and isotropic simple Markov fields.

We call a random field simple Markov if it satisfies (1.1), when for closed D, $\mathcal{B}(D)$ is defined by $\sigma(X(t),\ t\in D)$.

THEOREM 2.1. *The only Gaussian, homogeneous, isotropic and mean continuous simple Markov field is* $X=const.$ *a.s., if* $\nu>1$.

This will follow from the next two lemmas.

LEMMA 2.1. *The covariance of such a field satisfies*

$$R(t)=c(r)r^{-(\nu-1)}\int_{|s|=r}R(t-s)dO(s)\ \text{for}\ |t|\geqq r,\ r\ \text{small enough,} \tag{2.1}$$

with $\lim_{r\to 0} c(r)=1/\Omega_\nu$ (Ω_ν*=surface area of the unit sphere).*

PROOF. Choose a t with $|t|=r$, and put

$$c(r)=\frac{R(t)r^{\nu-1}}{\int_{|s|=r}R(t-s)dO(s)}\quad \text{(for } r \text{ small enough, this is finite).}$$

Since the field is isotropic, we have, if $|t'|=|t|=r$, $R(t)=R(t')$ and

$$\int_{|s|=r}R(t-s)dO(s)=\int_{|s|=r}R(t'-s)dO(s)\,.$$

Therefore

$$E(X(0)|\mathcal{B}(|t|=r))=c(r)r^{-(\nu-1)}\int_{|t|=r}X(t)dO(t)\,,$$

and from the Markov property we find at once (2.1). Continuity of $R(t)$ shows that $c(r)\to 1/\Omega_\nu$ for $r\to 0$. q. e. d.

(2.1) is a kind of mean value property, so the following lemma is not surprising.

LEMMA 2.2. *The covariance of such a field satisfies* $\Delta R=k\cdot R$ *in* $\boldsymbol{R}^\nu\setminus\{0\}$ *in* *the distribution sense with a constant k.*

PROOF. Let $\varphi\in C_0^\infty$, with $0\notin\operatorname{supp}\varphi$. Then by (2.1) for r small enough

$$\int R(t)\varphi(t)dt=c(r)r^{-(\nu-1)}\int R(t)\Big(\int_{|s|=r}\varphi(t+s)dO(s)\Big)dt.$$

Now develop φ in a Taylor series, and observe that

$$\int_{|s|=r}s_i dO(s)=0,\ \int_{|s|=r}s_i s_j dO(s)=k\cdot r^{\nu+1}\delta_{i,j};$$

hence

$$\int R(t)\varphi(t)dt=c(r)\Omega_\nu\int R(t)\varphi(t)dt+c(r)r^2k\int R(t)\Delta\varphi(t)dt+o(r^2),$$

or

$$\frac{1-c(r)\Omega_\nu}{r^2}\int R(t)\varphi(t)dt=c(r)k\int R(t)\Delta\varphi(t)dt+o(1).$$

As $r\to 0$, the limit of the right hand side exists and is $\neq 0$, so

$$\int R(t)\Delta\varphi(t)dt=k'\int R(t)\varphi(t)dt.$$

q. e. d.

PROOF OF THEOREM 2.1. By Lemma 2.2 the distribution $\Delta R-kR$ is concentrated on the origin. By Fourier transformation and rotation symmetry we get for the spectral measure dF

$$(k+|x|^2)dF(x)=p(|x|^2)dx,\ p \text{ a polynomial.}$$

(First both sides are equal as distributions, but then they must also be equal as signed Radon measures.)

Uniqueness of the Lebesgue decomposition shows then that the density of the absolute continuous part of dF is $p(|x|^2)/(k+|x|^2)$. Because dF is finite, this implies $p=0$ if $\nu\geqq 2$. Therefore for $\nu\geqq 2$, dF is singular and concentrated on $\{k+|x|^2=0\}$. Using rotation invariance of dF, we get that if $k=0$, $R(t)=$ constant, i. e. $X(t)=$constant a. s., or if $k=-c^2<0$,

$$R(t)=\text{constant}\int_{|x|=c}e^{itx}dO(x)=\text{constant}\,J_{\nu/2-1}(c|t|)/(c|t|)^{\nu/2-1},$$

where J_n is the Bessel function of order n.

The field with the second covariance is not simple Markov as follows: Let r be a zero of $R(|t|)$, then $X(0)$ is independent of any $X(t)$ with $|t|=r$, i. e.

$$E(X(0)|\mathcal{B}(|t|=r))=0\neq E(X(0)|\mathcal{B}(|t|\geqq r)).$$

Wong [21] and Yadrenko [22] did not pay attention to this last fact.

q. e. d.

3. A regular Markov field which is not G-Markov.

It is well known that Markovity does not imply G-Markovity, but in all current counterexamples this is only due to the presence of a deterministic part.

Example. Let $X(t)$, $t\in(0, 1)$, be a Gaussian process such that $EX(t)=0$ and $EX(t)X(s)=1-|t-s|$. This field is (simple) Markov, but not G-Markov (see Slepian [18]). $X(0)=\lim_{t\to 0} X(t)$ and $X(1)=\lim_{t\to 1} X(t)$ exist, and it is easily seen that $E(X(t)|\mathcal{B}((0, r)\cup(1-r, 1)))$ converges to $tX(1)+(1-t)X(0)$ as r goes to zero. Therefore, the regular part is $X(t)-tX(0)-(1-t)X(1)$ which is G-Markov by an easy calculation.

In [12], section 3, Molchan has stated that a regular Gaussian Markov field is G-Markov. If one tries to prove this by taking a limit $O_n\uparrow O$, $\bar{O}_n$ compact, one sees that it fails. One would need that for a subspace $G\subset H$, $G+H(O_n^c)\downarrow G$ which is known not to be true in general. In fact, we have the following counterexample:

Let $Z_0, Z_1, \cdots$ be Gaussian, $EZ_i=0$, $EZ_iZ_j=\delta_{i,j}$ and put $Y_k=Z_{k+1}-Z_k/2$. Then we have $\text{lin}(Y_k, Y_{k+1}, \cdots)+\text{lin}(Z_h)=\text{lin}(Z_k, Z_{k+1}, \cdots)$ if $h\geqq k$. But $\text{lin}(Y_k, Y_{k+1}, \cdots)\neq\text{lin}(Z_k, Z_{k+1}, \cdots)$ since $\sum_{n=k}^{\infty}\frac{Z_n}{2^n}\perp Y_h$ for any $h\geqq k$.

Now define the random field $X(t)$, $t\in\boldsymbol{R}$, by

$$X(t)=\begin{cases}[Z_n(t-n)+\sum_{k=n+1}^{\infty}\frac{Y_k}{k!}(t-n)^{k+1}](n+1-t) & \text{if } n\leqq t\leqq n+1\\ (n+1+t)\sum_{k=n}^{\infty}\frac{Z_k}{k!}(t+n)^k & \text{if } -(n+1)\leqq t\leqq -n.\end{cases}$$

$X(t)$ is a.s. continuous and regular. Since X is analytic in every $(k, k+1)$, $k\in\boldsymbol{Z}$, we find

$$H(\{t\})=\text{lin}(Z_n, Y_{n+1}, Y_{n+2}, \cdots) \quad \text{for } n<t<n+1$$
$$H(\{t\})=\text{lin}(Z_{n-1}, Z_n, \cdots) \quad \text{for } n=t$$
$$H(\{t\})=\text{lin}(Z_n, Z_{n+1}, \cdots) \quad \text{for } -(n+1)\leqq t<-n\,.$$

Therefore it follows that for $0\leqq u<t<v$ and for $u<t<v\leqq 0$ $E(X(t)|\mathcal{B}(\{u, v\}))=X(t)$. If $u<t<v$ and $u<0<v$, then we have $H(\{u, v\})=H((-\infty, u]\cup[v, \infty))$. Therefore X is Markov with respect to all finite intervals. Finally X is not Markov with respect to the halflines because

$$E\Big(X\Big(\frac{3}{2}\Big)\Big|\mathcal{B}\Big(\Big(-\infty, \frac{1}{2}\Big]\Big)\Big)=X\Big(\frac{3}{2}\Big),$$

but

$$E\left(X\left(\frac{3}{2}\right)\middle|\mathscr{B}\left(\left\{\frac{1}{2}\right\}\right)\right)\neq X\left(\frac{3}{2}\right),$$

since $E(Z_1|Z_0, Y_1, \cdots)=E(Z_1|Y_1, \cdots)\neq Z_1$.

REMARK. With the help of this counterexample, we can easily construct counterexamples for any ν. For instance, if $\nu=2$, put $\hat{X}(t)=X(t_1)(1-t_2)^2$ for $|t_2|\leqq 1$ and 0 elsewhere. Then $\hat{X}(t)$ is regular Markov, but not G-Markov.

4. Gaussian Markov fields with discrete parameter.

Let $X(t)$, $t\in T\subset \boldsymbol{Z}^\nu$, be a Gaussian random field with $EX(t)=0$, $EX(t)X(s)=R(t, s)$, which has the Markov property for all sets $\{t\}$, i.e. there exist coefficients $a(t, s)$ such that

$$E(X(t)|\mathscr{B}(T\setminus\{t\}))=\sum_{s\in\partial t} a(t, s)X(s)\,. \tag{4.1}$$

We put $a(t, s)=0$ for $s\notin\partial t$, so we can do the summation over T.

Since the conditional expectation is an orthogonal projection, (4.1) is equivalent to

$$R(t, s)-\sum_r a(t, r)R(r, s)=c(t)\delta_{t,s} \tag{4.2}$$

with some $c(t)\geqq 0$. $c(t)$ is the error of interpolation at t. We note that $c(t)>0$ iff X is non singular with respect to $\{t\}$. By (4.2) we have for any t and any $s\neq t$:

$$-E[X(t)-\sum_r a(t, r)X(r)][X(s)-\sum_q a(s, q)X(q)]=a(t, s)c(s)=a(s, t)c(t). \tag{4.3}$$

We deal here with the following problems:

1) Is a solution of (4.1) Markov? If yes, calculate the conditional expectation and the conditional distribution for all $C\subset T$. (See Theorems 4.1 and 4.3.)
2) What is the general structure of a solution of (4.1)? (See Theorem 4.2.)
3) Given $a(t, s)$, does a solution of (4.1) exist? (See Theorems 4.4 and 4.5.)
4) Same as 1) for G-Markovity. (See Theorems 4.6 and 4.7.)

Rozanov [17], Chay [1], Williams [20] and Spitzer [19] all gave solutions to some of the problems 1)-3) in special cases (either X homogeneous or $a(t, s)$ a substochastic kernel). We solve here 1) and 2) in the general case. 4) is treated here for the first time.

In analogy to discrete potential theory, we call a function f: $D\cup\partial D\rightarrow \boldsymbol{R}$ harmonic in D if for any $t\in D$, $f(t)=\sum_s a(t, s)f(s)$. Clearly, X is a solution of (4.1) with $c(t)\equiv 0$ iff X is a.s. a harmonic function in T.

THEOREM 4.1. *A solution X of* (4.1) *is Markov with respect to all finite $C\subset\{t; c(t)>0\}$. Any other solution Y of* (4.1) *is also Markov with respect to the same sets (regardless, if the $c(t)$'s belonging to Y are >0 or not). The condi-*

tional expectation of $Y(t)$, $t\in C$, given $Y(s)$, $s\notin C$, is given by (4.8), (4.6) *and* (4.5) (*in particular, the coefficients depend only on $a(t, s)$*).

PROOF. Choose some coefficients $h_C'(t, s)$ such that

$$E(X(t)|\mathscr{B}(\partial C))=\sum_{s\in\partial C} h_C'(t, s)X(s)$$

and put

$$R_C(t, s)=E[X(t)-\sum_r h_C'(t, r)X(r)][X(s)-\sum_q h_C'(s, q)X(q)].$$

Then we get easily

$$R(t, s)-\sum_r h_C'(t, r)R(r, s)=\begin{cases} R_C(t, s) & t\in C,\ s\in C \\ 0 & t\in C,\ s\in\partial C. \end{cases} \tag{4.4}$$

Now we show that $\dfrac{R_C(t, s)}{c(s)}$ is the inverse matrix of $(\delta_{t,s}-a(t, s))_{t,s\in C}$.

$$\sum_{s\in C}\frac{R_C(t, s)}{c(s)}a(s, q)=\sum_{s\in C}R(t, s)\frac{a(q, s)}{c(q)}-\sum_{\substack{s\in C\\ r\in\partial C}}h_C'(t, r)R(r, s)\frac{a(q, s)}{c(q)}$$

$$=\frac{1}{c(q)}[R(t, q)-c(q)\delta_{t,q}-\sum_{s\in\partial C}R(s, t)a(q, s)-\sum_{r\in\partial C}h_C'(t, r)R(r, q)$$

$$+\sum_{\substack{r\in\partial C\\ s\in\partial C}}h_C'(t, r)a(q, s)R(s, r)]=\frac{R_C(t, q)}{c(q)}-\delta_{t,q}.$$

(We have used (4.2)-(4.4) repeatedly.)

So, if we put $g_C(t, s)=\dfrac{R_C(t, s)}{c(s)}$, we have

$$\sum_{r\in C}g_C(t, r)(\delta_{r,s}-a(r, s))=\sum_{r\in C}(\delta_{t,r}-a(t, r))g_C(r, s)=\delta_{t,s}. \tag{4.5}$$

Finally, define

$$h_C(t, r)=\sum_{s\in C}g_C(t, s)a(s, r) \qquad t\in C,\ r\notin C. \tag{4.6}$$

Then, by an easy calculation, we get the identity

$$h_C(t, s)=a(t, s)+\sum_{r\in C}a(t, r)h_C(r, s). \tag{4.7}$$

Now let Y be an arbitrary solution of (4.1). Writing $Z(t)$ for $E(Y(t)|\mathscr{B}(C^c))$, we get for $t\in C$

$$\sum_s a(t, s)Z(s)=E(\sum_s a(t, s)Y(s)|\mathscr{B}(C^c))=E(E(Y(t)|\mathscr{B}(T\setminus\{t\}))|\mathscr{B}(C^c))=Z(t),$$

i.e. Z is harmonic in C and has boundary values $Y(t)$. But it follows from (4.7) and (4.5) that $\sum_{r\in\partial C}h_C(t, r)Y(r)$ with boundary values $Y(r)$ is the unique solution of the 'Dirchlet problem' (i.e. the problem to find a harmonic function in C with given boundary values). This means that

$$E(Y(t)|\mathcal{B}(C^c))=\sum_{s\in\partial C} h_C(t,\ s)Y(s)\,. \tag{4.8}$$

q. e. d.

It is well known that a solution of (4.1) is in general not Markov. Chay [1] gave a singular counterexample, we give here even a regular counterexample.

Example. Let $T=\boldsymbol{Z}^2$ with the usual neighbourhood relation, and let $EX(t)X(s)=0$ for all $s\neq t$, except for the 8 points $t_1, \cdots, t_8$ on a 3×3 square. Take U, V with $E(UV)=0$, U and V also independent of $X(t)$, $t\neq t_1, \cdots, t_8$, and put $X(t_k)=U\cos\frac{k}{4}\pi+V\sin\frac{k}{4}\pi$.

Obviously, this field is regular, and it is easily seen that it is a solution of (4.1) for some $a(t, s)$ because out of $X(t_{k-1})$ and $X(t_{k+1})$ we can calculate U and V. We have $c(t_k)=0$ $k=1, \cdots, 8$, and X is not Markov with respect to $\{t_1, t_2, t_3\}$ since $X(t_8)=X(t_4)=U$.

In the following, we denote the set $\{t\,;\,c(t)=0\}$ by N. We will also use notations such as a-solution of (4.1) and a-harmonic if it is necessary to specify the kernel $a(t, s)$.

The structure Theorem 4.2 will follow after we have proved two lemmas.

LEMMA 4.1. *A solution X of* (4.1) *has a decomposition $X=Y+Z$, where Y and Z are independent solutions of* (4.1), *Y is a.s. a harmonic function and $Z(t)=0$ a.s. for any $t\in N$.*

PROOF. Let $Y(t)=E(X(t)|\mathcal{B}(N))$, $Z(t)=X(t)-Y(t)$. Then $EZ(t)Y(s)=0$ for any t and s. By (4.3) we have for $t\in N$: $\sum_s a(t, s)Y(s)=\sum_s a(t, s)X(s)=X(t)=Y(t)$. That Y is harmonic in N^c, follows from the properties of conditional expectations like in the proof of Theorem 4.1. Finally $E[(Z(t)-\sum_s a(t, s)Z(s))\cdot Z(r)]=E[(X(t)-\sum_s a(t, s)X(s))\cdot X(r)]$. This means that Z is a solution of (4.1) and that $c_x(t)=c_z(t)$. q. e. d.

LEMMA 4.2. *A solution X of* (4.1) *with $c(t)>0$ for any t has a unique decomposition $X=Y+Z$ where Y and Z are independent solutions of* (4.1), *Y is a.s. a harmonic function and Z is regular. $Y(t)$ is given by $\lim_{C\uparrow T}\sum_s h_C(t, s)X(s)$.*

PROOF. Let $X=Y+Z$ be the decomposition in singular and regular parts. Because Y and Z are independent, and because for any t and $s\neq t$ $Y(s)$ and $Z(s)\in H_X(\{t\}^c)$, Y and Z are solutions of (4.1). Moreover, since Y is singular, $Y(t)=E(Y(t)|Y(s),\ s\neq t)=\sum_s a(t, s)Y(s)$. By the uniqueness of the solution of the Dirichlet problem, we have $Y(t)=\sum_s h_C(t, s)Y(s)$ for any C. By (4.8) $E(Z(t)|Z(s), s\notin C)=\sum_s h_C(t, s)Z(s)$, and, since Z is regular, the left hand side converges to 0 as $C\uparrow T$ in quadratic mean. Together, this shows that $Y(t)=\lim_{C\uparrow T}\sum_s h_C(t, s)X(s)$; in particular, the decomposition is unique. q. e. d.

THEOREM 4.2. *A solution X of* (4.1) *has a unique decomposition $X=Y+Z$, where Y and Z are independent solutions of* (4.1), *Y is a. s. a harmonic function, and Z is regular, $Z(t)=0$ a. s. on N. Moreover Z is a Markov field.*

PROOF. By $a'(t, s)$ we denote the restriction of $a(t, s)$ to $N^c\times N^c$. Let $X=Y_1+Z_1$ be a decomposition following Lemma 4.1. Then $Z_1(t)$, $t\in N^c$, is an a'-solution with $c(t)>0$ for $t\in N^c$, so we can apply Lemma 4.2 to obtain $Z_1(t)=Y_2(t)+Z(t)$, where Z is a regular a'-solution and Y_2 is an a'-harmonic function.

Put $Z(t)=Y_2(t)=0$ for $t\in N$, then by (4.3), Z is a regular a-solution and Y_2 is an a-harmonic function. Since $Z(t)\in H_{Z_1}$, we find that $Z(t)$ is independent of $Y_1(t)$. Together $X=(Y_1+Y_2)+Z$ is such a decomposition. For uniqueness, let $X=Y_1+Z_1$ and $X=Y_2+Z_2$ be two such decompositions. Then $Z_1=Z_2+(Y_2-Y_1)$. Because $Y_1=Y_2$ on N, Y_2-Y_1 is an a'-harmonic function. By uniqueness in Lemma 4.2, we find $Y_1=Y_2$ on N^c. That Z is Markov, is an easy consequence of Theorem 4.1. q. e. d.

THEOREM 4.3. *Let X be a solution of* (4.1), *and let C be a finite set with $C\subset\{t; c(t)>0\}$. Then the conditional distribution of $X(t)$, $t\in C$, given $X(t)$, $t\notin C$, has the density*

$$\text{constant}\cdot\exp\Big(-\frac{1}{2}\Big[\sum_{t\in C}\frac{x_t^2}{c(t)}-\sum_{\substack{t\in C\\ s\in C}}\frac{a(t, s)}{c(t)}x_tx_s-2\sum_{\substack{t\in C\\ s\in\partial C}}\frac{a(t, s)}{c(t)}x_tx_s\Big]\Big). \qquad (4.9)$$

PROOF. We use the fact that a conditional distribution of a Gaussian family is again Gaussian with the conditional expectation as mean and the conditional covariance as covariance matrix. In our case $E(X(t)|\mathscr{B}(C^c))=\sum_s h_C(t, s)X(s)$, and the conditional covariance is $R_C(t, s)=g_C(t, s)c(s)$.

By (4.5), we see that $\frac{1}{c(t)}(\delta_{t,s}-a(t, s))$ is the inverse matrix of $R_C(t, s)$. Therefore $-2\log$ of the density of the conditional distribution is up to a constant

$$\sum_{\substack{t\in C\\ s\in C}}\frac{1}{c(t)}(\delta_{t,s}-a(t, s))(x_t-\sum_r h_C(t, r)x_r)(x_s-\sum_q h_C(s, q)x(q)).$$

The coefficients of the quadratic terms of this density and of the density in (4.9) are the same, so we have only to show equality of the linear terms (the constant terms we can take to the constant). But this follows immediately from (4.7). q. e. d.

REMARK. The conditional distribution (4.9) is formally the same as the conditional distribution of a Gibbs field (see Preston [15]). In the latter case, the question of uniqueness of a field belonging to a given set of conditional distributions has received much attention. In the Gaussian case, for $c(t)>0$ for any t, we have in general not uniqueness, since we may always add a harmonic function without changing the $a(t, s)$ and $c(t)$, and in many cases harmonic functions exist, especially also for $\nu=1$.

For the proof of existence of non trivial solutions of (4.1), we have to impose conditions on the $a(t, s)$.

THEOREM 4.4 (Rozanov [17]). *There exists a regular homogeneous solution of* (4.1) *with* $c(t)\not\equiv 0$ *iff*

$$a(t, s)=a(s, t)=a(t-s),\ P(x)=1-\sum_k a(k)e^{ikx}\geqq 0,\ \int_{[-\pi,\pi]^\nu} P(x)^{-1}dx<\infty\,. \tag{4.10}$$

PROOF. Homogeneity implies that $c(t)$ is constant, so by (4.3) $a(t, s)=a(s, t)$. Moreover, $c(t)>0$ for any t implies that the $a(t, s)$ are unique, so by homogeneity, $a(t, s)=a(t-s)$.

By Lemma 1.2 X has a spectral density $f\not\equiv 0$, so (4.2) $\Leftrightarrow\int e^{itx}P(x)f(x)dx=c\delta_{t,0}\Leftrightarrow P(x)f(x)=\text{const}$. This shows necessity of (4.10). Sufficiency follows at once since $P(x)^{-1}$ is the spectral density of a regular field. q. e. d.

In the inhomogeneous case, we have only sufficient conditions which are too strong, compared with the homogeneous case.

THEOREM 4.5. *Let* $a(t, s)$ *satisfy the following two conditions: There exists a function* $c(t)>0$ *for any* t *such that*

$$\left.\begin{aligned} &a(t, s)c(s)=a(s, t)c(t)\ \text{and}\ \sum_{n=0}^{\infty} b^n(t, s)<\infty\,, \\ &\text{where}\ b(t, s)=|a(t, s)|\ \text{and}\ b^{n+1}(t, s)=\sum_r b^n(t, r)b(r, s)\,. \end{aligned}\right\} \tag{4.11}$$

Then there exists a solution X *of* (4.1) *with* $c_X(t)=c(t)$.

PROOF. Following Williams [20], we use the method of power series for the construction of the solution of (4.1). Let

$$g'(t, s)=\delta_{t,s}+\frac{1}{2}a(t, s)+\frac{1\cdot 3}{2\cdot 4}a^2(t, s)+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}a^3(t, s)+\cdots=\sum_{k=0}^{\infty}d_k a^k(t, s)$$

(formally $g'(t, s)=(\delta_{t,s}-a(t, s))^{-1/2}$). Then we have by a change in the order of summation (all series are absolutely convergent)

$$\sum_r g'(t, r)g'(r, s)=\sum_r(\sum_n d_n a^n(t, r))(\sum_k d_k a^k(r, s))=\sum_{n,k} d_n d_k a^{n+k}(t, s)$$

$$=\sum_m a^m(t, s)\sum_{k+n=m} d_k d_n=\sum_{m=0}^{\infty} a^m(t, s).$$

Now let $U(t)$ be the field with $EU(t)U(s)=\delta_{t,s}c(s)$. Then by the above calculation, $X(t)=\sum_s g'(t, s)U(s)$ converges in quadratic mean, and $R(t, s)=EX(t)X(s)=\sum_{m=0}^{\infty} a^m(t, s)c(s)$. From this it follows easily that $R(t, s)$ is a solution of (4.2). q. e. d.

REMARK. Suppose that $a(t, s)$ and $c(t)\not\equiv 0$ satisfy (4.3) and $a'(t, s)$=restriction of $a(t, s)$ to $N^c\times N^c$ satisfies (4.11). If we construct $X(t)$, $t\notin N$, as in Theorem 4.5 and put $X(t)=0$ for $t\in N$, then X is a non trivial solution of (4.1).

Since the X constructed in Theorem 4.5 is not singular, we know from Theorem 4.2 that the regular part $\neq 0$ is also a solution. But X is even regular, as the following lemma shows.

LEMMA 4.3. *The X constructed in Theorem 4.5 is regular.*

PROOF.
$$E(\sum_s h_C(t, s)X(s))^2=EX(t)^2-E(X(t)-\sum_s h_C(t, s)X(s))^2$$
$$=R(t, t)-g_C(t, t)c(t).$$

But $g_C(t, s)$ is equal to $\sum_{n=0}^{\infty} a_C^n(t, s)$, where a_C is the restriction of a to $C\times C$, since this power series satisfies (4.5). Therefore $R(t, t)-g_C(t, t)c(t)$ goes to 0 as $C\uparrow T$, so by Lemma 4.2, X is regular. q. e. d.

For G-Markovity there is nothing similar to Theorem 4.1 because the Dirichlet problem has no unique solution for unbounded domains.

Example. Let $X(t)$, $t\in \boldsymbol{Z}$, be the field with covariance $e^{-|t-s|}$, and let Y be independent of $X(t)$, $EY^2=1$. Put $a(t, s)=e/(1+e^2)$ if $|t-s|=1$, and $a(t, s)=0$ elsewhere.

Then $X_1(t)=X(t)$, $X_2(t)=X(t)+Ye^t$, $X_3=X(t)+Y(e^t+e^{-t})$ are solutions of (4.1) with $c(t)>0$ for any t.

X_1 is Markov with respect to $(0, \infty)$: $E(X_1(t)|\mathscr{B}((-\infty, 0]))=e^{-t}X_1(0)$. X_2 is also Markov with respect to $(0, \infty)$, but the coefficient in the conditional expectation is different: $E(X_2(t)|\mathcal{B}((-\infty, 0]))=\frac{e^t+e^{-t}}{2}X_2(0)$. X_3 is not Markov with respect to $(0, \infty)$.

In view of section 3, we cannot pretend neither that the regular part is G-Markov. We show in the following theorems that the regular parts in some cases are G-Markov.

THEOREM 4.6. *If the $a(t, s)$ satisfy (4.10) with $P(x)\neq 0$ for any x, then the field with spectral density $P(x)^{-1}$ is G-Markov.*

PROOF. Let $\varphi_t(x)$ be the function in $L_2^+\left(\frac{dx}{P}\right)$ which corresponds to $X(t)-E(X(t)|\mathcal{B}(D^c))$, and develop $\varphi_t(x)P(x)^{-1}$ in a Fourier series:

$$\varphi_t(x)P(x)^{-1}=\sum_s g_D(t, s)e^{isx} \text{ (in } L_2(dx)).$$

Since $P(x)$ and $P(x)^{-1}$ are bounded, $\varphi_t(x)=\sum_s g_D(t, s)e^{isx}P(x)$ in $L_2\left(\frac{dx}{P}\right)$, and $\sum_s g_D(t, s)e^{isx}$ converges also in $L_2\left(\frac{dx}{P}\right)$.

From the properties of conditional expectation:

$$\int \varphi_t(x)e^{-irx}\frac{dx}{P(x)}=(2\pi)^\nu g_D(t, r)=0 \qquad \text{if } r\notin D.$$

Therefore

$$\begin{aligned} e^{itx}-\sum_s g_D(t, s)e^{isx}P(x) &= e^{itx}-\sum_{s\in D} g_D(t, s)e^{isx}+\sum_k\sum_{s\in D} g_D(t, s)a(k)e^{i(k+s)x} \\ &= \sum_{s\in D\cup\partial D}[\delta_{t,s}-g_D(t, s)+\sum_k a(k)g_D(t, s-k)]e^{isx}. \end{aligned}$$

On the other hand $e^{irx}P(x)$, $r\in D$, is orthogonal on $H(D^c)$, so

$$\int [e^{itx}-\varphi_t(x)]e^{-irx}dx=0$$

for any $r\in D$. This shows that

$$e^{itx}-\varphi_t(x)=\sum_{s\in\partial D}\left(\sum_k a(k)g_D(t, s-k)\right)e^{isx},$$

or

$$E(X(t)|\mathcal{B}(D^c))=\sum_{s\in\partial D}\left(\sum_k a(k)g_D(t, s-k)\right)X(s). \tag{4.12}$$

q. e. d.

THEOREM 4.7. *The field X constructed in the proof of Theorem 4.5 is G-Markov.*

PROOF. Let $a_D(t, s)$ be the restriction of a to $D\times D$, and put

$$g_D(t, s)=\sum_{n=0}^{\infty} a_D^n(t, s) \quad \text{and} \quad h_D(t, s)=\sum_{r\in D} g_D(t, r)a(r, s). \tag{4.13}$$

Then by a change in the order of summation

$$R(t, s)-\sum_{r} h_D(t, r)R(r, s)=\begin{cases} 0 & t\in D,\ s\notin D \\ g_D(t, s)c(s) & t\in D,\ s\in D\,. \end{cases} \tag{4.14}$$

By (4.11) and (4.14) $\sum_{s,r}|h_D(t, s)|\,|R(s, r)|\,|h_D(t, r)|<\infty$, and this implies that for any $\varepsilon>0$ and for $C_1\supset C_2$, $E(\sum_{s\in C_1\setminus C_2} h_D(t, s)X(s))^2<\varepsilon$ when C_2 is big enough, i.e. $\sum_{s} h_D(t, s)X(s)$ converges in mean square. Finally by (4.14)

$$E(X(t)|\mathscr{B}(D^c))=\sum_{s\in\partial D} h_D(t, s)X(s)\,. \tag{4.15}$$

q. e. d.

REMARK. The expressions (4.8), (4.12) and (4.15) for the conditional expectation (and also the ones given by Williams [20] and Spitzer [19]) are formally the same: The coefficients depend on the inverse of $(\delta_{t,s}-a(t, s))$ by a formula like (4.5). However, their derivation and the $a(t, s)$ and X for which they are valid differ from case to case.

The identities (4.4), (4.5), (4.6), (4.7) and (4.14) are well known in discrete potential theory, but here they are valid for functions which do not appear in discrete potential theory. In this sense, the theory of Markov fields shows the existence of a potential theory with not necessarily positive $a(t, s)$.

5. Gaussian Markov fields with continuous parameter.

In this whole section we suppose X to be mean continuous with $EX(t)=0$, $EX(t)X(s)=R(t, s)$.

5.1. The Markov property expressed in the RKH.

The Reproducing Kernel Hilbert space (RKH) $\mathscr{H}$ is defined as follows: It consists of functions $f: T\to \boldsymbol{R}$, $f(\cdot)=E(XX(\cdot))$, $X\in H$, with the inner product $\langle f_1, f_2\rangle=EX_1X_2$ if $f_i(\cdot)=E(X_iX(\cdot))$, $i=1, 2$. The subspaces $\mathscr{H}_O$, $\mathscr{H}_D$ consist of the functions $f(\cdot)=E(XX(\cdot))$, $X\in H_O$, $X\in H_D$, respectively.

LEMMA 5.1. *The RKH has the following properties:*

i) *All functions in $\mathscr{H}$ are continuous.*

ii) *$A: X\to E(XX(\cdot))$ is an unitary operator from H on $\mathscr{H}$.*

iii) *$f(t)=\langle f, R(t, \cdot)\rangle$ for any $t\in T$ and $f\in\mathscr{H}$.*

The proof is obvious.

The RKH of a Markov field is characterized by

THEOREM 5.1. *A random field is Markov iff the RKH has the following two properties*:

If $f_1, f_2 \in \mathcal{H}$ *with* $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset$ *and* $\operatorname{supp} f_1$ *compact, then* $\langle f_1, f_2 \rangle = 0$. (5.1)

If $f \in \mathcal{H}$, $f = f_1 + f_2$ *with* $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset$ *and* $\operatorname{supp} f_1$ *compact, then* f_1 *and* $f_2 \in \mathcal{H}$. (5.2)

It is G-Markov iff (5.1) *and* (5.2) *are satisfied for* f_1, f_2 *such that* $(\operatorname{supp} f_1)_\varepsilon \cap (\operatorname{supp} f_2)_\varepsilon = \emptyset$ *for some* $\varepsilon > 0$.

PROOF. Markov implies (5.1): Choose an open, precompact O with $\operatorname{supp} f_1 \subset O$ and $\operatorname{supp} f_2 \subset \bar{O}^c$. Then $0 = f_1(t) = \langle f_1, R(t, \cdot) \rangle$ in a neighbourhood of O^c, which means that $f_1 \perp \mathcal{H}(O^c)$. By the same argument $f_2 \perp \mathcal{H}(\bar{O})$. But iii) of Lemma 1.3 shows that $\mathcal{H}(O^c)^\perp \subset \mathcal{H}(\bar{O})$ (by Lemma 5.1, we can state Lemma 1.3 also with $\mathcal{H}(\bar{O})$, $\mathcal{H}(O^c)$ and $\mathcal{H}(\partial O)$). Therefore $\langle f_1, f_2 \rangle = 0$.

Markov implies (5.2): Take O as above. Let f_n be the projection of f on $\mathcal{H}(\bar{O}_{1/n})$. Then $f(t) - f_n(t) = 0$ in $\bar{O}_{1/n}$.

Since $f = 0$ in a neighbourhood of ∂O, $f_n \perp \mathcal{H}(\partial O)$. From Hilbert space theory it follows that f_n converges to f_0=orthogonal projection of f on $\mathcal{H}(\bar{O})$ as n goes to infinity. Since also $f_0 \perp \mathcal{H}(\partial O)$, we have by iii) of Lemma 1.3 that $f_0 \perp \mathcal{H}(O^c)$, i.e. $f_0 = 0$ on O^c. Above we have proved that $f_0 = f$ in $\bar{O}$. Together $f_1 = f_0 \in \mathcal{H}$.

(5.1) and (5.2) imply Markov: Let O be open, precompact. First we show that $\mathcal{H}(\bar{O})^\perp \subset (\mathcal{H}(O^c) \ominus \mathcal{H}(\partial O))$. So let $f \perp \mathcal{H}(\bar{O})$, which means that $\operatorname{supp} f \subset O^c$. Let f_n be the orthogonal projection of f on $\mathcal{H}(O^c_{1/n})$. Then by (5.1) $\|f - f_n\|^2 = \langle f, f - f_n \rangle = 0$, i.e. $f = f_n$. Letting n go to infinity, we find $f \in \mathcal{H}(O^c)$. Since $\mathcal{H}(\partial O) \subset \mathcal{H}(\bar{O})$, we have $f \in \mathcal{H}(O^c) \ominus \mathcal{H}(\partial O)$.

Now take an $f \in \mathcal{H}(\bar{O}) \cap (\mathcal{H}(O^c) \ominus \mathcal{H}(\partial O))$. If we can show that $f = 0$, then we will have proved iii) of Lemma 1.3.

Let f_n be the orthogonal projection of f on $\mathcal{H}(\partial O_{1/n})$, and $g_n = f - f_n$. g_n is zero on $\partial O_{1/n}$, so we can write $g_n = h_n + k_n$ with $\operatorname{supp} h_n \subset O$ and $\operatorname{supp} k_n \subset \bar{O}^c$. Because of (5.2) $h_n \in \mathcal{H}$ and $k_n \in \mathcal{H}$. Now $f \in \mathcal{H}(\bar{O}_{1/n})$ and $f_n \in \mathcal{H}(\bar{O}_{1/n})$, therefore also $g_n \in \mathcal{H}(\bar{O}_{1/n})$.

Furthermore $h_n \perp \mathcal{H}(O^c_{1/n})$, and from the result proved before, $\mathcal{H}(O^c_{1/n})^\perp \subset \mathcal{H}(\bar{O}_{1/n})$. Therefore also $k_n \in \mathcal{H}(\bar{O}_{1/n})$. But since $k_n \perp \mathcal{H}(\bar{O}_{1/n})$, $k_n = 0$. By a similar argument, $h_n = 0$, i.e. $f = f_n$. Letting n go to infinity, we find $f \in \mathcal{H}(\partial O)$. Therefore $f = 0$.

The arguments for G-Markovity are the same. q.e.d.

REMARK. Theorem 5.1 is similar to Theorem 3.3 of Pitt [13]. However, he requests that $\langle f_1, f_2 \rangle = 0$ if $(\operatorname{supp} f_1)^\circ \cap (\operatorname{supp} f_2)^\circ = \emptyset$ which is not very convenient to check. Kono [5] has investigated in detail the relations between the Markov property and the properties of the RKH. Theorem 5.1 follows from his results.

The following is an application of Theorem 5.1 to homogeneous fields.

5.2. Homogeneous Markov fields.

DEFINITION. A function $f: \boldsymbol{C}^{\nu} \to \boldsymbol{C}$ is called of infra-exponential type if for any $\varepsilon>0$ there exists a constant C such that for any $z \in \boldsymbol{C}^{\nu}$ $|f(z)| \leqq C \exp(\varepsilon |z|)$.

THEOREM 5.2. *Let X be a Gaussian homogeneous field with spectral density $\Delta = P^{-1}$ where P is the restriction of an entire function of infra-exponential type and moreover*

$$P(x) \leqq C \exp(T(|x|)) \tag{5.3}$$

for any $x \in \boldsymbol{R}^{\nu}$ where T is monotone increasing and

$$\int_1^{\infty} T(x) x^{-2} dx < \infty . \tag{5.4}$$

Then X is a Markov field.

For the proof of this theorem we will need

LEMMA 5.2 (Roumieu [16]). *If $T: \boldsymbol{R}^{+} \to \boldsymbol{R}^{+}$ is monotone increasing and satisfies $\int_1^{\infty} T(x) x^{-2} dx < \infty$, then for any $\varepsilon > 0$ there exists $\varphi \in C_0^{\infty}(\boldsymbol{R}^{\nu})$ with* supp $\varphi \subset$ $[-\varepsilon, \varepsilon]^{\nu}$, *$\varphi \geqq 0$ and $\varphi \not\equiv 0$ such that*

$$|\hat{\varphi}(x)| \leqq \text{const.} \exp(-2T(|x|)) . \tag{5.5}$$

The proof of this lemma is purely analytical. In order not to disturb the reader, we give the proof at the end of this section.

REMARK. By Lemma 1.2, Remark 1.3, and Lemma 5.2 we see that (5.4) implies that X is regular and non singular with respect to any C.

PROOF OF THEOREM 5.2. H is unitary to $L_2^{+}\left(\frac{dx}{P(x)}\right)$, so to any $f \in \mathscr{H}$ corresponds an $u \in L_2^{+}\left(\frac{dx}{P}\right)$ and vice versa. By iii) of Lemma 5.1, the correspondence is given by $f(t) = \int u(x) e^{-itx} \frac{dx}{P(x)}$, or $f = \left(\frac{u}{P}\right)^{\wedge}$.

Therefore any $\varphi \in C_0^{\infty}(\boldsymbol{R}^{\nu})$ satisfying (5.5) is in $\mathscr{H}$, because $\check{\varphi}(x) P(x) \in L_2^{+}\left(\frac{dx}{P}\right)$. Moreover, $\text{lin}(\varphi(s+\cdot), s \in \boldsymbol{R}^{\nu}) = \mathscr{H}$: Take an f which is orthogonal to all $\varphi(s+\cdot)$, then $\int u(x) \overline{\check{\varphi}(x)} e^{isx} dx = 0$ for any $s \in \boldsymbol{R}^{\nu} \Leftrightarrow u \bar{\check{\varphi}} = 0$ a. e. $\Leftrightarrow u = 0$ a. e.

The inner product in $\mathscr{H}$ is easily calculated for functions in C_0^{∞} satisfying

(5.5). Define $P'\varphi=(2\pi)^{-\nu}(P\check{\varphi})^{\wedge}$. Then by Parsevals identity

$$\langle\psi, \varphi\rangle=\int\psi(t)P'\varphi(t)dt. \tag{5.6}$$

Now choose $\rho_\varepsilon\in C_0^\infty$ satisfying (5.5) with supp $\rho_\varepsilon=[-\varepsilon, \varepsilon]^\nu$, ρ_ε even, $\geqq 0$ and $\int\rho_\varepsilon(t)dt=1$. Then $|\hat{\rho}_\varepsilon(x)|\leqq 1$ and $\lim_{\varepsilon\to 0}\hat{\rho}_\varepsilon(x)=1$ for any x. Then for an arbitrary $f\in\mathcal{H}$, we have $f*\rho_\varepsilon\in\mathcal{H}$ and $f*\rho_\varepsilon$ goes to f because, if u is the corresponding element in $L_2^+\left(\frac{dx}{P}\right)$, $u\hat{\rho}_\varepsilon\in L_2^+\left(\frac{dx}{P}\right)$ and $\left(\frac{u}{P}\hat{\rho}_\varepsilon\right)^\wedge=f*\rho_\varepsilon$. Moreover, for any g $L_2(dt)$ with compact support, $g*\rho_\varepsilon\in\mathcal{H}$ because $\hat{g}$ is bounded.

Now we prove that (5.1) is satisfied. Let $f_1, f_2\in\mathcal{H}$ with disjoint supports and supp f_1 compact. Then $f_2=\lim_{n\to\infty}\varphi_n$ with $\varphi_n\in C_0^\infty$ satisfying (5.5). The convergence is uniform in t, because

$$|f_2(t)-\varphi_n(t)|=|\langle f_2-\varphi_n, R(t, \cdot)\rangle|\leqq R(0, 0)^{1/2}\|f_2-\varphi_n\|.$$

$f_1*\rho_\varepsilon$ is in C_0^∞ and satisfies (5.5). By the definition of P', $P'(f_1*\rho_\varepsilon)=f_1*(P'\rho_\varepsilon)$ and the Theorem of Paley-Wiener shows that $\text{supp}(P'\rho_\varepsilon)\subset\text{supp}\,\rho_\varepsilon$ (see Gelfand-Shilov [2], chap. III, section 4.4, p. 161). But then by (5.6) and uniform convergence $\langle f_2, f_1*\rho_\varepsilon\rangle=\lim_{n\to\infty}\langle\varphi_n, f_1*\rho_\varepsilon\rangle=\int f_2(t)P'(f_1*\rho_\varepsilon)(t)dt$. For ε small enough, the last integral is zero, so by letting ε go to zero, we have $\langle f_1, f_2\rangle=0$, which proves (5.1).

Finally, to prove (5.2), let f be given as in (5.2). $f_1*\rho_\varepsilon$ is then in $\mathcal{H}$ because supp f_1 is compact. For ε, ε' small enough, we have by (5.1)

$$\|f_1*\rho_\varepsilon-f_1*\rho_{\varepsilon'}\|^2\leqq\|f*\rho_\varepsilon-f*\rho_{\varepsilon'}\|^2,$$

so $f_1*\rho_\varepsilon$ converges to a $g\in\mathcal{H}$ as $\varepsilon\to 0$. But $g(t)=\lim_{\varepsilon\to 0}f_1*\rho_\varepsilon(t)=f_1(t)$. q. e. d.

We already saw in the case of discrete parameters (Theorem 4.6) that the field might not be G-Markov if the inverse of the spectral density has zeros. This is also the case here.

THEOREM 5.3. *If P satisfies in addition to* (5.3) *and* (5.4) *also*

$$|P(x)|\geqq C'\exp(-T(|x|)),\quad C'\neq 0 \tag{5.7}$$

(*with the same T as in* (5.4)), *then the field with spectral density $P(x)^{-1}$ is even G-Markov.*

PROOF. We claim that $f*\rho_\varepsilon\in L_2(dt)$ for any $f\in\mathcal{H}$: This follows from Parsevals identity, because

$$\int|(f*\rho_\varepsilon)^\vee|^2dt=\int|u(x)|^2|\hat{\rho}_\varepsilon(x)|^2P(x)^{-2}dx\leqq C\int|u(x)|^2P(x)^{-1}dx\,.$$

Next we claim that to any $\varepsilon>0$ there is a constant C such that for any $f\in L_2(dt)$ with compact support $\|f*\rho_\varepsilon\|_{\mathcal{H}}\leqq C\|f\|_{L_2}$ (As shown in the proof of Theorem 5.2, $f*\rho_\varepsilon$ is in $\mathcal{H}$.): $\check{f}\hat{\rho}_\varepsilon P$ is the element in $L_2^+\left(\frac{dx}{P}\right)$ which corresponds to $f*\rho_\varepsilon$, so

$$\|f*\rho_\varepsilon\|_{\mathcal{H}}^2=\int|\check{f}(x)|^2|\hat{\rho}_\varepsilon(x)|^2P(x)dx\leqq C\int|f(t)|^2dt.$$

Finally, we choose a $\psi\in C_0^\infty$ with $|\psi|\leqq 1$ and $\psi(t)=1$ on $\{t\in\boldsymbol{R}^\nu;\ |t|\leqq 1\}$ and put $\psi_n(t)=\psi\left(\frac{t}{n}\right)$.

Now let f_1, f_2 in $\mathcal{H}$ with $(\mathrm{supp}\, f_1)_\delta\cap(\mathrm{supp}\, f_2)_\delta=\emptyset$. Because $\langle f_1*\rho_\varepsilon, f_2\rangle$ goes to $\langle f_1, f_2\rangle$ as ε goes to zero, it is sufficient to prove that $\langle f_1, f_2\rangle=0$ if f_1 is in $L_2(dt)$. Since the field is Markov, we have for ε small enough $\langle(f_1\cdot\psi_n)*\rho_\varepsilon, f_2\rangle=0$. Now $\lim_{n\to\infty}(f_1\cdot\psi_n)*\rho_\varepsilon(t)=f_1*\rho_\varepsilon(t)$ for any $t\in\boldsymbol{R}^\nu$, and $\|(f_1\cdot\psi_n)*\rho_\varepsilon\|\leqq C\|f_1\cdot\psi_n\|_{L_2}\leqq C\|f_1\|_{L_2}$ (C is independent of n). So by weak convergence, $\langle f_1*\rho_\varepsilon, f_2\rangle=0$ and letting ε go to zero, $\langle f_1, f_2\rangle=0$.

(5.2) is proved by the same approximation argument. q. e. d.

Finally we give the *proof of Lemma* 5.2: It is no restriction to suppose that $T(x)=0$ near 0. Put $\bar{T}(x)=n$ on $\{x;\ n-1\leqq 2T(ex)<n\}$, and let m_k be the point where $\bar{T}$ jumps from $k-1$ to k: $m_k=\inf\{x;\ \bar{T}(x)\geqq k\}$. Then

$$\sum_{k=1}^{n}\frac{1}{m_k}=\sum_{k=2}^{n}(k-1)\left(\frac{1}{m_{k-1}}-\frac{1}{m_k}\right)+\frac{n}{m_n}\leqq\int_{m_1}^{\infty}\bar{T}(x)x^{-2}dx<\infty.$$

Put $M_0=1$, $M_k=m_kM_{k-1}$ and $M(x)=\log\sup_k\frac{x^k}{M_k}$. It is then well known that there exists a positive function f in $C_0^\infty(\boldsymbol{R}^1)$ with $\mathrm{supp}\, f=[-\varepsilon, \varepsilon]$ and $|f^{(k)}(t)|\leqq M_k/\nu^k$ because $\sum_k\frac{M_{k-1}}{M_k}=\sum_k\frac{1}{m_k}<\infty$ (see Mandelbrojt [9], Theorem 4.1. V, p. 103).

Take such an f and put $\varphi(t_1, \cdots, t_\nu)=\prod_{k=1}^{\nu}f(t_k)$. By partial integration under the Fourier integral, we find that there is a constant C_0 such that for any n $\frac{|x|^n}{M_n}|\hat{\varphi}(x)|\leqq C_0$, so by the definition of $M(x)$ we have $|\hat{\varphi}(x)|\leqq 2C_0\exp(-M(|x|))$.

On the other hand

$$M(x)=\sup_k\ \log\frac{x^kM_0\cdots M_{k-1}}{M_0M_1\cdots M_k}=\sup_k\sum_{n=1}^{k}\log\frac{x}{m_n}$$

$$=\sum_{n,\, m_n\leqq x}\log\frac{x}{m_n}\geqq\sum_{n,\, m_n\leqq x/e}1=\bar{T}\left(\frac{x}{e}\right)>2T(x).$$

q. e. d.

We should like to thank Prof. Okabe who kindly read the first version of this paper and gave valuable advice for the improvement of our results.

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(Received October 4, 1977)

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