Finite groups having 2-local subgroups $E_{16}\cdot L_4(2)$, II

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1. Introduction.

In this paper, the following theorem is proved:

THEOREM. *Let G be a finite group having 2-local subgroups isomorphic to a (nontrivial) split extension of an elementary abelian group E_{16} of order 16 by $L_4(2)$. Then $G/O(G)$ is isomorphic to one of the following groups: $E_{16}\cdot L_4(2)$, $L_5(2)$, $\mathrm{Aut}(L_5(2))$, M_{24}, A_{16}, A_{17}, S_{16}, S_{17}, A_{18}, or A_{19}.*

An initial work on a finite (fusion simple) group G having 2-local subgroups isomorphic to a split extension of an elementary abelian group E_{16} of order 16 by $L_4(2)\cong A_8$ was done by Kiernan [8]. Among other things, he has shown that if the order of Sylow 2-subgroups of G is less than 2^{13}, then G is isomorphic to $E_{16}\cdot L_4(2)$, M_{24} or $L_5(2)$. In [6], the first author treated the general cases. The main result of [6] is that if the order of a Sylow 2-subgroup T of G is at least 2^{13}, then T is of type A_{16} or A_{18}. In [12], the second author has partially classified the structure of fusion simple groups having Sylow 2-subgroups of type A_{16}. The result of [12] easily determines the structure of G if T is of type A_{16}. The main part of this paper is devoted to the case that T is of type A_{18}.

§2 is a collection of the precise statements of the assumed results [8], [6], [12], and [13]. In §2, the case that T is of type A_{16} is completely handled. The remaining sections §4∼§9 will be devoted to the case that T is of type A_{18}. In §4, we prove that if z is an involution in the center $Z(T)$ of T, then $C_G(z)/O(C_G(z))$ involves $C_{A_{18}}((1,2)(3,4)\cdots(15,16))$. In §5, we determine the precise structure of $C_G(z)/O(C_G(z))$. If all simple groups with an involution x satisfying $O(C_G(x))\neq 1$ are classified, we may stop our argument there and conclude that $G/O(G)\cong A_{18}$ or A_{19} (see the remark in section 5). In view of the fact that such a classification has not quite been completed at the time of writing, we shall give a brief proof of the precise structure of $O(C_G(z))$. The structure of G will then be determined by a result of Kondo [9].

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Our notation is standard.

$E_{p^n}\cdot X$; a split extension of an elementary abelian group E_{p^n} of order p^n by $X\subseteqq GL(n, p)$.

$W(X)$: the Weyl group of type X.

2. Assumed results.

THEOREM 2.1 (Kiernan [8], Harada [6]). *Let G be a finite group having a 2-local group $N=N_G(A)$ such that*

(i) $A\cong E_{16}$, *A is a Sylow 2-subgroup of $C_G(A)$,*

(ii) $N/O(N)\cong E_{16}\cdot L_4(2)$, *and*

(iii) $G\neq N\cdot O(G)$.

Then if the order of Sylow 2-subgroups of G is less than 2^{13}, $G/O(G)\cong M_{24}$, $L_5(2)$ or Aut$(L_5(2))$ [8], *and if the order of them is at least 2^{13}, they are of type A_{16} or A_{18}* [6]. *In particular, $|G|_2=2^{10}$, 2^{11}, 2^{14}, or 2^{15}.*

THEOREM 2.2 (Yamaki [12]). *If the Sylow 2-subgroups of a fusion-simple group G are of type A_{16}, then $G\cong A_{16}$, A_{17}, the split extension of an elementary abelian group of order 2^8 by A_9 or G has the involution fusion pattern of $\Omega_9(3)$.*

THEOREM 2.3 (Zappa, Yoshida [13] see also [7]). *Let G be a finite group with a Sylow p-subgroup P. Let A be a weakly closed elementary abelian subgroup of P. Then $G'G^p\cap A=N'N^p\cap A$ where $N=N_G(A)$.*

3. Case I: The Sylow 2-subgroups are of Type A_{16}.

In the balance of the paper we operate under the following assumption and notation.

G is a finite group having a 2-local subgroup $N=N_G(A)$ such that

(i) $A\cong E_{16}$, *A is a Sylow 2-subgroup of $C_G(A)$,*

(ii) $N/O(N)\cong E_{16}\cdot L_4(2)$, *and*

(iii) $O(G)=1$ *and* $G\neq N$.

T denotes a Sylow 2-subgroup of G containing a Sylow 2-subgroup of $N_G(A)$.

THEOREM 3.1. *If the Sylow 2-subgroups of G are of type A_{16}, then $G\cong A_{16}$ or A_{17}.*

PROOF. Suppose G contains a normal subgroup K of index 2. Set $N_1=N\cap K$. Then $N_1/O(N_1)\cong N/O(N)$. As $|K|_2=2^{13}$, Theorem 2.1 yields a contradiction. Hence G is fusion simple and now Theorem 2.2 is applicable. Since A is a Sylow 2-subgroup of $C_G(A)$, G can not be an extension of an elementary abelian group of order 2^8 by A_9. Thus all we need to show is that G does not have the involution fusion pattern of $\Omega_9(3)$.

In [12], it is shown that $Z(T)\cong Z_2$, $Z_2(T)\cong Z_2\times Z_2$, the involution of $Z(T)$ is not conjugate in G to any involution of $Z_2(T)-Z(T)$ [12; Lemma 2.2], and if G has the involution fusion pattern of $\Omega_9(3)$, then every involution of $T-C_T(Z_2(T))$ is not conjugate to the involution of $Z(T)$ [12; Theorem 6.5]. Since all involutions of $A^\#$ are conjugate in N and $C_T(A)=A$, $A\subsetneqq C_T(Z_2(T))$. But then G can not have the involution fusion pattern of $\Omega_9(3)$. This completes the proof.

4. Case II: The Sylow 2-subgroups are of Type A_{18}. The "approximate" structure of the centralizer of the involution of $Z(T)$.

In the remaining sections of this paper we assume that the Sylow 2-subgroup T of G is of type A_{18}. The main result of this section is:

THEOREM 4.1. *If z is an involution of $Z(T)$, then $C_G(z)/O(C_G(z))$ possesses a section isomorphic to $C_{A_{18}}((1, 2)(3, 4)\cdots(15, 16))$, which is a split extension of an elementary abelian group of order 2^8 by S_8.*

The proof of the theorem will be completed in a series of lemmas and propositions. We need the following omnibus lemma about the structure of T.

LEMMA 4.2. *The following condition holds.*

(a) $T\cong(D_8\int Z_2)\int Z_2$. *More precisely, T is generated by involutions a_i, b_j, u, v, $1\leqq i, j\leqq 4$ with the relations:*

$$\langle a_i, b_i\rangle\cong\langle u, v\rangle\cong D_8,\ 1\leqq i\leqq 4,$$

$$[\langle a_i, b_i\rangle, \langle a_j, b_j\rangle]=1,\ 1\leqq i\neq j\leqq 4,$$

$$[u, \langle a_i, b_i\rangle]=1,\ i=3, 4,\ [u, a_1]=a_1a_2,\ [u, b_1]=b_1b_2,$$

$$[v, a_1]=a_1a_4,\ [v, b_1]=b_1b_4,\ [v, a_2]=a_2a_3,\ [v, b_2]=b_2b_3.$$

Set $(a_ib_i)^2=z_i$, $(1\leqq i\leqq 4)$, $(uv)^2=t$, $a=a_1a_2a_3a_4$, $b=b_1b_2b_3b_4$, $z=z_1z_2z_3z_4$, $Z=\langle z_1, z_2, z_3, z_4\rangle$, $D=\langle a_i, z_i|1\leqq i\leqq 4\rangle$, $E=\langle b_i, z_i|1\leqq i\leqq 4\rangle$, *and* $J_i=\langle a_i, b_i\rangle$ $(1\leqq i\leqq 4)$. *Then* $Z(T)=\langle z\rangle$.

(b) *T has precisely two conjugacy classes of self-centralizing elementary abelian subgroups of order* 16. *The classes are represented by* $A=A_1=\langle a, z, v, t\rangle$ *and* $A_2=\langle b, z, v, t\rangle$. *$A_1$ is conjugate to A_2 by an element of* $\mathrm{Aut}(T)$.

(c) $N_T(A)=\langle A, a_1a_2, a_2a_3, z_1z_2, z_2z_3, u, b\rangle$. *$N_T(A)$ contains the unique extra special subgroup* $Q=\langle b, a, t, v, z_1z_3, z_1z_2\rangle=\langle b, a\rangle*\langle t, z_1z_3\rangle*\langle vt, z_1z_2\rangle$ *isomorphic to* $D_8*D_8*D_8$.

(d) $N_T(Q)=\langle Q, a_1a_2, a_2a_3, b_1b_2, b_2b_3, u, z_1\rangle$. *$N_T(Q)$ is of order 2^{13} and con-*

tains a unique elementary abelian normal subgroup F of the following properties:

(i) $|F\cap Q|=16$,

(ii) $|F\cap A|=2$.

In fact $F=\langle b_1b_2, b_2b_3, b_3b_4, z_1, z_2, z_3, z_4\rangle$.

(e) *The subgroup F of* (d) *satisfies:*

(i) $F\lhd T$,

(ii) $E=C_T(F)=\langle F, b_1\rangle$ *is an elementary abelian subgroup of order* 2^8,

(iii) T *splits over E and T/E is of type* S_8.

PROOF. Omitted.

We shall keep the notation of Lemma 4.2 in the balance of the paper. In particular, $A=\langle a, z, v, t\rangle$. The following lemma is a restatement of [6; Lemma 4.1] which was essentially due to Kiernan [8].

LEMMA 4.3. *The structure of* $N_G(Q)/\langle z\rangle(N_G(Q))$ *is uniquely determined.* $N_G(Q)/O(N_G(Q))$ *is a split extension of an elementary abelian subgroup of order* 2^7 *by* $E_8\cdot L_3(2)$. *In particular,* $|N_G(Q)|=2^{13}$.

Set $C=C_G(z)\cap N_G(F)$. By Lemma 4.2 (e), $T\subseteqq C$.

LEMMA 4.4. *If* $\bar{C}=C/O(C)$, *then* $C_{\bar{C}}(\bar{F})=\bar{E}$, $\bar{C}/\bar{E}$ *has Sylow 2-subgroups of type* S_8, *and* $\bar{C}/\bar{E}$ *contains a subgroup isomorphic to* $E_8\cdot L_3(2)$.

PROOF. By Lemma 4.2 (d) and Lemma 4.3, $N_T(Q)$ is a Sylow 2-subgroup of $N_G(Q)$. By Lemma 4.3, $N_T(Q)$ contains an elementary abelian subgroup F_1 of order 2^7 with the property $(N_G(F_1)\cap N_G(Q))O(N_G(Q))=N_G(Q)$. We shall show that $F_1=F=\langle b_1b_2, b_2b_3, b_3b_4, z_1, z_2, z_3, z_4\rangle$ which was defined in Lemma 4.2 (d). We know that $N_N(Q)/O(N_N(Q))$ is an extension of Q by $E_8\cdot L_3(2)$ [6; § 3]. So $(N\cap N_G(Q))N_T(Q)=N_G(Q)$. Let σ be a 7-element in $N\cap N_G(Q)-O(N_G(Q))$. Then $\sigma/\langle\sigma^7\rangle$ acts fixed-point free on $A/\langle z\rangle$. As $F_1^\sigma\subseteqq F_1O(N_G(Q))$ and $O(N_G(Q))\subseteqq N$, we may assume $F_1^\sigma=F_1$. Hence if $F_1\cap A\supset\langle z\rangle$, $F_1\supset A$. This is impossible, as A is self-centralizing in T. If $|F_1\cap Q|\leqq 2^3$, then $|F_1\cap Q|=2$ must hold, as $\sigma/\langle\sigma^7\rangle$ acts fixed-point-free on $Q/\langle z\rangle$. Clearly then the 2-rank of $\mathrm{Out}(D_8*D_8*D_8)\cong S_8$ is at least 6, which is false. Thus $|F_1\cap Q|=16$. Now Lemma 4.2 (d) shows that F_1 is uniquely determined in T: i.e., $F_1=F=\langle b_1b_2, b_2b_3, b_3b_4, z_1, z_2, z_3, z_4\rangle$. Since T is a Sylow 2-subgroup of G, $\bar{C}/C_{\bar{C}}(\bar{F})$ has Sylow 2-subgroups of type S_8 and $\bar{E}$ is a Sylow 2-subgroup of $C_{\bar{C}}(\bar{F})$ by Lemma 4.2 (e). Since $|E/F|=2$, $C_C(F)=O(C)\cdot E$. Hence $\bar{E}=C_{\bar{C}}(\bar{F})$. The last statement of the lemma is a direct consequence of Lemma 4.3 and $F_1=F$.

LEMMA 4.5. *If* $C_1=C_G(z)\cap N_G(E)$ *and* $\bar{C}_1=C_1/O(C_1)$, *then* $\bar{C}_1/\bar{E}\cong S_8$.

PROOF. Since $(\bar{C}_1/\bar{E})'$ has Sylow 2-subgroups of type A_8 or S_8, the main theorems of [4, Theorem A*, Theorem B*] and [5, Theorem A] are applicable. Since $\bar{C}_1/\bar{E}$ contains a subgroup isomorphic to $E_8 \cdot L_3(2)$, we conclude that if $\tilde{C}_1=\bar{C}_1/\bar{E}$, $\tilde{C}_1/O(\tilde{C}_1)\cong S_8$, S_9, A_{10} or A_{11}. Since $\tilde{C}_1$ acts on $\bar{E}/\langle\bar{z}\rangle$, $|\tilde{C}_1|_{2'}$ divides $3^4\cdot 5\cdot 7^2\cdot 31\cdot 127$. Thus $\tilde{C}_1'O(\tilde{C}_1)=\tilde{L}\times O(\tilde{C}_1)$ where $\tilde{L}\cong A_8$, A_9, A_{10} or A_{11}. As A_9 does not act on an elementary abelian group of order 2^7 [6, Lemma 2.8], $\tilde{L}\cong A_8$.

Suppose $O(\tilde{C}_1)\neq 1$. Then $\bar{T}$ acts on $C_{\bar{E}}(O(\tilde{C}_1))\neq 1\neq[\bar{E}, O(\tilde{C}_1)]$. Therefore, $Z(\bar{T})$ must be of order at least 4, which is not true by Lemma 4.1 (a). Thus $O(\tilde{C}_1)=1$, which completes the proof of the lemma.

LEMMA 4.6. *Under the notation of Lemma* 4.4, $\bar{C}_1\cong C_{A_8}((1, 2)(3, 4)\cdots(15, 16))$ *holds.*

PROOF. We first show that $\bar{C}_1$ does not act irreducibly on $\bar{E}/\langle\bar{z}\rangle$. Let $\bar{\sigma}$ be an element of order 7 of $\overline{C\cap C_1}$. Then $\bar{F}/\langle\bar{z}\rangle=[\bar{E}/\langle\bar{z}\rangle, \bar{\sigma}]$. Therefore, $\bar{F}/\langle\bar{z}\rangle$ is invariant under the conjugation by $N_{\tilde{C}_1}(\langle\bar{\sigma}\rangle)$. Clearly $\langle\bar{E}, N_{\tilde{C}_1}(\langle\bar{\sigma}\rangle), \overline{C_1\cap N(Q)}\rangle=\bar{C}_1$ and so $\bar{F}\lhd\bar{C}_1$. Thus $\bar{C}_1$ normalizes the chain $\bar{E}\supset\bar{F}\supset\langle\bar{z}\rangle\supset 1$. If $\bar{T}_1=\bar{C}_1'\cap\bar{T}$, then $Z(\bar{T}_1)\cong Z_2$. Therefore $\bar{C}_1$ does not have a 2-dimensional invariant space in $\bar{E}$. Thus, the action of $\bar{C}_1$ on $\bar{E}$ is indecomposable and uniserial: i.e., $\bar{E}\supset\bar{F}\supset\langle\bar{z}\rangle\supset 1$ is the unique composition series with the operator $\bar{C}_1$. We shall show that $\bar{E}-\bar{F}$ contains an element which has exactly eight conjugates under the action of $\bar{C}_1/\bar{E}$.

Let $\bar{x}$ be an element of $\bar{E}-\bar{F}$. Then by the structure of $\bar{T}$, $|\bar{T}:C_{\bar{T}}(\bar{x})|\geqq 8$. The equality holds if $x=b_1$ for example. Suppose that every element of $\bar{E}-\bar{F}$ has more than eight conjugates. Then $120>|\tilde{C}_1: C_{\tilde{C}_1}(\bar{b}_1)|>8$, as $\tilde{C}_1=\bar{C}_1/\bar{E}$ does not act transitively on $\bar{E}-\bar{F}$. We conclude that $|C_{\tilde{C}_1}(\bar{b}_1)|=2^4\cdot k$ with k odd and $3^2\cdot 5\cdot 7>k>3\cdot 7$. Thus $k=5\cdot 7$, $3^2\cdot 5$, $3^2\cdot 7$, or $3\cdot 5\cdot 7$. Since A_5, A_6, A_7, A_8, and $L_2(7)$ are the only nonsolvable simple groups involved in S_8, we conclude easily that $k=3^2\cdot 5$ and so $\bar{b}_1$ has 56 conjugates. Since $128-56$ is not divisible by 16, there must exist another element $\bar{b}'$ such that $|\tilde{T}: C_{\tilde{T}}(\bar{b}')|=8$ and so $\bar{b}'$ also has 56 conjugates. Thus $\bar{E}-\bar{F}$ has an orbit Ω of length $16=128-2\cdot 56$. Let $\bar{x}\in\Omega$. Then $|C_{\tilde{C}_1}(\bar{x})|=2^3\cdot 3^2\cdot 5\cdot 7$. One can conclude easily that $C_{\tilde{C}_1}(\bar{x})\cong A_7$. Let $\bar{\sigma}$ be an element of $C_{\tilde{C}_1}(\bar{x})$ of order 7. Then $C_{\bar{E}}(\bar{\sigma})=\langle\bar{z}, \bar{x}\rangle$. $N_{\tilde{C}_1}(\langle\bar{\sigma}\rangle)$ acts on $\langle\bar{z}, \bar{x}\rangle$ nontrivially, as otherwise $C_{\tilde{C}_1}(\bar{x})\cong S_7$. Hence, $\bar{x}\sim\bar{x}\bar{z}$. On the other hand, one can check directly that no element y of $E-F$ is conjugate to yz in T. This contradiction shows that $\bar{E}-\bar{F}$ contains an element having precisely eight conjugates under the action of $\tilde{C}_1$. Since the eight conjugate must span $\bar{E}$, the representation of $\tilde{C}_1$ on $\bar{E}$ is the natural permutation representation. This completes the proof.

5. The Structure of $C_G(z)/O(C_G(z))$.

In this section, we show that $C_G(z)/O(C_G(z))$ is isomorphic to the corresponding group in A_{18} (or in A_{19}).

If G has a normal subgroup K of index 2, then $K\cap N/O(K\cap N)\cong E_{16}\cdot L_4(2)$ and so $K/O(K)\cong A_{16}$ or A_{17} by Theorem 2.1 and Theorem 3.1. So *we henceforth assume that G does not possess a normal subgroup of index* 2.

Set $I=C_G(z)$ and $\bar{I}=I/O(I)$. As in § 4, we set $C_1=I\cap N_G(E)$. By Lemma 4.6, the structure of $C_1/O(C_1)$ is uniquely determined. Hence we may assume that $\bar{I}$ contains a subgroup generated by involutions $\bar{a}_i$, $\bar{b}_i$, $\bar{z}_i$ $(1\leqq i\leqq 4)$ and $\bar{\sigma}_j$ $(1\leqq j\leqq 3)$ satisfying the following relations:

$$(\bar{a}_j\bar{\sigma}_j)^3\equiv(\bar{\sigma}_j\bar{a}_{j+1})^3\equiv 1,\ [\bar{\sigma}_j, \bar{b}_{j+1}]\equiv\bar{b}_j\bar{b}_{j+1}\bar{z}_j,\ 1\leqq j\leqq 3,$$

$$[\bar{a}_i, \bar{b}_j]\equiv\bar{z}_i,\ (1\leqq i\leqq 4),\ \mathrm{mod}\ O(\bar{C}_1)$$

with all other commutators of pairs of generators being trivial mod $O(\bar{C}_1)$ (*Note.* $O(C_1)$ may not be in $O(I)$.). We also put $\bar{u}=\bar{\sigma}_1(\bar{a}_1\bar{a}_2)\bar{\sigma}_1$, $\bar{t}=\bar{u}\bar{\sigma}_3(\bar{a}_3\bar{a}_4)\bar{\sigma}_3$, $\bar{\xi}=\bar{u}\bar{\sigma}_2(\bar{a}_2\bar{a}_3)\bar{\sigma}_2$ and $\bar{v}=\bar{\xi}^{-1}\bar{t}\bar{\xi}$. If we choose representatives a_i, b_i, u, v suitably, we may assume that $T=\langle a_i, b_i, u, v|1\leqq i\leqq 4\rangle$ is a Sylow 2-subgroup of G. We may also assume that ξ is a 3-element in $I\cap N_G(J)$, $J=\langle a_i, b_j|1\leqq i\leqq 4\rangle\cong D_8\times D_8\times D_8\times D_8$, and that ξ: $a_1\to a_2\to a_3$, $b_1\to b_2\to b_3$, $a_4\to a_4$, $b_4\to b_4$. We note that J is so called the Thompson subgroup of T. The conjugacy of elements in $Z(J)$ is controlled by $N_G(J)$.

LEMMA 5.1. *The following condition holds*:

(i) *The representatives of the conjugacy classes of involutions in* $N_I(E)$ *are the following*:

$$z_1,\ a_1,\ b_1,\ z_1z_2,\ b_1z_2,\ a_1z_2,\ a_1b_2,\ a_1a_2,\ z_1z_2z_3,\ b_1z_2z_3,\ a_1a_2b_3,\ a_1a_2z_3,\ a_1z_2z_3,$$

$$a_1a_2a_3,\ a_1b_2z_3,\ z,\ b_1z,\ a_1z,\ a,\ a_1a_2a_3z,\ a_1a_2z,\ a_1b_2z,\ a_1a_2a_3b_4 \text{ and } a_1a_2b_3z.$$

(ii) $N_G(J)$ *acts on the set* $\{z_1, z_2, z_3, z_4\}$ *and any two of* z_1, z_1z_2, $z_1z_2z_3$ *and* z *are not conjugate in* G.

(iii) *If* $D=\langle a_i, z_i|1\leqq i\leqq 4\rangle$; *then* $O(N_G(D)/C_G(D))$ *is trivial or an elementary abelian group of order* 81. *If* $O(N_G(D)/C_G(D))=1$, *then a Sylow 2-subgroup of* $N_G(D)/C_G(D)$ *normalizes no nontrivial normal subgroup of odd order. If* $O(N_G(D)/C_G(D))=\langle\tilde{x}_i|\tilde{x}_i^3=1,\ 1\leqq i\leqq 4\rangle\cong Z_3\times Z_3\times Z_3\times Z_3$, *then we may assume that* $\tilde{x}_i$: $z_i\to a_i\to a_iz_i$ $(1\leqq i\leqq 4)$, $[\tilde{x}_i, \langle a_j, z_j\rangle]=1$ $(i\neq j)$, $\tilde{b}_i\tilde{x}_i\tilde{b}_i=\tilde{x}_i^{-1}$ $(1\leqq i\leqq 4)$, $[\tilde{b}_j, \tilde{x}_i]=1$ $(i\neq j)$, $[\tilde{\xi}, \tilde{x}_4]=1$ *and* $\tilde{\xi}$: $\tilde{x}_1\to\tilde{x}_2\to\tilde{x}_3$, *where* x_i*'s are suitable* 3-*elements*.

PROOF. (i) As $N_I(E)/O(N_I(E))\cong W(B_8)$, we may apply [10; (1.3)].
(ii) (resp. (iii)) follows from [9; (2.2)] (resp. [12; Lemma 4.4]).

LEMMA 5.2. $N_G(E)/C_G(E)\cong S_9$.

PROOF. Set $\bar{N}=N_G(E)/C_G(E)$. Then $\bar{T}=\bar{D}\langle\bar{u}, \bar{v}\rangle$ is of type S_8 and $\bar{T}'\langle\bar{u}, \bar{v}\rangle$ is of type A_8. Since $\bar{a}_1$ centralizes a space of dimension 7 of E but no involution of $\bar{T}'\langle\bar{u}, v\rangle$ has this property, $\bar{a}_1$ is not conjugate to any element of $\bar{T}'\langle\bar{u}, \bar{v}\rangle$. Hence $\bar{N}\supset O^2(\bar{N})$ by the Thompson transfer lemma. As $\bar{N}$ contains S_8, [4] yields $O^2(\bar{N})/O(O^2(\bar{N}))\cong A_8$, or A_9 and so $\bar{N}/O(\bar{N})\cong S_8$ or S_9. From $|GL_8(2)|_{2'}=3^5\cdot 5^2\cdot 7^2\cdot 17\cdot 31\cdot 127$, we must have $\bar{N}'O(\bar{N})=O(\bar{N})\times\bar{L}$ where $\bar{L}\cong A_8$ or A_9. Since $\bar{L}$ contains a subgroup isomorphic to A_8 which comes from $N(E)\cap C_G(z)$, the same proof as in Lemma 4.5 applies to show that $O(\bar{N})=1$.

Suppose $\bar{N}\cong S_8$. Then $\bar{N}=\overline{N\cap C_G(z)}$ and so N has a normal subgroup of index 2 not containing E. As E is weakly closed subgroup of T, Theorem 2.3 shows that $G\supset O^2(G)$. But this is not our case. This completes the proof.

LEMMA 5.3. *The structure of $N_G(E)/O(N_G(E))$ is uniquely determined. Moreover, renaming the generators if necessary, we may assume that* $z_1\sim b_1$, $z_1z_2\sim b_1z_2$, $z_1z_2z_3\sim b_1z_3z_4$, $z\sim b_1z$, $a_1z_2\sim a_1b_2$, $a_1a_2z_3\sim a_1a_3b_4$, $a_1z_2z_3\sim a_1b_2z_3$, $a_1b_2z\sim a_1z$, $a_1a_3b_3z\sim a_1a_2z$, $a_1a_2a_3b_3\sim a_1a_2a_3z$.

PROOF. As T splits over E, so does $\bar{N}=N_G(E)/O(N_G(E))$ over $\bar{E}$. Since A_9 can not act nontrivially on an elementary abelian group of order 2^7 [6, Lemma 2.8], $\bar{N}$ is irreducible on $\bar{E}$. We already know that $\overline{N\cap C_G(z)/E}\cong S_8$ and so $\bar{E}$ has 9 conjugates $\{\bar{z}, \bar{x}_i, \bar{x}_2, \cdots, \bar{x}_8\}$ under the action of $\bar{N}/\bar{E}\cong S_9$. As $\overline{N\cap C_G(a)}/\bar{E}\cong S_8$ is transitive on $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_8\}$, $\bar{x}_i\notin\bar{F}$ for all $1\leqq i\leqq 8$. Since $\langle\bar{z}\rangle$ and $\bar{F}$ are the only $\overline{N\cap C_G(z)}$ invariant proper subgroups of $\bar{E}$, $\langle\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_8\rangle=\bar{E}$ and $\bar{z}=\bar{x}_1+\cdots+\bar{x}_8$. Clearly then the action of $\bar{N}$ on $\bar{E}$ is obtained by the natural permutation representation of S_9 on a 9-dimensional space modulo the unique 1-dimensional trivial space. Thus the structure of N is uniquely determined. The fusion of involution may be obtained by a direct computation. We omit the detail.

PROPOSITION 5.4. $N_G(D)/C_G(D)\cong S_3\wr S_4$. *Moreover, renaming the generators if necessary, we may assume* $z_1\sim a_1\sim b_1$, $z_1z_2\sim a_1a_2\sim a_1z_2\sim b_1z_2\sim a_1b_3$, $z_1z_2z_3\sim a_1a_2z_3\sim a_1z_2z_3\sim a_1a_2a_3\sim b_1z_3z_4\sim a_1a_2b_3\sim a_1b_2z_3$, $z\sim a_1z\sim a\sim a_1a_2a_3z\sim a_1a_2z\sim b_1z\sim a_1b_2z\sim a_1a_3b_4\sim a_1a_2b_3z$.

PROOF. Set $\bar{N}=N_G(D)/C_G(D)$. By Lemma 5.1 (iii), $|O(\bar{N})|=3^4$ or 1. In the former case, [12; Lemma 4.5] yields $\bar{N}\cong S_3\wr S_4$, since $\xi\in N_G(D)-C_G(D)$. Suppose $O(\bar{N})=1$. The representatives $\bar{x}$ of the conjugacy classes of involutions in $\bar{T}$ and the order of $|[\bar{x}, D]|$ are given below (Table I).

$\|[\bar{x}, D]\|$	$\bar{x}$
2	$\bar{b}_1$
4	$\overline{b_1b_2}$, $\overline{b_1b_3}$, $\bar{u}$
8	$\overline{b_2b_3b_4}$, $\overline{b_1ut}$
16	$\bar{b}$, $\overline{ub_3b_4}$, $\bar{t}$, $\overline{vt}$

Table I

By the table above, one sees that $\bar{b}_1$ is not conjugate to any involution of $\langle\overline{b_1b_2}, \overline{b_2b_3}, \overline{b_3b_4}, \bar{t}, \bar{u}, \bar{v}\rangle$. Hence $O^2(\bar{N})\subset\bar{N}$.

Since $N_G(E)/C_G(E)\cong S_9$ and $D\nsubseteq N_G(E)'$, we see that $\overline{N\cap N_G(E)}$ is an extension of an elementary abelian group of order 16 by S_4. If $|\bar{N}: O^2(\bar{N})|\geqq 4$, then the index must be exactly four and $\bar{Q}=\langle\overline{b_1b_2}, \overline{b_2b_3}, \overline{b_3b_4}, \bar{v}, \bar{t}\rangle=\langle\overline{vb_1b_2}, \overline{tb_1b_3}\rangle*\langle\overline{vtb_1b_2}, \overline{tb_2b_3}\rangle\cong Q_8*Q_8$ is a Sylow 2-subgroup of $O^2(\bar{N})$. Hence $\langle\bar{b}\rangle=Z(\bar{N})$. Since $\overline{b_1b_2}\not\sim\bar{t}$ in $\bar{N}$, 9 does not divide $|N_{\bar{N}}(\bar{Q})/C_{\bar{N}}(\bar{Q})|$. Thus by [4; proposition 3.1] applied to $O^2(\bar{N})\langle\bar{v}\rangle$, we conclude that $\bar{N}=\overline{N\cap N_G(E)}$. But then $N_G(D)$ contains a normal subgroup of index 2 which does not contain D. As D is weakly closed in T, Theorem 2.3 yields a contradiction.

Thus we have shown $|\bar{N}: O^2(\bar{N})|=2$. By Table 1, $\langle\overline{b_1b_2}, \overline{b_2b_3}, \overline{b_3b_4}, \bar{t}, \bar{u}, \bar{v}\rangle$, which is of type A_8, must be a Sylow 2-subgroup of $O^2(\bar{N})$. Since 9 does not divide $|N_{\bar{N}}(\bar{Q})/C_{\bar{N}}(\bar{Q})|$, $\bar{N}\cong S_8$ or S_9 must hold by the main theorem of [4].

Suppose $\bar{N}\cong S_8$. We shall show that $N_G(D)/O(N_G(D))\cong C_{A_{18}}((1, 2)\cdots(15, 16))$. By the argument in Lemma 4.6, it suffices to show that $\bar{N}$ centralizes a nontrivial subgroup of D. We know that $\overline{N\cap N_G(E)}=C_{\bar{N}}(\bar{b})$ centralizes $z\in D$. On the other hand, $\langle\bar{b}, \bar{t}, \bar{v}\rangle$ is a self-centralizing elementary abelian subgroup of order 8 all of whose involutions are conjugate in $\bar{N}$. Hence $N_{\bar{N}}(\langle\bar{b}, \bar{t}, \bar{v}\rangle)/\langle\bar{b}, \bar{t}, \bar{v}\rangle\cong L_3(2)$. Since $C_D(\langle\bar{b}, \bar{t}, \bar{v}\rangle)=\langle z\rangle$, $C_{\bar{N}}(z)\supseteqq\langle C_{\bar{N}}(\bar{b}), N_{\bar{N}}(\langle\bar{b}, \bar{t}, \bar{v}\rangle)\rangle=\bar{N}$, as desired. Thus $N_G(D)/O(N_G(D))\cong C_{A_{18}}((1, 2)\cdots(15, 16))$. But then D is not contained in some normal subgroup of N of index 2. Theorem 2.3 again yields a contradiction.

Suppose $\bar{N}\cong S_9$. Then N' has the Sylow 2-subgroups of type A_{16} and $\bar{N}'\cong A_9$. So we may apply [12; Theorem 4.9] to obtain $z\not\sim a$ in G. This conflicts with $A=\langle z, a, v, t\rangle$ and $N_G(A)/C_G(A)\cong A_8$. Hence $\bar{N}\cong S_3\wr S_4$ is the unique possibility. The fusion pattern of involutions of G follows from [9].

Now we are in the position to prove:

PROPOSITION 5.5. $C_G(z)/O(C_G(z))\cong C_{A_{18}}((1, 2)\cdots(15, 16))$.

PROOF. Set $I=C_G(z)$ and $Q=T'\langle u, v\rangle$. We have $T\triangleright Q$, $T/Q\cong Z_2\times Z_2$ and $T=\langle Q, a_1, b_1\rangle$. Since $v\sim a=a_1a_2a_3a_4\in T'$ and $u\sim a_1a_2x$, $x\in T$, Q is contained in $O^2(I)$. As $N_I(E)/O(N_I(E))\cong C_{A_{18}}((1, 2)\cdots(15, 16))$, $\bar{b}_1\notin O^2(I)$ by Theorem 2.3. Moreover, by Lemma 5.1 (iii) and Proposition 5.4, $N_I(D)/C_I(D)$ is an extension of E_{16} by S_4. In particular, $N_I(E)$ covers $N_I(D)/O(N_I(D))$. As $a_1\notin N_I(E)'$, $a_1\notin O^2(I)$ again by Theorem 2.3.

Suppose that $Q\langle a_1b_1\rangle$ is a Sylow 2-subgroup of $O^2(I)$. Then a_1b_2 is conjugate to an element of Q and by Lemma 5.1 and Proposition 5.4, we may assume that $a_1b_2\sim z_1z_2$, b_1z_2, a_1z_2 or a_1a_2 in I. It would then follow that $z\sim a_1b_2z\sim(a_1b_2)^hz$ with $h\in I$. Hence, $z\sim z_1z_2z$, b_1z_2z, a_1z_2z or a_1a_2z. None of the four conjugacies above is possible by Proposition 5.4. Hence Q is a Sylow 2-subgroup of $O^2(I)$.

We next show that $F=Q\cap E$ is strongly closed in Q with respect to I. The involutions of F split into four conjugacy classes under the action of $N_I(E)$. Moreover, a_1a_2 and $a_1a_2z_3$ are the representatives of conjugacy classes in $N_I(E)$ of involutions in $Q-F$ which are not conjugate to z in G. If $(a_1a_2)^x=z_1z_2$ (resp. $(a_1a_2z_3)^x=z_1z_2z_3$) for some $x\in I$, then $(a_1a_2z)^x=z_3z$ (resp. $(a_1a_2z_3z)^x=z_4$). This is impossible by Proposition 5.4. Hence F is strongly closed, as desired. Since $N_I(E)/C_I(E)\cong S_8$, $FO(I)\triangleleft I$ by [3]. Put $\bar{I}=I/O(I)$. Then $C_{\bar{I}}(\bar{F})\triangleleft\bar{I}$ and $\bar{E}\in\mathrm{Syl}_2(C_{\bar{I}}(\bar{F}))$. As $|E:F|=2$, $\bar{E}=C_{\bar{I}}(\bar{F})\triangleleft\bar{I}$ and $\bar{I}/\bar{E}\cong S_8$. This completes the proof.

REMARK. When all simple groups having involution z with $O(C_G(z)\neq 1$ are classified, we may quote the result to show that our group G is isomorphic to A_{18} or A_{19}.

LEMMA 5.6. $C_G(z_1)/O(C_G(z_1))\cong C_{A_{18}}((1, 2)(3, 4))$ *or* $C_{A_{19}}((1, 2)(3, 4))$.

PROOF. Put $C=C_G(z_1)$ and $\bar{C}=C/O(C)$. $C_T(z_1)=J_1\times J_2\times(J_3\times J_4)\langle ut\rangle$ is a Sylow 2-subgroup of C. By Theorem 6.1 (the proof is independent to the previous sections) and Proposition 5.4 $\langle\bar{a}_1, \bar{z}_1\rangle$ is strongly closed in $C_{\bar{T}}(\bar{z}_1)$ with respect to $\bar{C}$ and hence $\langle\bar{a}_1, \bar{z}_1\rangle\triangleleft\bar{C}$ by [2]. Gaschütz's theorem yields $C_{\bar{C}}(\langle a_1, z\rangle)=\langle\bar{a}_1, \bar{z}_1\rangle\times\bar{X}$ where $\bar{X}$ is a group with Sylow 2-subgroups of Type A_{14}. Since $\bar{X}$ contains $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{\sigma}_2$, $\bar{\sigma}_3$, $\bar{\sigma}_4$ and $\bar{X}$ has the involution fusion pattern of A_{14}, $\bar{X}\cong A_{14}$ or A_{15} by [1, 11]. Thus $\bar{C}=(\langle\bar{a}_1, \bar{z}_1\rangle\times\bar{X})\langle\bar{b}_1\bar{b}_2\rangle\cong C_{A_{18}}((1, 2)(3, 4))$ or $C_{A_{19}}((1, 2)(3, 4))$. The lemma is proved.

LEMMA 5.7. $C_G(z_1z_2)/O(C_G(z_1z_2))\cong C_{A_m}((1, 2)\cdots(7, 8))$ *if and only if* $C_G(z_1)/O(C_G(z_1))\cong C_{A_m}((1, 2)(3, 4))$ *where* $m=18$ *or* 19.

PROOF. Put $C=C_G(z_1z_2)$. By Lemma 5.6 $\bar{C}=C/O(C)$ contains a subgroup isomorphic to A_{10} or A_{11} in which $(\bar{J}_3\times\bar{J}_4)\langle\bar{u}\bar{t}\rangle$ is a Sylow 2-subgroup. Since all involutions in $\langle\bar{z}_1, \bar{z}_2, \bar{b}_1\bar{b}_2\rangle-\langle\bar{z}_1\bar{z}_2\rangle$ are conjugate in $\bar{C}$ it follows from the structure of $N_C(E)$ and $N_C(D)$ that $\langle \bar{z}_1, \bar{z}_2, \bar{b}_1\bar{b}_2\rangle$ is strongly closed in $C_T(z_1z_2)$ with respect to C. The result follows from [2].

LEMMA 5.8. *$C_G(z_1z_2z_3)/O(C_G(z_1z_2z_3))\cong C_{A_m}((1, 2)\cdots(11, 12))$ if and only if $C_G(z_1)/O(C_G(z_1))\cong C_{A_m}((1, 2)(3, 4))$ where $m=18$ or 19.*

PROOF. Put $C=C_G(z_1z_2z_3)$ and $\bar{C}=C/O(C)$. By Proposition 5.4 J_4 is strongly closed in $C_T(z_1z_2z_3)\in \mathrm{Syl}_2(C)$ with respect to C and $\langle J_4^C\rangle$ has dihedral Sylow 2-subgroups. Since $\overline{C_C(z_1)}\supseteq\langle \bar{J}_4^{\bar{C}}\rangle$ and $\overline{C_C(z_1)}^{(\infty)}\cong A_6$ or A_7 by Lemma 5.6 we have $\langle \bar{J}_4^{\bar{C}}\rangle\cong A_6$ or A_7 by [3]. As $\langle \bar{J}_4^{\bar{C}}\rangle\lhd\bar{C}$ the result follows from Lemma 5.6. The lemma is proved.

6. Localization of 2-fusion.

Let G be a finite group with a Sylow 2-subgroup T of Type A_{18} and X be a subgroup of G. The purpose of this section is to prove

THEOREM 6.1. *Let P be a Sylow 2-subgroup of X. Suppose $P=T$, $J\langle u\rangle$ or $J\langle u, t\rangle$. Then the fusion of the subsets of P in X is controlled by $N_X(D)\cup N_X(E)\cup C_X(Z(P))$.*

We carry out the proof in a sequence of lemmas. Let $\mathcal{H}(P)$ be the set of subgroups H of P satisfying the conditions:

(1) $H=P\cap Q$ is a tame Sylow intersection for some $Q\in \mathrm{Syl}_2(X)$,

(2) $C_P(H)\subseteq H$,

(3) $H\in \mathrm{Syl}_2(O_{2',2}(N_X(H)))$,

(4) $H=P$ or $N_X(H)/H$ is 2-isolated.

Let $\mathcal{F}(P)$ be the set of all pairs (H, R) with $H\in\mathcal{H}(P)$ and $R=N_X(H)$ if $H=C_P(\Omega_1(Z(H)))$ or $R=N_X(H)\cap C_X(\Omega_1(Z(H)))$ if $H\subset C_P(\Omega_1(Z(H)))$. Let $\mathcal{F}'(P)$ be the set of pairs $(H, C_X(H))$ where H satisfies (1) but not all of (2)-(4). Then $\mathcal{F}(P)\cup\mathcal{F}'(P)$ is an inductive family. Put $N=N_X(H)/H$ and $L=\Omega_1(Z(H))$. Suppose $H\subset P$. Then by (4) either $N_P(H)/H$ has 2-rank 1 or $N/O(N)$ contains a normal subgroup of odd index isomorphic to one of the groups $L_2(2^n)$ $(n>2)$, $U_3(2^n)$ $(n>2)$, $S_z(2^{2n+1})$ $(n>1)$. If $H\subset C_P(L)$, then (2) yields $R=N_X(H)\cap C_X(L)\subseteq C_X(\Omega_1(Z(P)))=C_X(Z(P))$. We shall prove this theorem by surveying these subgroups $H\in\mathcal{H}(P)$ such that $N_X(H)\not\subseteq N_X(D)\cup N_X(E)$. Since D and E are weakly closed in T, $D\not\subseteq H$ and $E\not\subseteq H$. Now $H\subseteq C_P(x)$ for some involution $x\in P-D\cup E$ by (2). Our argument depends upon only the structure of P and hence we can assume that x is a representative of Aut(P)-associated classes of involutions. Put $H_0=C_P(x)$.

Case 1. $P=T$.

It is $C_T(x) \subseteqq C_T(a_1b_2)$ for $x \in \{a_1b_2b_3, a_1b_2b_3b_4, a_1b_2z_3\}$ and $C_T(x) \subseteqq C_T(a_1b_3)$ for $x \in \{a_1b_3z_2, a_1a_4b_3, a_1b_3z_4\}$. Thus we can assume that x is one of the following elements.

x	$C_T(x)$	$Z(C_T(x))$
u	$\langle u, t\rangle(J_3 \times J_4) \times \langle a_1a_2, b_1b_2\rangle$	$\langle u, z_1z_2, z_3z_4\rangle$
v	$\langle v, t\rangle(\langle a_1a_4, b_1b_4\rangle \times \langle a_2a_3, b_2b_3\rangle)$	$\langle v, z\rangle$
t	$\langle v, u\rangle(\langle a_1a_2, b_1b_2\rangle \times \langle a_3a_4, b_3b_4\rangle)$	$\langle t, z\rangle$
a_1b_2	$\langle tu\rangle(\langle a_1, z_1, b_2, z_2\rangle \times J_3 \times J_4)$	$\langle a_1, z_1, b_2, z_2, z_3z_4\rangle$
a_1b_3	$\langle a_1, z_1\rangle \times J_2 \times \langle b_3, z_3\rangle \times J_4$	$\langle a_1, b_3, Z\rangle$
$a_1a_2b_3$	$\langle u\rangle(\langle a_1, z_1, a_2, z_2, b_3, z_3, b_4, z_4\rangle \times J_4)$	$\langle a_1a_2, z_1z_2, b_2, z_3, z_4\rangle$
$a_1a_2b_3b_4$	$\langle u, t\rangle\langle a_1, z_1, a_2, z_2, b_3, z_3, b_4, z_4\rangle$	$\langle a_1a_2, z_1z_2, b_2b_3, z_3z_4\rangle$
$a_1b_2b_3a_4$	$\langle v\rangle\langle a_1, z_1, b_2, z_2, b_3, z_3, a_4, z_4\rangle$	$\langle a_1a_4, z_1z_4, b_2b_3, z_2z_3\rangle$

LEMMA 6.2. *H is not an elementary abelian group of order 2^8.*

PROOF. Suppose $H \cong E_{256}$. Then $H \lhd J$ and $(N_T(H): J) \geqq 2$. This is impossible by (4).

LEMMA 6.3. $H \not\subseteqq C_T(u)$.

PROOF. As $[z_1, H_0] = \langle z_1z_2\rangle \subseteqq H$, z_1 stabilizes $H_0 \supset H_0' \supset 1$ and $H \subset H_0$. It follows that $z_1z_2 \notin H'$ and $\langle a_1a_2, b_1b_2\rangle \not\subseteqq H$. We can assume $b_1b_2 \notin H$ and $H \subseteqq C_{H_0}(a_1a_2) = C$. As $[t, C_C(z_3)] \cong D_8$ and $[z_1, C_C(z_3)] \cong Z_2$, $z_3 \not\equiv L$. Since $\Omega_1(Z(C)) \cong E_{16}$ and $C' \cong E_8$, $H \subset C$. Thus we may assume $H \subseteqq \langle u, t\rangle(\langle a_1a_2, z_1z_2\rangle \times \langle a_3a_4, b_3b_4\rangle) \cong E_{16} \times D_8$. This is also impossible since all involutions in N are conjugate.

LEMMA 6.4. $H \not\subseteqq C_T(v)$.

PROOF. As $[z_1, H_0] = \langle z_1z_2\rangle \subseteqq H$, z_1 stabilizes $H_0 \supset H_0' \supset 1$ and $H \subset H_0$. Put $F = \langle v\rangle \times \langle a_1a_4, b_1b_4\rangle \times \langle a_2a_3, b_2b_3\rangle$ and assume $H \subseteqq F$. As $F' \subseteqq Z(F) \subseteqq H$, $H \lhd F$. If $H \cong E_{32}$, then $[w, H] \cong Z_2 \times Z_2$ for some involution $w \in F - H$, a contradiction since $[z_1z_2, H] = \langle z\rangle$. Since z_1z_2 stabilizes $F \supset F' \supset 1$, $H \cong E_8 \times D_8$. As $H' \neq 1$, $L_2(4)$ acts trivially on L, a contradiction. Now $H \cap tF \neq \varnothing$. If some conjugate of t is contained in L, $H \cong E_{16}$ or $Z_2 \times Z_2 \times D_8$. Clearly they are impossible. It follows that $tF \cap L = \varnothing$ and $L \subseteqq \langle v, t, a, b\rangle$. As $H \subset H_0$, $L \ni a, b, ta$ or tb. Since z_1z_2 cannot stabilize any critical chain of H, L contains ta or tb. It follows that H is a subgroup of $Z_2 \times Z_2 \times D_8$, a contradiction.

LEMMA 6.5. *$H = C_T(t)$ and $H \not\subset C_T(t)$.*

PROOF. If $H = H_0$, then $N_X(H) \subseteqq N_X(H'') = C_X(z)$. Thus $H \subset H_0$. By Lemmas 6.3 and 6.4 $L \cap (\langle a_1a_2, b_1b_2\rangle \times \langle a_3a_4, b_3b_4\rangle) \neq 1$ and we can assume that a or

$a_1a_2b_3b_4$ is contained in L. Suppose $H \subseteqq C_{H_0}(a)$. If $H=C_{H_0}(a)$, then $[\langle b, z_1z_2\rangle, H] \subseteqq H$ and $L_2(4)$ acts trivially on $L \cong E_8$, a contradiction. Lemma 6.2 yields $vJ(C_{H_0}(a)) \cap H \neq \emptyset$ and $H=C_{H_0}(a)$, a contradiction. If $H \subseteqq C_{H_0}(a_1a_2b_3b_4)$, then $u \in L$ which is impossible by Lemma 6.3.

LEMMA 6.6. (i) $H \not\subseteqq C_T(a_1b_2)$, (ii) $H \not\subseteqq C_T(a_1b_3)$, (iii) $H \not\subseteqq C_T(a_1a_2b_3)$, (iv) $H \not\subseteqq C_T(a_1a_2b_3b_4)$, (v) $H \not\subseteqq C_T(a_1b_2b_3a_4)$.

PROOF. (i) It is $[\langle b_1, a_2\rangle, H] \subseteqq H$ and $L_2(4)$ is involved in N. This is impossible since $|C_H(b_1)| \neq |C_H(b_1a_2)|$. (ii) As $[\langle b_1, a_3, vt\rangle, H_0] \subseteqq H_0$, $H \subset H_0$. Lemma 6.2 yields $H \cong E_{64} \times D_8$, a contradiction. (iii) Since $[\langle a_3, b_1b_2\rangle, H_0] \subseteqq H_0$ and $|C_{H_0}(a_3)| \neq |C_{H_0}(b_1b_2)|$, $H \subset H_0$ and $H \not\supseteqq \langle z_1, z_2\rangle$. As $C_{H_0}(z_1)=C_{H_0}(z_2)$, $uJ(C_T(a_1a_2b_3)) \cap L \neq \emptyset$ which contradicts Lemma 6.3. (iv), (v) By the similar way as (i)-(iii) we can prove (iv) and (v).

Case 2. $P=J\langle u, t\rangle=\langle u\rangle(J_1 \times J_2) \times \langle ut\rangle(J_3 \times J_4)$.

Clearly $H \ncong E_{256}$. For $y=a_1b_3a_4$, $b_1a_2z_3$, $b_1a_2a_3$, $b_1a_2a_3a_4$, uz_3, ua_3, ub_3a_4, t, $C_P(y)$ is contained in $C_P(b_1a_2)$, $C_P(u)$ or $C_P(a_1b_3b_4)$. Thus we can assume that x is one of the following elements.

x	$C_P(x)$	$Z(C_P(x))$
b_1a_2	$\langle b_1, z_1, a_2, z_2\rangle \times (J_3 \times J_4)\langle ut\rangle$	$\langle b_1, z_1, a_2, z_2, z_3z_4\rangle$
u	$\langle a_1a_2, b_1b_2\rangle \times \langle u\rangle \times (J_3 \times J_4)\langle ut\rangle$	$\langle z_1z_2, u, z_3z_4\rangle$
a_1b_3	$\langle a_1, z_1\rangle \times J_2 \times \langle b_3, z_3\rangle \times J_4$	$\langle a_1, b_3, Z\rangle$
$a_1b_3b_4$	$\langle a_1, z_1\rangle \times J_2 \times \langle b_3, z_3, b_4, z_4\rangle\langle ut\rangle$	$\langle a_1, z_1, z_2, b_3b_4, z_3z_4\rangle$
$a_1a_2b_3b_4$	$\langle a_1, z_1, a_2, z_2\rangle\langle u\rangle \times \langle b_3, z_3, b_4, z_4\rangle\langle ut\rangle$	$\langle a_1a_2, z_1z_2, b_3b_4, z_3z_4\rangle$
$a_1a_2b_3a_4$	$\langle a_1, z_1, a_2, z_2\rangle\langle u\rangle \times \langle b_3, z_3, a_4, z_4\rangle$	$\langle a_1a_2, z_1z_2, b_3, z_3, a_4, z_4\rangle$
a_1a_2ut	$\langle a_1, z_1, a_2, z_2\rangle\langle u\rangle \times \langle a_3a_4, b_3b_4\rangle \times \langle ut\rangle$	$\langle a_1a_2, z_1z_2, z_3z_4, ut\rangle$

LEMMA 6.7. (i) $H \not\subseteqq C_P(b_1a_2)$, (ii) $H \not\subseteqq C_P(u)$, (iii) $H \not\subseteqq C_P(a_1b_3)$, (iv) $H \not\subseteqq C_P(a_1b_3b_4)$, (v) $H \not\subseteqq C_P(a_1a_2b_3b_4)$, (vi) $H \not\subseteqq C_P(a_1a_2b_3a_4)$, (vii) $H \not\subseteqq C_P(a_1a_2ut)$.

PROOF. (i) As $[\langle a_1, b_2\rangle, H] \subseteqq H$, $|C_H(a_1)|=|C_H(a_1b_2)|$. This is impossible. (ii) As $[z_1, H_0]=\langle z_1z_2\rangle \subseteqq H_0'$, z_1 stabilizes $H_0 \supseteqq H_0' \supseteqq 1$ and hence $H \subset H_0$. As $\langle z_1z_2\rangle \not\subseteqq H'$ we may assume $H \subseteqq C_P(\langle u, a_1a_2\rangle)=\langle a_1a_2, z_1z_2, u\rangle \times (J_3 \times J_4)\langle ut\rangle$. This is impossible since $\langle a_1, b_1b_2\rangle \cong D_8$ normalizes H. (iii) As $[\langle b_1, a_3\rangle, H] \subseteqq H$, $|C_H(b_1)|=|C_H(b_1a_3)|$, a contradiction. (iv)-(vii) By the similar way to (i)-(iii) we can get a contradiction. The lemma is proved.

Case 3. $P=J\langle u\rangle=\langle u\rangle(J_1 \times J_2) \times J_3 \times J_4$.

Clearly $H \ncong E_{256}$. It is $C_P(a_1a_2b_3b_4) \subseteqq C_P(a_1a_2b_3)$ and $C_P(b_1a_2a_3)=C_P(b_1a_2a_3z_4)$. Thus we may assume that x is one of the following elements.

x	$C_P(x)$	$Z(C_P(x))$
a_1b_3	$\langle a_1, z_1\rangle\times J_2\times\langle b_3, z_3\rangle\times J_4$	$\langle a_1, b_3, Z\rangle$
$a_1a_3b_4$	$\langle a_1, z_1\rangle\times J_2\times\langle a_3, z_3, b_4, z_4\rangle$	$\langle a_1, a_3, b_4, Z\rangle$
$a_1a_2b_3$	$\langle a_1, z_1, a_2, z_2\rangle\langle u\rangle\times\langle b_3, z_3\rangle\times J_4$	$\langle a_1a_2, z_1z_2, b_3, z_3, z_4\rangle$
$a_1a_2a_3b_4$	$\langle a_1, z_1, a_2, z_2\rangle\langle u\rangle\times\langle a_3, z_3, b_4, z_4\rangle$	$\langle a_1a_2, z_1z_2, a_3, b_4, z_3, z_4\rangle$
$b_1a_3a_4$	$\langle b_1, z_1\rangle\times J_2\times\langle a_3, z_3, a_4, z_4\rangle$	$\langle b_1, a_3, a_4, Z\rangle$
$b_1a_2a_3$	$\langle b_1, z_1, a_2, z_2, a_3, z_3\rangle\times J_4$	$\langle b_1, a_2, a_3, Z\rangle$

By the similar way to Case 2 we can prove Theorem 6.1. So we omit the proof.

7. Subgroups of the minimal counter example.

Let G be a finite group with a Sylow 2-subgroup T of Type A_{18}.

LEMMA 7.1. *Put* $\overline{N_G(D)}=N_G(D)/C_G(D)$ *and* $\widetilde{N_G(E)}=N_G(E)/C_G(E)$. *Let* $\bar{H}$ *(resp.* $\tilde{H}$*) be a subgroup of* $\overline{N_G(D)}$ *(resp.* $\widetilde{N_G(E)}$*) containing* $\bar{T}$ *(resp.* $\tilde{T}$*). Then* (i) $\bar{H}\cong\bar{T}$, $\bar{T}\langle\xi\rangle$, $(S_3\wr Z_2)\wr Z_2$ *or* $S_3\wr S_4$. (ii) $\tilde{H}=\tilde{T}$, $\tilde{T}\langle\tilde{\xi}\rangle$, S_8, S_9 *or* $S_4\wr Z_2$.

PROOF. (i) The representatives $\bar{x}$ of $\bar{T}$-orbit on $O(\overline{N_G(D)})$ are $\bar{x}_1$, $\bar{x}_1\bar{x}_2$, $\bar{x}_1\bar{x}_2\bar{x}_3$ and $\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$. If $\bar{H}\cap O(\overline{N_G(D)})\neq 1$, then $\bar{H}\supseteq O(\overline{N_G(D)})$. If $\bar{H}\cap O(\overline{N_G(D)})=1$, then $\bar{H}=\bar{T}$ or $\bar{T}\langle\xi\rangle$. (ii) The result follows from [3] and Lemma 5.3.

LEMMA 7.2. *Let* H *be a subgroup of* G *with a Sylow* 2*-subgroup* $J\langle u\rangle$. *Then* (i) $\overline{N_H(D)}=N_H(D)/C_H(D)\cong\overline{J\langle u\rangle}$, $\overline{J\langle u, \xi\rangle}$, $D_8\times S_3\times Z_2$, $Z_2\times S_4\times S_3$, $(S_3\wr S_3)\times Z_2$, $(S_3\wr Z_2)\times Z_2\times Z_2$, $(S_3\wr Z_2)\times S_3\times Z_2$, $D_8\times S_3\times S_3$, $(S_3\wr S_3)\times S_3$ *or* $(S_3\wr Z_2)\times S_3\times S_3$. (ii) $\widetilde{N_H(E)}=N_H(E)/C_H(E)\cong\widetilde{J\langle u\rangle}$, $Z_2\times S_7$, $Z_2\times S_6$, $Z_2\times Z_2\times S_5$, $Z_2\times Z_2\times S_4$, $S_3\times S_6$, $S_3\times Z_2\times D_8$ *or* $S_3\times Z_2\times S_4$.

PROOF. (i) Let $\bar{x}$ be the representative of $\overline{J\langle u\rangle}$-orbit of $\langle\bar{x}_i\,|\,1\leqq i\leqq 4\rangle$. Then we have Table II which shows $\langle x_i\,|\,1\leqq i\leqq 4\rangle\cap\overline{N_H(D)}$.

$(\overline{J\langle u\rangle} : C(\bar{x})\cap\overline{J\langle u\rangle})$	$\bar{x}$
2	$\bar{x}_3$, $\bar{x}_4$
2^2	$\bar{x}_1$, $\bar{x}_1\bar{x}_2$, $\bar{x}_3\bar{x}_4$
2^3	$\bar{x}_1\bar{x}_3$, $\bar{x}_1\bar{x}_4$, $\bar{x}_1\bar{x}_2\bar{x}_3$, $\bar{x}_1\bar{x}_2\bar{x}_4$
2^4	$\bar{x}_1\bar{x}_3\bar{x}_4$, $\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$

Table II

(ii) By Burnside's argument $z_4^H\cap Z(J\langle u\rangle)=z_4^H\cap Z(J)=\{z_4\}$. Note that $\{\tilde{a}_i\,|\,1\leqq i\leqq 4\}=\{\tilde{x}\in\widetilde{J\langle u\rangle}\,|\,\tilde{x}^2=1, |[\tilde{x}, E]|=2\}$. Since $[\tilde{a}_4, E]=\langle z_4\rangle$, $\tilde{a}_4{}^{\widetilde{N_G(E)}}\cap\widetilde{J\langle u\rangle}=\{\tilde{a}_4\}$ and $\widetilde{N_H(E)}$

$=(C(\tilde{a}_4)\cap\widetilde{N_H(E)})O(\widetilde{N_H(E)})$ by Z^*-theorem. It follows that $O(\widetilde{N_H(E)})\cong Z_3$ or 1. If $O(\widetilde{N_H(E)})=1$, then $\widetilde{N_H(E)}=C(\tilde{a}_4)\cap\widetilde{N_H(E)}\subseteq Z_2\times S_7$. If $O(\widetilde{N_H(E)})\cong Z_3$, then $\widetilde{N_H(E)}\subseteq S_3\times S_6$. Now the result follows immediately.

LEMMA 7.3. *Let K be a subgroup of G with a Sylow 2-subgroup $J\langle u, t\rangle$. Then* (i) $\overline{N_K(D)}=N_K(D)/C_K(D)\cong\overline{J\langle u, t\rangle}$, $D_8\times(S_3\int Z_2)$ *or* $(S_3\int Z_2)\times(S_3\int Z_2)$. (ii) $\widetilde{N_K(E)}=N_K(E)/C_K(E)\cong\widetilde{J\langle u, t\rangle}$, $D_8\times S_4$, $D_8\times S_5$, $S_4\times S_4$ *or* $S_4\times S_5$.

PROOF. (i) Straightforward since $\xi\notin N_K(D)$. (ii) It is $Z(J\langle u, t\rangle)=\langle z_1z_2, z_3z_4\rangle$ and $z\not\sim z_1z_2$ by Lemma 5.1. It follows that $N_K(J\langle u, t\rangle)\subseteqq C_K(Z(J\langle u, t\rangle))$ and $z_1z_2\not\sim z_3z_4$ by Burnside's argument. As $N_K(J)/JC_K(J)\cong Z_2\times Z_2$, $z_1\sim z_2\not\sim z_3\sim z_4$ in K. Therefore $\{z_1z_2, z_3z_4\}^K\cap\{z_1z_3, z_1z_4, z_2z_3, z_2z_4\}=\emptyset$. Since $\{\tilde{y}\in\widetilde{J\langle u, t\rangle}\,|\,\tilde{y}^2=1, |[\tilde{y}, E]|=2\}=\{\tilde{u}, \tilde{u}\tilde{a}_1\tilde{a}_2, \tilde{u}\tilde{t}, \tilde{u}\tilde{t}\tilde{a}_3\tilde{a}_4, \tilde{a}_i\tilde{a}_j|i\neq j\}$, $\langle\tilde{a}_1\tilde{a}_2, \tilde{u}\rangle$ and $\langle\tilde{a}_3\tilde{a}_4, \tilde{u}\tilde{t}\rangle$ are strongly closed in $\widetilde{J\langle u, t\rangle}$ with respect to $\widetilde{N_K(E)}$. As $\tilde{a}_i$ $(1\leqq i\leqq 4)$ are the only involutions in $\widetilde{J\langle u, t\rangle}$ which centralize 7-dimensional subspaces, $\langle\tilde{a}_1, \tilde{u}\rangle$ and $\langle\tilde{a}_3, \tilde{u}\tilde{t}\rangle$ are strongly closed in $\widetilde{J\langle u, t\rangle}$. By [3], $\langle\tilde{a}_1, \tilde{u}\rangle^{\widetilde{N_K(E)}}\times\langle\tilde{a}_3, \tilde{u}\tilde{t}\rangle^{\widetilde{N_K(E)}}\subseteq S_4\times S_5$. The lemma is proved.

Henceforth we assume that G is a minimal counter-example to our theorem. We shall determine the possible structures of the proper subgroups of G containing $J\langle u\rangle$ which depend only on $N(E)$ and $N(D)$ by Lemma 5.3, Proposition 5.5 and Theorem 6.1.

PROPOSITION 7.4. *Let H be a proper subgroup of G containing T. Put $\bar{H}=H/O(H)$. Then $\bar{H}$ is isomorphic to S_{16}, S_{17}, A_{18}, A_{19}, $S_8\int Z_2$, $S_9\int Z_2$, a subgroup of $N_G(E)/O(N_G(E))$ or a subgroup of $N_G(D)/O(N_G(D))$.*

PROOF. By Proposition 5.5 and Lemma 7.1 we can assume $H\neq C_H(z)O(H)$. If $\bar{H}$ is fusion simple, then by the minimality of G, $\bar{H}\cong A_{18}$ or A_{19}. Thus we can assume $O^2(\bar{H})\subset\bar{H}$. If $N_H(D)\subseteqq N_H(E)$ (resp. $N_H(E)\subseteqq N_H(D)$), then E (resp. D) is strongly closed in T with respect to H by Theorem 6.1. Proposition 5.4 and Lemma 7.1 yield $DO(H)\triangleleft H$ or $EO(H)\triangleleft H$ by [2]. Assume $N_H(D)\not\subseteqq N_H(E)$ and $N_H(E)\not\subseteqq N_H(D)$. If $N_H(E)/C_H(E)\cong S_8$ and $N_H(D)/C_H(D)\cong S_3\int S_4$, then $\bar{H}\cong S_{16}$ or S_{17} by [12]. Finally suppose that $N_H(D)/C_H(D)\cong(S_3\int Z_2)\int Z_2$ and $N_H(E)/C_H(E)$ $\cong S_4\int Z_2$. Then $a_1a_2E\sim uE\sim ua_1a_2E$ in $N_H(E)$ and $\mathrm{Foc}_{\bar{H}}(\bar{T})=\langle\bar{b}_1\bar{b}_2, \bar{b}_2\bar{b}_3, \bar{b}_3\bar{b}_4, \bar{u}, \bar{t}\rangle\bar{D}$. The element $\bar{b}_2\bar{b}_3$ is not fused into $\bar{D}\langle\bar{b}_1\bar{b}_2, \bar{b}_3\bar{b}_4, \bar{u}, \bar{t}\rangle\cong$ Type $A_8\times$ Type A_8 so that $(\bar{H}: \bar{O}^2(\bar{H}))=8$. Since the direct factors of a Sylow 2-subgroup of $O^2(\bar{H})$ are strongly closed by Theorem 6.1, [3, 4] yields $O^2(\bar{H})\cong A_8\times A_8$ or $A_9\times A_9$. It follows that $\bar{H}\cong S_8\int Z_2$ or $S_9\int Z_2$. The proof is complete.

We can prove Propositions 7.5 and 7.6 by the similar argument to that of Proposition 7.4. So we will omit their proofs.

PROPOSITION 7.5. *Let H be a subgroup of G with a Sylow 2-subgroup $J\langle u\rangle$. Then $O^2(H)/O(O^2(H))$ is isomorphic to one of the following groups*: (i) *a subgroup of $N_H(D)/O(N_H(D))$ or of $N_H(E)/O(N_H(E))$*, (ii) A_i, $A_i\times A_j$, $5\leqq i\leqq 15$, $j=4, 5$, (iii) $A_i\times A_k$, $i=12, 13$, $k=6, 7$, (iv) $A_i\times A_j\times A_k$, $5\leqq i\leqq 11$, $j, k=4, 5$, (v) $A_i\times A_j\times A_k\times A_m$, $i, j, k=4, 5$, $m=5, 6, 7$, (vi) $O^2(W(D_k))\times A_i$, $k=5, 7$, $i=4, 5$, (vii) $O^2(W(B_6))\times A_j$, $4\leqq j\leqq 7$, (viii) $O^2(W(D_5))\times A_i\times A_j$, $i, j=4, 5$.

PROOF. All possibilities of $N_H(D)$ and $N_H(E)$ are given in Lemma 7.2. Noting Proposition 5.5 we can apply Theorem 6.1 and [2, 3].

PROPOSITION 7.6. *Let H be a subgroup of G with a Sylow 2-subgroup $J\langle u, t\rangle$. Then $O^2(H)/O(O^2(H))$ is isomorphic to one of the following groups*: (i) *a subgroup of $N_H(D)/O(N_H(D))$ or of $N_H(E)/O(N_H(E))$*, (ii) A_i, $A_i\times A_j$, $8\leqq i\leqq 11$, $j=8, 9$, (iii) $A_i\times A_j\times A_k$, $8\leqq i\leqq 11$, $j, k=4, 5$, (iv) $O^2(W(B_4))\times A_i$, $4\leqq i\leqq 11$, (v) $O^2(W(D_5))\times A_j$, $4\leqq j\leqq 9$.

PROOF. See Lemma 7.3, Theorem 6.1 and [2, 3].

PROPOSITION 7.7. (i) *If H is a proper subgroup of G which covers $C_G(z)/O(C_G(z))$, $C_G(z_4)/O(C_G(z_4))$, $C_G(z_1z_2)/O(C_G(z_1z_2))$ or $C_G(z_1z_2z_3)/O(C_G(z_1z_2z_3))$ then H is 3-stable.* (ii) *Let H be a proper subgroup of G having T, $J\langle u\rangle$ or $J\langle u, t\rangle$ as a Sylow 2-subgroup. Then $\bar{H}=H/O(H)$ has the unique $J\langle u\rangle$-invariant subgroup $\bar{W}$ of odd order and $|\bar{W}|=1$ or 3 and $\bar{W}$ centralizes $\bar{D}\langle\bar{b}_1\bar{b}_2, \bar{b}_2\bar{b}_3, \bar{b}_3\bar{b}_4\rangle$.* (iii) *Any two maximal $J\langle u\rangle$-invariant 3-subgroup of G whose intersection is non-trivial are conjugate by an element of $C_G(J\langle u\rangle)$.*

PROOF. (i) Let x be an element of order 3 in H. Then H is 3-stable if we can find an element y conjugate to x in H such that $\langle x, y\rangle$ is not a 3-group and involves no $SL(2, 3)$. By our assumption T, $J\langle u\rangle$ or $J\langle u, t\rangle$ is a Sylow 2-subgroup of H. The possible structures of $O^2(H)/O(O^2(H))$ are given in Propositions 7.4, 7.5 and 7.6. In each case we can easily verify that H is 3-stable. (ii) The result follows from Propositions 7.4, 7.5 and 7.6. (iii) See [4; Proposition 8.6].

8. Existence of D-signalizer functors.

In this section we construct D-signalizer functors. Our arguments are entirely the same as those in [4] and we omit the details.

Put $I=I_1=C_G(z)$, $I_2=C_G(z_4)$, $I_3=C_G(z_1z_2z_3)$ and $I_4=C_G(z_3z_4)$. Then $T\in \mathrm{Syl}_2(I_1)$, $J\langle u\rangle\in \mathrm{Syl}_2(I_2)\cap \mathrm{Syl}_2(I_3)$ and $J\langle u, t\rangle\in \mathrm{Syl}_2(I_4)$. It is $T\supseteqq J\langle u, t\rangle\supseteqq J\langle u\rangle\supseteqq J$ and $Z(J\langle u\rangle)=\langle z_1z_2, z_3, z_4\rangle$.

LEMMA 8.1. *If* $I_2/O(I_2)\cong C_{A_{18}}((1,2)(3,4))$, *then* O *is a D-signalizer functor.*

PROOF. The result follows from Lemmas 5.5, 5.6, 5.7 and 5.8.

In the balance of this section assume $I_2/O(I_2)\cong C_{A_{19}}((1,2)(3,4))$. It follows from Proposition 5.5 that the order of $O(I_1)$ is divisible by 3.

LEMMA 8.2. *If the order of* $O(I_2)$ *is divisible by* 3, *then the order of* $O(I_j)$ $(j=3,4)$ *is also divisible by* 3.

PROOF. As $\langle z_3, a_3\rangle\langle x_3\rangle\subseteqq I_2$, $|C_G(z_3)\cap O(I_2)|$ is divisible by 3. As $C_G(z_3)\cap O(I_2)\subseteqq I_4$, $O(I_2)\cap C_G(z_3)\subseteqq O(I_4)$ by Lemma 5.7. It follows that the order of $O(I_4)$ is divisible by 3. Lemma 5.8 yields $O(I_3)\supseteqq O(I_4)\cap C_G(z_2)$. The lemma is proved.

LEMMA 8.3. *If* $O(I_2)$ *is a* 3'-*group, then* $|O(I_1)|$ *is divisible by* 3 *only to the first power and* $O(I_j)$ $(j=3,4)$ *is also a* 3'-*group. In particular* $O^3(O(I_1))=O_{3'}(O(I_1))$ *and* $(O(I_1):O_{3'}(O(I_1)))=3$.

PROOF. Let $R\in \mathrm{Syl}_3(O(I_1))$ such that $[S, R]\subseteqq R$ and $|R|\geqq 3^2$. Put $\bar{K}=N_{I_1}(R)/\langle z\rangle$. Then $\langle \bar{z}_1\bar{z}_2, \bar{z}_2\bar{z}_3\rangle\lhd\bar{T}$ and $\xi:\bar{z}_1\bar{z}_2\to\bar{z}_2\bar{z}_3\to\bar{z}_1\bar{z}_3$. Now we can apply directly the proof of [4; Proposition 10.2].

LEMMA 8.4. *If* $O(I_2)$ *is a* 3'-*group and for each* $x\in D^\#$ *we set* $\theta(C_G(x))=O^3(O(C_G(x)))$, *then* θ *is a D-signalizer functor.*

PROOF. See [4; Proposition 10.3].

Assume that the order of $O(I_2)$ is divisible by 3. Then a 3-local subgroup K_i of G is said to be a covering group of i-th kind if K_i satisfies the conditions: i) $K_i/O(K_i)\cong A_{19}$ and ii) K_i covers $I_i/O_{3'}(O(I_i))$.

LEMMA 8.5. *If* $O(I_2)$ *is not a* 3'-*group, then* G *possesses covering* 3-*local subgroups of all kinds.*

PROOF. By Propositions 7.7–7.9, the proof of [4; Proposition 9.3] can be applied directly for our case.

Let K_i be a covering 3-local subgroup of i-th kind. Then we may assume $T\subseteqq K_1$, $J\langle u\rangle\subseteqq K_2\cap K_3$ and $J\langle u, t\rangle\subseteqq K_4$.

LEMMA 8.6. *If* $O(I_2)$ *is not a* 3'-*group and for* $x\in D^\#$ *we set* $\theta(C_G(x))=O_{3'}(O(C_G(x)))(O(K_1)\cap C_G(x))$ *or* $O(C_G(x))$ *according as* $x\sim z$ *or* $x\not\sim z$, *then* θ *is a D-signalizer functor.*

PROOF. See [4; Proposition 10.4].

9. Proof of the theorem.

We follow the notation in section 3 and assume $O(G)=1$. For a subgroup $B\subseteqq D$ we put $W_B=\langle\theta(C_G(x))|x\in B^{\#}\rangle$. Then $W_B=W_D$ provided $m(B)\geqq 2$.

LEMMA 9.1. *$\theta(C_G(x))=1$ for each $x\in D^{\#}$.*

PROOF. As $N_G(D)=(N_G(D)\cap K_1)C_G(D)$, $N_G(D)\subseteqq N_G(W_D)$. As $z_4\sim a_4\sim a_4z_4$, $N_G(\langle a_4, z_4\rangle)\subseteqq N_G(W_D)$. It is $O(I_j)\subseteqq N_G(W_D)$ for $j=2, 3, 4$ and $I_2\subseteqq N_G(W_D)$ by Lemma 5.6. $O(I_1)=(C_G(z_4)\cap O(I_1))(C_G(a_4)\cap O(I_1))(C_G(a_4z_4)\cap O(I_1))$ yields $O(I_1)\subseteqq N_G(W_D)$. It follows from Proposition 7.4 that $N_G(W_D)/O(N_G(W_D))\cong A_{18}$ or A_{19}. Now $I_1\subseteqq N_G(W_D)$ and $N_G(W_D)$ is a strongly embedded subgroup. If follows that $W_D=1$. The lemma is proved.

LEMMA 9.2. (i) *If $I_2/O(I_2)\cong C_{A_{18}}((1, 2)(3, 4))$, then $G\cong A_{18}$.* (ii) *If $I_2/O(I_2)\cong C_{A_{19}}((1, 2)(3, 4))$, then $G\cong A_{19}$.*

PROOF. (i) The result follows from [10] and Lemma 9.1. (ii) Since $I_1=O(I_1)(N_G(E)\cap I_1)$ and $O(I_1)\cap N_G(E)=1$, Sylow 3-subgroups of I_1 are elementary abelian of order 3^3. Hence by Proposition 5.5 and [10] $G\cong A_{19}$. The lemma is proved.

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